Limits of renewal processes and Pitman-Yor distribution

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Abstract

We consider a renewal process with regularly varying stationary and weakly dependent steps, and prove that the steps made before a given time t, satisfy an interesting invariance principle. Namely, together with the age of the renewal process at time t, they converge after scaling to the Pitman–Yor distribution. We further discuss how our results extend the classical Dynkin–Lamperti theorem.

1 Introduction

By one of the main results in renewal theory, it is known that the age of a renewal process has a limiting distribution, given that its steps have finite mean. When the steps are iid and regularly varying with infinite mean, the limiting distribution is determined by the Dynkin–Lamperti theorem. In this article, we aim to understand the limiting behavior of the whole path of such a renewal process before a given time t. Moreover, we do that under milder conditions, that is, we keep the regular variation assumption, but allow certain degree of dependence between the steps of the renewal process. In the rest of the article, we assume that the steps form a stationary sequence (Y_n) of nonnegative random variables which are regularly varying with index $\alpha \in (0, 1)$. By one characterization of regular variation, see Resnick [19, section 1.2], this means that there exists a sequence of nonnegative real numbers (d_n) such that

$$nP(Y \in d_n \, \cdot \,) \xrightarrow{v} \mu(\,\cdot\,), \tag{1.1}$$

as $n \to \infty$, where $\xrightarrow{v}$ denotes vague convergence of measures on $(0, \infty)$ and the limiting measure satisfies $\mu(x, \infty) = x^{-\alpha}$ for all $x > 0$.

It will be useful in the sequel to extend (d_n) to a function on $[0, \infty)$ by denoting $d(t) = d_{\lfloor t \rfloor}$, for $t \geq 0$, with $d_0 = 1$. It is known that d has an asymptotic inverse, $\widetilde{d}$ say, see Seneta [20, p. 21], in the sense that

$$d(\widetilde{d}(t)) \sim \widetilde{d}(d(t)) \sim t, \tag{1.2}$$

as $t \to \infty$. One can show that $\widetilde{d}$ is a regularly varying function with index α.

Denote by

$$\tau(t) = \inf\{k : Y_1 + \cdots + Y_k > t\}, \quad \text{for } t \geq 0, \tag{1.3}$$

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the first passage time of the level t by the random walk with steps (Y_n). Our main goal is to describe the asymptotics of all the steps in the renewal process before the passage time $\tau(t)$, i.e. of the random variables

$$\frac{Y_i}{t}, \quad i = 1, \ldots, \tau(t) - 1, \tag{1.4}$$

including the age of the renewal process at the passage time, that is

$$A^{(t)} = \frac{t - \sum_{i < \tau(t)} Y_i}{t}. \tag{1.5}$$

For iid steps (Y_n), the proof of the following classical theorem can be found in Bingham et al. [6, section 8.6.2].

Theorem 1.1 (Dynkin–Lamperti). *Suppose that (Y_n) is iid sequence of random variables, then (Y_n) satisfies (1.1) with the tail index $\alpha \in (0, 1)$ if and only if*

$$A^{(t)} \xrightarrow{d} A,$$

as $t \to \infty$, where the random variable on the right hand side has a generalized arcsine distribution with the density

$$q_\alpha(u) = \frac{\sin \pi \alpha}{\pi} u^{-\alpha} (1 - u)^{\alpha - 1}, \quad u \in [0, 1]. \tag{1.6}$$

It turns out that the direct implication of this theorem holds for certain dependent renewal processes too. More importantly, in all such cases one can describe the joint asymptotic behavior of the random variables in (1.4) and (1.5), and show that they, when ordered, form a sequence which converges towards the so-called Pitman–Yor distribution. As far as we know, this result is new even in the iid case.

The paper is organized as follows: in Section 2 we consider Pitman–Yor distribution on the interval partitions from the perspective of point processes theory. We further present two limiting theorems about stationary strongly mixing sequences (Y_n) satisfying (1.1) which are likely to be of independent interest. These theorems are used in Section 3 to determine the asymptotic distribution of the steps Y_i/t, $i = 1, \ldots, \tau(t) - 1$ and the age of the renewal process $A^{(t)}$. We also exhibit how this result extends the classical Dynkin–Lamperti theorem and discuss corresponding assumptions. This immediately yields the joint asymptotic distribution for the ranked lengths of excursions in a simple symmetric random walk, cf. Csáki and Hu [7]. More technical proofs and results concerning Skorohod's topology and convergence of point measures are postponed to the Appendix.

2   Point processes and Pitman-Yor distribution

In a remarkable series of papers: [16], [15], [14]; Perman, Pitman and Yor describe the distribution of jumps of stable subordinators on a given time interval. In particular, Pitman and Yor in [16], use such jumps to introduce a new family of distributions on interval partitions and relate them to the classical arcsine laws for Brownian motion. Recall that a stable subordinator $(S(t))_{t \geq 0}$ is a Lévy process with the Laplace transform given by the formula

$$Ee^{-\lambda S(t)} = \exp \left[-t \int_0^\infty \left(1 - e^{-\lambda x} \right) \mu'(dx) \right], \quad \lambda \geq 0,$$

with the Lévy measure

$$\mu'(dx) = c_\alpha x^{-\alpha - 1} dx,$$

for some $\alpha \in (0,1)$ and a constant $c_\alpha > 0$ which turns out to be unimportant in the sequel. So without loss of generality we typically assume $c_\alpha = 1$, i.e. $\mu' = \mu$. The subordinator $(S(t))$ has the distribution of the inverse local times of d–dimensional Bessel process, for $d = 2(1-\alpha)$, with the case $\alpha = 1/2$ corresponding to the Brownian motion. In other words, jumps of the process $(S(t))$ correspond to the lengths of excursions of the Brownian motion or, more generally Bessel process, away form the origin. By Ito's representation $(S(t))$ can be constructed from a Poisson process N on the space $[0,\infty) \times (0,\infty]$ with intensity measure equal to $\mathrm{Leb} \times \mu'$, so that

$$N = \sum_i \delta_{T_i, P_i} \ \text{ and } \ S(t) = \sum_{T_i \leq t} P_i, \quad \text{for } t \in [0,\infty). \tag{2.1}$$

We alternatively say that N is a Poisson random measure and denote this by $N \sim \mathrm{PRM}(\mathrm{Leb} \times \mu')$. By the construction, $(S(t))$ is a nondecreasing element of the space of càdlàg functions $D[0,\infty)$. Denote by $z^{\leftarrow}$ the right continuous generalized inverse of a function $z \in D[0,\infty)$, i.e.

$$z^{\leftarrow}(u) = \inf\{s \in [0,\infty) : z(s) > u\}, \quad u \geq 0.$$

The generalized inverse of the process $S(t)$ is the process

$$L(s) = S^{\leftarrow}(s) = \inf\{t : S(t) > s\}, \quad s \geq 0.$$

It is well defined and continuous at any $s \geq 0$, and for the reasons explained above it is called local time process by Bertoin in [4, p. 22]. If we denote by $\mathcal{Z}$ the closure of the range of the process $(S(t))$, the maximal open subintervals in the set $\mathcal{Z}^c \cap (0,s)$, $s > 0$, correspond to the jumps of the subordinator before it crosses over level s. Their lengths are $(S(t) - S(t-))$, $t < L(s)$, which are equal to

$$P_i, \quad T_i < L(s),$$

above, together with the last incomplete jump which has the length

$$A_s = s - \sum_{T_i < L(s)} P_i, \tag{2.2}$$

see Bertoin [4]. Considering these points in descending order we arrive at the sequence

$$\boldsymbol{V}(s) = (V_1(s), V_2(s), V_3(s), \ldots).$$

Observe that the distribution of $\boldsymbol{V}(s)$ corresponds to the distribution of the point process

$$\sum_i \delta_{V_i(s)} = \delta_{A_s} + \sum_{T_i < L(s)} \delta_{P_i}.$$

Clearly, normalization of the infinite sequence $\boldsymbol{V}(s)$ by $s > 0$, produces a random sequence which sums up to one. An extraordinary observation of Pitman and Yor [16], Theorem 1.1 was that

$$\frac{\boldsymbol{V}(s)}{s} \overset{d}{=} \frac{\boldsymbol{V}(S(t))}{S(t)}, \quad \text{for all } s, t \in (0,\infty). \tag{2.3}$$

This is surprising, since the sequence on the right hand side has no special "last interval" as in (2.2). Namely, since S is right continuous and increasing at t, $A_{S(t)} = 0$ with probability one, i.e. there is no incomplete jump at level $S(t)$. In particular, $\boldsymbol{V}(S(t))/S(t)$ is produced by ordering and scaling the points

$$P_i, \quad T_i \leq t.$$

Due to the identity (2.3), it suffices to describe the distribution of $\boldsymbol{V}(1)$, thus we denote

$$\boldsymbol{V}(1) = (V_1(1), V_2(1), V_3(1), \ldots) =: (D_1, D_2, D_3, \ldots) .$$

The distribution of this sequence corresponds to the distribution of the point process

$$M^{(\alpha)} = \sum_{i=1}^{\infty} \delta_{D_i} .$$

Recall that for a sequence of strictly positive numbers (x_n) such that $s = \sum_{i=1}^{\infty} x_i < \infty$, its size–biased permutation is simply a random reordering of the same sequence, say (x_{σ_n}), where $P(\sigma_1 = i) = x_i/s$, and for distinct integers $i_1, \ldots, i_k > 0$

$$P\left(\sigma_k = i_k \mid \sigma_1 = i_1, \ldots, \sigma_{k-1} = i_{k-1}\right) = \frac{x_{i_k}}{s - \left(x_{i_1} + \cdots + x_{i_{k-1}}\right)} .$$

It turns out to be easier to describe the distribution of the size–biased permutation of the sequence $\boldsymbol{V}(1)$, say

$$(U_1, U_2, U_3, \ldots) ,$$

although clearly

$$\sum_i \delta_{U_i} = \sum_i \delta_{D_i} = M^{(\alpha)} .$$

Perman [13] proved that

$$U_i = \xi_i \prod_{j=1}^{i-1} (1 - \xi_j), \quad j = 1, 2, 3, \ldots$$

for a sequence of independent random variables $\xi_i = 1, 2, 3, \ldots$, such that $\xi_i \sim \text{Beta}(1 - \alpha, i\alpha)$. We call the distribution of the sequence $\boldsymbol{V}(1)$, or equivalently of the point process $M^{(\alpha)}$, the Pitman–Yor distribution with parameter α. This distribution has further natural extension to two parameter family of distributions on the interval partitions, see Pitman and Yor [17]. That family found important applications in nonparametric Bayesian statistics, e.g. see Teh and Jordan [21] and references therein. Observe also, that the arcsine laws for the fraction of time Brownian motion spends in the upper halfplane at a fixed time t or at inverse local time $L(s)$, can be seen as corollaries of the results in [16].

In the course of showing (2.3), Pitman and Yor showed that U_1 above actually has the distribution of the final interval length A_1 from (2.2). This distribution is the same as the generalized arcsine distribution of the random variable A in Theorem 1.1. These results allowed Perman [14] to describe the density of $\sup\{P_i : T_i < L(1)\}$, which corresponds to the longest excursion of the d–dimensional Bessel process completed by the time 1, and of $D_1 = \max\{\sup\{P_i : T_i < L(1)\}, A_1\}$, which has the same interpretation but includes the last a.s. incomplete excursion.

It is well known that iid sequence (Y_n) satisfies (1.1), if and only if the following convergence of point processes holds

$$N_t := \sum_{i \geq 1} \delta_{\left(\frac{i}{\bar{d}(t)}, \frac{Y_i}{t}\right)} \xrightarrow{d} N = \sum_{i \geq 1} \delta_{(T_i, P_i)} , \tag{2.4}$$

where N denotes a Poisson process on the space $[0, \infty) \times (0, \infty]$ with intensity measure $\text{Leb} \times \mu$, see Resnick [19], Proposition 3.21. The convergence of point processes in (2.4) and throughout is to be understood with respect to the vague topology on the space of Radon point measures on $[0, \infty) \times (0, \infty]$, denoted by $M_p = M_p([0, \infty) \times (0, \infty])$.

If (1.1) and (2.4) hold for a general stationary sequence (Y_n), then it necessarily has the extremal index equal to 1, see Leadbetter et al. [11, p. 67] for instance. In other words, the partial maxima in the sequence (Y_n) behave as if the sequence was iid, i.e. $M_n = \max\{Y_1, \ldots, Y_n\}$ satisfies $M_n/d_n \to \Phi_\alpha$, as $n \to \infty$ where Φ_α denotes the standard Fréchet distribution, i.e. $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, $x > 0$. Next theorem, proved in the Appendix, claims that the opposite is also true. Namely, strongly mixing sequences which satisfy (1.1) and have extremal index equal to 1, necessarily satisfy (2.4). Observe that the theorem holds for all $\alpha > 0$, and not merely on the interval $(0,1)$ which is of our main interest in this paper.

Theorem 2.1. *Suppose that (Y_n) is a stationary strongly mixing sequence of nonnegative regularly varying random variables with tail index $\alpha > 0$. Then*

$$N_t \xrightarrow{d} N,$$

as $t \to \infty$, where N is $\mathrm{PRM}(\mathrm{Leb} \times \mu)$ if and only if (Y_n) has extremal index equal to 1.

For iid steps, (2.4) and a subtle adaption of continuous mapping argument, see the proof of Theorem 2.2 below, yield

$$S_t(\,\cdot\,) = \sum_{i=1}^{\lfloor \widetilde{d}(t)\cdot \rfloor} \frac{Y_i}{t} \xrightarrow{d} S(\,\cdot\,) = \sum_{T_i \leq \cdot} P_i, \tag{2.5}$$

in $D[0, \infty)$ with respect to Skorohod's J_1 metric, see Resnick [18], Chapter 7, cf. Theorem 2.2 below. Moreover, for $\alpha \in (0,1)$

$$S(t) = \sum_{T_i \leq t} P_i = \int_{[0,t] \times (0,\infty]} y N(dt, dy),$$

takes finite value with probability 1 for all $t > 0$. Observe that

$$S_t^{\leftarrow}(1) = \inf\left\{ s : \sum_{i=1}^{\lfloor \widetilde{d}(t)s \rfloor} Y_i > t \right\} = \frac{\inf\left\{ k \in \mathbb{N} : \sum_{i=1}^{k} Y_i > t \right\}}{\widetilde{d}(t)} = \frac{\tau(t)}{\widetilde{d}(t)}.$$

Denote $L^{(t)} = S_t^{\leftarrow}(1) = \tau(t)/\widetilde{d}(t)$ and recall $L(u) = S^{\leftarrow}(u) = \inf\{t : S(t) > u\}$. For simplicity denote $L = L(1)$. One can again adapt the continuous mapping argument, see theorem below, to show that (2.5) implies

$$L^{(t)} = \frac{\tau(t)}{\widetilde{d}(t)} = S_t^{\leftarrow}(1) \xrightarrow{d} S^{\leftarrow}(1) = L.$$

Moreover, in the following theorem we show that this convergence holds jointly with the convergence in (2.5), whenever (2.4) holds.

Theorem 2.2. *Suppose that (Y_n) is a stationary strongly mixing sequence of nonnegative regularly varying random variables with extremal index equal to one and the tail index $\alpha \in (0,1)$. Then, as $t \to \infty$*

$$(N_t, S_t, L^{(t)}) \xrightarrow{d} (N, S, L), \tag{2.6}$$

in the product space $M_p \times D[0, \infty) \times \mathbb{R}$ and the corresponding product topology (of vague, J_1 and Euclidean topologies).

Proof. By Theorem 2.1

$$N_t \xrightarrow{d} N,$$

as $t \to \infty$. We will first prove the convergence of the other two components in (2.6) by an application of the continuous mapping argument. Since all the components are obtained by a transformation of the point process N_t, one can easily see that the convergence is joint.

The proof of $S_t \xrightarrow{d} S$ in J_1 topology is standard. One could first observe that the functional $\psi^{\varepsilon,t} : M_p \to D[0,t]$, given by

$$\psi^{\varepsilon,t}(s) = \sum_{t_i \leq s} x_i \mathbb{I}_{\{x_i > \varepsilon\}} \, , \quad \text{for} \quad m = \sum_i \delta_{t_i, x_i} \text{ and } s \in [0,t] \, ,$$

is a.s. continuous with respect to the distribution of the limiting point process N and chosen topologies. Then one can simply follow the lines of the proof of Theorem 7.1 in Resnick [18] to let $\varepsilon \to 0$, and finally apply lemma 16.3 in Billingsley [5] to extend the convergence from $D[0,t]$ to $D[0,\infty]$.

By the continuous mapping argument, see Lemma 4.1 in the Appendix, it follows that

$$L^{(t)} = S_t^{\leftarrow}(1) \xrightarrow{d} S^{\leftarrow}(1) = L \, .$$

$\square$

The random variable L in (2.6) represents the first passage time of the level one by the α–stable subordinator S. Its distribution is known in the literature as a Mittag–Leffler distribution.

3   Main theorem

Our main result extends the sufficiency part of the Dynkin–Lamperti theorem in a couple of ways. We first show that one can describe the limiting distribution of not merely the age of the renewal process at time t, but also the behavior of all other large steps before that time. By doing that, we obtain the Pitman–Yor distribution as the limiting distribution for the steps after appropriate normalization. We also show that the statement of Dynkin–Lamperti theorem about iid regularly varying random variables can be generalized to cover all regularly varying sequences with non–clustering extremes considered in the previous section. For simplicity, denote

$$A^{(t)} = 1 - S_t \left(L^{(t)} - \right) = \frac{t - \sum_{i < \tau(t)} Y_i}{t}$$

and $A = A_1 = 1 - S(L-)$.

Theorem 3.1. *Suppose that (Y_n) is a stationary strongly mixing sequence of nonnegative regularly varying random variables with extremal index equal to one and the tail index $\alpha \in (0,1)$. Then, as $t \to \infty$,*

$$\delta_{A^{(t)}} + \sum_{i < \tau(t)} \delta_{Y_i/t} \xrightarrow{d} M^{(\alpha)} \, , \tag{3.1}$$

where $M^{(\alpha)}$ represents a Pitman-Yor point process with parameter α. Moreover, the convergence above is joint with

$$A^{(t)} \xrightarrow{d} A \, ,$$

as $t \to \infty$, where A has the generalized arcsine distribution given in (1.6).

For a measure ν on a measurable space $(\$, \mathcal{S})$, by $\nu|_B$ we denote the restriction of the measure ν on the set $B \in \mathcal{S}$ given by $\nu|_B(C) = \nu(B \cap C)$, $C \in \mathcal{S}$. Abusing this

notation somewhat, for any time period $A \subseteq [0, \infty)$ and an arbitrary point measure $n \in M_p([0, \infty) \times (0, \infty])$, we write

$$n\Big|_A \text{ for } n\Big|_{A \times (0, \infty]} . \tag{3.2}$$

Proof. By Theorem 2.2, $(N_t, S_t) \xrightarrow{d} (N, S)$ in the appropriate product topology, as $t \to \infty$. Moreover, the limit (N, S) a.s. satisfies the regularity assumption of Lemma 4.1 and Theorem 4.2 below. Therefore, this convergence is joint with the convergence in

$$A^{(t)} \xrightarrow{d} A .$$

By Theorem 4.2, as $t \to \infty$,

$$N_t\Big|_{[0, L^{(t)})} = \sum_{i < \tau(t)} \delta_{\left(\frac{i}{d(t)}, \frac{Y_i}{t} \right)} \xrightarrow{d} N\Big|_{[0, L)} = \sum_{T_i < L} \delta_{(T_i, P_i)} .$$

In particular for $f \in C_K^+((0, \infty])$, where C_K^+ denotes the family of nonnegative continuous functions with compact support

$$E\left[\exp\left\{ -\sum_{i < \tau(t)} f(Y_i/t) \right\} \right] \to E\left[\exp\left\{ -\sum_{T_i < L} f(P_i) \right\} \right] ,$$

as $t \to \infty$. Since the corresponding Laplace functionals converge, we conclude that

$$\sum_{i < \tau(t)} \delta_{Y_i/t} \xrightarrow{d} \sum_{T_i < L} \delta_{P_i} .$$

Because, this holds jointly with $A^{(t)} \xrightarrow{d} A$. We conclude that

$$\delta_{A^{(t)}} + \sum_{i < \tau(t)} \delta_{Y_i/t} \xrightarrow{d} \delta_A + \sum_{T_i < L} \delta_{P_i} = M^{(\alpha)} .$$

$\square$

Remark 3.2. The strong mixing assumption in the theorem is actually unnecessarily strong, one could alternatively consider any stationary sequence (Y_n) which satisfies (2.4). In the extreme value theory it is known that this holds under milder conditions cf. Basrak et al. [1].

Remark 3.3. An interesting implication of Theorem 3.1 concerns the lengths of excursions of the simple symmetric random walk during the first n steps. They are known to be independent and regularly varying with index $\alpha = 1/2$. Therefore, theorem can be applied to deduce and extend results in Csáki and Hu [7] about the asymptotic distribution of these excursions, cf. Basrak and Špoljarić [3].

The value $A^{(t)}$ in Theorem 3.1 is called the undershoot or the age of the renewal process at time t. Similarly, one could define the overshoot at t as

$$B^{(t)} = S_t\left(L^{(t)} \right) - 1 = \frac{\sum_{i \leq \tau(t)} Y_i - t}{t} .$$

Recall that $L^{(t)}$ represents the scaled first passage time. Recall also that $A = A_1$ represents the undershoot of the stable subordinator S at the first passage time $L = S^{\leftarrow}(1)$ from (2.2). Denote by $B = B_1 = S(L) - 1$ the corresponding overshoot. The joint density of the random vector (A, B) can be found in Theorem 5.16 of Kyprianou [10] or

in Theorem 8.6.3 Bingham et al. [6]. The random variable L, on the other hand, has the Mittag–Leffler distribution as in Theorem 2.2. The joint distribution of all three random variables A, B and L is more involved, see Kyprianou [10, chapters 5 and 7].

Straightforward application of Theorem 2.2 and Lemma 4.1 yields the following corollary which should be compared with Dynkin–Lamperti theorem, see Theorem 8.6.3 of Bingham et al. [6]. Note however that the corollary admits weak dependence between the steps of the renewal process.

Corollary 3.4. *Under the assumptions of Theorem 3.1, the convergence in* (3.1) *is joint with*

$$\left(A^{(t)}, B^{(t)}, L^{(t)}\right) \xrightarrow{d} (A, B, L) \,,$$

as $t \to \infty$.

4 Appendix

Proof. (of Theorem 2.1) As we explained before the theorem, it remains to show sufficiency. Assume that (Y_n) is strongly mixing with the extremal index equal to 1.

Denote by $\alpha(n)$ the mixing coefficients of the sequence (Y_n). Observe that they are in general unrelated to the index of regular variation α. Then set $l_n = \lfloor \max\{1, n^{0.1}\} \rfloor$, clearly $l_n = o(n)$, $l_n \to \infty$. Introduce also the sequence $r_n = \lfloor \max\{1, n\sqrt{\alpha(l_n)}, n^{2/3}\} \rfloor$, and observe $r_n \to \infty$. Consider an arbitrary function $f \in C_K^+([0, \infty) \times (0, \infty])$, where C_K^+ denotes the family of continuous nonnegative functions with relatively compact support. Without loss of generality, suppose that the support of f lies in $[0, M] \times [l, \infty]$ for some constants $M, l > 0$. By Proposition 1.34 in Krizmanić [9], the sequence (r_n) satisfies the following condition for every such f

$$E\left[\exp\left\{-\sum_{i=1}^{\infty} f\left(\frac{i}{n}, d_n^{-1} Y_i\right)\right\}\right] - \prod_{k=1}^{\lfloor Mn/r_n \rfloor} E\left[\exp\left\{-\sum_{i=1}^{r_n} f\left(\frac{kr_n}{n}, d_n^{-1} Y_i\right)\right\}\right] \to 0. \quad (4.1)$$

as $n \to \infty$, which corresponds to the condition $\mathcal{A}'(a_n)$ introduced in Basrak et al. [1].

Observe that

$$\lim_{n\to\infty} (P(Y \le d_n u))^n = \lim_{n\to\infty} \left(1 - \frac{nP(Y > d_n u)}{n}\right)^n = e^{-u^{-\alpha}} > 0, \quad (4.2)$$

for any $u > 0$. Note that the sequences (l_n) and (r_n) satisfy $r_n = o(n)$, $n\alpha(l_n) = o(r_n)$, $l_n = o(r_n)$. According to Theorem 2.1, in O'Brien [12], the extremal index θ of the sequence (Y_n) satisfies

$$\theta = \lim_{n\to\infty} P(M_{r_n} \le d_n u \mid Y_0 > d_n u)$$

for any fixed $u > 0$. Since, by assumption, $\theta = 1$, we obtain

$$P(M_{r_n} > d_n u \mid Y_0 > d_n u) = P\left(\max_{1 \le i \le r_n} Y_i > d_n u \,\middle|\, Y_0 > d_n u\right) \to 0 \,,$$

as $n \to \infty$. Hence, by stationarity, for every $u > 0$,

$$\limsup_{n\to\infty} P\left(\max_{1 \le |i| \le r_n} Y_i > d_n u \,\middle|\, Y_0 > d_n u\right) = 0. \quad (4.3)$$

Consequently (Y_n) is jointly regularly varying in the sense of Basrak and Segers [2]. Moreover, its tail sequence is trivial.

We observe next that by (4.1) and (4.3), the point processes

$$\widetilde{N}_n := \sum_{i \geq 1} \delta_{\left(\frac{i}{n}, \frac{Y_i}{d_n}\right)} \tag{4.4}$$

in M_p satisfy the assumptions of Theorem 2.3 in [1], adjusting the state space from $[0,1] \times (0,\infty]$ used there, to the case $[0,M] \times (0,\infty]$ we need here. This means, in particular, that for all $M > 0$, $u > 0$

$$\widetilde{N}_n \bigg|_{[0,M] \times (u,\infty]} \xrightarrow{d} N^{(u)} = \sum_i \delta_{(T_i^{(u)}, uZ_i)} \bigg|_{[0,M] \times (u,\infty]},$$

as $n \to \infty$, where

$$\sum_i \delta_{T_i^{(u)}, Z_i}$$

is a homogeneous Poisson process on $[0,\infty) \times (1,\infty]$ with intensity $u^{-\alpha}\mathrm{Leb} \times \widetilde{\mu}$, where $\widetilde{\mu}$ denotes the probability measure obtained by restricting measure μ to the interval $(1,\infty]$. However, if we denote by $N = \sum_i \delta_{T_i,Z_i}$ a Poisson process on $[0,\infty) \times (0,\infty]$ with intensity $\mathrm{Leb} \times \mu$, then for all $M > 0$, $u > 0$

$$N^{(u)} \stackrel{d}{=} N \bigg|_{[0,M] \times (u,\infty]}.$$

In particular,

$$\widetilde{N}_n \xrightarrow{d} N.$$

Consider the mapping from $M_p \times (0,\infty)^2$ to M_p, given by

$$(m, a, b) \mapsto m^{a,b}, \tag{4.5}$$

where

$$m^{a,b}(I \times J) = m(I/a \times J/b),$$

for all measurable sets I, J. Vague convergence theory as presented in section 3.4 of Resnick [19], shows that this mapping is continuous. This turns to be useful, since

$$N_t(I, J) = \widetilde{N}_{\lfloor \widetilde{d}(t) \rfloor} \left(\frac{\lfloor \widetilde{d}(t) \rfloor}{\widetilde{d}(t)} I \times \frac{d(\widetilde{d}(t))}{t} J \right). \tag{4.6}$$

Observe that

$$\widetilde{N}_{\lfloor \widetilde{d}(t) \rfloor} \xrightarrow{d} N \ \text{ and } \ \left(\frac{\lfloor \widetilde{d}(t) \rfloor}{\widetilde{d}(t)}, \frac{d(\widetilde{d}(t))}{t} \right) \to (1,1)$$

as $t \to \infty$. Hence, from (4.6) and (1.2), one can conclude

$$N_t \xrightarrow{d} N,$$

as $t \to \infty$ as well.

$\square$

Denote by $n, n_t,\ t > 0$, arbitrary Radon point measures in $M_p([0,\infty) \times (0,\infty])$. One can always write

$$n_t = \sum_i \delta_{v_i^t, y_i^t}, \ n = \sum_i \delta_{v_i, y_i}$$

for some sequences (v_i^t), (y_i^t), (v_i) and (y_i) of positive real numbers. Denote further the corresponding cumulative sum functions of the point measures n, n_t, $t > 0$, by

$$s_t(u) = \int_{[0,u] \times (0,\infty)} y \, n_t(dv, dy), \ u \geq 0,$$

and

$$s(u) = \int_{[0,u] \times (0,\infty)} y \, n(dv, dy), \ u \geq 0,$$

Assume that these values are finite for each $u > 0$, but tend to ∞ as $u \to \infty$. This makes s_t and s well defined, unbounded, nondecreasing elements of the space of càdlàg functions $D[0, \infty)$. Their right continuous generalized inverses (or hitting time functions) we denote by $s^{\leftarrow}$ and $s_t^{\leftarrow}$, recall that $s^{\leftarrow}(u) = \inf\{v \in [0,\infty) : s(v) > u\}$, $u \geq 0$. We will use the following abbreviations in the sequel

$$\tau_t = s_t^{\leftarrow}(1) \text{ and } \tau = s^{\leftarrow}(1).$$

It is well known that J_1 convergence

$$s_t \xrightarrow{J_1} s,$$

in general does not imply convergence of s_t towards s at a given point. However, the following technical lemma shows that under some regularity conditions, it implies the convergence of both s_t and its left limit at the first passage time τ_t. It is a consequence of Theorem 13.6.4 in Whitt [22] which has a weaker assumption that s_t converge towards s in M_2 topology.

Lemma 4.1. *Assume that*

$$s_t \xrightarrow{J_1} s, \tag{4.7}$$

and suppose that $s(w) < 1$ for each $w < \tau$. Then

$$(\tau_t, s_t(\tau_t-), s_t(\tau_t)) \to (\tau, s(\tau-), s(\tau)). \tag{4.8}$$

Theorem 4.2. *Assume that*

$$(n_t, s_t) \to (n, s), \tag{4.9}$$

in the product topology (of vague and J_1 topologies) as $t \to \infty$. Assume further that $0 < s(\tau-) < 1 < s(\tau)$ and $n(\{w\} \times (0,\infty)) \leq 1$ for all $w \geq 0$. Then

$$n_t\big|_{[0,\tau_t]} \xrightarrow{v} n\big|_{[0,\tau]} \quad \text{and} \quad n_t\big|_{[0,\tau_t)} \xrightarrow{v} n\big|_{[0,\tau)}. \tag{4.10}$$

Proof. By the definition of the vague convergence, it is enough to show the convergence in the state space $[0, \infty) \times (u, \infty]$ for some number $u > 0$ in an arbitrary neighborhood of 0. Because n is a Radon measure, one can always find $u > 0$ which is arbitrarily close to 0 and satisfies $n([0,\infty) \times \{u\}) = 0$ and $u < s(\tau) - s(\tau-)$. Since s is a càdlàg function, there exists $\varepsilon_0 > 0$, such that

$$s(\tau + \varepsilon_0) - s(\tau) < u/3. \tag{4.11}$$

Since s_t and s are monotone functions, by Theorem 2.15 a) in Jacod and Shiryayev [8], chapter VI, (4.9) implies that the set of points $G \subseteq [0,\infty)$ such that $s_t(d) \to s(d)$ for $d \in G$ is dense in $[0,\infty)$. Take $0 < \varepsilon < \varepsilon_0$, such that $n(\{\tau + \varepsilon\} \times (0,\infty)) = 0$, and such that

$$n\big|_{[0,\tau] \times (u,\infty)} = n\big|_{[0,\tau+\varepsilon] \times (u,\infty)} = \sum_{j=1}^{k} \delta_{v_j, y_j}. \tag{4.12}$$

Since G is dense and s is continuous at $\tau + \varepsilon$, one can show that $\tau + \varepsilon \in G$ as well. By Proposition 3.13 in Resnick [19], there exists a constant t_0', such that for all $t > t_0'$

$$n_t\big|_{[0,\tau+\varepsilon]\times(u,\infty)} = \sum_{j=1}^{k} \delta_{v_j^t, y_j^t}\,.$$

and

$$(v_j^t, y_j^t) \to (v_j, y_j)\,, \quad j = 1, \ldots, k\,, \tag{4.13}$$

as $t \to \infty$. Since $\tau_t \to \tau$ by Lemma 4.1, there also exist t_0'', such that for all $t > t_0'$

$$\tau_t < \tau + \varepsilon\,.$$

For $t > t_0' \vee t_0''$

$$n_t\big|_{[0,\tau+\varepsilon]\times(u,\infty)} = n_t\big|_{[0,\tau_t]\times(u,\infty)} + n_t\big|_{(\tau_t,\tau+\varepsilon]\times(u,\infty)}\,. \tag{4.14}$$

But we will show that the second point process on the right hand side above equals zero for all t large enough.

Because $s_t(\tau + \varepsilon) \to s(\tau + \varepsilon)$, $s_t(\tau_t) \to s(\tau)$, by (4.11) there exists t_0''' such that for $t > t_0'''$

$$s_t(\tau + \varepsilon) - s_t(\tau_t) < \frac{2}{3}u. \tag{4.15}$$

Now for $t > t_0' \vee t_0'' \vee t_0'''$

$$n_t\left((\tau_t, \tau+\varepsilon]\times(u,\infty)\right) = \sum_{v_i^t \in (\tau_t,\tau+\varepsilon]} \delta_{y_i^t}(u,\infty) \tag{4.16}$$

$$\leq \frac{1}{u} \sum_{v_i^t \in (\tau_t,\tau+\varepsilon]} y_i^t \delta_{y_i^t}(u,\infty) \leq \frac{1}{u}\left(s_t(\tau+\varepsilon) - s_t(\tau_t)\right) < \frac{2}{3}\,. \tag{4.17}$$

Since n_t is a point measure, $n_t\left((\tau_t, \tau+\varepsilon]\times(u,\infty)\right) = 0$ for all t large enough. So, by Proposition 3.13 in [19]

$$n_t\big|_{[0,\tau_t]} \xrightarrow{v} n\big|_{[0,\tau]}\,. \tag{4.18}$$

Suppose that the rightmost point of the point measure on the right hand side above has the coordinates

$$(\tau, j) \text{ where } j = s(\tau) - s(\tau-) > 0\,,$$

because $n(\{\tau\} \times (0,\infty)) \leq 1$. Then, by (4.13) there exist points (σ_t, j_t) of the point measures on the left hand side of (4.18) such that

$$(\sigma_t, j_t) \to (\tau, j) = (\tau, s(\tau) - s(\tau-))\,.$$

By the assumption, there exists $\varepsilon > 0$ such that $s(\tau) > 1 + \varepsilon$. Note that there also exists $d \in G$, $d < \tau$ such that

$$s(\tau-) - \frac{\varepsilon}{4} < s(d)\,.$$

For all sufficiently large t now $\sigma_t > d$, and

$$s_t(\sigma_t) \geq s_t(d) + j_t > s(d) - \frac{\varepsilon}{4} + j - \frac{\varepsilon}{4} > s(\tau-) + j - \frac{3}{4}\varepsilon > 1\,.$$

Therefore $\sigma_t = \tau_t$ for all sufficiently large t. Recall that we fixed a constant $u > 0$ such that $n([0,\infty) \times \{u\}) = 0$. Then for all large enough t

$$n_t\big|_{[0,\tau_t)\times(u,\infty]} = n_t\big|_{[0,\tau_t]\times(u,\infty]} - \delta_{\tau_t, j_t}\,.$$

Thus (4.18) together with $(\tau_t, j_t) \to (\tau, j)$, implies

$$n_t\big|_{[0,\tau_t)\times(u,\infty]} = n_t\big|_{[0,\tau_t]\times(u,\infty]} - \delta_{\tau_t,j_t} \xrightarrow{v} n\big|_{[0,\tau]\times(u,\infty]} - \delta_{\tau,j} = n\big|_{[0,\tau)\times(u,\infty]},$$

as $t \to \infty$.

$\square$

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