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Symmetric $1-$dependent colorings of the integers

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Abstract

In a recent paper, we constructed a stationary $1-$dependent $4-$coloring of the integers that is invariant under permutations of the colors. This was the first stationary finitely dependent $q-$coloring for any q. When the analogous construction is carried out for $q > 4$ colors, the resulting process is not finitely dependent. We construct here a process that is symmetric in the colors and $1-$dependent for every $q \geq 4$. The construction uses a recursion involving Chebyshev polynomials evaluated at $\sqrt{q}/2$.

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1 Introduction

By a (proper) $q-$coloring of the integers, we mean a sequence $(X_i : i \in \mathbb{Z})$ of $[q]-$valued random variables satisfying $X_i \neq X_{i+1}$ for all i (where $[q] := \{1, \ldots, q\}$). The coloring is said to be stationary if the (joint) distribution of $(X_i : i \in \mathbb{Z})$ agrees with that of $(X_{i+1} : i \in \mathbb{Z})$, and $k-$dependent if the families $(X_i : i \leq m)$ and $(X_i : i > m + k)$ are independent of each other for each m. In [4], we gave a construction of a stationary $1-$dependent $4-$coloring of the integers that is invariant under permutations of the colors. When the same construction is carried out for $q > 4$ colors, the resulting distribution is not $k-$dependent for any k. Of course, the $1-$dependent $4-$coloring is also a $1-$dependent $q-$coloring for every $q > 4$, and one may obtain other $1-$dependent $q-$colorings by splitting a color into further colors using an independent source of randomness. However, these colorings are not symmetric in the colors. We give here a modification of the process of [4] for every $q \geq 4$ that is symmetric in the colors and $1-$dependent. Here is our main result.

Theorem 1. For each integer $q \geq 4$, there exists a stationary $1-$dependent $q-$coloring of the integers that is invariant in law under permutations of the colors and under the reflection $(X_i : i \in \mathbb{Z}) \mapsto (X_{-i} : i \in \mathbb{Z})$.

This paper is part of a series ([3], [4], [5]) that began with a question raised by Benjamini, Weiss and Holroyd, and studied by Schramm in 2008. The question is: for which k, q does there exist a stationary $k-$dependent $q-$coloring of the integers. It is easy to see that there is no such $2-$coloring for any k, and Schramm proved, among other things, that there is no $1-$dependent $3-$coloring. Proofs of the latter fact are given in [4] and [5]. In [4], we constructed a $2-$dependent $3-$coloring and a $1-$dependent

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$4-$coloring of the integers. These are arguably the first genuinely natural finitely dependent processes that are not block factors, i.e., that cannot be written in the form $X_i = f(U_{i+1}, \ldots, U_{i+r})$ for some function f of r variables and an i.i.d. sequence U_i. (A coloring is finitely dependent if it is $k-$dependent for some k.) It is shown in [3] that while it is not a block factor the four coloring is a finitary factor of an i.i.d. process.

Block factors are of course examples of finitely dependent processes. Some history of finitely dependent processes that cannot be written as block factors is given in [4]. The ones constructed in the present paper provide additional examples, which hopefully will lead to a better understanding of such processes.

Part of the motivation for this paper involves the problem of constructing random colorings on other graphs. In [4], we used our $\mathbb{Z}^1$ construction to build $1-$dependent 4^d-colorings of $\mathbb{Z}^d$. These are stationary and symmetric in the d coordinates separately, but are not invariant under all automorphisms of $\mathbb{Z}^d$ if $d > 1$. Specifically, if $d = 2$, they are not invariant under reflection about 45 degree lines.

As pointed out by R. Lyons (private communication) a similar construction applies to homogeneous trees, but again, the resulting coloring is not fully automorphism invariant. A $1-$dependent coloring of the infinite $3-$regular tree requires at least 7 colors, and on the $4-$regular tree, at least 10 colors (see [4]). We have no example of a fully automorphism invariant coloring of a regular tree but any such coloring, when restricted to a copy of $\mathbb{Z}^1$, must be a coloring of $\mathbb{Z}^1$ that requires more than 4 colors. So, one can regard our present construction as being a first step toward building colorings on more general graphs.

It is natural to ask whether our colorings are unique in any sense. We do not know the answer, but will say something more about this in the final section. Unfortunately, we have no probabilistic construction of our new colorings. Hopefully, such a construction will emerge in the future.

Finitely dependent colorings cannot be written as block factors or as functions of finite state Markov chains. These statements are proved in Sections 4 and 5 of [4] respectively. The former is a consequence of an earlier result in [6], where it appears in a different form, motivated by applications in distributed computing. Further consequences and extensions appear in [2] and [5]. For example, block factors must contain arbitrarily long constant sequences with positive probability.

The fact that colorings cannot be constructed from various stochastic processes that are reasonably well understood makes it necessary to employ more difficult and less obvious techniques. Our construction is given in the next section. Sections 3 and 4 provide some preliminary results and the proof of Theorem 1 respectively. Open problems are discussed in the final section.

2 The construction

For $x = (x_1, x_2, \ldots, x_n) \in [q]^n$, we will write $\mathbb{P}(X_{i+1} = x_1, \ldots, X_{i+n} = x_n) = P(x)$ for any $i \in \mathbb{Z}$. To motivate the construction, we begin by noting that the finite-dimensional distributions P of the $4-$coloring in [4] are defined recursively by $P(\emptyset) = 1$ and

$$P(x) = \frac{1}{2(n+1)} \sum_{i=1}^{n} P(\widehat{x}_i) \tag{2.1}$$

for proper $x \in [4]^n$, where $\widehat{x}_i$ is obtained from x by deleting the ith entry in x. Of course, even if x is proper, $\widehat{x}_i$ may not be. So the definition is completed by setting $P(x) = 0$ for x's that are not proper.

For general $q \geq 4$, we will now allow the coefficients in the defining sum to depend on i as well as n. Considering many special cases, and the constraints imposed by the

1−dependence requirement, we were led to define

$$P(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(n-2i+1) P(\widehat{x}_i) \tag{2.2}$$

for proper $x \in [q]^n$, in terms of two sequences C and D. Again motivated by computations in special cases, we take

$$\begin{aligned} C(n) &= T_n\big(\sqrt{q}/2\big), && n \geq 0; \\ D(n) &= \sqrt{q}\, U_{n-1}\big(\sqrt{q}/2\big), && n \geq 1, \end{aligned}$$

where T_n and U_n are the Chebyshev polynomials of the first and second kind respectively. (For more on these computations, see the final section.)

There are several standard equivalent definitions of Chebyshev polynomials. One is

$$T_n(u) = \cosh(nt) \quad \text{and} \quad U_n(u) = \frac{\sinh[(n+1)t]}{\sinh(t)}, \quad \text{where } u = \cosh(t). \tag{2.3}$$

A variant definition using trigonometric functions (e.g. (22:3:3-4) of [7]) is easily seen to be equivalent by taking t imaginary; the hyperbolic function version is convenient for arguments $u \geq 1$. Another definition is

$$T_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} u^{n-2k}(u^2-1)^k \quad \text{and} \quad U_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} u^{n-2k}(u^2-1)^k.$$

That this is equivalent to (2.3) follows from e.g. (22:3:1-2) of [7].

If x is not a proper coloring, we take $P(x) = 0$ as before. We extend both sequences C and D to all integer arguments by taking $C(n)$ and $D(n)$ to be even and odd functions of n respectively (in accordance with (2.3)). In particular, $D(0) = 0$.

Observe that $C(n)$ and $D(n)$ are strictly positive for $q \geq 4$ and $n \geq 1$, and therefore $P(x)$ is strictly positive for all proper x. Note also that $C(n-2i+1)/D(n+1)$ is a rational function of q; therefore so is $P(x)$. (The factors of $\sqrt{q}$ cancel). When $q = 4$ we have $C(n) = 1$ and $D(n) = 2n$, and so (2.2) reduces to (2.1) in this case. As we will see, the fact that the coefficients in (2.2) depend on i substantially complicates the verifications of the required properties of P.

Here are a few examples of cylinder probabilities generated by (2.2).

$$P(1) = \frac{1}{q}, \quad P(12) = \frac{1}{q(q-1)}, \quad P(121) = \frac{1}{q^2(q-1)}, \quad P(123) = \frac{1}{q^2(q-2)},$$
$$P(1212) = \frac{q-3}{q^2(q-1)(q^2-3q+1)}, \quad P(1234) = \frac{1}{q^2(q^2-3q+1)}.$$

3 Preliminary results

Chebyshev polynomials satisfy a number of standard identities. They lead to identities satisfied by the sequences C and D. The first three in the proposition below are examples of this. The fourth is a consequence of the third one. Before stating them, we record some values of C and D to facilitate checking computations here and later.

$$C(0) = 1, \quad C(1) = \frac{\sqrt{q}}{2}, \quad C(2) = \frac{q-2}{2}, \quad C(3) = \frac{\sqrt{q}(q-3)}{2}, \quad C(4) = \frac{q^2-4q+2}{2}.$$
$$D(0) = 0, \quad D(1) = \sqrt{q}, \quad D(2) = q, \quad D(3) = \sqrt{q}(q-1), \quad D(4) = q(q-2).$$

Proposition 2. For $j, k, \ell, m, n \in \mathbb{Z}$, the following identities hold.

$$2C(m)C(n) = C(m+n) + C(n-m). \tag{3.1}$$

$$\frac{q-4}{2q} D(m)D(n) = C(m+n) - C(n-m). \tag{3.2}$$

$$2C(m)D(n) = D(m+n) + D(n-m). \tag{3.3}$$

$$C(j+k)D(k+\ell) = C(k)D(j+k+\ell) - C(\ell)D(j). \tag{3.4}$$

Proof. The first three parts are immediate consequences of (22:5:5-7) in [7], or 22.7.24-26 in [1], if m and n are nonnegative. None of the identities is changed by changing the sign of either m or n. Therefore, they hold for all m and n. Alternatively, the identities may be checked directly from (2.3) using the product formulae for hyperbolic functions. For (3.4), replace the products of C's and D's by sums of D's using (3.3), and then use the fact that D is an odd function. $\square$

Next we verify some identities that involve both the sequences C and D and the measure P defined by (2.2). For the statement of the second part of the next result, let

$$Q(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(2i) P(\widehat{x}_i) \quad \text{and} \quad Q^*(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(2n-2i+2) P(\widehat{x}_i)$$

for $x \in [q]^n$. The first part of Proposition 3 is needed in proving the second part, which plays a key role in the proof of consistency and $1-$dependence of P. Note the similarity between the left side of (3.5) and the right side of (2.2).

Proposition 3. If $n \geq 1$, and x is a proper coloring of length n, then

$$\sum_{i=1}^{n} D(n-2i+1) P(\widehat{x}_i) = 0; \tag{3.5}$$

$$Q(x) = Q^*(x) = P(x)C(n+1). \tag{3.6}$$

Proof. For the first statement, let R be the set of proper colorings, and $\widehat{x}_A$ be obtained by deleting the entries x_i for $i \in A$ from x. The proof of (3.5) is by induction on n, the length of x. The identity is easily seen to be true if $n \leq 2$. Suppose that (3.5) is true for all x of length $n-1$, and let $x \in R$ have length n. For those i with $\widehat{x}_i \in R$, applying (3.5) gives

$$\sum_{j=1}^{i-1} D(n-2j) P(\widehat{x}_{i,j}) + \sum_{j=i+1}^{n} D(n-2j+2) P(\widehat{x}_{i,j}) = 0. \tag{3.7}$$

On the other hand, if $\widehat{x}_i \notin R$, then $1 < i < n$ and

$$P(\widehat{x}_{i,j}) = 0 \text{ if } |j-i| > 1 \text{ and } P(\widehat{x}_{i-1,i}) = P(\widehat{x}_{i,i+1}). \tag{3.8}$$

The left side of (3.5) for x can be written, using the definition of $P(\widehat{x}_i)$ and then (3.3), as

$$\begin{aligned}
&= \frac{1}{D(n)} \sum_{\substack{1 \leq i \leq n: \\ \widehat{x}_i \in R}} D(n-2i+1) \left[\sum_{1 \leq j < i} C(n-2j) P(\widehat{x}_{i,j}) + \sum_{i < j \leq n} C(n-2j+2) P(\widehat{x}_{i,j}) \right] \\
&= \frac{1}{2D(n)} \sum_{\substack{1 \leq j < i \leq n: \\ \widehat{x}_i \in R}} [D(2n-2i-2j+1) + D(2j-2i+1)] P(\widehat{x}_{i,j}) \\
&\quad + \frac{1}{2D(n)} \sum_{\substack{1 \leq i < j \leq n: \\ \widehat{x}_i \in R}} [D(2n-2i-2j+3) + D(2j-2i-1)] P(\widehat{x}_{i,j}).
\end{aligned} \tag{3.9}$$

Rewriting, and ignoring the $2D(n)$ in the denominator, gives

$$\begin{aligned}\sum_{i=1}^{n}\mathbf{1}[\widehat{x}_i\in R]\Bigg[&\sum_{j=1}^{i-1}[D(2n-2i-2j+1)+D(2j-2i+1)]P(\widehat{x}_{i,j})\\&+\sum_{j=i+1}^{n}[D(2n-2i-2j+3)+D(2j-2i-1)]P(\widehat{x}_{i,j})\Bigg].\end{aligned}\tag{3.10}$$

We must show that (3.7) and (3.8) imply that (3.10) is zero.

We would like to write (3.10) as a linear combination of expressions that vanish because of (3.7) and (3.8) as follows.

$$\sum_{\substack{1\le i\le n:\\ \widehat{x}_i\in R}}\alpha_i\left[\sum_{j=1}^{i-1}D(n-2j)P(\widehat{x}_{i,j})+\sum_{j=i+1}^{n}D(n-2j+2)P(\widehat{x}_{i,j})\right]+\sum_{\substack{1\le i\le n:\\ \widehat{x}_i\notin R}}\sum_{j=1}^{n}\beta_{i,j}P(\widehat{x}_{i,j}),\tag{3.11}$$

where $\beta_{i,i}=\beta_{i,i-1}+\beta_{i,i+1}=0$. If $1\le i<j\le n$, the coefficient of $P(\widehat{x}_{i,j})$ in (3.10) is

$$\begin{aligned}&\mathbf{1}[\widehat{x}_j\in R]\left[D(2n-2i-2j+1)+D(2i-2j+1)\right]\\&+\mathbf{1}[\widehat{x}_i\in R]\left[D(2n-2i-2j+3)+D(2j-2i-1)\right].\end{aligned}\tag{3.12}$$

The coefficient of $P(\widehat{x}_{i,j})$ in (3.11) is

$$\mathbf{1}[\widehat{x}_j\in R]\alpha_jD(n-2i)+\mathbf{1}[\widehat{x}_i\in R]\alpha_iD(n-2j+2)+\mathbf{1}[\widehat{x}_i\notin R]\beta_{i,j}+\mathbf{1}[\widehat{x}_j\notin R]\beta_{j,i}.\tag{3.13}$$

We need to choose the α's and β's so that (3.12) and (3.13) agree. If $\widehat{x}_i,\widehat{x}_j\in R$, this says

$$D(2n-2i-2j+1)+D(2n-2i-2j+3)=\alpha_jD(n-2i)+\alpha_iD(n-2j+2)$$

since D is an odd function. It may sound unreasonable to expect to solve this system, since there are n unknowns and $\binom{n}{2}$ equations. However, D satisfies relations that make this possible. Solving the equations for small n suggests trying $\alpha_i=2C(n-2i+1)$. The fact that this choice solves these equations for all choices of n,i,j then follows from (3.3) and the fact that D is odd. If $\widehat{x}_i\notin R$ and $\widehat{x}_j\notin R$, (3.12) and (3.13) agree if $\beta_{i,j}+\beta_{j,i}=0$. If $\widehat{x}_i\in R$ and $\widehat{x}_j\notin R$, they agree if

$$D(2n-2i-2j+3)+D(2j-2i-1)=\alpha_iD(n-2j+2)+\beta_{j,i}.$$

Using (3.3) again gives $\beta_{j,i}=2D(2j-2i-1)$. Similarly, if $\widehat{x}_i\notin R$ and $\widehat{x}_j\in R$, they agree if $\beta_{i,j}=2D(2i-2j+1)$. With these choices, β is anti-symmetric, and $\beta_{k,k-1}=2D(1)$ and $\beta_{k,k+1}=2D(-1)$, so $\beta_{k,k-1}+\beta_{k,k+1}=0$ as required. This completes the induction argument.

For (3.6), use the definition of P to write the right side of (3.6) as

$$\frac{C(n+1)}{D(n+1)}\sum_{i=1}^{n}C(n-2i+1)P(\widehat{x}_i).$$

Using (3.1), this becomes

$$\frac{1}{2}Q(x)+\frac{1}{2}Q^*(x).$$

Therefore, we need to prove that

$$Q^*(x)-Q(x)=\frac{1}{D(n+1)}\sum_{i=1}^{n}\left[C(2n-2i+2)-C(2i)\right]P(\widehat{x}_i)=0.$$

But by (3.2), this follows from (3.5). □

4 Proof of the main result

We will often write $x_1 x_2 \cdots x_n$ instead of $(x_1, x_2, \dots, x_n)$ below. If $x \in [q]^m$ and $y \in [q]^n$, let xy denote the word $x_1 \cdots x_m y_1 \cdots y_n \in [q]^{m+n}$.

Proof of Theorem 1. We first need to show that the finite dimensional distributions defined in (2.2) are consistent, i.e., that

$$\sum_{a\in[q]} P(xa) = P(x), \qquad x \in [q]^n,\ n \geq 0. \tag{4.1}$$

This is true if x is not proper, since then xa is also not proper, and so both sides vanish. For proper x, the proof is by induction on n. Note that for $a \in [q]$,

$$P(a) = \frac{C(0)}{D(2)} = \frac{1}{q},$$

so $\sum_{a\in[q]} P(a) = 1$. This gives (4.1) for $n = 0$. Suppose it holds for all $x \in [q]^{n-1}$ with $n \geq 1$. Then for proper $x \in [q]^n$, using the induction hypothesis in the second equality,

$$\begin{aligned}\sum_{a\in[q]} P(xa) &= \sum_{a\neq x_n} \frac{1}{D(n+2)}\left[\sum_{i=1}^{n} C(n-2i+2)P(\widehat{x}_i a) + C(-n)P(x)\right]\\ &= \frac{1}{D(n+2)}\left[\sum_{i=1}^{n} C(n-2i+2)P(\widehat{x}_i) - C(-n+2)P(x) + (q-1)C(-n)P(x)\right].\end{aligned}$$

The middle term in the second line accounts for the missing term $a = x_n$ when the inductive hypothesis is applied to the case $i = n$ (since $\widehat{x}_n x_n = x$). Using $(j,k,\ell) = (1, n-2i+1, 2i)$ in (3.4) gives

$$\frac{C(n-2i+2)}{D(n+2)} = \frac{C(n-2i+1)}{D(n+1)} - \frac{C(2i)D(1)}{D(n+2)D(n+1)}.$$

Therefore

$$\sum_{a\in[q]} P(xa) = P(x) - \frac{D(1)Q(x)}{D(n+2)} - \frac{C(n-2)}{D(n+2)}P(x) + (q-1)\frac{C(n)}{D(n+2)}P(x).$$

This is $P(x)$, as required, by (3.6) and the fact that

$$(q-1)C(n) = C(n-2) + C(n+1)D(1),$$

which is obtained by taking $(j,k,\ell) = (2, -n, n+1)$ in (3.4), and then canceling a factor of $\sqrt{q}$.

Invariance of the measure under permutations of colors and translations is immediate from the definition. Invariance under reflection amounts to checking $P(x) = P(x_n \cdots x_1)$, which follows from the fact that the coefficients of $\widehat{x}_i$ and $\widehat{x}_{n-i+1}$ in (2.2), which are $C(n-2i+1)$ and $C(-n+2i-1)$ respectively, are equal by the symmetry of C.

For $1-$dependence, we need to show that for $x \in [q]^m$ and $y \in [q]^n$ with $m, n \geq 0$,

$$P(x * y) = P(x)P(y),$$

where the * means that there is no constraint at the single site between x and y. This is again true if x or y is not proper since then both sides are zero. For proper x and y, the proof is by induction, but now on $m + n$. The statement is immediate if $m = 0$ or $n = 0$. So, we take $m \geq 1$ and $n \geq 1$.

There are two cases, according to whether or not xy is a proper coloring, i.e., whether x_m and y_1 are equal or different. Assume first that $x_m = y_1$. Without loss of generality, take their common value to be 1. Then using the definition of P, including the fact that $P(xy) = 0$,

$$\begin{aligned} P(x*y) = \sum_{a\in[q]} P(xay) = \frac{1}{D(n+m+2)} \sum_{a\neq 1} \Bigg[\sum_{i=1}^{m} C(n+m-2i+2) P(\widehat{x}_i ay) \\ + C(n-m)P(xy) + \sum_{j=1}^{n} C(n-m-2j)P(xa\widehat{y}_j)\Bigg] \\ = \frac{1}{D(n+m+2)} \left[\sum_{i=1}^{m} C(n+m-2i+2)P(\widehat{x}_i * y) + \sum_{j=1}^{n} C(n-m-2j)P(x*\widehat{y}_j)\right]. \quad (4.2) \end{aligned}$$

Using the induction hypothesis, this becomes

$$P(x*y) = \frac{1}{D(n+m+2)} \left[P(y)\sum_{i=1}^{m} C(n+m-2i+2)P(\widehat{x}_i) + P(x)\sum_{j=1}^{n} C(n-m-2j)P(\widehat{y}_j)\right].$$

Taking $(j,k,l) = (n+1, m-2i+1, 2i)$ in (3.4) gives

$$\frac{C(n+m-2i+2)}{D(n+m+2)} = \frac{C(m-2i+1)}{D(m+1)} - \frac{C(2i)D(n+1)}{D(m+1)D(n+m+2)}.$$

Similarly, replacing (j,k,l) in (3.4) by $(m+1, 2j-n-1, 2n-2j+2)$ gives

$$\frac{C(m+2j-n)}{D(n+m+2)} = \frac{C(2j-n-1)}{D(n+1)} - \frac{C(2n-2j+2)D(m+1)}{D(n+1)D(n+m+2)}.$$

Therefore, since $C(\cdot)$ is even,

$$P(x*y) = P(y)\left[P(x) - \frac{D(n+1)}{D(n+m+2)}Q(x)\right] + P(x)\left[P(y) - \frac{D(m+1)}{D(n+m+2)}Q^*(y)\right].$$

By (3.6),

$$P(x*y) = P(x)P(y)\left[2 - \frac{C(m+1)D(n+1) + C(n+1)D(m+1)}{D(n+m+2)}\right].$$

By (3.3), we see that the expression in brackets above is 1, as required.

Assume now that $x_m \neq y_1$, say $x_m = 1$ and $y_1 = 2$. Then

$$\begin{aligned} P(x*y) = \sum_{a\in[q]} P(xay) = \frac{1}{D(n+m+2)} \sum_{a\neq 1,2} \Bigg[\sum_{i=1}^{m} C(n+m-2i+2) P(\widehat{x}_i ay) \\ + C(n-m)P(xy) + \sum_{j=1}^{n} C(n-m-2j)P(xa\widehat{y}_j)\Bigg] \\ = \frac{1}{D(n+m+2)} \left[\sum_{i=1}^{m} C(n+m-2i+2)P(\widehat{x}_i * y) + \sum_{j=1}^{n} C(n-m-2j)P(x*\widehat{y}_j)\right] \quad (4.3) \end{aligned}$$

as in the previous case. However, in the previous case, the term $P(xy)$ dropped out because xy was not a proper coloring. In this case, the term $(q-2)C(n-m)P(xy)$ is cancelled by the terms $-P(xy)C(n-m+2)$ and $-P(xy)C(n-m-2)$, which arise from

$$\sum_{a\neq 1,2} P(\widehat{x}_m ay) = P(\widehat{x}_m * y) - P(xy) \text{ and } \sum_{a\neq 1,2} P(xa\widehat{y}_1) = P(x*\widehat{y}_1) - P(xy).$$

The fact that the overall coefficient of $P(xy)$ vanishes is a consequence of (3.1) with $m = 2$, since $2C(2) = q - 2$. The rest of the proof is the same as in the case $x_m = y_1$ above. □

5 Open problems

A number of open problems related to colorings are listed at the end of [4]. We list here several that are most natural in the present context.

1. Is our construction the unique S_q-symmetric, $1-$dependent, $q-$coloring of $\mathbb{Z}$? To give some information about this, we will discuss the computations that led to the choices of $C(n), D(n)$ in Section 2. One can think of a $1-$dependent coloring as a nonnegative solution to an infinite set of nonlinear equations. The unknowns are the cylinder probabilities $P(x)$, which are assumed to be translation invariant, symmetric in the colors, and invariant under reflection. The equations are quadratic, since they represent the $1-$dependence requirement. Solving the equations for cylinders of length at most three leads to the unique solution that is given at the end of Section 2. However, when moving to cylinders of length four, one obtains the one parameter family of solutions

$$P(1212) = \frac{\alpha}{q^2(q-1)}, \quad P(1213) = \frac{1-\alpha}{q^2(q-1)(q-2)}, \quad P(1231) = \frac{1}{q^2(q-1)(q-2)},$$
$$P(1234) = \frac{1}{q^2(q-1)}\left[\frac{\alpha}{q-3} + \frac{1-\alpha}{q-2}\right].$$

Thus one gets uniqueness for some cylinder probabilities along the way. But, if one is to show uniqueness for all cylinder probabilities, it appears that one must consider the full infinite set of equations. We used the unique cylinder probabilities to “guess" the expressions for $C(n), D(n)$ that we used, and then checked that the required properties of $P(x)$ are true in general.

2. Find a fully automophism invariant $1-$dependent coloring of $\mathbb{Z}^d$.

3. Find a fully automophism invariant $1-$dependent coloring of the $d-$regular tree.

4. Can our colorings be expressed as finitary factors of i.i.d. processes? Can this be done with a finite mean coding radius? (The first of these questions is answered affirmatively for $q = 4$ in [3].)

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