Electron. Commun. Probab. **20** (2015), no. 17, 1–9.
DOI: 10.1214/ECP.v20-4027
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The mean number of sites visited by a random walk pinned at a distant point

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Abstract

This paper concerns the number Z_n of sites visited up to time n by a random walk S_n having zero mean and moving on the two dimensional square lattice $\mathbb{Z}^2$. Asymptotic evaluation of the conditional expectation of Z_n for large n given that $S_n = x$ is carried out under some exponential moment condition. It gives an explicit form of the leading term valid uniformly in (x, n), $|x| < cn$.

Keywords: Range of random walk; pinned random walk; Cramér transform; local central limit theorem.
AMS MSC 2010: Primary 60J65, Secondary 60J45.
Submitted to ECP on December 26, 2014, final version accepted on February 22, 2015.

1 Introduction and main results

This paper is a continuation of the paper [12] by the present author, where the expectation of the cardinality of the range of a pinned random walk is studied when the random walk of prescribed length is pinned at a point within a parabola of space-time variables. In this paper we deal with the case when it is outside a parabola at which the walk is pinned and compute the asymptotic form of the (conditional) expectation. To this end we derive a local limit theorem valid outside parabolas by using Cramér transform.

The random number, denoted by Z_n, of the distinct sites visited by a random walk in the first n steps is one of typical characteristics or functionals of the random walk paths. The expectation of Z_n may be regarded as the total heat emitted from a site at the origin which is kept at the unit temperature. The study of Z_n is traced back to Dvoretzky and Erdös [2] in which the law of large numbers of Z_n is obtained for simple random walk. Nice exposition of their investigation and an extension of it is found in [10]. For the pinned walk the expectation of Z_n is computed by [12], [4]. Corresponding problems for Brownian sausage have also been investigated (often earlier) (cf. [11], [3] for free motions and [6], [7], [14] for bridges).

Let $S_n = X_1 + \cdots + X_n$ be a random walk on the two-dimensional square lattice $\mathbb{Z}^2$ starting at the origin. Here the increments X_j are i.i.d. random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ taking values in $\mathbb{Z}^2$. The random walk is supposed to be irreducible and having zero mean: $E[X] = 0$. Here and in what follows we write X for a random variable having the same law as X_1.

For $\lambda \in \mathbb{R}^2$, put

$$\phi(\lambda) = \log E[e^{\lambda \cdot X}]$$

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and for $\mu \in \mathbb{R}^2$ let $m(\mu)$ be the value of λ determined by

$$\nabla\phi(\lambda)\Big|_{\lambda=m(\mu)} = \mu : \tag{1.1}$$

$m(\mu)$ is well defined if μ is an interior point of the image set $\nabla\phi(\Xi)$ of

$$\Xi = \{\lambda : E[\,|X|e^{\lambda\cdot X}] < \infty\}.$$

Since $\nabla\phi(0) = 0$, if the interior of Ξ contains the origin, then so does the interior of $\nabla(\Xi)$. Let $f_0(n)$ be the probability that the walk returns to the origin for the first time at the n-th step $(n \geq 1)$ and define

$$H(\mu) = \sum_{k=1}^{\infty} f_0(k)\Big(1 - e^{-k\phi(m(\mu))}\Big).$$

Let Z_n $(n = 1, 2, \ldots)$ denote the cardinality of the set of sites visited by the walk up to time n, namely

$$Z_n = \sharp\{S_1, S_2, \ldots, S_n\}.$$

Let Q be the covariance matrix of X and $|Q|$ be the determinant of Q.

Theorem 1. Suppose that $\phi(\lambda) < \infty$ in a neighborhood of the origin and let K be a compact set contained in the interior of Ξ. Then,

$$H(\mu) = \frac{\pi\sqrt{|Q|}}{-\log|\mu|} + O\bigg(\frac{1}{(\log|\mu|)^2}\bigg) \qquad \text{as} \quad |\mu| \to 0, \tag{1.2}$$

and, uniformly for $\mathbf{x} \in \mathbb{Z}^2$ satisfying $\mathbf{x}/n \in \nabla\phi(K)$ and $|\mathbf{x}| \geq \sqrt{n}$,

$$E\Big[Z_n \,\Big|\, S_n = \mathbf{x}\Big] = nH(\mathbf{x}/n) + O\bigg(\frac{n}{(\log n) \vee (\log|\mathbf{x}/n|)^2}\bigg) \qquad \text{as} \quad n \to \infty. \tag{1.3}$$

Example 1. For symmetric simple random walk we have $e^{\phi(\lambda)} = \frac{1}{2}\cosh\alpha + \frac{1}{2}\cosh\beta$ for $\lambda = (\alpha, \beta)$. Given $\mathbf{x}/n = \mu + o(1)$, the leading term $nH(\mathbf{x}/n)$ in (1.3) may be computed from

$$H(\mu) = 1 - \sum_{j=1}^{\infty} \frac{f_0(2j)2^{2j}}{(\cosh\alpha + \cosh\beta)^{2j}}, \quad \mu = \nabla\phi(\lambda) = \frac{(\sinh\alpha, \sinh\beta)}{\cosh\alpha + \cosh\beta}.$$

The derivative of H along a circle centered at the origin directed counter-clockwise is given by

$$\nabla H(\mu) \cdot \mu^{\perp} = C_0(\mu)\mu_1\mu_2(\mu_2^2 - \mu_1^2),$$

where $\mu^{\perp} = (\mu_2, -\mu_1)$ and $C_0(\mu)$ is a smooth positive function of $\mu \neq 0$. (See Appendix (B).)

We see shortly that the behavior of the probability $P_0[S_n = \mathbf{x}]$ differs greatly in different directions of $\mathbf{x}$ as soon as $|\mathbf{x}|/n^{3/4}$ gets large even if Q is isotropic. (See Proposition 2 below.) According to Theorem 1.2, in contrast to this, the leading term of $E[Z_n \,|\, S_n = \mathbf{x}]$ as $\mathbf{x}/n \to 0$ as well as that of $H(\mu)$ as $\mu \to 0$ is rotation invariant; only when $|\mathbf{x}|/n$ is bounded away from zero, $E[Z_n \,|\, S_n = \mathbf{x}]$ in general becomes dependent on directions of $\mathbf{x}$.

The case $|\mathbf{x}| = O(\sqrt{n})$ is studied in [12] under certain mild moment conditions. If we assume the rather strong moment condition $E[|X|^4] < \infty$, the result is presented as follows: for each $a_\circ > 0$ it holds that uniformly for $|\mathbf{x}| < a_\circ\sqrt{n}$, as $n \to \infty$

$$E[Z_n|S_n = \mathbf{x}] = 2\pi\sqrt{|Q|}\,n\int_{e^{c_\circ}n}^{\infty} W(u)du + \frac{4\sqrt{|Q|}\,\tilde{x}^2}{(\log n)^2}\left(\log^+ \frac{n}{|x|_+^2} + O(1)\right) + \frac{o(1) + b_3 O(|x|)}{\log n}, \tag{1.4}$$

where $W(\lambda) = \int_0^\infty([\log t]^2 + \pi^2)^{-1}e^{-\lambda t}\,dt\,(\lambda > 0)$ and $\tilde{x} = Q^{-1/2}x$. We have the asymptotic expansion $\int_\lambda^\infty W(u)du = (\log\lambda)^{-1} - \gamma(\log\lambda)^{-2} + (\gamma^2 - \frac{1}{6}\pi^2)(\log\lambda)^{-3} + \cdots \quad (\lambda \to \infty)$, where $\gamma = 0.5772\ldots$ (Euler's constant).

Brownian analogue of (1.4) is given in [14], the proof being similar but rather more involved than for the random walk case.

Remark 1. By a standard argument we have

$$1 - \sum_1^\infty e^{-k\lambda}f_0(k) = \left(\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\frac{d\theta}{1 - e^{-\lambda}E[e^{i\theta\cdot X}]}\right)^{-1} \qquad (\lambda > 0).$$

Substitution from $E[e^{i\theta\cdot X}] = e^{\phi(i\theta)}$ and $\lambda = \phi(m(\mu))$ therefore yields

$$\frac{1}{H(\mu)} = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\frac{d\theta}{1 - \exp\{-\phi(m(\mu)) + \phi(i\theta)\}} \qquad (\mu \neq 0). \tag{1.5}$$

Remark 2. For $d \geq 3$ the results analogous to (1.4) are obtained by the same method. Here only a result of [12] for the case $d = 3$ is given:

Suppose $d = 3$ and $E[|X|^4] < \infty$. Then uniformly for $|\mathbf{x}| < a_\circ\sqrt{n}$, as $n \to \infty$

$$E[Z_n|S_n = x] = q_0 n + \frac{q_0^2|\tilde{x}|}{2\pi\sqrt{|Q|}} + O\left(\frac{1}{1 + |x|}\right) + b_3 O(1) + \frac{o(1) + b_3 O(|x|)}{\sqrt{n}},$$

where $q_0 = P[S_n \neq 0 \text{ for all } n \geq 1]$.

Remark 3. For random walks of continuous time parameter the asymptotic form of the expectation are deduced from those of the embedded discrete time walks by virtue of the well-known purely analytic result as given in [5].

For the proof of Theorem 1 we derive a local limit theorem, an asymptotic evaluation of the probability $P[S_n = \mathbf{x}]$, denoted by $q^n(\mathbf{x})$, for large n, that is sharp uniformly for the space-time region $\sqrt{n} \leq |\mathbf{x}| < \varepsilon n$ (with some $\varepsilon > 0$) (Lemma 3). As a byproduct of it we obtain the following proposition which lucidly exhibits what happens for variables $\sqrt{n} < |x| << n$ with n large: if all the third moments vanish, then the ratio of the probabilities $q^n(\mathbf{x})$ among directions of $\mathbf{x}$ with the same modulus $|\mathbf{x}|$ can be unbounded as $|\mathbf{x}|/n^{3/4}$ gets large; if not, this may occur as $|\mathbf{x}|/n^{2/3}$ gets large. This result though not directly used in the proof of Theorem 1 is interesting by itself.

Proposition 2. Uniformly in $\mathbf{x}$, as $n \to \infty$ and $|\mathbf{x}|/n \to 0$,

$$q^n(\mathbf{x}) = \frac{\nu\mathbf{1}(q^n(\mathbf{x}) \neq 0)}{2\pi n\sigma^2}e^{-x\cdot Q^{-1}\mathbf{x}/2n}\left(1 + O\Big(\frac{|\mathbf{x}| + 1}{n}\Big)\right) \times \exp\left\{n\kappa_3\Big(\frac{\mathbf{x}}{n}\Big) + n\kappa_4\Big(\frac{\mathbf{x}}{n}\Big) + O\Big(\frac{|\mathbf{x}|^5}{n^4}\Big)\right\},$$

where $\kappa_3(\mu) = \frac{1}{6}E[(Q^{-1}X\cdot\mu)^3]$ and κ_4 is a homogeneous polynomial of degree 4. If all the third moments of X vanish, then

$$\kappa_4(\mu) = -\frac{1}{8}[Q^{-1}(\mu)]^2 + \frac{1}{24}E[(Q^{-1}X\cdot\mu)^4].$$

Example 2. For the same simple random walk as in Example 1 it follows from Proposition 2 that

$$q^n(\mathbf{x}) = \frac{4e^{-|\mathbf{x}|^2/n}}{\pi n}\Big(1 + O\Big(\frac{|\mathbf{x}|+1}{n}\Big)\Big)\exp\Big\{-\frac{|\mathbf{x}|^4 + 4(x_1x_2)^2}{6n^3} + O\Big(\frac{|\mathbf{x}|^5}{n^4}\Big)\Big\}$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$ with $n + x_1 + x_2$ even. This formula, however, can be obtained rather directly if one notices that in the frame obtained by rotating the original one by a right angle the two components in the new frame are symmetric simple random walks on $\mathbb{Z}/\sqrt{2}$ that are independent of each other and use an expansion of transition probability of these walks as given in [8] (Section VII.6, problem 14).

2 Proof of Theorem 1

2.1. Proof of (1.2).

The arguments involved in this subsection partly prepares for the proof of (1.3).

By definition $\lambda = m(\mu)$ is the inverse function of

$$\mu = \nabla\phi(\lambda) = \frac{E[Xe^{X\cdot\lambda}]}{E[e^{X\cdot\lambda}]} = Q\lambda + \frac{1}{2}E[(X\cdot\lambda)^2X] + O(|\lambda|^3),$$

so that

$$\lambda = m(\mu) = Q^{-1}\mu - \frac{1}{2}E[(X\cdot Q^{-1}\mu)^2Q^{-1}X] + O(|\mu|^3). \tag{2.1}$$

The Taylor expansion of ϕ about the origin up to the thid order is given by

$$\phi(\lambda) = \frac{1}{2}Q(\lambda) + \frac{1}{6}E[(X\cdot\lambda)^3] + O(|\lambda|^4), \tag{2.2}$$

hence for $|\mu|$ small enough,

$$\phi(m(\mu)) = \frac{1}{2}Q^{-1}(\mu) - \frac{1}{3}E[(Q^{-1}X\cdot\mu)^3] + O(|\mu|^4). \tag{2.3}$$

Here $Q(\lambda) = \lambda\cdot Q\lambda$, the quadratic form determined by the matrix Q and similarly $Q^{-1}(\mu) = \mu\cdot Q^{-1}\mu$.

Now we compute $H(\mu)$ by using (1.5). From (2.3) and $\phi(i\theta) = -\frac{1}{2}Q(\theta) + O(|\theta|^3)$ (for θ small) it follows that

$$1 - e^{-\phi(m(\mu))+\phi(i\theta)} = \frac{1}{2}[Q^{-1}(\mu) + Q(\theta)] + O(|\mu|^3 + |\theta|^3).$$

Substitution into (1.5) and a simple computation show

$$\begin{aligned}\frac{1}{H(\mu)} &= \frac{2}{(2\pi)^2}\int_{[-\pi,\pi]^2}\frac{d\theta}{Q^{-1}(\mu) + Q(\theta) + O(|\mu|^3 + |\theta|^3)} \\ &= \frac{-1}{2\pi|Q|^{1/2}}\log Q^{-1}(\mu) + O(1).\end{aligned}$$

Noting $\log Q^{-1}(\mu) = 2\log|\mu| + O(1)$ we obtain (1.2).

2.2. A local limit theorem.

Let $q(\mathbf{x})$ denote the probability law of the increment of the walk: $q(\mathbf{x}) = P[X = \mathbf{x}]$. Let $\mu = \nabla\phi(\lambda)$ with λ in the interior of Ξ and define

$$q_\mu(\mathbf{x}) = \frac{1}{E[e^{m(\mu)\cdot X}]}e^{m(\mu)\cdot\mathbf{x}}q(\mathbf{x})$$

($m(\mu)$ is defined by (1.1)) so that q_μ is a probability on $\mathbb{Z}^2$ with the mean

$$\sum \mathbf{x} q_\mu(\mathbf{x}) = \nabla\phi(m(\mu)) = \mu.$$

Let q^n and q_μ^n be the n-fold convolution of q and q_μ, respectively. Then

$$q^n(\mathbf{x}) := P[S_n = \mathbf{x}] = (E[e^{m(\mu)\cdot X}]\,)^n e^{-m(\mu)\cdot\mathbf{x}} q_\mu^n(\mathbf{x}). \tag{2.4}$$

Let Q_μ denote the covariance matrix of the probability q_μ and $Q_\mu^{-1}(\mathbf{x})$ the quadratic form determined by Q_μ^{-1}.

Lemma 3. Let K be a compact set contained in the interior of Ξ (as in Theorem 1). Then uniformly for $\mathbf{y} \in \mathbb{Z}^2 - n\mu$ and for $\mu \in \nabla\phi(K)$, as $n \to \infty$

$$q_\mu^n(n\mu + \mathbf{y}) = \frac{\nu \mathbf{1}(q^n(n\mu+\mathbf{y}) \neq 0)}{2\pi n \sigma_\mu^2} e^{-Q_\mu^{-1}(\mathbf{y})/2n}\Big[1 + P_\mu^{n,N}(\mathbf{y})\Big] + O\bigg([\mathbf{y}^2 \vee n]^{-N/2}\bigg).$$

Here N may be an arbitrary positive integer, ν is the period of the walk S_n, $\mathbf{1}(\mathcal{S})$ is 1 or 0 according as the statement $\mathcal{S}$ is true or false, σ_μ^2 denotes the square root of the determinant of Q_μ and

$$P_\mu^{n,N}(\mathbf{y}) = n^{-1/2} P_1^\mu(\mathbf{y}/\sqrt{n}\,) + \cdots + n^{-N/2} P_N^\mu(\mathbf{y}/\sqrt{n}\,),$$

where P_j^μ is a polynomial of degree at most $3j$ determined by the moments of q_μ^n and odd for odd j.

Proof. This lemma may be a standard result. In fact it is reduced to the usual local central limit theorem as follows. Let $\psi_\mu(\theta)$ be the characteristic function of q_μ and put $\tilde\psi_\mu(\theta) = \sum_{\mathbf{x}} q_\mu(\mathbf{x}) e^{i\theta\cdot(\mathbf{x}-\mu)}$, so that

$$\psi_\mu(\theta) := \sum_{\mathbf{x}} q_\mu(\mathbf{x}) e^{i\theta\cdot\mathbf{x}} = \tilde\psi_\mu(\theta) e^{i\mu\cdot\theta}.$$

Hence

$$\begin{aligned} q_\mu^n(n\mu + \mathbf{y}) &= \frac{1}{(2\pi)^2}\int_T [\psi_\mu(\theta)]^n e^{-i(n\mu+\mathbf{y})\cdot\theta} d\theta \\ &= \frac{1}{(2\pi)^2}\int_T [\tilde\psi_\mu(\theta)]^n e^{-i\mathbf{y}\cdot\theta} d\theta, \end{aligned} \tag{2.5}$$

where $T = [-\pi, pi) \times [-\pi, pi)$. Since $\nabla\tilde\psi_\mu(0) = 0$, the Hessian matrix of $\tilde\psi_\mu$ at zero equals Q_μ and $\sum p_\mu(\mathbf{x})|\mathbf{x}|^{2N} < \infty$ for all $N > 0$, the usual procedure to derive the local limit theorem (see [9]; also Appendix (A) for the case $\nu > 1$ if necessary) shows that the right-most member equals that of the formula of the lemma. □

Define $\Lambda \subset \mathbb{Z}^2$ by

$$\Lambda = \{\mathbf{x} \in \mathbb{Z}^2 : q^{\nu n}(\mathbf{x}) \neq 0 \text{ for some } n\}. \tag{2.6}$$

Plainly Λ is a subgroup of $\mathbb{Z}^2$. Take an $\xi \in \mathbb{Z}^2$ with $q(\xi) > 0$ and put $\Lambda_k = \Lambda + k\xi$, the shift of Λ by $k\xi$. Λ_k does not depend on the choice of ξ and is periodic in k of period ν. It holds that $P[S_n \in \Lambda_k] > 0$ only if $n = k \bmod(\nu)$. In the formula of Lemma 3 the trivial factor $\mathbf{1}(q^n(n\mu+\mathbf{y}) \neq 0)$ may be replaced by $\mathbf{1}(n\mu + \mathbf{y} \in \Lambda_n)$; also, for each $k \in \mathbb{Z}$, $q_\mu^n(n\mu+\mathbf{y})$ may be replaced by $q_\mu^n((n-k)\mu + \mathbf{y})$, hence by $q_\mu^{n+k}(n\mu + \mathbf{y})$. Thus we can reformulate Lemma 3 as in the following

Corollary 4. Let K be a compact set contained in the interior of Ξ. Then for each $k \in \mathbb{Z}$, uniformly for $\mathbf{y} \in \mathbb{Z}^2 - n\mu$ and for $\mu \in \nabla\phi(K)$, as $n \to \infty$

$$q_\mu^{n+k}(n\mu + \mathbf{y}) = \frac{\nu \mathbf{1}(n\mu+\mathbf{y} \in \Lambda_{n+k})}{2\pi n \sigma_\mu^2} e^{-Q_\mu^{-1}(\mathbf{y})/2n}\Big[1 + P_\mu^{n,N}(\mathbf{y})\Big] + O\bigg([\mathbf{y}^2 \vee n]^{-N/2}\bigg),$$

with the same notation as in Lemma 3.

Proof of Proposition 2. In Lemma 3 we take $\mu = \mathbf{x}/n$. It follows that with $\lambda = m(\mu)$

$$Q_\mu = \nabla \log \phi(\lambda) + [\nabla \phi(\lambda)/\phi(\lambda)]^2 = Q + O(|\mu|),$$

so that $\sigma_\mu^2 = \sigma^2 + O(|\mu|)$. In view of (2.4) and Lemma 3 we have only to compute asymptotic form of

$$E[e^{m(\mu)\cdot X}]^n e^{-m(\mu)\cdot \mathbf{x}} = \exp\{n[\phi(m(\mu)) - m(\mu)\cdot \mu]\}.$$

By (2.1) and (2.3)

$$\phi(m(\mu)) - m(\mu)\cdot\mu = -\frac{1}{2}Q^{-1}(\mu) + \frac{1}{6}E[(Q^{-1}X\cdot\mu)^3] + \kappa_4(\mu) + O(|\mu|^5)$$

for $|\mu|$ small enough, where $\kappa_4(\mu)$ is a polynomial of degree 4.

Assume that all the third moments of X vanish. Then, in place of (2.1) and (2.2) we have

$$\lambda = m(\mu) = Q^{-1}\mu + b(\mu). \tag{2.7}$$

with $b(\mu) = O(|\mu|^3)$ and

$$\phi(\lambda) = \frac{1}{2}Q(\lambda) - \frac{1}{8}[Q(\lambda)]^2 + \frac{1}{24}E[(X\cdot\lambda)^4] + O(|\lambda|^5), \tag{2.8}$$

respectively. Substituting these formulae into $m(\mu)\cdot\mu - \phi(m(\mu))$ we observe that the term involving $b(\mu)$ disappears from the fourth order term by cancellation and hence that

$$\phi(m(\mu)) - m(\mu)\cdot\mu = -\frac{1}{2}Q^{-1}(\mu) - \frac{1}{8}[Q(\lambda)]^2 + \frac{1}{24}E[(Q^{-1}X\cdot\mu)^4] + O(|\mu|^5),$$

in which we find the explicit form of $\kappa_4(\mu)$ as presented in the proposition. □

2.3. Proof of (1.3).

The proof is based on the identity

$$E[Z_n; S_n = \mathbf{x}] = nq^n(\mathbf{x}) - \sum_{k=1}^{n-1} f_0(k) q^{n-k}(\mathbf{x})(n-k) \tag{2.9}$$

(cf. [12], Lemma 1.1) as well as Corollary 4. Let $q^n(\mathbf{x}) \neq 0$. Remembering $E[e^{m(\mu)\cdot X}] = e^{\phi(m(\mu))}$ we obtain from (2.4) that

$$\frac{q^{n-k}(\mathbf{x})(n-k)}{q^n(\mathbf{x})n} = e^{-k\phi(m(\mu))}\frac{q_\mu^{n-k}(\mathbf{x})(n-k)}{q_\mu^n(\mathbf{x})n}.$$

On writing $\mu := \mathbf{x}/n$ and $\mathbf{x} = (n-k)\mu + k\mu$, Corollary 4 gives

$$\begin{aligned} q_\mu^{n-k}(\mathbf{x})(n-k) \quad &= \quad \frac{\nu \mathbf{1}(\mathbf{x}\in\Lambda_{n-k})}{2\pi\sigma_\mu^2} e^{-Q_\mu^{-1}(k\mu)/2(n-k)}\Big[1 + P_\mu^{n-k,N}(k\mu)\Big] \\ &\qquad + O([|k\mu|^2 \vee (n-k)]^{-N}) \end{aligned}$$

and

$$q_\mu^n(\mathbf{x})n = \frac{\nu}{2\pi\sigma_\mu^2}\Big[1 + O(1/n)\Big].$$

Let $1/\sqrt{n} \le |\mu|$ and $\mu \in \nabla\phi(K)$. Noting that σ_μ^2 is then bounded away from zero for $\mu \in \nabla\phi(K)$ we see

$$\begin{aligned} \frac{q^{n-k}(\mathbf{x})(n-k)}{q^n(\mathbf{x})n} \quad &= \quad \mathbf{1}(\mathbf{x}\in\Lambda_{n-k})e^{-k\phi(m(\mu))}e^{-Q_\mu^{-1}(k\mu)/2(n-k)}\Big[1 + O(1/\sqrt{n})\Big] \\ &\qquad + O\Big(e^{-k\phi(m(\mu))}n^{-N}\Big). \end{aligned} \tag{2.10}$$

Since $\sum_{k>n^{1/3}} f_0(k) = O(1/\log n)$, it follows that

$$E[Z_n; S_n = \mathbf{x}] = nq^n(\mathbf{x})\left[\sum_{k=1}^{n^{1/3}} f_0(k)\Big(1 - \mathbf{1}(\mathbf{x} \in \Lambda_{n-k})e^{-k\phi(c(\mathbf{x}/n))}\Big) + O\Big(\frac{1}{\log n}\Big)\right].$$

Under the condition $q^n(\mathbf{x}) \neq 0$, it follows from $f_0(k) \neq 0$ that $\mathbf{x} \in \Lambda_{n-k}$. Hence

$$E[Z_n \mid S_n = \mathbf{x}] = n\sum_{k=1}^{\infty} f_0(k)\Big(1 - e^{-k\phi(c(\mathbf{x}/n))}\Big) + O\Big(\frac{n}{\log n}\Big). \tag{2.11}$$

We still need to obtain the error bound $O(n/|\log \mu|^2)$ instead of $O(n/\log n)$. To this end, on applying the asymptotic formula

$$f_0(k) = \frac{2\pi|Q|^{1/2}}{k(\log k)^2} + O\Big(\frac{1}{k(\log k)^3}\Big)$$

(cf. [13]) we see, on the one hand, that for $0 < \phi < 1/2$

$$\sum_{k>\delta/\phi} f_0(k)e^{-k\phi} = O\Big(\frac{1}{(\log \phi)^2}\Big), \tag{2.12}$$

where δ is an arbitrarily fixed positive constant, and by using (2.3), on the other hand, we see

$$\phi(m(\mu)) > c|\mu|^2 \geq c/n$$

(the second inequality is nothing but our present supposition that $|\mathbf{x}| \geq \sqrt{n}$). As in a similar way to the derivation of (2.11) we deduce from (2.9) with the help of (2.12) as well as of (2.10) that

$$\frac{E[Z_n;\, S_n = \mathbf{x}]}{nq^n(\mathbf{x})} = \sum_{k=1}^{\infty} f_0(k)\Big(1 - e^{-k\phi(c(\mathbf{x}/n))}\Big) + O\Big(\frac{1}{(\log|\mu|)^2}\Big),$$

if it is true that as $\mu \to 0$

$$\sum_{k<c/2\phi(m(\mu))} f_0(k)e^{-k\phi(m(\mu))}(1 - e^{-Q^{-1}(k\mu)/2(n-k)}) = O(1/(\log|\mu|)^2). \tag{2.13}$$

Since $c/2\phi(m(\mu)) \leq n/2$, the sum on the left-hand side of (2.13) is at most a constant multiple of

$$\begin{aligned}\sum_{k<c/2\phi(m(\mu))} f_0(k)\frac{Q^{-1}(k\mu)}{n} \quad &= \quad \frac{Q^{-1}(\mu)}{n}\sum_{k<c/2\phi(m(\mu))} f_0(k)k^2 \\ &\leq \quad \frac{c'|\mu|^2}{n}\frac{k^2}{(\log k)^2}\bigg|_{k=c/2\phi(m(\mu))} = O\Big(\frac{1}{n(\log|\mu|)^2}\Big).\end{aligned}$$

verifying (2.13) (with a better bound).

Thus we have proved (1.3) and hence Theorem 1.

3 Appendix

(A) In the case when the period ν is larger than 1 the evaluation of the integral in (2.5) is reduced to that for the case $\nu = 1$ by consideration of a property of its integrand that reflects the periodicity. By an elementary algebra one can find a point $\eta \in \mathbb{R}^2$ that satisfies that for $j = 0, 1, \ldots, \nu - 1$,

$$\eta \cdot \mathbf{x} - j\nu^{-1} \in \mathbb{Z} \quad \text{if} \quad \mathbf{x} \in \Lambda_j$$

(Λ_j is defined shortly after (2.6)). From this relation it follows that

$$\psi(\theta + 2\pi k\eta) = \psi(\theta)e^{i2\pi k/\nu} \quad (k = 0, \dots, \nu - 1).$$

Now consider the expression $q_\mu^n(x) = (2\pi)^{-2}\int_T [\psi(\theta)]^n e^{-ix\cdot\theta}d\theta$. Observe that if $x \in \Lambda_j$,

$$[\psi(\theta + 2\pi k\eta)]^n e^{-ix\cdot(\theta+2\pi k\eta)} = [\psi(\theta)]^n e^{-ix\cdot\theta}e^{i2\pi(n-j)k/\nu} \quad (k = 0, \dots, \nu - 1)$$

and the right-hand sides equal $[\psi(\theta)]^n e^{-ix\cdot\theta}$ for all k if $n - j$ equals zero in mod ν, while their sum over k vanishes otherwise. Choosing $\varepsilon > 0$ small enough, we may replace $2\pi\eta$ by a unique $\eta_k \in [-1-\varepsilon, 1+\varepsilon]$ such that $\eta_k - \eta \in \mathbb{Z}^2$ and apply the usual method for evaluation of Fourier integral.

(B) Put $\psi(\lambda) = E[e^{\lambda\cdot X}]$, so that $\mu = \nabla\phi(\lambda) = E[Xe^{\lambda\cdot X}]/\psi(\lambda)$. At $\lambda = m(\mu)$ we have

$$\nabla^2\phi(\lambda) = E[X^2 e^{\lambda\cdot X}]/\psi(\lambda) - \mu^2 = Q_\mu,$$

where μ^2 is understood to be 2×2 matrix: $\mu^2 = (\mu_i\mu_j)_{1\le i,j\le 2}$, and similarly for X^2 and ∇^2. Since $id = \nabla m(\mu)\frac{\partial\mu}{\partial\lambda} = \nabla m(\mu)\nabla^2\phi(m(\mu)) = \nabla m(\mu)Q_\mu$, it holds that

$$\nabla m(\mu) = Q_\mu^{-1}.$$

Therefore, from the defining formula of H we have

$$\nabla H(\mu) = C(\mu)\nabla(\phi \circ m)(\mu) = C(\mu)Q_\mu^{-1}\mu,$$

where

$$C(\mu) = \sum_{k=1}^\infty kf_0(k)e^{-k\phi(m(\mu))}.$$

Let E^μ designate the expectation w.r.t. q_μ, i.e.,

$$E^\mu[\cdot] = \Big[E[\cdot\, e^{\lambda\cdot X}]/\psi(\lambda)\Big]_{\lambda=m(\mu)}.$$

Then

$$\begin{aligned} Q_\mu^{-1}\mu &= \frac{1}{\det Q_\mu}\begin{bmatrix} E^\mu[X_2^2] - \mu_2^2 & \mu_1\mu_2 - E^\mu[X_1X_2] \\ \mu_1\mu_2 - E^\mu[X_1X_2] & E^\mu[X_1^2] - \mu_1^2 \end{bmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\ &= \frac{1}{\det Q_\mu}\begin{bmatrix} \mu_1 E^\mu[X_2^2] - \mu_2 E^\mu[X_1X_2] \\ \mu_2 E^\mu[X_1^2] - \mu_1 E^\mu[X_1X_2] \end{bmatrix}. \end{aligned}$$

Hence

$$\nabla H(\mu)\cdot\mu^\perp = \frac{1}{\det Q_\mu}C(\mu)\Big(\mu_1\mu_2 E^\mu[X_2^2 - X_1^2] + (\mu_1^2 - \mu_2^2)E^\mu[X_1X_2].\Big).$$

For the simple random walk in Example 1, $\det Q_\mu = (\cosh\alpha + \cosh\beta)^{-2}$, $E^\mu[X_1X_2] = 0$ and

$$\begin{aligned} E^\mu[X_2^2 - X_1^2] = \frac{\cosh\beta - \cosh\alpha}{2\psi(m(\mu))} &= \frac{\sinh^2\beta - \sinh^2\alpha}{(\cosh a + \cosh\beta)^2} \\ &= \mu_2^2 - \mu_1^2 \qquad (\lambda = m(\mu)), \end{aligned}$$

showing the last formula of Example 1 with $C_0(\mu) = C(\mu)(\cosh\alpha + \cosh\beta)^2$.

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