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Functional limit theorems for divergent perpetuities in the contractive case*

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Abstract

Let $\big(M_k, Q_k\big)_{k\in\mathbb{N}}$ be independent copies of an $\mathbb{R}^2$-valued random vector. It is known that if $Y_n := Q_1 + M_1Q_2 + \ldots + M_1 \cdot \ldots \cdot M_{n-1}Q_n$ converges a.s. to a random variable Y, then the law of Y satisfies the stochastic fixed-point equation $Y \overset{d}{=} Q_1 + M_1Y$, where (Q_1, M_1) is independent of Y. In the present paper we consider the situation when $|Y_n|$ diverges to ∞ in probability because $|Q_1|$ takes large values with a high probability, whereas the multiplicative random walk with steps M_k's tends to zero a.s. Under a regular variation assumption we show that $\log|Y_n|$, properly scaled and normalized, converge weakly in the Skorokhod space equipped with the J_1-topology to an extremal process. A similar result also holds for the corresponding Markov chains. Proofs rely upon a deterministic result which establishes the J_1-convergence of certain sums to a maximal function and subsequent use of the Skorokhod representation theorem.

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1 Introduction

Let $\big(M_k, Q_k\big)_{k\in\mathbb{N}}$ be independent copies of a random vector $\big(M, Q\big)$ with arbitrary dependence of the components, and let X_0 be a random variable which is independent of $\big(M_k, Q_k\big)_{k\in\mathbb{N}}$. Then the sequence $\big(X_n\big)_{n\in\mathbb{N}_0}$ defined by

$$X_n = M_nX_{n-1} + Q_n, \quad n \in \mathbb{N}, \tag{1.1}$$

is a homogeneous Markov chain. In view of the representation

$$\begin{aligned} X_n &= \Psi_n(X_{n-1}) = \Psi_n \circ \ldots \circ \Psi_1(X_0) \\ &= Q_n + M_nQ_{n-1} + \ldots + M_nM_{n-1} \cdot \ldots \cdot M_2Q_1 + M_nM_{n-1} \cdot \ldots \cdot M_1X_0 \end{aligned}$$

for $n \in \mathbb{N}$, where $\Psi_n(t) := Q_n + M_nt$ for $n \in \mathbb{N}$, $(X_n)_{n\in\mathbb{N}}$ is nothing else but the *forward* iterated function system. Closely related is the *backward* iterated function system

$$Y_n := \Psi_1 \circ \ldots \circ \Psi_n(0) = Q_1 + M_1Q_2 + \ldots + M_1M_2 \cdot \ldots M_{n-1}Q_n, \ \ n \in \mathbb{N}.$$

In the case that $X_0 = 0$ a.s. it is easily seen that X_n has the same law as Y_n for each fixed n.

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Put

$$\Pi_0 := 1, \ \ \Pi_n := M_1 M_2 \cdot \ldots \cdot M_n, \ \ n \in \mathbb{N}$$

and assume that

$$\mathbb{P}\{M = 0\} = 0 \quad \text{and} \quad \mathbb{P}\{Q = 0\} < 1 \tag{1.2}$$

and

$$\mathbb{P}\{Q + Mr = r\} < 1 \quad \text{for all} \quad r \in \mathbb{R}. \tag{1.3}$$

Then according to Theorem 2.1 in [8] the series $\sum_{k \geq 1} \Pi_{k-1} Q_k$ is absolutely a.s. convergent provided that

$$\lim_{n\to\infty} \Pi_n = 0 \ \text{ a.s. and } \ I := \int_{(1,\infty)} \frac{\log x}{A(\log x)} \mathbb{P}\{|Q| \in \mathrm{d}x\} < \infty, \tag{1.4}$$

where $A(x) := \mathbb{E}(\log^- |M| \wedge x)$, $x > 0$. The sum Y, say, of the series is then called *perpetuity*.

It is also well-known what happens in the 'trivial cases' when at least one of conditions (1.2) and (1.3) does not hold.

(a) If $\mathbb{P}\{M = 0\} > 0$, then $\tau := \inf\{k \in \mathbb{N} : M_k = 0\} < \infty$ a.s., and the perpetuity trivially converges, the limit being an a.s. finite random variable $\sum_{k=1}^{\tau} \Pi_{k-1} Q_k$. Plainly, its law is a unique invariant measure for (X_n).

(b) If $\mathbb{P}\{Q = 0\} = 1$, then $\sum_{k\geq 1} \Pi_{k-1} Q_k = 0$ a.s.

(c) If $\mathbb{P}\{Q + Mr = r\} = 1$ for some $r \in \mathbb{R}$, then either δ_r is a unique invariant probability measure for (X_n) or every probability law is an invariant measure, or every symmetric around r probability law is an invariant measure (see Theorem 3.1 in [8] for the details).

Under assumptions (1.2), (1.3) and (1.4) the Markov chain $\big(X_n\big)$ has a unique invariant probability measure which is the law of the perpetuity. Equivalently, the law of Y is a unique solution to the stochastic fixed-point equation

$$Y \overset{d}{=} Q + MY, \tag{1.5}$$

where the vector (M, Q) is assumed independent of Y, sometimes called the *random difference equation*. Equation (1.5) appears in diverse areas of both applied and pure mathematics and various properties of Y have attracted considerable attention. Papers [1, 8, 18] give pointers to relevant literature.

For (X_n) defined by (1.1) we write X_n^v to indicate that $X_0 = v$ for $v \in \mathbb{R}$. If the first part of (1.4) is in force we infer $|X_n^v - X_n^w| = |\Pi_n||v - w| \to 0$ a.s. as $n \to \infty$, for any $v, w \in \mathbb{R}$. Therefore, the case when $\lim_{n\to\infty} \Pi_n = 0$ a.s. will be called *contractive*.

In the present paper we are interested in the case when conditions (1.2), (1.3) and

$$\lim_{n\to\infty} \Pi_n = 0 \quad \text{a.s. and} \quad I = \infty \tag{1.6}$$

hold, i.e., the model is still contractive, yet the second condition in (1.4) is violated. By Theorem 2.1 in [8] (Y_n) is then a *divergent perpetuity* in the sense that $|Y_n| = |\sum_{k=1}^n \Pi_{k-1} Q_k| \overset{\mathrm{P}}{\to} \infty$ as $n \to \infty$. The purpose of the present paper is to prove functional limit theorems for the Markov chains (X_n) and for the divergent perpetuities (Y_n) under the aforementioned assumptions. It is noteworthy that unlike some previous papers on limit theorems for perpetuities we allow M and Q to take values of both signs.

As far as we know Grincevičius [9] was the first to prove a limit theorem for Y_n in the case $\mathbb{E}\log|M| = 0$ under the assumption that $M > 0$ a.s. Also, weak convergence of one-dimensional distributions of divergent perpetuities has been investigated in [3, 10, 13, 15] under various assumptions on M and Q. To the best of our knowledge, (a) functional limit theorems for divergent perpetuities have not been obtained so far; (b)

[13] is the only contribution to case (1.6) which deals with one-dimensional convergence. We would like to stress that outside the area of limit theorems we are only aware of two papers [12] and [19] which investigate case (1.6). Unlike (1.6) the critical non-contractive case $\mathbb{E}\log|M| = 0$ has received more attention in the literature, see [2, 4, 5, 6, 9, 10, 15].

Assuming that the tail of $\log^-|M|$ is lighter than that of $\log^+|Q|$ we state two functional limit theorems thereby covering a variety of situations. In particular, we do not require finiteness of $\mathbb{E}\log|M|$. Under (1.6) the complementary case is also possible where the tail of $\log^-|M|$ is not lighter than that of $\log^+|Q|$. Assuming that $|M| \le 1$ a.s. take, for instance, $\mathbb{P}\{-\log|M| > x\} \sim x^{-\alpha}\log^\delta x$, $x \to \infty$ for any $\delta \ge 0$ and some $\alpha \in (0,1)$, and $\mathbb{P}\{\log|Q| \in \mathrm{d}x\} = \alpha x^{-\alpha-1}\,\mathbb{1}_{(1,\infty)}\,\mathrm{d}x$. However this situation is beyond the scope of the present work.

For $c > 0$ and $\alpha > 0$, let $N^{(c,\alpha)} := \sum_k \varepsilon_{(t_k^{(c,\alpha)},\, j_k^{(c,\alpha)})}$ be a Poisson random measure on $[0,\infty) \times (0,\infty]$ with mean measure $\mathbb{LEB} \times \mu_{c,\alpha}$, where $\varepsilon_{(t,\,x)}$ is the probability measure concentrated at $(t,x) \in [0,\infty) \times (0,\infty]$, $\mathbb{LEB}$ is the Lebesgue measure on $[0,\infty)$, and $\mu_{c,\alpha}$ is a measure on $(0,\infty]$ defined by

$$\mu_{c,\alpha}\big((x,\infty]\big) = cx^{-\alpha}, \quad x > 0.$$

Let $D := D[0,\infty)$ denote the Skorokhod space of right-continuous functions defined on $[0,\infty)$ with finite limits from the left at positive points. Throughout the paper we use '$\Rightarrow$' to denote weak convergence in the Skorokhod space D equipped with the J_1-topology. We write '$\Rightarrow$ in S' to denote weak convergence in a space S other than D. Also, we stipulate hereafter that the supremum over the empty set is equal to zero.

Theorem 1.1 treats the situation in which both M_k's and Q_k's affect the limit behavior of the processes in question, whereas in the situation of Theorem 1.5 only the contribution of Q_k's persists in the limit.

Theorem 1.1. *Assume that*

$$\mathbb{E}\log|M| = -a \in (-\infty, 0), \tag{1.7}$$

that

$$\lim_{x\to\infty} x\mathbb{P}\{\log|Q| > x\} = c \tag{1.8}$$

for some $c > 0$. If

$$\mathbb{P}\{Y_k = 0\} = 0 \tag{1.9}$$

for each $k \in \mathbb{N}$, then

$$\frac{\log\big|Y_{[n\cdot]+1}\big|}{an} \Rightarrow \sup_{t_k^{(c/a,1)} \le \cdot} \big(-t_k^{(c/a,1)} + j_k^{(c/a,1)}\big), \quad n \to \infty, \tag{1.10}$$

and if

$$\mathbb{P}\{X_k = 0\} = 0 \tag{1.11}$$

for each $k \in \mathbb{N}$, then

$$\frac{\log\big|X_{[n\cdot]+1}\big|}{an} \Rightarrow g(\cdot) + \sup_{t_k^{(c/a,1)} \le \cdot} \big(t_k^{(c/a,1)} + j_k^{(c/a,1)}\big), \quad n \to \infty, \tag{1.12}$$

where $g(t) := -t$, $t \ge 0$.

Remark 1.2. Conditions (1.9) and (1.11) ensure that the paths of $\log\big|Y_{[n\cdot]+1}\big|$ and $\log\big|X_{[n\cdot]+1}\big|$ belong to D. While a simple sufficient condition for (1.9) to hold is continuity of the law of Q, (1.11) holds if either $X_0 = 0$ a.s. and the law of Q is continuous or the law of X_0 is continuous. Condition (1.9) ((1.11)) is not needed if we (a) replace $\log$ with $\log^+$ in (1.10) ((1.12)); (b) consider weak convergence in $D(0,\infty)$ rather than D. The same remark also concerns Theorem 1.5 given below.

Remark 1.3. Since $X_n \overset{d}{=} Y_n$ for each $n \in \mathbb{N}$ provided that $X_0 = 0$ a.s., the one-dimensional distributions of the limit processes in (1.10) and (1.12) must coincide. Moreover, they can be explicitly computed and are given by

$$\mathbb{P}\Big\{\sup_{t_k^{(c/a,1)}\le u}\big(-t_k^{(c/a,1)}+j_k^{(c/a,1)}\big)\le x\Big\}$$
$$=\mathbb{P}\big\{-u+\sup_{t_k^{(c/a,1)}\le u}\big(t_k^{(c/a,1)}+j_k^{(c/a,1)}\big)\le x\big\}=\Big(\frac{x}{x+u}\Big)^{c/a} \quad (1.13)$$

for $x \ge 0$ and $u > 0$.

Indeed, for $x \ge 0$, the probability on the left-hand side equals

$$\mathbb{P}\big\{N^{(c/a,1)}\big((t,y):t\le u,-t+y>x\big)=0\big\}=\exp\big(-\mathbb{E}N^{(c/a,1)}\big((t,y):t\le u,-t+y>x\big)\big)$$

because $N^{(c/a,1)}\big((t,y):t\le u,-t+y>x\big)$ is a Poisson random variable. It remains to note that

$$\begin{aligned}\mathbb{E}N^{(c/a,1)}\big((t,y):t\le u,-t+y>x\big) &= \int_0^u\int_{[0,\infty)}\mathbb{1}_{\{-t+y>x\}}\,\mu_{c/a,\,1}(\mathrm{d}y)\mathrm{d}t\\ &= (c/a)\int_0^u(x+t)^{-1}\mathrm{d}t\\ &= (c/a)(\log(x+u)-\log x).\end{aligned}$$

Remark 1.4. Theorem 5(ii) in [13] states that, for fixed $a > 0$,

$$\lim_{n\to\infty}\mathbb{P}\Big\{\log\Big(\sum_{k=0}^n e^{-ak}|Q_{k+1}|\Big)\le anx\Big\}=\Big(\frac{x}{x+1}\Big)^{c/a},\quad x\ge 0 \quad (1.14)$$

provided that

$$\lim_{x\to\infty}x\big(1-\mathbb{E}\exp(-e^{-x}|Q|)\big)=c\in(0,\infty).$$

By an Abelian-Tauberian argument the last relation is equivalent to (1.8). This implies that convergence (1.14) follows from (1.10) and (1.13).

Theorem 1.5. *Suppose that* $\mathbb{P}\{M=0\}=0$, $\lim_{n\to\infty}\Pi_n=0$ *a.s., and that*

$$\mathbb{P}\{\log|Q|>x\}\sim x^{-\alpha}\ell(x),\quad x\to\infty \quad (1.15)$$

for some $\alpha\in(0,1]$ *and some* ℓ *slowly varying at* ∞. *Let* (b_n) *be a sequence of positive numbers which satisfy* $\lim_{n\to\infty}n\mathbb{P}\{\log|Q|>b_n\}=1$. *In the case* $\alpha=1$ *assume additionally*[1] *that* $\lim_{x\to\infty}\ell(x)=+\infty$. *In the case* $\mathbb{E}\log^-|M|=\infty$ *assume that*

$$\lim_{x\to\infty}\frac{\mathbb{E}\big(\log^-|M|\wedge x\big)}{x\mathbb{P}\{\log|Q|>x\}}=0. \quad (1.16)$$

If condition (1.9) *holds, then*

$$\frac{\log\big|Y_{[n\cdot]+1}\big|}{b_n}\ \Rightarrow\ \sup_{t_k^{(1,\,\alpha)}\le\cdot}j_k^{(1,\,\alpha)},\quad n\to\infty, \quad (1.17)$$

and if condition (1.11) *holds, then*

$$\frac{\log\big|X_{[n\cdot]+1}\big|}{b_n}\ \Rightarrow\ \sup_{t_k^{(1,\,\alpha)}\le\cdot}j_k^{(1,\,\alpha)},\quad n\to\infty. \quad (1.18)$$

[1] Among other things this implies $\mathbb{E}\log^+|Q|=\infty$.

Remark 1.6. Theorem 5(iii) in [13] states that, for fixed $a > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big\{ \log \Big(\sum_{k=0}^{n} e^{-ak} |Q_{k+1}| \Big) \le b_n x \Big\} = \exp(-x^{-\alpha}), \quad x \ge 0 \tag{1.19}$$

provided that the function $x \mapsto 1 - \mathbb{E} \exp(-e^{-x}|Q|)$ is regularly varying at ∞ with index $-\alpha$, $\alpha \in (0,1)$, and (b_n) satisfies $n\big(1 - \mathbb{E}\exp(-e^{-b_n}|Q|)\big) = 1$. By an Abelian theorem,

$$1 - \mathbb{E}\exp(-e^{-x}|Q|) \ \sim \ \mathbb{P}\{\log|Q| > x\}, \quad x \to \infty.$$

Therefore, (1.19) follows from (1.17) after noting that

$$\mathbb{P}\big\{ \sup_{t_k^{(1,\alpha)} \le u} j_k^{(1,\alpha)} \le x \big\} = \mathbb{P}\big\{ N^{(1,\alpha)}\big((t,y) : t \le u, y > x\big) = 0 \big\} = \exp(-ux^{-\alpha}), \quad x \ge 0 \tag{1.20}$$

for each $u > 0$.

The rest of the paper is structured as follows. In Section 2 we state and prove Theorem 2.1, a deterministic result which is our key tool for dealing with the functional limit theorems. With this at hand, Theorem 1.1 and Theorem 1.5 are then proved in Section 3 and Section 4, respectively.

2 Main technical tool

Denote by M_p the set of Radon point measures ν on $[0,\infty) \times (0,\infty]$ which satisfy

$$\nu([0,T] \times [\delta,\infty]) < \infty \tag{2.1}$$

for all $\delta > 0$ and all $T > 0$. The M_p is endowed with the vague topology. Denote by M_p^* the set of $\nu \in M_p$ which satisfy

$$\nu([0,T] \times (0,\infty]) < \infty$$

for all $T > 0$. Define the mapping G from $D \times M_p$ to D by[2]

$$G(f,\nu)(t) := \begin{cases} \sup\limits_{k:\ \tau_k \le t} (f(\tau_k) + y_k), & \text{if } \tau_k \le t \text{ for some } k, \\ f(0), & \text{otherwise,} \end{cases}$$

where $\nu = \sum_k \varepsilon_{(\tau_k,\, y_k)}$. Also, for each $n \in \mathbb{N}$, we define the mapping F_n from $D \times M_p^*$ to D by

$$F_n(f,\nu)(t) := \begin{cases} c_n^{-1} \log^+ \big| \sum_{k:\, \tau_k \le t} \pm \exp(c_n(f(\tau_k) + y_k)) \big|, & \text{if } \tau_k \le t \text{ for some } k, \\ f^+(0), & \text{otherwise,} \end{cases}$$

where the signs $+$ and $-$ are arbitrarily arranged, and (c_n) is some sequence of positive numbers.

Theorem 2.1. *For $n \in \mathbb{N}$, let $f_n \in D$ and $\nu_n \in M_p$. Let $\big(\tau_k^{(n)}, y_k^{(n)}\big)$ be the points of ν_n, i.e., $\nu_n = \sum_k \varepsilon_{(\tau_k^{(n)},\, y_k^{(n)})}$. Assume that f_0 is continuous with $f_0(0) = 0$ and*

(A1) $\nu_0(\{0\} \times (0,\infty]) = 0$ *and* $\nu_0((r_1,r_2) \times (0,\infty]) \ge 1$ *for all positive* r_1 *and* r_2 *such that* $r_1 < r_2$*;*

[2] Assumption (2.1) ensures that $G(f,\nu) \in D$. If (2.1) does not hold, $G(f,\nu)$ may lose right-continuity.

(A2) $\nu_0 = \sum_k \varepsilon_{\left(\tau_k^{(0)},\, y_k^{(0)}\right)}$ *does not have clustered jumps, i.e.,* $\tau_k^{(0)} \neq \tau_j^{(0)}$ *for* $k \neq j$*;*

(A3) if not all the signs under the sum defining F_n *are the same, then*

$$f_0(\tau_i^{(0)}) + y_i^{(0)} \neq f_0(\tau_j^{(0)}) + y_j^{(0)} \quad \text{for } i \neq j \tag{2.2}$$

and

$$\sup_{\tau_k^{(0)} \le T,\, y_k^{(0)} \le \gamma} \left(f_0(\tau_k^{(0)}) + y_k^{(0)} \right) > 0 \tag{2.3}$$

for each $T > 0$ *such that* $\nu_0(\{T\}, (0,\infty]) = 0$ *and small enough* $\gamma > 0$*;*

(A4) $\lim_{n\to\infty} c_n = \infty$ *and*

$$\lim_{n\to\infty} c_n^{-1} \log \#\{k : \tau_k^{(n)} \le T\} = 0 \tag{2.4}$$

for each $T > 0$ *such that* $\nu_0(\{T\}, (0,\infty]) = 0$*;*

(A5) $\lim_{n\to\infty} f_n = f_0$ *in* D *in the* J_1*-topology.*

(A6) $\lim_{n\to\infty} \nu_n = \nu_0$ *in* M_p*.*

Then

$$\lim_{n\to\infty} F_n(f_n, \nu_n) = G(f_0, \nu_0) \tag{2.5}$$

in D *in the* J_1*-topology.*

Proof. It suffices to prove convergence (2.5) in $D[0,T]$ for any $T > 0$ such that $\nu_0(\{T\} \times (0,\infty]) = 0$ because the last condition ensures that $G(f_0, \nu_0)$ is continuous at T.

If all the signs under the sum defining F_n are the same, then

$$G(f_n, \nu_n)(t) \le F_n(f_n, \nu_n)(t) \le c_n^{-1} \log^+ \#\{k : \tau_k^{(n)} \le t\} + G(f_n, \nu_n)(t)$$

for all $t \in [0,T]$. In this case, (2.5) is a trivial consequence of Theorem 1.3 in [11] which treats the convergence $\lim_{n\to\infty} G(f_n, \nu_n) = G(f_0, \nu_0)$ in D.

In what follows we thus assume that not all the signs are the same. Let $\rho = \{0 = s_0 < s_1 < \cdots < s_m = T\}$ be a partition of $[0,T]$ such that

$$\nu_0(\{s_k\} \times (0,\infty]) = 0, \quad k = 1, ..., m.$$

Pick now $\gamma > 0$ so small that

$$\nu_0((s_k, s_{k+1}) \times (\gamma, \infty]) \ge 1, \quad k = 0, ..., m-1 \tag{2.6}$$

and that $\sup_{\tau_k^{(0)} \le T,\, y_k^{(0)} > \gamma} (f_0(\tau_k^{(0)}) + y_k^{(0)}) > 0$. The latter is possible because $\sup_{\tau_k^{(0)} \le T} (f_0(\tau_k^{(0)}) + y_k^{(0)}) > 0$ as a consequence of (2.3).

Condition (A6) implies that $\nu_0([0,T] \times (\gamma,\infty]) = \nu_n([0,T] \times (\gamma,\infty]) = p$ for large enough n and some $p \ge 1$. Denote by $(\bar{\tau}_i, \bar{y}_i)_{1\le i\le p}$ an enumeration of the points of ν_0 in $[0,T] \times (\gamma,\infty]$ with $\bar{\tau}_1 < \bar{\tau}_2 < \ldots < \bar{\tau}_p$ and by $(\bar{\tau}_i^{(n)}, \bar{y}_i^{(n)})_{1\le i\le p}$ the analogous enumeration of the points of ν_n in $[0,T] \times (\gamma,\infty]$. Then

$$\lim_{n\to\infty} \sum_{i=1}^{p} (|\bar{\tau}_i^{(n)} - \bar{\tau}_i| + |\bar{y}_i^{(n)} - \bar{y}_i|) = 0$$

and more importantly

$$\lim_{n\to\infty} \sum_{i=1}^{p} (|f_n(\bar{\tau}_i^{(n)}) - f_0(\bar{\tau}_i)| + |\bar{y}_i^{(n)} - \bar{y}_i|) = 0 \tag{2.7}$$

because (A5) and the continuity of f_0 imply that $\lim_{n\to\infty} f_n = f_0$ uniformly on $[0,T]$.

Define λ_n to be continuous and strictly increasing functions on $[0,T]$ with $\lambda_n(0)=0$, $\lambda_n(T)=T$, $\lambda_n(\bar\tau_i^{(n)})=\bar\tau_i$ for $i=1,\dots,p$, and let λ_n be linearly interpolated elsewhere on $[0,T]$. For $n\in\mathbb{N}$ and $t\in[0,T]$, set

$$V_n(t) := \sum_{\bar\tau_i=\lambda_n(\bar\tau_i^{(n)})\le t} \pm\exp\big(c_n(f_n(\bar\tau_i^{(n)})+\bar y_i^{(n)})\big)$$

and

$$W_n(t) := \sum_{\lambda_n(\tau_k^{(n)})\le t} \pm\exp\big(c_n(f_n(\tau_k^{(n)})+y_k^{(n)})\big) - V_n(t).$$

With this at hand we have

$$\begin{aligned} d_T(F_n(f_n,\nu_n),G(f_0,\nu_0)) &\le \sup_{t\in[0,\,T]}|\lambda_n(t)-t| \\ &+ c_n^{-1}\sup_{t\in[0,\,T]}\Big|\log^+\big|W_n(t)+V_n(t)\big|-\log^+\big|V_n(t)\big|\Big| \\ &+ \sup_{t\in[0,\,T]}\Big|c_n^{-1}\log^+\big|V_n(t)\big|-\sup_{\bar\tau_i\le t}(f_0(\bar\tau_i)+\bar y_i)\Big| \\ &+ \sup_{t\in[0,T]}\Big|\sup_{\bar\tau_i\le t}\big(f_0(\bar\tau_i)+\bar y_i\big)-\sup_{\tau_k^{(0)}\le t}\big(f_0(\tau_k^{(0)})+y_k^{(0)}\big)\Big|, \end{aligned} \tag{2.8}$$

where d_T is the standard Skorokhod metric on $D[0,T]$.

We treat the terms on the right-hand side of (2.8) separately.

1st term. The relation $\lim_{n\to\infty}\sup_{t\in[0,\,T]}|\lambda_n(t)-t|=0$ is easily checked.

2nd term. We denote the second term by $I_n(\gamma)$ and use inequality

$$|\log^+|x|-\log^+|y||\le\log(1+|x-y|),\quad x,y\in\mathbb{R}$$

which yields

$$\begin{aligned} I_n(\gamma) &\le c_n^{-1}\sup_{t\in[0,T]}\log\big(1+\big|W_n(t)\big|\big) \\ &\le c_n^{-1}\log\Big(1+\sum_{\lambda_n(\tau_k^{(n)})\le T,\,\tau_k^{(n)}\ne\bar\tau_i^{(n)}}\exp\big(c_n(f_n(\tau_k^{(n)})+y_k^{(n)})\big)\Big) \\ &\le c_n^{-1}\log\Big(1+\#\big\{k:\tau_k^{(n)}\le T,\,\tau_k^{(n)}\ne\bar\tau_i^{(n)}\big\} \\ &\times \sup_{\tau_k^{(n)}\le T,\,\tau_k^{(n)}\ne\bar\tau_i^{(n)}}\exp\big(c_n(f_n(\tau_k^{(n)})+y_k^{(n)})\big)\Big) \\ &\le c_n^{-1}\log\#\big\{k:\tau_k^{(n)}\le T\big\}+\sup_{\tau_k^{(n)}\le T,\,\tau_k^{(n)}\ne\bar\tau_i^{(n)}}\big(f_n(\tau_k^{(n)})+y_k^{(n)}\big) \\ &+ \Big(c_n\#\big\{k:\tau_k^{(n)}\le T,\,\tau_k^{(n)}\ne\bar\tau_i^{(n)}\big\}\Big)^{-1} \\ &\times \exp\Big(-c_n\sup_{\tau_k^{(n)}\le T,\,\tau_k^{(n)}\ne\bar\tau_i^{(n)}}\big(f_n(\tau_k^{(n)})+y_k^{(n)}\big)\Big) \end{aligned} \tag{2.9}$$

having utilized $\log(1+x)\le\log x+1/x$, $x>0$ and that $\lambda_n(\tau_k^{(n)})\le T$ iff $\tau_k^{(n)}\le T$. The first term on the right-hand side of (2.9) converges to zero in view of (2.4). As for the second, we apply Theorem 1.3 in [11] to infer

$$\begin{aligned}\sup_{\tau_k^{(n)}\le T,\,\tau_k^{(n)}\neq\bar\tau_i^{(n)}}\big(f_n(\tau_k^{(n)})+y_k^{(n)}\big) &= \sup_{\tau_k^{(n)}\le T,\,y_k^{(n)}\le\gamma}\big(f_n(\tau_k^{(n)})+y_k^{(n)}\big)\\ &\to \sup_{\tau_k^{(0)}\le T,\,y_k^{(0)}\le\gamma}\big(f_0(\tau_k^{(0)})+y_k^{(0)}\big), \qquad (2.10)\end{aligned}$$

as $n\to\infty$. The latter goes to zero as $\gamma\to 0$ because $f_0(0)=0$ by assumption. Finally, the last term on the right-hand side of (2.9) tends to zero as $n\to\infty$ for the exponential factor tends to zero as a consequence of (2.10) and the assumption $\sup_{\tau_k^{(0)}\le T,\,y_k^{(0)}\le\gamma}(f_0(\tau_k^{(0)})+y_k^{(0)})>0$. Summarizing we have proved that $\lim_{\gamma\to 0}\limsup_{n\to\infty} I_n(\gamma)=0$.

3rd term. Denote the third term of (2.8) by J_n. We shall use the inequality

$$J_n\le \sup_{t\in[0,\,T]}A_n(t)+c_n^{-1}\sup_{t\in[0,\,T]}\log^-|V_n(t)|,$$

where $A_n(t):=\Big|c_n^{-1}\log|V_n(t)|-\sup_{\bar\tau_i\le t}(f_0(\bar\tau_i)+\bar y_i)\Big|$, $t\in[0,T]$.

If $t\in[0,\bar\tau_1)$, then $A_n(t)=|f_n(0)-f_0(0)|\to 0$ as $n\to\infty$ by the definition of the functionals. Let now $t\in[\bar\tau_k,\bar\tau_{k+1})$, $k=1,\dots,p-1$ or $t\in[\bar\tau_p,T]$. Since all $\exp(f_0(\bar\tau_1)+\bar y_1),\dots,\exp(f_0(\bar\tau_k)+\bar y_k)$ are distinct by (2.2) and

$$\lim_{n\to\infty}\exp(f_n(\bar\tau_j^{(n)})+\bar y_j^{(n)})=\exp(f_0(\bar\tau_j)+\bar y_j),\quad j=1,\dots,k$$

by (2.7), we conclude that $\exp(f_n(\bar\tau_1^{(n)})+\bar y_1^{(n)}),\dots,\exp(f_n(\bar\tau_k^{(n)})+\bar y_k^{(n)})$ are all distinct, for large enough n. Denote by $a_{k,n}<\ldots<a_{1,n}$ their increasing rearrangement[3] and put

$$B_n(t):=c_n^{-1}\log\left|1\pm\left(\frac{a_{2,n}}{a_{1,n}}\right)^{c_n}\pm\ldots\pm\left(\frac{a_{k,n}}{a_{1,n}}\right)^{c_n}\right|.$$

Since $\lim_{n\to\infty}\bigg(\pm\big(\frac{a_{2,n}}{a_{1,n}}\big)^{c_n}\pm\ldots\pm\big(\frac{a_{k,n}}{a_{1,n}}\big)^{c_n}\bigg)=0$, there is an N_k such that

$$|B_n(t)|\le c_n^{-1}\qquad\text{for } n\ge N_k.$$

Summarizing we have

$$\sup_{t\in[0,\,T]}|B_n(t)|\le c_n^{-1}\quad\text{for all } n\ge\max(N_1,\dots,N_p). \qquad (2.11)$$

With these at hand we can proceed as follows

$$\begin{aligned}A_n(t) &= \Big|\sup_{\bar\tau_i\le t}\big(f_n(\bar\tau_i^{(n)})+\bar y_i^{(n)}\big)+B_n(t)-\sup_{\bar\tau_i\le t}\big(f_0(\bar\tau_i)+\bar y_i\big)\Big|\\ &\le \Big|\sup_{\bar\tau_i\le t}\big(f_n(\bar\tau_i^{(n)})+\bar y_i^{(n)}\big)-\sup_{\bar\tau_i\le t}\big(f_0(\bar\tau_i)+\bar y_i\big)|+|B_n(t)\Big|\\ &\le \sum_{i=1}^p\Big(\big|f_n(\bar\tau_i^{(n)})-f_0(\bar\tau_i)\big|+\big|\bar y_i^{(n)}-\bar y_i\big|\Big)+|B_n(t)|.\end{aligned}$$

In view of (2.7) and (2.11) the right-hand side tends to zero uniformly in $t\in[0,T]$ as $n\to\infty$.

We already know that

$$\lim_{n\to\infty}\sup_{t\in[0,\,T]}c_n^{-1}\log\big|V_n(t)\big|=\sup_{\bar\tau_i\le T}(f_0(\bar\tau_i)+\bar y_i).$$

[3] Although $a_{j,n}$'s depend on t we suppress this dependence for the sake of clarity.

Recalling that

$$\sup_{\bar\tau_i\le T}\big(f_0(\bar\tau_i)+\bar y_i\big)=\sup_{\tau_k^{(0)}\le T,\,y_k^{(0)}>\gamma}\big(f_0(\tau_k^{(0)})+y_k^{(0)}\big)>0$$

we infer $\lim_{n\to\infty}\sup_{t\in[0,\,T]}\big|V_n(t)\big|=+\infty$ and thereupon $\sup_{t\in[0,\,T]}\log^-\big|V_n(t)\big|=0$ for large enough n. Hence $\lim_{n\to\infty}J_n=0$.

4th term. In the proof of Theorem 1.3 in [11] it is shown that[4]

$$\sup_{t\in[0,T]}|\sup_{\bar\tau_i\le t}(f_0(\bar\tau_i)+\bar y_i)-\sup_{\tau_k^{(0)}\le t}(f_0(\tau_k^{(0)})+y_k^{(0)})|\le\omega_{f_0}(2|\rho|)+\gamma,$$

where $|\rho|:=\max_i(s_{i+1}-s_i)$ and $\omega_{f_0}(\varepsilon):=\sup_{|u-v|<\varepsilon,\,u,v\ge0}|f_0(u)-f_0(v)|$ is the modulus of continuity of f_0. Of course, the right-hand side of the last inequality tends to zero on sending $|\rho|$ and γ to zero.

Collecting pieces together and letting in (2.8) $n\to\infty$ and then $|\rho|$ and γ tend to zero we arrive at the desired conclusion

$$\lim_{n\to\infty}d_T(F_n(f_n,\nu_n),G(f_0,\nu_0))=0.$$

□

3 Proof of Theorem 1.1

Proof of (1.10). We first show that

$$\frac{\log^-|Y_{[n\cdot]+1}|}{an}\ \Rightarrow\ h(\cdot),\tag{3.1}$$

where $h(t)=0$, $t\ge0$. To this end, we intend to check that conditions (1.2), (1.3) and (1.6) hold. If they do, then, as $n\to\infty$, $|Y_n|\overset{\mathrm{P}}{\to}\infty$ by Theorem 2.1 in [8] and thereupon $\sup_{t\in[0,\,T]}|Y_{[nt]+1}|=\sup_{1\le k\le[nT]+1}|Y_k|\overset{\mathrm{P}}{\to}\infty$ for each $T>0$. This entails $\sup_{t\in[0,\,T]}\log^-|Y_{[nt]+1}|=0$ for each $T>0$ and large enough n which proves (3.1). Assumption (1.7) entails $\lim_{n\to\infty}\Pi_n=0$ a.s. and $\mathbb{P}\{M=0\}=0$. Condition $\mathbb{P}\{Q=0\}=0$ is a part of (1.9). Suppose $Q+Mr=r$ a.s. for some $r\in\mathbb{R}$. In view of $\mathbb{P}\{Q=0\}=0$ we have $r\neq0$ and then $|Q|/|r|=|1-M|\le1+|M|$ a.s. Since $\mathbb{E}\log(1+|M|)<\infty$ by (1.7) we must have $\mathbb{E}\log^+|Q|<\infty$. This contradiction completes the proof of (3.1).

For $k\in\mathbb{N}_0$, set $S_k:=\log|\Pi_k|$ and $\eta_{k+1}:=\log|Q_{k+1}|$. As a consequence of the strong law of large numbers,

$$\frac{S_{[n\cdot]}}{an}\Rightarrow g(\cdot),\ \ n\to\infty,\tag{3.2}$$

where $g(t):=-t$, $t\ge0$ (actually, in (3.2) the a.s. convergence holds, see Theorem 4 in [7]). According to Corollary 4.19 (ii) in [16] condition (1.8) entails

$$\sum_{k\ge0}\mathbb{1}_{\{\eta_{k+1}>0\}}\,\varepsilon_{(n^{-1}k,\,(an)^{-1}\eta_{k+1})}\ \Rightarrow\ N^{(c/a,1)},\ \ n\to\infty\tag{3.3}$$

in M_p, see Section 2 for the definition of M_p. Now relations (3.2) and (3.3) can be combined into the joint convergence

$$\Big((an)^{-1}S_{[n\cdot]},\sum_{k\ge0}\mathbb{1}_{\{\eta_{k+1}>0\}}\,\varepsilon_{(n^{-1}k,\,(an)^{-1}\eta_{k+1})}\Big)\ \Rightarrow\ \big(g(\cdot),N^{(c/a,1)}\big)\ \text{as}\ \ n\to\infty$$

[4] Condition (2.6) is only used in this part of the proof.

in $D \times M_p$. By the Skorokhod representation theorem there are versions which converge a.s. Retaining the original notation for these versions we want to apply Theorem 2.1 with $f_n(\cdot) = (an)^{-1} S_{[n\cdot]}$, $f_0 = g$, $\nu_n = \sum_{k\geq 0} \mathbb{1}_{\{\eta_{k+1}>0\}} \varepsilon_{\{n^{-1}k,\,(an)^{-1}\eta_{k+1}\}}$, $\nu_0 = N^{(c/a,1)}$, $c_n = an$ and the signs $\pm$ defined by $\operatorname{sgn}(\Pi_k Q_{k+1})$ to conclude

$$\frac{\log^+ |Y_{[n\cdot]+1}|}{an} \;\Rightarrow\; \sup_{t_k^{(c/a,1)}\leq \cdot} \big(- t_k^{(c/a,1)} + j_k^{(c/a,1)} \big).$$

Of course, this together with (3.1) proves (1.10).

Thus it remains to check that all the assumptions of Theorem 2.1 hold. We already know that conditions (A5) and (A6) are fulfilled. Condition (2.4) holds trivially. Further $N^{(c/a,1)}([0,T]\times[\delta,\infty]) < \infty$ a.s. for all $\delta > 0$ and all $T > 0$ because $\mu_{c/a,1}([\delta,\infty]) < \infty$. Plainly, $N^{(c/a,1)}(\{0\}\times(0,+\infty]) = 0$ a.s., and $N^{(c/a,1)}((r_1,r_2)\times(0,\infty]) \geq 1$ a.s. whenever $0 < r_1 < r_2$ because $\mu_{c/a,1}((0,\infty]) = \infty$. This gives (A1).

Next we check (2.2). Our argument is similar to that given on p. 223 in [17]. We fix any $T > 0$, $\delta > 0$ and use the representation

$$N^{(c/a,1)}([0,T]\times(\delta,\infty]\cap\cdot) = \sum_{k=1}^{N} \varepsilon_{(U_k,V_k)}(\cdot),$$

where (U_i) are i.i.d. with the uniform distribution on $[0,T]$, (V_j) are i.i.d. with $\mathbb{P}\{V_1 \leq x\} = (1-\delta/x)\,\mathbb{1}_{(\delta,\infty)}(x)$, and N has the Poisson distribution with parameter $Tc/(a\delta)$, all the random variables being independent. Since $-U_1 + V_1$ has a continuous distribution we have

$$\mathbb{P}\{N \geq 2, -U_k + V_k = -U_i + V_i \quad \text{for some } 1 \leq k < j \leq N\} = 0$$

which entails (2.2). An analogous argument leads to the conclusion that $N^{(c/a,1)}$ does not have clustered jumps a.s., i.e., (A2) holds. The last thing that needs to be checked is condition (2.3). Arguing as in Remark 1.3 we infer

$$\begin{aligned} &\mathbb{P}\Big\{ \sup_{t_k^{(c/a,1)}\leq T,\, j_k^{(c/a,\,1)}\leq\gamma} (-t_k^{(c/a,1)} + j_k^{(c/a,1)}) \leq 0 \Big\} \\ =\;& \exp\Big(- \mathbb{E}N^{(c/a,1)}\big((t,y): t\leq T, y\leq\gamma, y>t\big)\Big) \\ =\;& \exp\Big(-(c/a)\int_0^\gamma (t^{-1}-\gamma^{-1})\mathrm{d}t\Big) = 0 \end{aligned}$$

for any $T > 0$ and any $\gamma \in (0,T)$.

Proof of (1.12). Without loss of generality we assume that $X_0 = 0$ a.s. and use the representation

$$X_{[n\cdot]+1} = \Pi_{[n\cdot]+1} \sum_{k=0}^{[n\cdot]} \Pi_k^* Q_{k+1}^*, \tag{3.4}$$

where $\Pi_k^* := \Pi_k^{-1}$, $k \in \mathbb{N}_0$ and $Q_k^* := Q_k/M_k$ (with generic copy Q^*), $k \in \mathbb{N}$.

Observe that

$$\frac{\sup_{0\leq t\leq T} |S_{[nt]+1} - S_{[nt]}|}{n} = \frac{\max_{1\leq k\leq [nT]+1} \big|\log|M_k|\big|}{n} \overset{P}{\to} 0, \quad n\to\infty$$

for every $T > 0$, because $\lim_{x\to\infty} x\mathbb{P}\big\{\big|\log|M|\big| > x\big\} = 0$ as a consequence of $\mathbb{E}|\log|M|| < \infty$. This together with (3.2) proves

$$\frac{\log|\Pi_{[n\cdot]+1}|}{an} \Rightarrow g(\cdot), \quad n\to\infty, \tag{3.5}$$

where $g(t)=-t$, $t\geq 0$. Further, write, for $\varepsilon\in(0,1)$ and $x>0$,

$$\begin{aligned}\mathbb{P}\{\log|Q|>(1+\varepsilon)x\}-\mathbb{P}\{\log|M|>\varepsilon x\} &\leq \mathbb{P}\{\log|Q|-\log|M|>x\}\\ &\leq \mathbb{P}\{\log|Q|>(1-\varepsilon)x\}\\ &+ \mathbb{P}\{\log^-|M|>\varepsilon x\}. \end{aligned} \tag{3.6}$$

Multiplying the inequality by x, sending $x\to\infty$ and then $\varepsilon\to 0$ yields

$$\mathbb{P}\{\log|Q^*|>x\}=\mathbb{P}\{\log|Q|-\log|M|>x\}\ \sim\ \mathbb{P}\{\log|Q|>x\}\ \sim\ cx^{-1},\ \ x\to\infty.$$

Set $M^*:=1/M$. Conditions (1.2) and (1.3) with (M,Q) replaced by (M^*,Q^*) are easily checked. Also, we have $\lim_{n\to\infty}\Pi_n^*=\infty$ a.s. Hence $|\sum_{k=1}^n\Pi_{k-1}^*Q_k^*|\overset{\mathrm{P}}{\to}\infty$ as $n\to\infty$ by Theorem 2.1 in [8]. Arguing in the same way as in the proof of (1.10) we see that

$$\frac{\log^-|\sum_{k=0}^{[n\cdot]}\Pi_k^*Q_{k+1}^*|}{an}\ \Rightarrow\ h(\cdot),\quad n\to\infty.$$

An application of Theorem 2.1 gives[5]

$$\frac{\log^+|\sum_{k=0}^{[n\cdot]}\Pi_k^*Q_{k+1}^*|}{an}\ \Rightarrow\ \sup_{t_k^{(c/a,\,1)}\leq\cdot}\big(t_k^{(c/a,1)}+j_k^{(c/a,1)}\big),\quad n\to\infty.$$

Now (1.12) follows by a combination of the last two relations and (3.5).

4 Proof of Theorem 1.5

The proof proceeds along the lines of that of Theorem 1.1 but is simpler for the contribution of M_k's is negligible. Therefore we only provide details for fragments which differ principally from the corresponding ones in the proof of Theorem 1.1.

Observe that

$$\lim_{n\to\infty}\frac{b_n}{n}=+\infty \tag{4.1}$$

as follows from the definition of (b_n) and (1.15).

Proof of (1.17). As far as

$$\frac{\log^-|Y_{[n\cdot]+1}|}{b_n}\ \Rightarrow\ h(\cdot),\quad n\to\infty \tag{4.2}$$

is concerned which is the counterpart of (3.1) we have to check two things that are not obvious in the case when $\mathbb{E}\log^-|M|=\infty$: condition (1.3) and $I=\int_{(1,\infty)}\frac{\log x}{A(\log x)}\mathbb{P}\{|Q|\in \mathrm{d}x\}=\infty$.

Assume first that $\mathbb{P}\{Q+Mr=r\}=1$ for some $r\neq 0$. In view of $|Q-r|=|M||r|$, the tails of $\log^+|Q|$ and $\log^+|M|$ must exhibit the same asymptotics. However, this is not a case, for the tail of $\log^+|Q|$ is heavier than that of $\log^+|M|$.

Next, according to (1.16), for any $B>0$ there exists $x_0>0$ such that

$$\frac{\log x}{A(\log x)}\geq\frac{B}{\mathbb{P}\{|Q|>x\}}$$

whenever $x\geq x_0$. Hence,

$$I\geq B\int_{[x_0,\infty)}\frac{\mathbb{P}\{|Q|\in\mathrm{d}x\}}{\mathbb{P}\{|Q|>x\}}=\infty.$$

[5] We omit details which are very similar to but simpler than those appearing in the proof of (1.10).

Thus, (4.2) holds.

To proceed we recall the already used notation $S_k := \log|\Pi_k|$ and $\eta_{k+1} := \log|Q_{k+1}|$, $k \in \mathbb{N}_0$. According to Corollary 4.19 (ii) in [16] condition (1.15) entails

$$\sum_{k\geq 0} \mathbb{1}_{\{\eta_{k+1}>0\}}\, \varepsilon_{(n^{-1}k,\, b_n^{-1}\eta_{k+1})} \;\Rightarrow\; N^{(1,\alpha)}, \quad n\to\infty \tag{4.3}$$

in M_p. If we can prove that

$$\frac{S_{[n\cdot]}}{b_n} \;\Rightarrow\; h(\cdot), \quad n\to\infty, \tag{4.4}$$

where $h(t)=0$, $t\geq 0$, then relations (4.3) and (4.4) can be combined into the joint convergence

$$\Big(b_n^{-1}S_{[n\cdot]}, \sum_{k\geq 0} \mathbb{1}_{\{\eta_{k+1}>0\}}\, \varepsilon_{(n^{-1}k,\, b_n^{-1}\eta_{k+1})}\Big) \;\Rightarrow\; \big(h(\cdot), N^{(1,\alpha)}\big), \quad n\to\infty$$

in $D\times M_p$. By the Skorokhod representation theorem there are versions which converge a.s. Retaining the original notation for these versions we apply Proposition 2.1 with $f_n(\cdot)=b_n^{-1}S_{[n\cdot]}$, $f_0=h$, $\nu_n=\sum_{k\geq 0}\mathbb{1}_{\{\eta_{k+1}>0\}}\,\varepsilon_{\{n^{-1}k,\, b_n^{-1}\eta_{k+1}\}}$, $\nu_0=N^{(1,\alpha)}$, $c_n=b_n$ and the signs $\pm$ defined by $\operatorname{sgn}(\Pi_k Q_{k+1})$ which gives (1.17) with $\log$ replaced with $\log^+$. The latter in combination with (4.2) proves (1.17).

It only remains to check (4.4). To this end, it suffices to prove that

$$\frac{\sup_{0\leq t\leq T}|S_{[nt]}|}{b_n} = \frac{\max_{0\leq k\leq [nT]}|S_k|}{b_n} \overset{P}{\to} 0, \quad n\to\infty \tag{4.5}$$

for every $T>0$. Set

$$S_0^+ = S_0^- := 0,\; S_n^+ := \log^+|M_1| + \ldots + \log^+|M_n|,\; S_n^- := \log^-|M_1| + \ldots + \log^-|M_n|$$

for $n\in\mathbb{N}$. Since (b_n) is a regularly varying sequence and

$$\max_{0\leq k\leq [nT]}|S_k| \leq \max_{0\leq k\leq [nT]} S_k^+ + \max_{0\leq k\leq [nT]} S_k^- = S_{[nT]}^+ + S_{[nT]}^-,$$

(4.5) follows if we prove that $\lim_{n\to\infty}(S_n^{\pm}/b_n)=0$ in probability. While doing so, we treat two cases separately.

Case when $\mathbb{E}\log^-|M|<\infty$. Then necessarily $\mathbb{E}\log^+|M|<\infty$ for otherwise $\lim_{n\to\infty}\Pi_n=\infty$ a.s. Therefore we have $\lim_{n\to\infty} n^{-1}S_n^{\pm} = \mathbb{E}\log^{\pm}|M|$ by the strong law of large numbers. Invoking (4.1) proves (4.5).

Case when $\mathbb{E}\log^-|M|=\infty$. Condition (1.16) entails $\lim_{n\to\infty}\frac{n}{b_n}\mathbb{E}\big((\log^-|M|)\wedge b_n\big)=0$. Since

$$\frac{n}{b_n}\mathbb{E}\big((\log^-|M|)\wedge b_n\big) = n\mathbb{P}\{\log^-|M|>b_n\} + \frac{n}{b_n}\mathbb{E}\log^-|M|\,\mathbb{1}_{\{\log^-|M|\leq b_n\}},$$

we infer

$$\lim_{n\to\infty} n\mathbb{P}\{\log^-|M|>b_n\}=0 \tag{4.6}$$

and

$$\lim_{n\to\infty}\frac{n}{b_n}\mathbb{E}\big(\log^-|M|\,\mathbb{1}_{\{\log^-|M|\leq b_n\}}\big)=0. \tag{4.7}$$

Using (4.7) together with Markov's inequality proves

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n \log^-|M_k|\,\mathbb{1}_{\{\log^-|M_k|\leq b_n\}}}{b_n}=0 \;\text{ in probability.}$$

Since

$$\begin{aligned}\mathbb{P}\Big\{b_n^{-1}\sum_{k=1}^n \log^-|M_k| \neq b_n^{-1}\sum_{k=1}^n \log^-|M_k|\,1\!\!1_{\{\log^-|M_k|\le b_n\}}\Big\} &\le \sum_{k=1}^n \mathbb{P}\{\log^-|M_k|>b_n\}\\ &= n\mathbb{P}\{\log^-|M|>b_n\},\end{aligned}$$

(4.6) implies that the left-hand side tends to zero as $n\to\infty$. Therefore $\lim_{n\to\infty}(S_n^-/b_n)=0$ in probability.

Left with proving that $\lim_{n\to\infty}(S_n^+/b_n)=0$ in probability we suppose immediately that $\mathbb{E}\log^+|M|=\infty$ for the complementary case can be treated in exactly the same way as above. Since $\lim_{n\to\infty}S_n=-\infty$ a.s. by the assumption, Lemma 8.1 in [14] tells us that $\lim_{n\to\infty}S_n^+/S_n^-=0$ a.s. which together with $\lim_{n\to\infty}(S_n^-/b_n)=0$ in probability implies $\lim_{n\to\infty}(S_n^+/b_n)=0$ in probability. The proof of (4.4) is complete. Hence so is that of (1.17).

Proof of (1.18) follows the pattern of that of (1.12) but is simpler. Referring to (1.12) the only things that need to be checked are that

$$\frac{\log|\Pi_{[n\cdot]+1}|}{b_n}\Rightarrow h(\cdot),\quad n\to\infty,$$

where $h(t)=0$, $t\ge 0$, and that

$$\mathbb{P}\{\log|Q|-\log|M|>x\}\ \sim\ \mathbb{P}\{\log|Q|>x\}\ \sim\ x^{-\alpha}\ell(x),\ \ x\to\infty.$$

To prove the first of these, write

$$\frac{\sup_{0\le t\le T}|S_{[nt]+1}-S_{[nt]}|}{b_n}\le\frac{\sup_{0\le t\le T}|S_{[nt]+1}|}{b_n}+\frac{\sup_{0\le t\le T}|S_{[nt]}|}{b_n}$$

and use (4.5) to infer

$$\frac{\sup_{0\le t\le T}|S_{[nt]+1}-S_{[nt]}|}{b_n}\ \overset{P}{\to}\ 0,\ \ n\to\infty$$

for every $T>0$. To check the second we shall use (3.6).

Case $\mathbb{E}\log^-|M|<\infty$. We have $\lim_{x\to\infty}x\mathbb{P}\{\log^-|M|>\varepsilon x\}=0$ whereas $\lim_{x\to\infty}x\mathbb{P}\{\log|Q|>x\}=\infty$ (recall that in the case $\alpha=1$ we assume that $\lim_{x\to\infty}\ell(x)=\infty$). Therefore,

$$\lim_{x\to\infty}\frac{\mathbb{P}\{\log^-|M|>\varepsilon x\}}{\mathbb{P}\{\log|Q|>x\}}=0. \tag{4.8}$$

Since $\mathbb{E}\log^-|M|<\infty$ entails $\mathbb{E}\log^+|M|<\infty$, the same argument proves (4.8) for the tail of $\log^+|M|$.

Case $\mathbb{E}\log^-|M|=\infty$ *and* $\mathbb{E}\log^+|M|<\infty$. It suffices to check (4.8) which is a consequence (1.16).

Case $\mathbb{E}\log^-|M|=\mathbb{E}\log^+|M|=\infty$. We only have to prove that

$$\lim_{x\to\infty}\frac{\mathbb{P}\{\log^+|M|>\varepsilon x\}}{\mathbb{P}\{\log|Q|>x\}}=0.$$

Since $\lim_{n\to\infty}S_n=-\infty$ a.s. by the assumption, we have

$$\mathbb{E}\frac{\log^+|M|}{A(\log^+|M|)}<\infty$$

(see Proposition 2.6 in [8]). Therefore

$$\lim_{x\to\infty}\frac{x\mathbb{P}\{\log^+|M|>x\}}{\mathbb{E}(\log^-|M|\wedge x)}=0$$

and the desired relation follows by an application of (1.16).

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