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Gaussian integrability of distance function under the Lyapunov condition

Yuan Liu*

Abstract

In this note we give a direct proof of the Gaussian integrability of distance function as $\mu e^{\delta d^2(x,x_0)} < \infty$ for some $\delta > 0$ provided the Lyapunov condition holds for symmetric diffusion operators, which answers a question in Cattiaux-Guillin-Wu [6, Page 295]. The similar argument still works for diffusions processes with unbounded diffusion coefficients and for jump processes such as birth-death chains. An analogous discussion is also made under the Gozlan's condition arising from [9, Proposition 3.5].

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1 Introduction

In this note, we will investigate how to directly derive the Gaussian integrability from two kinds of criteria for the Talagrand's inequality W_2H (or T_2 in short), say the Lyapunov condition and Gozlan's condition presented in a symmetric diffusion Markov setting. Referring to Bakry-Gentil-Ledoux [2], we denote by E a complete connected Riemannian manifold of finite dimension, d the geodesic distance, $\mathrm{d}x$ the volume measure, $\mu(\mathrm{d}x) = e^{-V(x)}\mathrm{d}x$ a probability measure with $V \in C^2(E)$, $\mathrm{L} = \Delta - \nabla V \cdot \nabla$ the μ-symmetric diffusion operator, $\Gamma(f,g) = \nabla f \cdot \nabla g$ the carré du champ operator, and $\mathcal{E}$ the associated Dirichlet form, which satisfy the formula for integration by parts

$$\int \nabla f \cdot \nabla g \mathrm{d}\mu = -\int f \mathrm{L} g \mathrm{d}\mu, \quad f \in \mathcal{D}(\mathcal{E}), g \in \mathcal{D}(\mathrm{L}).$$

First of all, say $W \geqslant 1$ is a Lyapunov function if there exist two constants $b \geqslant 0$ and $c > 0$ such that for some $x_0 \in E$ and any $x \in E$

$$\mathrm{L}W \leqslant (-cd^2(x,x_0) + b)W. \tag{1.1}$$

More generally, to avoid assuming the integrability and second-order differentiability of W, it is convenient to introduce a locally Lipschitz function $U > 0$ such that in the sense of distribution

$$\mathrm{L}U + |\nabla U|^2 \leqslant -cd^2(x,x_0) + b, \tag{1.2}$$

*Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China.
E-mail: liuyuan@amss.ac.cn

which means that for any nonnegative $h \in C_c^\infty(E)$ holds

$$\int \left(\mathrm{L}U + |\nabla U|^2\right) h\mathrm{d}\mu := \int U\mathrm{L}h + |\nabla U|^2\, h\mathrm{d}\mu \leqslant \int \left(-cd^2(x,x_0) + b\right) h\mathrm{d}\mu.$$

When $W \in C^2(E)$, (1.1) and (1.2) are equivalent by taking $U = \log W$. And it is not hard to see that (1.1) implies a weaker version for some c', b' and R

$$\mathrm{L}W \leqslant -c'W + b'\mathbf{1}_{B(0,R)}. \tag{1.3}$$

The Lyapunov condition plays a powerful role in studying functional inequalities or estimating convergence rate of Markov processes, which even works as a substitute of curvature-dimension condition sometimes. Cattiaux-Guillin [4] gave a comprehensive review on this topic, and here we would like to take partial literature into account. A simple proof of the Poincaré inequality through (1.3) can be found in Bakry-Barthe-Cattiaux-Guillin [1]. The L^1 transport-information inequality W_1I was discussed further under (1.1) by Guillin-Léonard-Wu-Yao [12]. Then Cattiaux-Guillin-Wu [6] showed the Talagrand's inequality and logarithmic Sobolev inequality (LSI for short) provided (1.2), which was also applied to weighted LSIs for heavy tailed distributions by [7]. Most recently, Guillin-Joulin [10] obtained non-Gaussian concentration estimates by means of functional inequalities with some kind of Lyapunov condition yet.

According to [6, Lemma 3.5], it was proved that if (1.2) holds, there exist some $\delta > 0$ and $x_0 \in E$ such that

$$\int e^{\delta d^2(x,x_0)}\mathrm{d}\mu(x) < \infty, \tag{1.4}$$

which is necessary to derive W_2H. Their proof starts from (1.2) and the spectral gap to show W_1I due to [12]. It then follows a L^1 transport-entropy inequality W_1H by Guillin-Léonard-Wang-Wu [11], which is equivalent to (1.4) by Djellout-Guillin-Wu [8]. The strategy relies on a series of works on transport inequalities, thereupon the authors of [6] feel interested in finding a simple or direct proof of (1.4), see [6, Page 295].

Indeed, there exists an elementary proof, and we actually obtain

Proposition 1.1. *If (1.2) holds, then* $\mu e^{\delta d^2(x,x_0)} < \infty$ *for any* $\delta < \sqrt{c}$.

Remark 1.2. The upper bound for δ is sharp. For instance, let $\mathrm{d}\mu = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}|x|^2}\mathrm{d}x$ and $\mathrm{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x}$ associated to one-dimensional Ornstein-Uhlenbeck process, then $W = e^{\frac{1}{4}|x|^2}$ satisfies $\mathrm{L}W \leqslant (-\frac{1}{4}|x|^2 + \frac{1}{2})W$, which exactly gives $\delta < \sqrt{c} = \frac{1}{2}$.

Remark 1.3. A weak version $\mathrm{L}W \leqslant (-cd^p(x,x_0) + b)W$ with $p < 2$ is not enough to derive the Gaussian integrability, since $W = \exp\left(\frac{1}{2}(1+|x|^2)^q\right)$ with $2(q-1) = p$ fulfills (1.2) with respect to $\mathrm{d}\mu = \frac{1}{Z}e^{-(1+|x|^2)^{\frac{p}{2}}}\mathrm{d}x$, where Z is a normalized factor.

The same argument can be extended to diffusion processes with unbounded diffusion coefficients. Define an infinitesimal generator in $\mathbb{R}^m$

$$\mathrm{L}_a = \frac{1}{2}\sum_{i,j=1}^m a^{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^m b^i(x)\frac{\partial}{\partial x_i},$$

where $A = (a^{ij})_{i,j=1}^m$ is symmetric positive-definite and $b^i = \frac{1}{2}\left(\sum_{j=1}^m \frac{\partial a^{ij}}{\partial x_j} - a^{ij}\frac{\partial V}{\partial x_j}\right)$ so that L_a admits an invariant probability measure $\mathrm{d}\mu(x) = e^{-V}\mathrm{d}x$. Then define the Carrédu champ operator by means of

$$\Gamma_a(f,g) = \frac{1}{2}\left[\,\mathrm{L}_a(fg) - f\mathrm{L}_a g - g\mathrm{L}_a f\,\right] = \frac{1}{2}\langle\nabla f, A\nabla g\rangle,$$

which satisfies the integration by parts formula for $f, g \in C_c^\infty(\mathbb{R}^m)$

$$-\int f \mathrm{L}_a g \mathrm{d}\mu = \int \Gamma_a(f,g)\mathrm{d}\mu =: \mathcal{E}_a(f,g).$$

Thanks to the Lyapunov type criterion by Stroock-Varadhan [13, Theorem 10.2.1], it can be quickly derived that L_a corresponds to a non-explosive diffusion process provided that (1.1) holds by substituting L to L_a

$$\mathrm{L}_a W \leqslant (-cd^2(x,x_0)+b)W \tag{1.5}$$

with $\lim_{|x|\to\infty} W = \infty$.

However, if a^{ij} is unbounded, (1.5) is not enough to get the Gaussian integrability for μ. Consider one-dimensional case, when $a^{ij} = a(x) = o(|x|^4)$, we take $V = \frac{x^2}{2\sqrt{a}}$ and $W = e^{\delta V}$ for small $\delta > 0$ so that (1.5) holds but V has a growth rate slower than quadratic. On the other hand, when $a(x) = O(|x|^4)$ or grows even faster, (1.5) is useless to yield a Poincaré type inequality so that we have no effective calculus on the integrability of $e^{\delta d^2(x,x_0)}$. For this reason, a stronger condition is necessary.

Proposition 1.4. *Let $\lambda_{\max}$ be the maximal eigenvalue of A satisfying $\mu\lambda_{\max} < \infty$. Suppose there exists a Lyapunov function $W \geqslant 1$ with two constants $b \geqslant 0$ and $c > 0$ such that for some $x_0 \in \mathbb{R}^m$ and any $x \in \mathbb{R}^m$*

$$\mathrm{L}_a W \leqslant (-cd^2(x,x_0)+b)\lambda_{\max} W. \tag{1.6}$$

Then $\mu\left(e^{\delta d^2(x,x_0)}\lambda_{\max}\right) < \infty$ for any $\delta < \sqrt{c}$.

Remark 1.5. (1.6) is natural, for instance, if there exist V and W satisfying (1.1) over $\mathbb{R}$, then (1.6) follows automatically provided that $\lim\limits_{|x|\to\infty} \frac{a'W'}{aW|x|^2} = 0$. Moreover, there is no need to assume $\lambda_{\max} \geqslant \lambda > 0$ uniformly on $\mathbb{R}^m$.

Another possible extension is about jump processes (see Bass [3]). To clarify the effect from jumps part, we simply consider the infinitesimal generator of the form

$$\mathrm{L}_\nu = \int_{\mathbb{R}^m - \{0\}} \left[f(x+y) - f(x) - \nabla f \cdot y \mathbf{1}_{0<|y|<1}(y)\right] \nu(x, \mathrm{d}y),$$

Where ν satisfies $\int_{\mathbb{R}^m-\{0\}} \min\{1, |y|^2\}\nu(x,\mathrm{d}y) < \infty$. Suppose that L_ν admits an invariant probability measure μ, and the Carrédu champ operator

$$\begin{aligned}\Gamma_\nu(f,g) &= \frac{1}{2}\left[\, \mathrm{L}_\nu(fg) - f\mathrm{L}_\nu g - g\mathrm{L}_\nu f \,\right] \\ &= \frac{1}{2}\int_{\mathbb{R}^m-\{0\}} [f(x+y)-f(x)]\,[g(x+y)-g(x)]\,\nu(x,\mathrm{d}y)\end{aligned}$$

fulfills the integration by parts formula for $f, g \in C_c^\infty(\mathbb{R}^m)$

$$-\int f\mathrm{L}_\nu g \mathrm{d}\mu = \int \Gamma_\nu(f,g)\mathrm{d}\mu =: \mathcal{E}_\nu(f,g).$$

Then define an intrinsic (pseudo)metric according to Sturm [14, Definition 6.5]

$$\rho(x,y) := \sup\{f(x) - f(y) : \Gamma_\nu(f,f) \leqslant 1\},$$

which gives $\Gamma_\nu(\rho(x,x_0), \rho(x,x_0)) \leqslant 1$ if $\rho(x,x_0) \in \mathcal{D}(\mathcal{E}_\nu)$. For convenience, we also require that $\lim_{|x|\to\infty} \rho(x,x_0) = \infty$.

The setting includes discrete Markov chains. For example, consider a birth-death process on $\mathbb{N}$ with strictly positive birth rates b_i and death rates d_i except $d_0 = 0$.

Let $r_0 = 1$ and $r_i = \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}$ for $i \geqslant 1$, we can take $\nu(i,y) = b_i \delta_1(y) + d_i \delta_{-1}(y)$ and $\mu(i) = \frac{r_i}{r_0 + r_1 + \cdots}$ provided the series converges, and then $\mathcal{E}_\nu$ has an alternative expression $\mathcal{E}_\nu(f,g) = \sum_{i=0}^{\infty}[f(i+1) - f(i)][g(i+1) - g(i)]b_i \mu_i$, which determines the intrinsic metric $\rho(i,j) = b_i^{-\frac{1}{2}} + b_{i+1}^{-\frac{1}{2}} + \cdots b_{j-1}^{-\frac{1}{2}}$ for $i \leqslant j$.

Proposition 1.6. *Suppose there exist some $x_0 \in \mathbb{R}^m$ and a constant $K > 0$ such that for all $x \in \mathbb{R}^m$ and all $y \in \mathrm{Supp}\nu$*

$$\left|\rho^2(x+y, x_0) - \rho^2(x, x_0)\right| \leqslant K. \tag{1.7}$$

Suppose also there exists a Lyapunov function $W \geqslant 1$ with two constants $b \geqslant 0$ and $c > 0$ such that for any $x \in \mathbb{R}^m$

$$\mathrm{L}_\nu W \leqslant (-c\rho^2(x, x_0) + b)W. \tag{1.8}$$

Then $\mu e^{\delta \rho^2(x,x_0)} < \infty$ for $\delta < C \min\{\sqrt{c}, K^{-1}\}$ with some multiple $C \in (0,1]$.

Remark 1.7. For a birth-death process referring to Cattiaux-Guillin-Wang-Wu [5], let $b_i = d_i = i^a \log^\alpha(i+1)$ with $a \geqslant 2$ and $\alpha \in \mathbb{R}$ except $b_0 = 1$, let $W = 1 + i^\gamma$ with $0 < \gamma < 1$, then $\mu(i) \asymp b_i^{-1}$ and $\mathrm{L}_\nu W \leqslant -ci^{a-2}\log^\alpha(i+1)W$. Take $a = 2, \alpha = 1, \gamma = \frac{1}{2}$ explicitly, it follows $\rho(i,0) \asymp \log^{\frac{1}{2}}(i+1)$ satisfying (1.7-1.8) and then $\mu e^{\delta\rho^2} < \infty$ for any $\delta < 1$. The combination of (1.7) and (1.8) is necessary. When $a = 2, \alpha < 1, \gamma = \frac{1}{2}$, (1.7) still holds, but (1.8) fails and so does the Gaussian integrability. On the other hand, let $b_i = (i+1)^{\frac{1}{2}}$ and $d_i = ib_i$, then $\mu(i) \asymp (i!b_i)^{-1}$, $\rho(i,0) \asymp i^{\frac{3}{4}}$, and (1.8) holds for $W = 2^i$, but (1.7) fails and so does the Gaussian integrability again.

We further investigate another criterion for transport inequalities. According to Gozlan [9, Proposition 3.5], let μ be a probability on R^m, suppose there exists $\omega \in C^3(\mathbb{R})$ with $\omega'(0) > 0$, $\left|\frac{\omega^{(3)}}{\omega'^3}\right| \leqslant M$ for some constant M, and

$$\liminf_{|x|\to\infty} \frac{1}{u^2}\sum_{i=1}^{m}\left[\frac{1}{10}\left(\frac{\partial V}{\partial x_i}\right)^2\left(\frac{x}{u}\right) - \frac{\partial^2 V}{\partial x_i^2}\left(\frac{x}{u}\right)\right]\frac{1}{\omega'(x_i)^2} > mM \tag{1.9}$$

for some constant $u > 0$, then a transport-entropy inequality holds with the cost function $d_\omega(x,y) = \left(\sum_{i=1}^{m}|\omega(x_i) - \omega(y_i)|^2\right)^{\frac{1}{2}}$. An interesting case is to set

$$\omega(t) = \int_0^t \sqrt{1+s^2}\mathrm{d}s = \frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left|t + \sqrt{1+t^2}\right|$$

satisfying $\omega'(0) = 1$ and $\left|\frac{\omega^{(3)}}{\omega'^3}(t)\right| = (1+t^2)^{-3} \leqslant 1$, which corresponds to W_2H.

In [6], it was pointed out that (1.9) is not comparable to the Lyapunov condition (1.2) in general. Using the similar argument, we still have

Proposition 1.8. *If the Gozlan's type condition holds, i.e.*

$$\liminf_{|x|\to\infty} \sum_{i=1}^{m}\left[\frac{23}{27}\left(\frac{\partial V}{\partial x_i}\right)^2(x) - \frac{\partial^2 V}{\partial x_i^2}(x)\right]\frac{1}{1+x_i^2} \geqslant m, \tag{1.10}$$

then $\mu e^{\delta|x|^2} < \infty$ for any $\delta < \frac{2(\sqrt{m}-\sqrt{m-1})}{3\sqrt{3m}}$.

Remark 1.9. To yield the Gaussian integrability, or equivalently W_1H, the original constant $\frac{1}{10}$ in (1.9) can be increased to arbitrary $a < 1 - \frac{4}{27}\frac{m-1}{m}$. So it is convenient to take $a = \frac{23}{27}$. Except $m = 1$, it is unlikely to allow a approaching 1, according to the estimates in Lemma 3.1 below.

The next two sections will supply the proofs of all propositions respectively.

2 Proofs of Proposition 1.1, 1.4 and 1.6

Under the Lyapunov condition (1.2), [6, Lemma 3.4] asserts

$$\int h^2(x)d^2(x,x_0)\mathrm{d}\mu(x) \leqslant \frac{1}{c}\int |\nabla h|^2\mathrm{d}\mu + \frac{b}{c}\int h^2\mathrm{d}\mu, \ \ \forall h\in\mathcal{D}(\mathcal{E}), \tag{2.1}$$

basically via the same technique as in [1, Page 64].

Proof of Proposition 1.1. Let $\beta_n = \int d^{2n}(x,x_0)\mathrm{d}\mu$, which satisfies a recursion by using (2.1) that

$$\begin{aligned}\beta_n &= \int d^{2(n-1)}(x,x_0)d^2(x,x_0)\mathrm{d}\mu \\ &\leqslant \frac{1}{c}\int |\nabla d^{n-1}(x,x_0)|^2\mathrm{d}\mu + \frac{b}{c}\beta_{n-1} = \frac{(n-1)^2}{c}\beta_{n-2} + \frac{b}{c}\beta_{n-1}.\end{aligned} \tag{2.2}$$

Since $\beta_0 = 1$ and $\beta_1 \leqslant \frac{b}{c}$, we get the integrability of all $d^{2n}(x,x_0)$.

Combining the Hölder inequality with (2.2) gives

$$\beta_n = \int d^{n+1}(x,x_0)d^{n-1}(x,x_0)\mathrm{d}\mu \leqslant \beta_{n+1}^{\frac{1}{2}}\beta_{n-1}^{\frac{1}{2}} \leqslant \left(\frac{n^2}{c}\beta_{n-1} + \frac{b}{c}\beta_n\right)^{\frac{1}{2}}\beta_{n-1}^{\frac{1}{2}},$$

which implies

$$\beta_n \leqslant \frac{\frac{b}{c} + \sqrt{\frac{b^2}{c^2} + \frac{4n^2}{c}}}{2}\beta_{n-1} \leqslant (\tfrac{b}{c} + \tfrac{n}{\sqrt{c}})\beta_{n-1}.$$

Taking any $\gamma > \frac{1}{\sqrt{c}}$ gives $\frac{b}{c} + \frac{n}{\sqrt{c}} \leqslant \gamma n$ for big n, which yields some $C > 0$ such that

$$\beta_n \leqslant C\gamma^n n!, \quad \forall n \geqslant 1.$$

Hence, for any $\delta < \gamma^{-1} < \sqrt{c}$, we have by the Fatou's lemma

$$\begin{aligned}\int e^{\delta d^2(x,x_0)}\mathrm{d}\mu \ &= \int \lim_{k\to\infty}\sum_{n=0}^{k}\left(\delta d^2(x,x_0)\right)^n/n!\ \mathrm{d}\mu \\ \leqslant \ \liminf_{k\to\infty}\int\sum_{n=0}^{k}\left(\delta d^2(x,x_0)\right)^n/n!\ \mathrm{d}\mu \ &= \ \liminf_{k\to\infty}\sum_{n=0}^{k}\delta^n\beta_n/n! \ \leqslant \ \frac{C}{1-\delta\gamma}.\end{aligned} \tag{2.3}$$

The proof is completed. □

The next proof is almost the same.

Proof of Proposition 1.4. Using the Lyapunov condition (1.6) with the technique from [1, Page 64] gives a similar inequality for $h\in\mathcal{D}(\mathcal{E}_a)$ as (2.1) that

$$\begin{aligned}\int h^2(x)d^2(x,x_0)\lambda_{\max}\mathrm{d}\mu &\leqslant \frac{1}{c}\int h^2\cdot\frac{-\mathrm{L}_a W}{W}\mathrm{d}\mu + \frac{b}{c}\int h^2\lambda_{\max}\mathrm{d}\mu \\ &= \frac{1}{c}\int \Gamma_a(\frac{h^2}{W}, W)\mathrm{d}\mu + \frac{b}{c}\int h^2\lambda_{\max}\mathrm{d}\mu \\ &= \frac{1}{c}\int \Gamma_a(h,h) - W^2\Gamma_a(\frac{h}{W},\frac{h}{W})\mathrm{d}\mu + \frac{b}{c}\int h^2\lambda_{\max}\mathrm{d}\mu \\ &\leqslant \frac{1}{c}\int |\nabla h|^2\lambda_{\max}\mathrm{d}\mu + \frac{b}{c}\int h^2\lambda_{\max}\mathrm{d}\mu.\end{aligned}$$

Let $\beta_n = \int d^{2n}(x,x_0)\lambda_{\max}\mathrm{d}\mu$, which satisfies

$$\begin{aligned}\beta_n &= \int d^{2(n-1)}(x,x_0)d^2(x,x_0)\lambda_{\max}\mathrm{d}\mu \\ &\leqslant \frac{1}{c}\int |\nabla d^{n-1}(x,x_0)|^2\lambda_{\max}\mathrm{d}\mu + \frac{b}{c}\beta_{n-1} = \frac{(n-1)^2}{c}\beta_{n-2} + \frac{b}{c}\beta_{n-1}.\end{aligned}$$

Following rest steps in the previous proof, we get the Gaussian integrability. □

For jump processes, we use a little different method.

Proof of Proposition 1.6. The strategy contains three steps.

Step 1. Denote $\rho_t(x) = \sqrt{\rho^2(x,x_0)+t}$ with a parameter $t>0$. Using the technique in [1, Page 64] again, we have by Condition (1.8) that for $h \in \mathcal{D}(\mathcal{E}_\nu)$

$$\begin{aligned}\int h^2\rho_t^2\mathrm{d}\mu &\leqslant \frac{1}{c}\int h^2\cdot\frac{-\mathrm{L}_a W}{W}\mathrm{d}\mu + \left(\frac{b}{c}+t\right)\int h^2\mathrm{d}\mu \\ &= \frac{1}{c}\int \Gamma_\nu\left(\frac{h^2}{W},W\right)\mathrm{d}\mu + \frac{b+ct}{c}\int h^2\mathrm{d}\mu \\ &= \frac{1}{c}\cdot\frac{1}{2}\int\int_{\mathbb{R}^m-\{0\}} - \left|h(x+y)\frac{W(x)^{\frac12}}{W(x+y)^{\frac12}} - h(x)\frac{W(x+y)^{\frac12}}{W(x)^{\frac12}}\right|^2 \\ &\qquad + |h(x+y)-h(x)|^2\nu(x,\mathrm{d}y)\mu(\mathrm{d}x) \ + \ \frac{b+ct}{c}\int h^2\mathrm{d}\mu \\ &\leqslant \frac{1}{c}\int \Gamma_\nu(h,h)\mathrm{d}\mu + \frac{b+ct}{c}\int h^2\mathrm{d}\mu.\end{aligned}$$

Step 2. Basically, our aim is to estimate $\int_\Omega e^{\delta\rho(x,x_0)^2}\mathrm{d}\mu(x)$ for any bounded domain Ω, while the integration by parts requires to regularize the characteristic function $\mathbf{1}_\Omega$. It is usually a routine but with a few tricks in this case.

Define a family of $\phi_r \in C^1(\mathbb{R}^+)$ with any $r>0$ and some constant $N>0$ as

$$\phi_r(s) = \begin{cases} 1, & s \leqslant r; \\ 2(\frac{s-r}{N})^3 - 3(\frac{s-r}{N})^2 + 1, & r < s < r+N; \\ 0, & s \geqslant r+N,\end{cases}$$

which satisfies $0 \leqslant \phi_r \leqslant 1$ and $|\phi_r'| \leqslant \frac{3}{2N}\mathbf{1}_{r<s<r+N}$.

Let $f = e^{\frac{\delta}{2}\rho_t^2}$ and $f_r = \phi_r(\rho_t^2)f$. Let $h_r = \frac{f_r}{\rho_t}$, we have by Step 1

$$\int f_r^2\mathrm{d}\mu = \int h_r^2\rho_t^2\mathrm{d}\mu \leqslant \frac{1}{c}\int\Gamma_\nu(h_r,h_r)\mathrm{d}\mu + \frac{b+ct}{c}\int h_r^2\mathrm{d}\mu. \tag{2.4}$$

For convenience, rewrite $h_r = \phi_r(\rho_t^2)\psi(\rho_t)$ by putting $\psi(s) := \frac{e^{\frac{\delta}{2}s^2}}{s}$.

Take $t = 2\delta^{-1}$ so that ψ is increasing on $[\sqrt{t},\infty)$. Using the mean value theorem respectively to ψ and ϕ_r yields that for any $x \in \mathbb{R}^m$ and $y \in \mathrm{Supp}\nu$, there exist ξ and ζ both falling between $\rho(x+y)$ and $\rho(x)$ such that

$$\begin{aligned}&|h_r(x+y)-h_r(x)| \\ \leqslant\ & \phi_r(\rho_t^2(x))\cdot|\psi(\rho_t(x+y))-\psi(\rho_t(x))| \\ &\quad + \psi(\rho_t(x+y))\cdot|\phi_r(\rho_t^2(x+y))-\phi_r(\rho_t^2(x))| \\ =\ & \phi_r(\rho_t^2(x))\cdot\left|\delta-\xi^{-2}\right|e^{\frac{\delta}{2}\xi^2}\cdot|\rho_t(x+y)-\rho_t(x)| \\ &\quad + \psi(\rho_t(x+y))\cdot|2\zeta\phi_r'(\zeta^2)|\cdot|\rho_t(x+y)-\rho_t(x)|,\end{aligned}$$

which implies by Condition (1.7) that

$$\begin{aligned}|h_r(x+y)-h_r(x)| \quad &\leqslant \quad \frac{\delta}{2}e^{\frac{\delta}{2}K}\cdot f_r(x)\cdot|\rho_t(x+y)-\rho_t(x)| \\ &\quad + \frac{3e^{\frac{\delta}{2}K}}{N}\cdot f(x)\mathbf{1}_{r-K<\rho_t^2(x)<r+N+K}\cdot|\rho_t(x+y)-\rho_t(x)|.\end{aligned}$$

Due to $\Gamma_\nu(\rho_t,\rho_t)\leqslant 1$, it follows

$$\begin{aligned}\Gamma_\nu(h_r,h_r) \quad &= \quad \frac{1}{2}\int_{\mathbb{R}^m-\{0\}}|h_r(x+y)-h_r(x)|^2\nu(x,\mathrm{d}y) \\ &\leqslant \quad \frac{1}{2}\delta^2e^{\delta K}f_r^2(x)+\frac{18e^{\delta K}}{N^2}f^2(x)\mathbf{1}_{r-K<\rho_t^2(x)<r+N+K} \\ &\leqslant \quad \frac{1}{2}\delta^2e^{\delta K}f_r^2(x)+\frac{18e^{\delta(2N+3K)}}{N^2}e^{\delta(r-N-K)}\mathbf{1}_{r-N-K<\rho_t^2(x)<r+N+K}.\end{aligned}$$

Let $\eta_1=\frac{\delta^2}{2c}e^{\delta K}$ and $\eta_2=\frac{18e^{\delta(2N+3K)}}{N^2c}$, inserting the above estimate into (2.4) gives

$$\begin{aligned}\int f_r^2\mathrm{d}\mu \quad &\leqslant \quad \eta_1\int f_r^2\mathrm{d}\mu+\frac{b+ct}{c}\int h_r^2\mathrm{d}\mu \\ &\quad + \eta_2e^{\delta(r-N-K)}\mu\{r-N-K<\rho_t^2<r+N+K\}.\end{aligned}$$

Step 3. Choose some big N and small δ so that $\eta_1+2\eta_2<1$. Since μ is a probability, there exists a sequence of $n_k\in\mathbb{N}$ such that for each $r_k=n_k(N+K)$

$$\mu\{r_k-N-K<\rho_t^2<r_k\}\geqslant\mu\{r_k<\rho_t^2<r_k+N+K\},$$

which implies

$$\begin{aligned}&e^{\delta(r-N-K)}\mu\{r_k-N-K<\rho_t^2<r_k+N+K\} \\ \leqslant \quad & 2\int f^2\mathbf{1}_{r_k-N-K<\rho_t^2\leqslant r_k}\mathrm{d}\mu \quad \leqslant \quad 2\int f_{r_k}^2\mathrm{d}\mu.\end{aligned}$$

It follows from Step 2

$$\int f_{r_k}^2\mathrm{d}\mu\leqslant(\eta_1+2\eta_2)\int f_{r_k}^2\mathrm{d}\mu+\frac{b+ct}{c}\int h_{r_k}^2\mathrm{d}\mu,$$

and thus

$$\int f_{r_k}^2\mathrm{d}\mu\leqslant\frac{b+ct}{c(1-\eta_1-2\eta_2)}\int h_{r_k}^2\mathrm{d}\mu=:C\int h_{r_k}^2\mathrm{d}\mu.$$

Recall $h_r=\frac{f_r}{\rho_t}$, fix a domain Ω with $\rho_t^2\geqslant 2C$ on Ω^c, which means for $r_k>\mathrm{diam}\Omega$

$$\int f_{r_k}^2\mathrm{d}\mu\leqslant C\int_\Omega\frac{f^2}{\rho_t^2}\mathrm{d}\mu+\frac{1}{2}\int_{\Omega^c}f_{r_k}^2\mathrm{d}\mu.$$

Consequently, we get $\int f^2\mathrm{d}\mu=\lim_{k\to\infty}\int f_{r_k}^2\mathrm{d}\mu\leqslant 2C\int_\Omega\frac{f^2}{\rho_t^2}\mathrm{d}\mu<\infty.$ □

3 Proof of Proposition 1.8

We firstly derive a Poincaré like inequality.

Lemma 3.1. *If the Gozlan's type condition (1.10) holds, there exist two constants λ_1 and λ_2 with big R such that for any $h\in\mathcal{D}(\mathcal{E})$*

$$\int h^2\mathrm{d}\mu\leqslant\lambda_1\int\sum_{i=1}^m\frac{|h_i'|^2}{1+x_i^2}\mathrm{d}\mu+\lambda_2\int_{B(0,R+1)}h^2\mathrm{d}\mu.$$

Proof. For convenience, denote $a = \frac{23}{27}$, $\mathrm{d}\nu_i = e^{-aV}\mathrm{d}x_i$ and

$$\mathrm{d}\hat{x}_i = \mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots \mathrm{d}x_m$$

so that $\mathrm{d}\mu = e^{-(1-a)V}\mathrm{d}\nu_i\mathrm{d}\hat{x}_i$. Define $\phi_r \in C^1(\mathbb{R}^n)$ as

$$\phi_r(x) = \begin{cases} 1, & |x| \leqslant r; \\ 2(|x|-r)^3 - 3(|x|-r)^2 + 1, & r < |x| < r+1; \\ 0, & |x| \geqslant r+1, \end{cases}$$

which satisfies $0 \leqslant \phi_r \leqslant 1$ and $|(\phi_r)'_i| \leqslant 6\frac{|x_i|}{|x|}\sqrt{1-\phi_r}$. The proof has three steps.

Step 1. For any $\varepsilon > 0$, there exists $R > 0$ by (1.10) such that for all $|x| \geqslant R$

$$\sum_{i=1}^m \left(a|V_i'|^2 - V_{ii}''\right)\frac{1}{1+x_i^2} \geqslant m - \varepsilon.$$

It follows for any $h \in \mathcal{D}(\mathcal{E})$

$$\begin{aligned} (m-\varepsilon)\int h^2 \mathrm{d}\mu \; &= \; (m-\varepsilon)\int h^2\phi_R + h^2(1-\phi_R)\mathrm{d}\mu \\ \leqslant \quad & (m-\varepsilon)\int h^2\phi_R\mathrm{d}\mu + \int h^2(1-\phi_R)\sum_{i=1}^m \left(a|V_i'|^2 - V_{ii}''\right)\frac{1}{1+x_i^2}\mathrm{d}\mu \\ = \quad & (m-\varepsilon)\int h^2\phi_R\mathrm{d}\mu + \sum_{i=1}^m \int \frac{h^2(1-\phi_R)e^{-(1-a)V}}{(1+x_i^2)}\left(a|V_i'|^2 - V_{ii}''\right)\mathrm{d}\nu_i\mathrm{d}\hat{x}_i. \end{aligned} \tag{3.1}$$

Set $U^{(i)} = \frac{h^2(1-\phi_R)e^{-(1-a)V}}{1+x_i^2}$. For the reader's convenience, recall the integration by parts formula satisfied by ν_i, we have

$$\begin{aligned} \int U^{(i)}\left(a|V_i'|^2 - V_{ii}''\right)\mathrm{d}\nu_i\mathrm{d}\hat{x}_i \; &= \; \int (U^{(i)})_i' V_i' \mathrm{d}\nu_i\mathrm{d}\hat{x}_i \\ = \quad & \int \Big[2hh_i'V_i'(1-\phi_R) - (\phi_R)_i'h^2V_i' - \frac{2x_i}{1+x_i^2}h^2V_i'(1-\phi_R) \\ & \qquad -(1-a)h^2|V_i'|^2(1-\phi_R)\Big]\frac{1}{1+x_i^2}\mathrm{d}\mu. \end{aligned}$$

Using the Cauchy-Schwarz inequality gives for any positive $\varepsilon_1, \varepsilon_2$ and ε_3

$$\begin{aligned} 2hh_i'V_i' \quad &\leqslant \quad \varepsilon_1 h^2|V_i'|^2 + \varepsilon_1^{-1}|h_i'|^2, \\ -(\phi_R)_i'h^2V_i' \quad &\leqslant \quad 6\frac{|x_i|}{|x|}\sqrt{1-\phi_R}\cdot h^2|V_i'| \; \leqslant \; 3\varepsilon_2 h^2|V_i'|^2(1-\phi_R) + 3\varepsilon_2^{-1}\frac{|x_i|^2}{|x|^2}h^2, \\ -\frac{2x_ih^2V_i'}{1+x_i^2} \quad &\leqslant \quad \varepsilon_3 h^2|V_i'|^2 + \frac{x_i^2h^2}{\varepsilon_3(1+x_i^2)^2}, \end{aligned}$$

which implies by combining the above estimates subject to $\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 = 1 - a$

$$\begin{aligned} & \int U^{(i)}\left(a|V_i'|^2 - V_{ii}''\right)\mathrm{d}\nu_i\mathrm{d}\hat{x}_i \\ \leqslant \quad & \int \frac{|h_i'|^2(1-\phi_R)}{\varepsilon_1(1+x_i^2)} + \frac{3|x_i|^2h^2}{\varepsilon_2(1+x_i^2)|x|^2} + \frac{x_i^2h^2(1-\phi_R)}{\varepsilon_3(1+x_i^2)^3}\mathrm{d}\mu. \end{aligned} \tag{3.2}$$

Step 2. Since $\frac{x_i^2}{(1+x_i^2)^3} \leqslant \frac{4}{27}$ for any x_i and there exists x_j with $|x_j|^2 \geqslant |x|^2/m$, we have

$$\sum_{i=1}^m \frac{x_i^2}{(1+x_i^2)^3} \leqslant \frac{4}{27}(m-1) + \frac{1}{(1+m^{-1}|x|^2)^2},$$

which implies

$$\sum_{i=1}^{m}\int \frac{x_i^2h^2(1-\phi_R)}{\varepsilon_3(1+x_i^2)^3}\mathrm{d}\mu \leqslant \left(\frac{4(m-1)}{27\varepsilon_3}+\frac{m^2}{\varepsilon_3R^4}\right)\int_{B(0,R)^c}h^2\mathrm{d}\mu. \tag{3.3}$$

We also have

$$\sum_{i=1}^{m}\int \frac{3|x_i|^2h^2}{\varepsilon_2(1+x_i^2)|x|^2}\mathrm{d}\mu \leqslant \frac{3}{\varepsilon_2}\int_{B(0,R)}h^2\mathrm{d}\mu+\frac{3m}{\varepsilon_2R^2}\int_{B(0,R)^c}h^2\mathrm{d}\mu. \tag{3.4}$$

Choose R (depending on ε and $\varepsilon_{1,2,3}$) so big that $\frac{m^2}{\varepsilon_3R^4}+\frac{3m}{\varepsilon_2R^2}\leqslant\varepsilon$, then combining (3.1-3.4) gives

$$\begin{aligned}(m-\varepsilon)\int h^2\mathrm{d}\mu \;\leqslant\; & \frac{1}{\varepsilon_1}\int\sum_{i=1}^{m}\frac{|h_i'|^2}{1+x_i^2}\mathrm{d}\mu \;+\\ & \left(m-\varepsilon+\frac{3}{\varepsilon_2}\right)\int_{B(0,R+1)}h^2\mathrm{d}\mu+\left(\frac{4(m-1)}{27\varepsilon_3}+\varepsilon\right)\int h^2\mathrm{d}\mu. \end{aligned}\tag{3.5}$$

Step 3. We have to decide the range of ε and $\varepsilon_{1,2,3}$. First of all, fix $\varepsilon_1<\frac{4}{27m}$, and take any ε_2 such that $\varepsilon_1+3\varepsilon_2<\frac{4}{27m}$ too. It follows

$$\frac{4(m-1)}{27\varepsilon_3}=\frac{4(m-1)}{27(1-a-\varepsilon_1-3\varepsilon_2)}<m,$$

so we can take any ε such that $\frac{4(m-1)}{27\varepsilon_3}+2\varepsilon<m$.

Now, using (3.5) yields

$$\int h^2\mathrm{d}\mu\leqslant\lambda_1\int\sum_{i=1}^{m}\frac{|h_i'|^2}{1+x_i^2}\mathrm{d}\mu+\lambda_2\int_{B(0,R+1)}h^2\mathrm{d}\mu, \tag{3.6}$$

where $\lambda_1=[\varepsilon_1(m-2\varepsilon-\frac{4(m-1)}{27\varepsilon_3})]^{-1}$ and $\lambda_2=\left(m-\varepsilon+3\varepsilon_2^{-1}\right)(m-2\varepsilon-\frac{4(m-1)}{27\varepsilon_3})^{-1}$. The proof is completed. □

Under the Gozlan's condition, we use a similar argument.

Proof of Proposition 1.8. Let $\beta_n=\int|x|^{2n}\mathrm{d}\mu$. Applying (3.6) to $h(x)=|x|^n$ yields

$$\begin{aligned}\beta_n \;&\leqslant\; \lambda_1\int\sum_{i=1}^{m}\frac{n^2x_i^2}{1+x_i^2}|x|^{2n-4}\mathrm{d}\mu+\lambda_2\int_{B(0,R+1)}|x|^{2n}\mathrm{d}\mu\\ &\leqslant\; \lambda_1mn^2\int|x|^{2n-4}\mathrm{d}\mu+\lambda_2(R+1)^2\int_{B(0,R+1)}|x|^{2n-2}\mathrm{d}\mu\\ &\leqslant\; \lambda_1mn^2\beta_{n-2}+\lambda_2(R+1)^2\beta_{n-1}, \end{aligned}\tag{3.7}$$

which implies all $\beta_n<\infty$.

For simplicity, abbreviate $\lambda'_1=\lambda_1m$ and $\lambda'_2=\lambda_2(R+1)^2$. Combining the Hölder inequality with (3.7) gives

$$\beta_n=\int|x|^{n+1}|x|^{n-1}\mathrm{d}\mu\leqslant\beta_{n+1}^{\frac12}\beta_{n-1}^{\frac12}\leqslant\left[\lambda'_1(n+1)^2\beta_{n-1}+\lambda'_2\beta_n\right]^{\frac12}\beta_{n-1}^{\frac12},$$

which implies

$$\beta_n\leqslant\frac{\lambda'_2+\sqrt{\lambda'^2_2+4\lambda'_1(n+1)^2}}{2}\beta_{n-1}\leqslant\left[\lambda'_2+\sqrt{\lambda'_1}(n+1)\right]\beta_{n-1}.$$

Choose any $\gamma > \sqrt{\lambda'_1}$, it follows $\lambda'_2 + \sqrt{\lambda'_1}(n+1) \leqslant \gamma n$ for big n, which yields a constant C such that for all n

$$\beta_n \leqslant C\gamma^n n!.$$

By the same argument as (2.3) for any $\delta < \gamma^{-1} < {\lambda'}_1^{-\frac{1}{2}}$, we have $\mu e^{\delta|x|^2} < \infty$.

Recall the constraints on all parameters (See Step 3 in the proof of Lemma 3.1), δ is allowed to be not greater than

$$\sup\left\{{\lambda'}_1^{-\frac{1}{2}} : \varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 = 1 - a,\ \varepsilon_1 < \frac{4}{27m},\ \varepsilon = \varepsilon_2 = 0\right\},$$

which achieves $\frac{2(\sqrt{m}-\sqrt{m-1})}{3\sqrt{3m}}$. The proof is completed. □

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