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Reflected backward stochastic differential equations driven by countable Brownian motions with continuous coefficients

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Abstract

In this note, we study one-dimensional reflected backward stochastic differential equations (RBSDEs) driven by Countable Brownian Motions with one continuous barrier and continuous generators. Via a comparison theorem, we provide the existence of minimal and maximal solutions to this kind of equations.

Keywords: Backward doubly stochastic differential equations; Countable Brownian Motions; comparison theorem; continuous and linear growth conditions.
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1 Motivation

Recently, Pengju Duan et al. [10] studied a new class of reflected backward stochastic differential equations driven by countable Brownian motions. That is :

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \sum_{j=1}^{\infty}\int_t^T g_j(s,Y_s,Z_s)\overleftarrow{dB}_s^j + K_T - K_t - \int_t^T Z_s dW_s. \tag{1.1}$$

Under the global Lipschitz continuity condition, they proved in [10] via Snell envelope and fixed point theorem, the existence and uniqueness of the solution for RBSDEs (1.1). Unfortunately, the global Lipschitz continuity condition can not be satisfied in certain models that limits the scope of the result of Pengju Duan et al. [10] for several applications (finance, stochastic control, stochastic games, SPDEs, etc,...).

In finance, this kind of RBSDEs is useful to describe the wealth/strategy of a particular investor who, trading continuously, has countable extra information which can not be detected in the market. In this case, ξ can be regarded as the contingent claim; Y_t is the wealth of the investor at time t, Z_t is his strategy at time t to make the wealth Y_t be measurable to the information inside or outside the market at time t and to allow the realization of the option $Y_T = \xi$, and K_t is the subsidy injected by the government in the market at time t to allow the wealths of investors to remain above a threshold price S_t at time t; $f(t,Y_t,Z_t)$ is the appreciation of the wealth/strategy at time t and $g_j(t,Y_t,Z_t)$ is the impact of the j^{th} additional information on the wealth/strategy at time t. For example, consider a particular investor who trades continuously in a financial market. Suppose that this investor has countable additional information which is not

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available in the market. If the appreciation of his wealth/strategy is $\sqrt{Y_t 1_{\{Y_t \geq 0\}}}$, then this investor has to solve the following RBSDEs :

$$Y_t = \xi + \int_t^T \sqrt{Y_s 1_{\{Y_s \geq 0\}}} ds + \sum_{j=1}^{\infty} \int_t^T g_j(s, Y_s, Z_s) \overleftarrow{dB}_s^j + K_T - K_t - \int_t^T Z_s dW_s.$$

Since $y \mapsto \sqrt{y 1_{\{y \geq 0\}}}$ is not Lipschitz in y, then we can not apply the existence result in Pengju Duan et al. [10] to get the existence of solution of the above RBSDE. Consequently, the problem of such an investor can not be solved at present. To correct this shortcoming, we relax in this paper the global Lipschitz continuity condition on the coefficients f to a continuity with sub linear condition and derive the existence of minimal and maximal solutions to RBSDEs (1.1).

The paper is organized as follows. In section 2, we give some notations and preliminaries Section 3 is devoted to the main result.

2 Preliminaries

The scalar product of the space $\mathbb{R}^d (d \geq 2)$ will be denoted by $< .,. >$ and the associated Euclidian norm by $\|.\|$.

Throughout this paper, T is a positive constant and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which, $\{B_t^j, 0 \leq t \leq T\}_{j=1}^{\infty}$ are mutual independent one-dimensional standard Brownian motions and $\{W_t, 0 \leq t \leq T\}$ is a $\mathbb{R}^d-$ valued standard Brownian motion which is independent of $\{B_t^j, 0 \leq t \leq T\}$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$ and define

$$\mathcal{F}_t = \left(\bigvee_{j=1}^{\infty} \mathcal{F}_{t,T}^{B^j} \right) \bigvee \mathcal{F}_t^W \bigvee \mathcal{N}, \quad 0 \leq t \leq T$$

where $\mathcal{F}_{s,t}^{\eta} = \sigma\{\eta_r - \eta_s, s \leq r \leq t\}$ for any η_t, and $\mathcal{F}_t^{\eta} = \mathcal{F}_{0,t}^{\eta}$. Since $\{\mathcal{F}_t^W, t \in [0,T]\}$ is an increasing filtration and $\{\mathcal{F}_{t,T}^{B^j}, t \in [0,T]\}$ is a decreasing filtration, the collection $\{\mathcal{F}_t, t \in [0,T]\}$ is neither increasing nor decreasing so that it does not constitute a filtration.

For any $n \in \mathbb{N}$, let $\mathcal{M}^2(0,T;\mathbb{R}^n)$ denote the set of (class of $d\mathbb{P} \otimes dt$ a.e. equal) n-dimensional jointly measurable random processes $\{\varphi_t; 0 \leq t \leq T\}$ which satisfy:

(i) $\|\varphi\|_{\mathcal{M}^2}^2 = \mathbb{E}(\int_0^T |\varphi_t|^2 dt) < \infty$

(ii) φ_t is $\mathcal{F}_t$-measurable, for a.e. $t \in [0,T]$.

We denote by $\mathcal{S}^2([0,T];\mathbb{R})$ the set of continuous one-dimensional random processes which satisfy:

(i) $\|\varphi\|_{\mathcal{S}^2}^2 = \mathbb{E}(\sup_{0 \leq t \leq T} | \varphi_t |^2) < \infty$

(ii) φ_t is $\mathcal{F}_t$-measurable, for any $t \in [0,T]$.

We set by $\mathcal{A}^2(0,T,\mathbb{R}_+)$ the space of a real positive continuous and increasing process, such that $\varphi_0 = 0$ and

(i) $\|\varphi\|_{\mathcal{A}^2}^2 = \mathbb{E}(|\varphi_T|^2) < \infty$

(ii) φ_t is $\mathcal{F}_t$-measurable, for a.e. $t \in [0,T]$.

Finally, we set $\mathcal{E}^2(0,T) = \mathcal{S}^2([0,T],\mathbb{R}) \times \mathcal{M}^2(0,T,\mathbb{R}^d) \times \mathcal{A}^2(0,T,\mathbb{R}_+)$.

Definition 2.1. *A solution of a* (1.1) *is a triplet of* $(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+)$*-valued process* $(Y_t, Z_t, K_t)_{0 \le t \le T}$*, which satisfies* (1.1)*, and*

(i) $(Y, Z, K) \in \mathcal{E}^2(0, T)$;
(ii) $Y_t \ge S_t$;
(iii) K *is a continuous and increasing process with* $K_0 = 0$ *and* $\int_0^T (Y_t - S_t) dK_t = 0.$

Definition 2.2. *A triplet of processes* $(\underline{Y}, \underline{Z}, \underline{K})$ *(resp.* $(\overline{Y}, \overline{Z}, \overline{K})$*) of* $\mathcal{E}^2(0, T)$ *is said to be a minimal (resp. a maximal) solution of* (1.1) *if for any other solution* (Y, Z, K) *of* (1.1)*, we have* $\underline{Y} \le Y$ *(resp.*$Y \le \overline{Y}$*).*

We consider the following assumptions:
Assume that the coefficients $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, the terminal value $\xi : \Omega \to \mathbb{R}$ and the obstacle $S : \Omega \times [0, T] \to \mathbb{R}$ satisfy the following conditions:

(H1) **(i)** ξ is a $\mathcal{F}_T$-measurable random variable such that, $\mathbb{E}(|\xi|^2) < \infty$,

(ii) $S \in \mathcal{S}^2([0, T])$ such that, $S_T \le \xi$ a.s.;

(H2) for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $f(t, y, z)$, $\{g_j(t, y, z)\}_{j=1}^{\infty}$ are $\mathcal{F}_t$-measurable such that

$$\mathbb{E} \int_0^T |f(s, 0, 0)|^2 ds + \sum_{j=1}^{\infty} \mathbb{E} \int_0^T |g_j(s, 0, 0)|^2 ds < +\infty;$$

(H3) for all $t \in [0, T]$ and (y_1, z_1), $(y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$| g_j(t, y_1, z_1) - g_j(t, y_2, z_2) |^2 \le C_j | y_1 - y_2 |^2 + \alpha_j || z_1 - z_2 ||^2$$

where $C_j > 0$ and $\alpha_j > 0$ are constants with $\sum_{j=1}^{\infty} C_j < \infty$ and $\alpha = \sum_{j=1}^{\infty} \alpha_j < 1$,

(H4) **(i)** for all $t \in [0, T]$ and (y_1, z_1), $(y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$| f(t, y_1, z_1) - f(t, y_2, z_2) |^2 \le C(| y_1 - y_2 |^2 + || z_1 - z_2 ||^2),$$

where $C > 0$ is a nonnogative constant;

(ii) for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $|f(t, y, z)| \le M(1 + |y| + ||z||)$,
where $M > 0$ is a nonnegative constant,

(H5) **(i)** for a.e. $(t, \omega) \in [0, T] \times \Omega$, the map $(y, z) \mapsto f(t, y, z)$ is continuous,

(ii) for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $|f(t, y, z)| \le \varphi(t) + M(|y| + ||z||)$,
where $\varphi \in \mathcal{M}^2(0, T, \mathbb{R})$ is a positive process and $M > 0$ is a nonnegative constant.

Lemma 2.3 (Pengju Duan et al. [10])**.** *Under assumptions* $(H1) - (H4)$*, there exists a unique solution* $(Y, Z, K) \in \mathcal{E}^2(0, T)$ *of* (1.1)*.*

Remark 2.4. The above result still holds when assumption (H4)**(ii)** is replaced by (H5)**(ii)**

3 The main results

In this section, our principal aim, is to prove an existence result for reflected BSDEs driven by countable Brownian motions under general continuous conditions. More precisely, we will derive the existence of a minimal and a maximal solution to equation (1.1) under assumptions (H1)-(H3) and (H5).

To this end, we establish first, the following comparison theorem, which extend the comparison theorem due to Aman and Owo [2] with RBSDEs driven by finite Brownian motions.

For $i = 1, 2$, let us consider the following RBSDEs driven by countable Brownian motions:

$$\begin{cases} (i)\;\; Y_t^i = \xi^i + \displaystyle\int_t^T f^i(s, Y_s^i, Z_s^i)ds + \sum_{j=1}^{\infty} \int_t^T g_j(s, Y_s^i, Z_s^i)\overleftarrow{dB}_s^j + \int_t^T dK_s^i - \int_t^T Z_s^i dW_s, \\ (ii)\;\; Y_t^i \geq S_t^i, \\ (iii)\;\; \displaystyle\int_0^T (Y_t^i - S_t^i)dK_t^i = 0. \end{cases} \tag{3.1}$$

Theorem 3.1 (Comparison Theorem). *Let assumptions $(H1) - (H3)$ hold. Assume that RBDSDEs* (3.1) $(i = 1, 2)$ *have solutions (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2), respectively. Assume moreover that:*

(i) $\xi^1 \leq \xi^2$ *a.s.,*

(ii) $S_t^1 \leq S_t^2$ *a.s., for all $t \in [0, T]$*

(iii) f^1 *satisfies $(H4)$**(i)** such that $f^1(t, Y^2, Z^2) \leq f^2(t, Y^2, Z^2)$ a.s.*

*(resp. f^2 satisfies $(H4)$**(i)** such that $f^1(t, Y^1, Z^1) \leq f^2(t, Y^1, Z^1)$ a.s.).*

Then, $Y_t^1 \leq Y_t^2$ a.s., for all $t \in [0, T]$.

Proof. Applying Itô's formula to $|(Y_t^1 - Y_t^2)^+|^2$, we have

$$\begin{aligned} & \mathbb{E}|(Y_t^1 - Y_t^2)^+|^2 + \mathbb{E}\int_t^T \mathbf{1}_{\{Y_s^1 > Y_s^2\}} \|Z_s^1 - Z_s^2\|^2 ds \\ = \;& \mathbb{E}|(\xi^1 - \xi^2)^+|^2 + 2\mathbb{E}\int_t^T (Y_s^1 - Y_s^2)^+ \left(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\right) ds \\ & +2\mathbb{E}\int_t^T (Y_s^1 - Y_s^2)^+ (dK_s^1 - dK_s^2) \\ & +\sum_{j=1}^{\infty} \mathbb{E}\int_t^T \mathbf{1}_{\{Y_s^1 > Y_s^2\}} |g_j(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)|^2 ds. \end{aligned}$$

From (i), $\mathbb{E}|(\xi^1 - \xi^2)^+|^2 = 0$ and from (iii), we have

$$f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \leq f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2).$$

Therefore, from Young inequality, and the fact that f^1 satisfies (H4)**(i)** and g verifies

(H3), we get

$$\begin{aligned}
&\mathbb{E}|(Y_t^1-Y_t^2)^+|^2+\mathbb{E}\int_t^T \mathbf{1}_{\{Y_s^1>Y_s^2\}}\|Z_s^1-Z_s^2\|^2ds\\
&\le\left(\frac{1}{\beta}+\beta C+\sum_{j=1}^{\infty}C_j\right)\mathbb{E}\int_t^T|(Y_s^1-Y_s^2)^+|^2ds\\
&\quad+(\beta C+\sum_{j=1}^{\infty}\alpha_j)\mathbb{E}\int_t^T \mathbf{1}_{\{Y_s^1>Y_s^2\}}\|Z_s^1-Z_s^2\|^2ds\\
&\quad+2\mathbb{E}\int_t^T(Y_s^1-Y_s^2)^+(dK_s^1-dK_s^2).
\end{aligned}$$

Since $Y_s^1>S_s^2\ge S_s^1$ on the set $\{Y_s^s>Y_s^2\}$ we derive that

$$\mathbb{E}\int_t^T(Y_s^1-Y_s^2)^+(dK_s^1-dK_s^2)=-\mathbb{E}\int_t^T(Y_s^1-Y_s^2)^+dK_s^2\le 0.$$

Hence,

$$\begin{aligned}
&\mathbb{E}|(Y_t^1-Y_t^2)^+|^2+\mathbb{E}\int_t^T \mathbf{1}_{\{Y_s^1>Y_s^2\}}\|Z_s^1-Z_s^2\|^2ds\\
&\le\left(\frac{1}{\beta}+\beta C+\sum_{j=1}^{\infty}C_j\right)\mathbb{E}\int_t^T|(Y_s^1-Y_s^2)^+|^2ds\\
&\quad+(\beta C+\alpha)\mathbb{E}\int_t^T \mathbf{1}_{\{Y_s^1>Y_s^2\}}\|Z_s^1-Z_s^2\|^2ds.
\end{aligned}$$

Consequently, choosing $0<\beta<\frac{1-\alpha}{C}$ and using Gronwall inequality, we obtain

$$\mathbb{E}|(Y_t^1-Y_t^2)^+|^2\le 0.$$

Thus $(Y_t^1-Y_t^2)^+=0$ $a.s.$ $i.e.$ $Y_t^1\le Y_t^2$ $a.s.,$ $\forall\, t\in[0,T].$ □

Also, to reach our objective, we need the following useful approximation lemma (see Lepeltier and San Martin [8], and K. Bahlali, S. Hamadène, B. Mezerdi [4] to appear for the proof).

For $n\in\mathbb{N}$ and any continuous function f, let consider the following sequences of functions: $\forall\,(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$,

$$\underline{f}_n(t,y,z)=\inf_{u,v\in\mathbb{Q}}\{f(t,u,v)+n(|y-u|+|z-v|)\}$$

and

$$\overline{f}_n(t,y,z)=\sup_{u,v\in\mathbb{Q}}\{f(t,u,v)-n(|y-u|+|z-v|)\}.$$

Lemma 3.2. *If f satisfies $(H5)$, then, for $n>M$ and $t\in[0,T]$, $(y,z),(y_i,z_i)\in\mathbb{R}\times\mathbb{R}^d$, $i=1,2$, we have*

- *(i)* $-\varphi(t)-M(|y|+|z|)\le\underline{f}_n(t,y,z)\le f(t,y,z)\le\bar{f}_n(t,y,z)\le\varphi(t)+M(|y|+|z|)$;
- *(ii)* $\underline{f}_n(t,y,z)$ *is non-decreasing in* n *and* $\overline{f}_n(t,y,z)$ *is non-increasing in* n *;*
- *(iii)* $|\psi(t,y_1,z_1)-\psi(t,y_2,z_2)|\le n(|y_1-y_2|+|z_1-z_2|)$, *with* $\psi\in\left\{\underline{f}_n,\overline{f}_n\right\}$;

(iv) *If* $(y_n, z_n) \to (y, z)$*, then* $\psi(t, y_n, z_n) \to f(t, y, z)$ *as* $n \to +\infty$*, with* $\psi \in \left\{\underline{f}_n, \overline{f}_n\right\}$*.*

Now, we are ready to establish the main result of this paper which is the following theorem.

Theorem 3.3. *Under assumptions* $(H1) - (H3)$ *and* $(H5)$*, there exists a minimal (resp. a maximal) solution* $(\underline{Y}, \underline{Z}, \underline{K}) \in \mathcal{E}^2(0, T)$ *(resp.* $(\overline{Y}, \overline{Z}, \overline{K}) \in \mathcal{E}^2(0, T)$ *of* (1.1).

Proof. We only prove that (1.1) has a minimal solution. The other case can be proved similarly. For fixed $(t, \omega) \in [0, T] \times \Omega$, it follows from $(H5)$, that $(y, z) \mapsto f(t, y, z)$ is continuous and with linear growth. Then, by Lemma 3.2, the associated sequences of functions $\underline{f}_n$ and $\overline{f}_n$ are Lipschitz functions. Therefore, since assumptions $(H1) - (H4)$**(i)** and $(H5)$**(ii)** hold, we get from Lemma 2.3, that there exists a unique solution $(Y^n, Z^n, K^n) \in \mathcal{E}^2(0, T)$ of the following RBDSDEs,

$$\begin{cases} (i)\;\; Y_t^n = \xi + \int_t^T \underline{f}_n(s, Y_s^n, Z_s^n)ds + \sum_{j=1}^{\infty} \int_t^T g_j(s, Y_s^n, Z_s^n)\overleftarrow{dB}_s^j + \int_t^T dK_s^n - \int_t^T Z_s^n dW_s, \\ (ii)\;\; Y_t^n \geq S_t, \\ (iii)\;\; \int_0^T (Y_t^n - S_t^n)dK_t^n = 0. \end{cases} \tag{3.2}$$

Since by Lemma 3.2, $\underline{f}_n \leq \underline{f}_{n+1}$, for all $n \geq M$, it follows from the comparison theorem (Theorem 3.1) that for every $n \geq M$

$$Y^n \leq Y^{n+1}, \;\; dt \otimes d\mathbb{P}\text{-a.s.} \tag{3.3}$$

To complete the proof, it suffices to show that the sequence (Y^n, Z^n, K^n) converges to a process $(\underline{Y}, \underline{Z}, \underline{K})$ which is the minimal solution of the RBDSDEs (1.1).
To this end, we will sketch the proof in two steps:

Step 1: A priori estimates
There exists a constant $C > 0$ independent of n such that

$$\sup_{n \geq M} \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T \|Z_t^n\|^2 dt + |K_T^n|^2\right) \leq C. \tag{3.4}$$

Indeed, applying Itô's formula to $|Y_t^n|^2$, we have

$$\begin{aligned} &\mathbb{E}|Y_t^n|^2 + \mathbb{E}\int_t^T \|Z_s^n\|^2 ds \\ &= \mathbb{E}|\xi|^2 + 2\mathbb{E}\int_t^T Y_s^n \underline{f}_n(s, Y_s^n, Z_s^n)ds + 2\mathbb{E}\int_t^T Y_s^n dK_s^n + \sum_{j=1}^{\infty} \mathbb{E}\int_t^T |g_j(s, Y_s^n, Z_s^n)|^2 ds \end{aligned}$$

From assumption $(H3)$, the proprieties of $\underline{f}_n$ and Young's inequality, for any $\theta > 0$, we have

$$2Y_s^n \underline{f}_n(s, Y_s^n, Z_s^n) \;\; \leq \;\; (1 + 2M + \frac{M^2}{\theta})\,|Y_s^n|^2 + \theta|Z_s^n|^2 + |\varphi(s)|^2\,,$$

$$|g_j(s, Y_s^n, Z_s^n)|^2 \leq (1 + \theta)C_j\,|Y_s^n|^2 + (1 + \theta)\alpha_j\,|Z_s^n|^2 + (1 + \frac{1}{\theta})\,|g_j(s, 0, 0)|^2\,.$$

Using again Young inequality, we have for any $\beta > 0$,

$$2\mathbb{E}\int_t^T Y_s^n dK_s^n = 2\mathbb{E}\int_t^T S_s dK_s^n \leq \frac{1}{\beta}\mathbb{E}\sup_{0\leq s\leq T}|S_s|^2 + \beta\mathbb{E}\left(K_T^n - K_t^n\right)^2.$$

Since

$$\begin{aligned} K_T^n - K_t^n &= Y_t^n - \xi - \int_t^T f(s, Y_s^n, Z_s^n)ds \\ &\quad - \sum_{j=1}^{\infty}\int_t^T g_j(s, Y_s^n, Z_s^n)\overleftarrow{dB}_s^j + \int_t^T Z_s^n dW_s, \quad t \in [0, T], \end{aligned}$$

we have, for any $t \in [0, T]$,

$$\begin{aligned} &\mathbb{E}\left(K_T^n - K_t^n\right)^2 \\ \leq\ & 5\mathbb{E}\left(|Y_t^n|^2 + |\xi|^2 + \left|\int_t^T \underline{f}_n(s, Y_s^n, Z_s^n)ds\right|^2 + \left|\sum_{j=1}^{\infty}\int_t^T g_j(s, Y_s^n, Z_s^n)\overleftarrow{dB}_s^j\right|^2 + \left|\int_t^T Z_s^n dW_s\right|^2\right) \\ \leq\ & 5\mathbb{E}\left(|Y_t^n|^2 + |\xi|^2 + 3T\int_t^T\left(M^2|Y_s^n|^2 + M^2|Z_s^n|^2 + |\varphi(s)|^2\right)ds\right) \\ &+5\mathbb{E}\left(\sum_{j=1}^{\infty}\int_t^T\left((1+\theta)C_j|Y_s^n|^2 + (1+\theta)\alpha_j|Z_s^n|^2 + (1+\frac{1}{\theta})|g_j(s,0,0)|^2\right)ds + \int_t^T|Z_s^n|^2ds\right), \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbb{E}|Y_t^n|^2 + \mathbb{E}\int_t^T\|Z_s^n\|^2ds \\ &\leq \mathbb{E}|\xi|^2 + \mathbb{E}\int_t^T\left((1 + 2M + \frac{M^2}{\theta})|Y_s^n|^2 + \theta|Z_s^n|^2 + |\varphi(s)|^2\right)ds + \frac{1}{\beta}\mathbb{E}\sup_{0\leq s\leq T}|S_s|^2 \\ &5\beta\mathbb{E}\left(|Y_t^n|^2 + |\xi|^2 + 3T\int_t^T\left(M^2|Y_s^n|^2 + M^2|Z_s^n|^2 + |\varphi(s)|^2\right)ds\right) \\ &+5\beta\sum_{j=1}^{\infty}\mathbb{E}\int_t^T\left((1+\theta)C_j|Y_s^n|^2 + (1+\theta)\alpha_j|Z_s^n|^2 + (1+\frac{1}{\theta})|g_j(s,0,0)|^2\right)ds + 5\beta\mathbb{E}\int_t^T|Z_s^n|^2ds \\ &+\sum_{j=1}^{\infty}\mathbb{E}\int_t^T\left((1+\theta)C_j|Y_s^n|^2 + (1+\theta)\alpha_j|Z_s^n|^2 + (1+\frac{1}{\theta})|g_j(s,0,0)|^2\right)ds. \end{aligned}$$

Consequently,

$$\begin{aligned} &\Big(1 - 5\beta\Big)\mathbb{E}|Y_t^n|^2 + \Big[1 - \theta - (1+\theta)\alpha - 5\beta(3TM^2 + (1+\theta)\alpha + 1)\Big]\mathbb{E}\int_t^T|Z_s^n|^2ds \\ \leq\ & \Big(1 + 5\beta\Big)\mathbb{E}|\xi|^2 + \Big[(1 + 2M + \frac{M^2}{\theta}) + (5\beta+1)(1+\theta)\sum_{j=1}^{\infty}C_j + 15\beta TM^2\Big]\mathbb{E}\int_t^T|Y_s^n|^2ds \\ &+(1 + 15\beta T)\mathbb{E}\int_t^T|\varphi(s)|^2ds + (1+\frac{1}{\theta})(1+5\beta)\sum_{j=1}^{\infty}\mathbb{E}\int_t^T|g_j(s,0,0)|^2ds + \frac{1}{\beta}\mathbb{E}\sup_{0\leq s\leq T}|S_s|^2. \end{aligned}$$

Choosing $\beta, \theta > 0$ such that, $\beta < \frac{1-\alpha}{5(3TM^2+\alpha+1)}, \quad \theta \leq \frac{1-\alpha-5\beta(3TM^2+\alpha+1)}{1+\alpha+5\beta\alpha}$, we derive that

$$\begin{aligned}\mathbb{E}\left|Y_t^n\right|^2 \quad &\leq \quad c_1\mathbb{E}\left|\xi\right|^2 + c_2\mathbb{E}\int_t^T \left|Y_s^n\right|^2 ds + c_3\mathbb{E}\int_t^T \left|\varphi(s)\right|^2 ds \\ &\quad + c_4\sum_{j=1}^{\infty}\mathbb{E}\int_t^T \left|g_j(s,0,0)\right|^2 ds + c_5\mathbb{E}\sup_{0\leq s\leq T}|S_s|^2,\end{aligned}$$

where $c_1 = \frac{1+5\beta}{1-5\beta}$, $c_2 = \frac{(1+2M+\frac{M^2}{\theta})+(5\beta+1)(1+\theta)\sum_{j=1}^{\infty}C_j+15\beta TM^2}{1-5\beta}$, $c_3 = \frac{1+15\beta T}{1-5\beta}$, $c_4 = \frac{(1+\frac{1}{\theta})(1+5\beta)}{1-5\beta}$, $c_5 = \frac{1}{\beta(1-5\beta)}$.
Applying Gronwall's inequality, we get

$$\begin{aligned}\mathbb{E}\left|Y_t^n\right|^2 \quad &\leq \quad \Bigg(c_1\mathbb{E}\left|\xi\right|^2 + c_3\mathbb{E}\int_t^T \left|\varphi(s)\right|^2 ds \\ &\quad + c_4\sum_{j=1}^{\infty}\mathbb{E}\int_t^T \left|g_j(s,0,0)\right|^2 ds + c_5\mathbb{E}\sup_{0\leq s\leq T}|S_s|^2\Bigg) e^{c_2 T}.\end{aligned}$$

Therefore, we have the existence of a constant $c = c(T, M, \alpha)$ such that

$$\begin{aligned}&\mathbb{E}\left(|Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds + |K_T^n|^2\right) \\ \leq \quad & c\mathbb{E}\left(|\xi|^2 + \int_t^T \left|\varphi(s)\right|^2 ds + \sum_{j=1}^{\infty}\int_t^T \left|g_j(s,0,0)\right|^2 ds + \sup_{0\leq s\leq T}|S_s|^2\right),\end{aligned}$$

which by Burkhölder-Davis-Gundy's inequality provides

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t^n|^2 + \int_0^T \|Z_s^n\|^2 ds + |K_T^n|^2\right) \\ &\leq c\mathbb{E}\left(|\xi|^2 + \int_0^T \left|\varphi(s)\right|^2 ds + \sum_{j=1}^{\infty}\int_0^T \left|g_j(s,0,0)\right|^2 ds + \sup_{0\leq s\leq T}|S_s|^2\right) < \infty.\end{aligned}$$

Step 2: Convergence to the minimal solution
We have from (3.3) and (3.4) the existence of a process $\underline{Y}$ such that $Y_t^n \nearrow \underline{Y}_t$ a.s. for all $t \in [0, T]$. Hence, it follows from Fatou's lemma together with the dominated convergence theorem that

$$\mathbb{E}\left(|\underline{Y}_t|^2\right) \leq C \text{ and } \lim_{n\to\infty}\mathbb{E}\left(\int_0^T |Y_s^n - \underline{Y}_s|^2 ds\right) = 0 \tag{3.5}$$

On the other hand, for all $n > m \geq M$, applying Itô's formula to $|Y_t^n - Y_t^m|^2$ and noting that $\int_t^T (Y_s^n - Y_s^m)(dK_s^n - dK_s^m) \leq 0$, we have

$$\begin{aligned}&\mathbb{E}|Y_t^n - Y_t^m|^2 + \int_t^T \|Z_s^n - Z_s^m\|^2 ds \\ &\leq 2\mathbb{E}\int_t^T (Y_s^n - Y_s^m)\left(\underline{f}_n(s, Y_s^n, Z_s^n) - \underline{f}_m(s, Y_s^m, Z_s^m)\right) ds \\ &+ \sum_{j=1}^{\infty}\mathbb{E}\int_t^T |g_j(s, Y_s^n, Z_s^n) - g_j(s, Y_s^m, Z_s^m)|^2 ds.\end{aligned} \tag{3.6}$$

Then, from Hölder's inequality, the uniform linear growth condition on the sequence $\underline{f}_n$ and the assumption $(H3)$, we have,

$$\begin{aligned}
&\mathbb{E}|Y_0^n - Y_0^m|^2 + (1-\alpha)\mathbb{E}\int_0^T \|Z_s^n - Z_s^m\|^2 ds \\
\leq\ & 2\Big(\mathbb{E}\int_0^T |Y_s^n - Y_s^m|^2 ds\Big)^{\frac{1}{2}}\Big(6\mathbb{E}\int_0^T \big(2|\varphi(s)|^2 + M^2(|Y_s^n|^2 + |Y_s^m|^2 + \|Z_s^n\|^2 + \|Z_s^m\|^2)\big)\, ds\Big)^{\frac{1}{2}} \\
&+ \left(\sum_{j=1}^{\infty} C_j\right)\mathbb{E}\int_0^T |Y_s^n - Y_s^m|^2 ds.
\end{aligned}$$

Therefore, by virtue of inequality (3.4) and the fact that $\varphi \in \mathcal{M}^2(0,T,\mathbb{R})$, we have the existence of a constant C such that, for all $n > m \geq M$,

$$(1-\alpha)\mathbb{E}\int_0^T \|Z_s^n - Z_s^m\|^2 ds \quad\leq\quad C\Big(\mathbb{E}\int_0^T |Y_s^n - Y_s^m|^2 ds\Big)^{\frac{1}{2}} + \left(\sum_{j=1}^{\infty} C_j\right)\mathbb{E}\int_0^T |Y_s^n - Y_s^m|^2 ds.$$

Thus from (3.5), $\{Z^n\}$ is a Cauchy sequence in the Banach space $\mathcal{M}^2(0,T,\mathbb{R}^d)$, and there exists an $\mathcal{F}_t$-jointly measurable process $\underline{Z}$ such that $\{Z^n\}$ converges to $\underline{Z}$ as $n \to \infty$.

Similarly, by Itô's formula together with Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t^n - Y_t^m|^2 \to 0, \text{ as } n, m \to \infty,$$

from which we deduce that $\mathbb{P}$-almost surely, Y^n converges uniformly to $\underline{Y}$ which is continuous, such that $\mathbb{E}\left(\sup_{0\leq t\leq T}|\underline{Y}_t|^2\right) \leq C$.

On the other hand, since $Z^n \to \underline{Z}$ in $\mathcal{M}^2(0,T,\mathbb{R}^d)$, along a subsequence which we still denote Z^n, $Z^n \to \underline{Z},\ dt \otimes d\mathbb{P}$ a.e., and there exists $\Pi \in \mathcal{M}^2(0,T,\mathbb{R})$ such that $\forall n \geq M,\ |Z^n| < \Pi,\ dt \otimes d\mathbb{P}$ a.e.. Therefore, by Lemma 3.2, we have

$$\underline{f}_n(t, Y_t^n, Z_t^n) \quad\longrightarrow\quad f(t, \underline{Y}_t, \underline{Z}_t) \quad dt \otimes d\mathbb{P} \text{ a.e., as } n \to \infty.$$

Then, by virtue of $(H5)$**(ii)** and (3.4), it follows from the dominated convergence theorem that

$$\mathbb{E}\int_t^T \Big|\underline{f}_n(s, Y_s^n, Z_s^n) - f(s, \underline{Y}_s, \underline{Z}_s)\Big|^2 ds \longrightarrow 0, \text{ as } n \to \infty. \tag{3.7}$$

Further, in view of Burkholder-Davis-Gundy inequality, $(H3)$ and (3.5), we obtain

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\sum_{j=1}^{\infty}\left(\int_t^T g_j(s, Y_s^n, Z_s^n)\overleftarrow{dB}_s^j - \int_t^T g_j(s, \underline{Y}_s, \underline{Z}_s)\overleftarrow{dB}_s^j\right)\right| \longrightarrow 0, \quad \text{as } n \to \infty \tag{3.8}$$

and

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\int_t^T Z_s^n dW_s - \int_t^T \underline{Z}_s dW_s\right| \longrightarrow 0, \quad \text{as } n \to \infty. \tag{3.9}$$

We get also, from Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|K_t^n-K_t^m|^2\\
\le\;& 5\mathbb{E}|Y_0^n-Y_0^m|^2+5\mathbb{E}\sup_{0\le t\le T}|Y_t^n-Y_t^m|^2+5T\mathbb{E}\int_0^T\left|\underline{f}_n(s,Y_s^n,Y_s^n)-\underline{f}_m(s,Y_s^m,Z_s^m)\right|^2ds\\
&+5\mathbb{E}\sup_{0\le t\le T}\left|\sum_{j=1}^{\infty}\int_0^t\left(g_j(s,Y_s^n,Y_s^n)-g_j(s,Y_s^m,Z_s^m)\right)\overleftarrow{dB}_s^j\right|^2\\
&+5\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t(Z_s^{m+n}-Z_s^n)dW_s\right|^2,
\end{aligned}$$

which, together with (3.7), (3.8), (3.9), provides

$$\mathbb{E}\sup_{0\le t\le T}|K_t^n-K_t^m|^2\longrightarrow 0,\quad as\;\; n,m\to\infty.$$

Consequently, there exists a $\mathcal{F}_t$-measurable process $\underline{K}$ with value in $\mathbb{R}_+$ such that

$$\mathbb{E}\sup_{0\le t\le T}|K_t^n-\underline{K}_t|^2\longrightarrow 0,\quad as\;\; n\to\infty.$$

Obviously, $\underline{K}_0=0$ and $\{\underline{K}_t;\;0\le t\le T\}$ is a non-decreasing and continuous process.

On the other hand, from the result of Saisho [11] (see p. 465), we have

$$\int_0^T(Y_s^n-S_s)dK_s^n\;\longrightarrow\int_0^T(\underline{Y}_s-S_s)d\underline{K}_s\quad\mathbb{P}-a.s.,\quad as\;\; n\to\infty.$$

Finally, passing to the limit in (3.2), we conclude that $(\underline{Y},\underline{Z},\underline{K})$ is a solution of (1.1).

Now, let $(Y,Z,K)\in\mathcal{E}_m^2(0,T)$ be any solution of (1.1). By virtue of Theorem 3.1, we have $Y^n\le Y,\;\forall n\ge M$. Therefore, due to the above step and taking the limit, we have $\underline{Y}\le Y$. The proof is complete. □

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