Extension of time for decomposition of stochastic flows in spaces with complementary foliations[*]

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Abstract

Let M be a manifold equipped (locally) with a pair of complementary foliations. In Catuogno, da Silva and Ruffino [4], it is shown that, up to a stopping time τ, a stochastic flow of local diffeomorphisms φ_t in M can be decomposed in diffeomorphisms that preserves this foliations. In this article we present techniques which allow us to extend the time of this decomposition. For this extension, we use two techniques: In the first one, assuming that the vector fields of the system commute with each other, we apply Marcus equation to jump nondecomposable diffeomorphisms. The second approach deals with the general case: we introduce a 'stop and go' technique that allows us to construct a process that follows the original flow in the 'good zones' for the decomposition, and remains paused in 'bad zones'. Among other applications, our results open the possibility of studying the asymptotic behaviour of each component.

1 Introduction

Given a stochastic flow φ_t of local diffeomorphisms in a differentiable manifold M, in many circunstances the decomposition of φ_t with components in subgroups of the group of diffeomorphisms $\mathrm{Diff}(M)$ provide interesting dynamical or geometrical information of the system. In the literature, this kind of decomposition has been studied in several frameworks and with different aimed subgroups; among others in Bismut [2], Kunita [7], [8], Ming Liao [10] and some of our previous work [3], [5], [15]. In the last few papers mentioned, geometrical conditions on a Riemannian manifold have been stated to guarantee the existence of the decomposition where the first component lies in the subgroups of isometries or affine transformations.

In particular, in Catuogno, da Silva and Ruffino [4], the authors consider a pair of complementary distributions in a differentiable manifold M, in the sense that each tangent space splits in a direct sum of two subspaces depending differentiably on M. These subspaces are called, by convenience, horizontal and vertical distributions. In that article it is shown that locally, up to a stopping time τ, a stochastic flow φ_t in M can be decomposed as $\varphi_t = \xi_t \circ \Psi_t$, where ξ_t is a diffusion in the group of diffeomorphisms

$\operatorname{Diff}(\Delta^H, M)$ generated by horizontal vector fields. On the other hand, Ψ_t is a process in the group of diffeomorphisms $\operatorname{Diff}(\Delta^V, M)$ generated by vertical vector fields. The infinite dimensional Lie group structure considered in this case is described in Milnor [11], Neeb [12] and Omori [14]. In [4] the authors present stochastic differential equations on the corresponding infinite dimensional Lie subgroups for the components ξ_t and Ψ_t.

The stopping time τ mentioned above, which restricts the time where the decomposition holds, appears due to an explosion in the equation of one of the components of the decomposition, with initial condition at the identity. This explosion phenomenon is related to the choice of the distributions, and it is not intrinsic to the original flow φ_t. There might be large intervals on time after τ such that the diffeomosphisms φ_t are decomposable for t in these intervals.

In this article we present techniques which allow us to extend the time of the decomposition in a simplified framework of [4]: We consider that the distributions are in fact integrable, hence the manifold M has locally a pair of complementary foliations. This is a natural structure since any coordinate systems on M generates this foliations. Besides, it means that locally the manifold is diffeomorphic to an open set in $\mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^{n-p}$, where p and $(n-p)$ are the dimensions of the horizontal and vertical foliations, respectively.

With this change of coordinates in mind, consider a diffeomorphism $\varphi : U \subset \mathbf{R}^p \times \mathbf{R}^{n-p} \to \varphi(U) \subset \mathbf{R}^p \times \mathbf{R}^{n-p}$, with U an open set of $\mathbf{R}^n$. One writes in coordinates $\varphi = (\varphi^1(x,y), \varphi^2(x,y))$. Then, by the inverse function theorem, the local decomposition exists if and only if the $(n-p) \times (n-p)$ matrix $\frac{\partial \varphi^2(x,y)}{\partial y}$ is invertible. That is, $\varphi = (\xi^1(x,y), Id_2) \circ (Id_1, \varphi^2(x,y))$, where Id_1 and Id_2 are the identities in $\mathbf{R}^p$ and in $\mathbf{R}^{n-p}$, respectively. With this notation, we call

Definition 1.1. *A diffeomorphism* $\varphi : U \subset \mathbf{R}^p \times \mathbf{R}^{n-p} \to \varphi(U) \subset \mathbf{R}^p \times \mathbf{R}^{n-p}$ *is* p-*decomposable in a neighborhood of* (x,y) *if* $\det \frac{\partial \varphi^2(x,y)}{\partial y} \neq 0$.

The explosion time τ for the decomposition of the flow is related to the fact that φ_τ is not p-decomposable. As a basic example to illustrate the explosion, consider the linear rotation, whose 1-decomposition is given by

$$\varphi_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \begin{pmatrix} \sec t & -\tan t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin t & \cos t \end{pmatrix}.$$

Hence, φ_t is not decomposable when $t = \frac{\pi}{2} + k\pi$, with $k \in \mathbf{Z}$. The stopping time as in [4] is given by $\tau = \frac{\pi}{2}$, nevertheless, the decomposition can be done in many other intervals after this point. Note that, in this example, immediately after this stopping time, $\cos t$ changes its sign and the second component Ψ_t revert orientation, hence, it is no longer in the subgroup of diffeomorphism generated by vertical vector fields, but it is rather in the subgroup of diffeomorphisms which preserves each vertical leaves of the foliation, not necessarily in the connected component of the identity. Hence, here $\operatorname{Diff}(\Delta^H, M)$ and $\operatorname{Diff}(\Delta^V, M)$ are extended to include nonpreserving orientation diffeomorphisms.

One of the motivations for the decomposition in [4] is that it shows how close the system is horizontal leaf-preserving, i.e., how close is the vertical component to the identity. Here, the motivation is to show that this decomposition extends further in time, even if one has to consider the nonpreserving orientation diffeomorphisms. Among other applications, our results open the possibility of studying the asymptotic behaviour of each component.

The extension of time is described using two techniques: the first one (Section 2) assumes that the vector fields of the system commute with each other. We apply Marcus equation to jump nondecomposable diffeomorphisms; essentially we construct a flow $\widetilde{\varphi}_t$

which is close to the original one and is p-decomposable for all $t \geq 0$. Approximation here holds except in a set of arbitrarily small probability (Theorem 2.3). The second approach (Section 3) deals with the general case: we introduce a 'stop and go' technique that allows us to construct a process that follows the original flow in the 'good zones' for the decomposition, and remains paused in 'bad zones', i.e., close to nondecomposable diffeomorphisms (Proposition 3.2).

2 Commuting vector fields

2.1 Marcus equation and preliminaries

For reader's convenience, we recall the definition of Marcus equation (see e.g. Kurtz, Pardoux and Protter [9], Applebaum [1], Fujiwara and Kunita [6]):

$$dx_t = \sum_{i=0}^{m} X^i(x_t) \circ dZ_t^i, \tag{2.1}$$

where the integrator $\{Z_s^i : s \geq 0\}$ is a semimartingale with jumps and X^i are smooth vector fields in $\mathbf{R}^d$ for all $i \in \{0, 1, \ldots, m\}$, with initial condition $x(0) = x_0$. The integral form is given by

$$x_t = x_0 + \sum_{i=0}^{m} \int_0^t X^i(X_s) \circ dZ_s^i =: x_0 + \int_0^t X(x_s) \circ dZ_s, \tag{2.2}$$

where $X(x_s)$ and Z_s are the matricial representation for the summation. The solution x_t is interpreted as an adapted stochastic process that satisfies the equation

$$
\begin{aligned}
x_t = x_0 + \int_0^t X(x_{s_-})\, dZ_s\ &+\ \frac{1}{2} \int_0^t X'X(x_s)\, d[Z,Z]_s^c\ + \\
&+ \sum_{0 < s \leq t} \left\{ \varphi(X\Delta Z_s, x_{s_-}) - x_{s_-} - X(x_{s_-})\Delta Z_s \right\},
\end{aligned}
\tag{2.3}
$$

in the following sense: the first term on the right hand side of equation (2.3) is a standard Itô integral of the predictable process $\{X(x_{t_-})\}$ with respect to the semimartingale Z_t. The second term is a Stieltjes integral with respect to the continuous part of the quadratic variation of Z_t. In the third term: $\varphi(X\Delta Z_s, x_{s_-})$ indicates the solution of the initial value problem in $t = 1$ with respect to the vector field $X\Delta Z_s$, and initial condition x_{s_-}. Regularity conditions on the vector fields X implies that there exists a unique stochastic flow of diffeomorphisms φ_t that is solution of equation (2.1). Conditions on the derivatives of the vector fields guarantee that the flow exists for all $t \geq 0$.

The main reason of using Marcus equation in this section is that, in these equations, the jumps of the solution occur in the direction of the deterministic flow given by the vector field $X\Delta Z_s$. This property is used in the proof of Theorem 2.1 to guarantee that, after a jump, the solution of the proposed approximation flow $(\widetilde{\varphi}_t)$ reaches again the original flow (φ_t).

Moreover, for an embedded submanifold M in an Euclidean space, if the vector fields of the equation (2.1) are in TM then, for each initial condition on M, the solution stays in M a.s., see [9]. This feature, a sort of support theorem, allows one to extend directly the result of Theorem 2.1 to manifolds.

We present a simple example that illustrates the technique in the proof of the next results. The idea is to use Marcus equation in order to construct a new flow $\widetilde{\varphi}_t$, which is decomposable for all $t \geq 0$ and simultaneously is close to the original flow φ_t. Consider

again the simple example of pure rotation flow

$$\varphi_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

which is solution of the linear equation $dx_t = Ax_t \, dt$, where A is skew-symmetric. We have to construct an integrator Z_t in such a way that the solution flow $\widetilde{\varphi}_t$ of the Marcus equation $dx_t = Ax_t \circ dZ_t$ jumps over the nondecomposable rotations, with the additional property that $\widetilde{\varphi}_t$ differs from φ_t for t in a set with Lebesgue measure arbitrarily small.

Fix $\varepsilon > 0$ and choose a sequence of points p_n just before the critical time for the decomposition:

$$p_n \in (\frac{\pi}{2} + n\pi - \frac{\varepsilon}{2^n}, \frac{\pi}{2} + n\pi)$$

for all nonnegative integer n. These points indicate the beginning of what we call in the proof of the theorem a 'red zone' for the corresponding critical point. Then, define

$$Z_t = \begin{cases} p_n & \text{if } t \in [p_n, (2n+1)\pi - p_n); \\ t & \text{otherwise.} \end{cases}$$

Z_t is a semimartingale with jumps, and the solution of $dx_t = Ax_t \circ dZ_t$ is given by

$$\widetilde{\varphi}_t = \begin{pmatrix} \cos(Z_t) & -\sin(Z_t) \\ \sin(Z_t) & \cos(Z_t) \end{pmatrix}.$$

We have that $\widetilde{\varphi}_t$ is 1-decomposable for all $t \geq 0$ and, by construction, it is arbitrarily close to φ_t. A similar idea is used to obtain results on decomposition of stochastic flows driven by Brownian motion (extensible to a larger class of semimartingales via representation, see e.g. Nualart [13]).

2.2 Main results

In this section, we consider the flow φ_t generated by a Stratonovich SDE:

$$dx_t = X^0(x_t) \, dt + \sum_{i=1}^{m} X^i(x_t) \circ dB_t^i, \tag{2.4}$$

where $(B_t^1, \ldots, B_t^m) \in \mathbf{R}^m$ is a Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$; X^i for $i \in \{0, 1, 2, \ldots, m\}$ are smooth vector fields in $\mathbf{R}^n$. We assume that the corresponding (deterministic) flows $\phi_t^{X^i}$ commute with each other. Given $u = (t_0, t_1, \ldots, t_m) \in \mathbf{R}^{m+1}$, let $\phi(u)$ be the composition of the deterministic flows, that is:

$$\phi(u) = \phi_{t_0}^{X_0} \circ \phi_{t_1}^{X_1} \circ \ldots \circ \phi_{t_m}^{X_m}.$$

We write $(x, y) \in \mathbf{R}^p \times \mathbf{R}^{n-p} = \mathbf{R}^n$ and $\phi(u)(x, y) = (\phi(u)^1, \phi(u)^2)$. For a given initial condition, consider $P = \{u \in \mathbf{R}^{m+1} : \det \frac{\partial \phi(u)^2}{\partial y} = 0\}$, the set of undecomposable generated diffeomorphisms.

Theorem 2.1. *Let φ_t be the stochastic flow generated by equation (2.4). Assume that P has zero Lebesgue measure. Then, given $\varepsilon > 0$ and $a > 0$, there exists a semimartingale $Z_t \in \mathbf{R}^{m+1}$ such that:*

1. the solution $\widetilde{\varphi}_t$ of the Marcus SDE $dx_t = \sum_{i=0}^{m} X^i(x_t) \circ dZ_t^i$ is p-decomposable for
   $t \geq 0$;

2. *the random set $C(\omega) = \{t \geq 0 : \varphi_t(\omega) \neq \widetilde{\varphi}_t(\omega)\}$ is Lebesgue measurable a.s;*
3. $\mathbf{P}[\mu(C) > a] \leq \varepsilon$.

Proof. Initially, fix $\varepsilon > 0$ and $a > 0$. For each $t \geq 0$, denote $U_t = (t, B_t^1, \ldots, B_t^n)$. As P has zero Lebesgue measure, given $\delta > 0$, there is an open set A_δ, such that $A_\delta \supseteq P$ and $\mu(A_\delta) < \delta$. By properties of Brownian motion, given $d > 0$, we have that:

$$\mathbf{P}\left[U_s \in A_\delta \ \forall s \in [t, t+d] \ \middle| \ U_s \in A_\delta \right] \to 0 \ \text{ when } \ \delta \to 0.$$

Therefore, for each $k \in \mathbf{N}$, there exists an open set A_k that contains P with Lebesgue measure sufficiently small such that:

$$\mathbf{P}\left[U_s \in A_k \ \forall s \in [t, t+\frac{a}{2^k}] \ \middle| \ U_s \in A_k \right] < \frac{\varepsilon}{2^k}.$$

We can assume that the sequence of sets $(A_k)_{k \in \mathbf{N}}$ is decreasing; besides, since $0 \notin P$ one can always assume that $0 \notin A_k$ for all $k \in \mathbf{N}$.

By Urysohn's lemma, there exist continuous functions $F_k : \mathbf{R}^m \to [0,1]$, such that $F_k^{-1}(1) = (A_k)^C$ and $F_k^{-1}(0) = P$. For each $k \in \mathbf{N}$, consider the following partition in $\mathbf{R}^{m+1}$:

- 'Green zone' $G_k := F_k^{-1}(1)$;
- 'Yellow zone' $Y_k = F_k^{-1}\left(\frac{1}{2}, 1\right)$;
- 'Red zone' $R_k = F_k^{-1}\left[0, \frac{1}{2}\right]$.

Red zone corresponds to a set where we do not allow the dynamics to get into (via stopping times). Green zone corresponds to the set where we allow the dynamics to go on freely; finally, the yellow zone is an intermediate set.

By construction, the origin of $\mathbf{R}^{m+1}$ belongs to green zone for all $k \in \mathbf{N}$, besides, this set is increasing with k. We define by induction the following sequences of stopping times: Take $\overline{T}_0 = 0$, $T_i = \inf\{t > \overline{T}_{i-1} : U_t \in R_i\}$ and $\overline{T}_i = \inf\{t > T_i : U_t \in G_i\}$. The yellow zone guarantees that $\overline{T}_{i-1} < T_i < \overline{T}_i$ for all $i \in \mathbf{N}$.

Define the semimartingale $\{Z_t : t \geq 0\}$ as follows:

$$Z_t = \begin{cases} U_t, & \text{if } \ t \in \left[\overline{T}_{k-1}, T_k\right) \ \text{for some } \ k \in N; \\ U_{T_k}, & \text{if } \ t \in \left[T_k, \overline{T}_k\right). \end{cases}$$

Note that Z_t is constant when U_t crosses the yellow zone and goes from the red to the green zone; otherwise, it coincides with U_t. Now, consider the Marcus SDE given by

$$dx_t = \sum_{i=0}^{m} X^i(x_t) \circ dZ_t^i,$$

whose solution flow is given by:

$$\widetilde{\varphi}_t = \phi_{Z_t} = \phi_{Z_t^0}^{X_0} \circ \phi_{Z_t^1}^{X_1} \circ \ldots \circ \phi_{Z_t^m}^{X_m}.$$

Hence, $\widetilde{\varphi}_t$ is p-decomposable for all $t \geq 0$.

For the second part of the statement, initially note that $C(\omega)$ is Lebesgue measurable a.s. since φ_t and $\widetilde{\varphi}_t$ are measurable with respect to the product σ-algebra in $\mathbf{R} \otimes \Omega$. Moreover, $\varphi_t(\omega) = \widetilde{\varphi}_t(\omega)$ if $U_t(\omega) = Z_t(\omega)$, therefore:

$$C(\omega) \subseteq \bigcup_{k \in \mathbf{N}} \left(T_k(\omega), \overline{T}_k(\omega)\right).$$
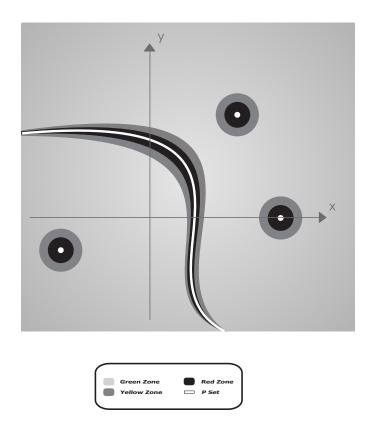
Figure 1: Sketch of possible green, yellow and red zones in $\mathbf{R}^2$.

Finally, a necessary condition to $\mu(C(\omega)) > a$ is that, for some $k \in \mathbf{N}$, we have $\overline{T}_k(\omega) - T_k(\omega) > \frac{a}{2^k}$. So, it follows that:

$$
\begin{aligned}
\mathbf{P}[\mu(C) > a] &\leq \mathbf{P}\left[\bigcup_{k \in \mathbf{N}} \left\{ \omega : \left(\overline{T}_k - T_k\right) > \frac{a}{2^k} \right\}\right] \\
&\leq \sum_{k \in \mathbf{N}} \mathbf{P}\left[\left(\overline{T}_k - T_k\right) > \frac{a}{2^k}\right] \\
&\leq \sum_{k \in \mathbf{N}} \frac{\varepsilon}{2^k} = \varepsilon.
\end{aligned}
\tag{2.5}
$$

$\square$

Remark 2.2. *Generically in the C^1 topology, when the SDE has just a single vector field, then the set $P \subset \mathbf{R}$ in the statement of the theorem has zero Lebesgue measure: In fact, the condition on the second component $\det \frac{\partial \phi_t^2}{\partial y} = 0$ is destroyed by small perturbation on their derivatives. Also for linear vector fields, $P \subset \mathbf{R}$ is discrete since $\det \frac{\partial \phi_t^2}{\partial y}$ is a nonvanishing analytical function.*

When P, defined above, has an arbitrary Lebesgue measure, a weaker result holds. One can still construct a semimartingale Z_t that is close to the process U_t, such that the corresponding generated flows $\widetilde{\varphi}_t$ and φ_t are again close of each other. But here one loses the probabilistic approach to control the Lebesgue measure of $C(\omega)$. We have, with the same notation as before:

Proposition 2.3. *Let φ_t be the stochastic flow generated by equation (2.4). Given $\varepsilon > 0$ and an open set $A \supseteq P$ such that $\mu(A \setminus P) < \varepsilon$, there exists a semimartingale Z_t which satisfies the following properties:*

1. *The corresponding solution flow $\widetilde{\varphi}_t$ driven by Z_t is p-decomposable for all $t \geq 0$;*

2. *If $U_t(\omega) \notin A$, then $Z_t(\omega) = U_t(\omega)$, hence $\widetilde{\varphi}_t(\omega) = \varphi_t(\omega)$.*

Proof. Since $0 \notin P$, we can assume (reducing A, if necessary), that $0 \notin A$. By Urysohn's lemma, there exists a continuous function $F : \mathbf{R}^{m+1} \to [0,1]$, such that $F^{-1}(1) = A^C$ and $F^{-1}(0) = P$. In an analogous way to the proof of Theorem 2.1, consider a partition of $\mathbf{R}^{m+1}$ as follows:

- 'Green zone' $G = F^{-1}(1)$;

- 'Yellow zone' $Y = F^{-1}\left(\frac{1}{2}, 1\right)$;

- 'Red zone' $R = F^{-1}\left[0, \frac{1}{2}\right]$.

We remark that here, different from the proof of the previous theorem, the green, yellow and red zones remain fixed during all the process. Again, by induction, consider the following sequences of stopping times: Take $\overline{T}_0 = 0$, $T_i = \inf\{t > \overline{T}_{i-1} : U_t \in R\}$ and $\overline{T}_i = \inf\{t > T_i : U_t \in G\}$. Define the process $\{Z_t : t \geq 0\}$ as follows:

$$Z_t = \begin{cases} U_t, & \text{if } t \in \left[\overline{T}_{k-1}, T_k\right) \text{ for some } k \in N; \\ U_{T_k}, & \text{if } t \in \left[T_k, \overline{T}_k\right). \end{cases}$$

Finally, note that if $U_t(\omega) \notin A$, then $U_t(\omega)$ is contained in the green zone. Therefore, in this case $U_t(\omega) = Z_t(\omega)$ and $\varphi_t(\omega) = \widetilde{\varphi}_t(\omega)$. $\qquad\square$

Example 2.4. *We present an example of P with nonzero Lebesgue measure. Let $f : \mathbf{R} \to [-\pi/2, \pi/2]$ be a smooth function such that $f(x) = -\pi/2$ if $x \leq -\pi/2$ and $f(x) = \pi/2$ if $x \geq \pi/2$. Consider the following deterministic ODE in $\mathbf{R}^3$:*

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -f'(z)y \\ f'(z)x \\ 1 \end{pmatrix}$$

whose solution at points in the horizontal plane $(x_0, y_0, 0)$ is given by

$$\varphi(t)(x_0, y_0, 0) = \begin{pmatrix} \cos f(t) & -\sin f(t) & 0 \\ \sin f(t) & \cos f(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ t \end{pmatrix}.$$

Consider the 1-decomposition of the flow in $\mathbf{R} \times \mathbf{R}^2$. The determinant of the submatrix in the dashed box above corresponds to $\det \frac{\partial \phi_t^2}{\partial y}$ in the definition of P. Hence, for any initial condition in the horizontal plane $(x, y, 0)$, we have that $P = [\pi/2, \infty)$. Geometrically, the dynamics of the horizontal plane $\{(x, y, 0); x, y \in \mathbf{R}\}$ is simply a rotation by $f(t)$ around the z-axis, with a constant velocity increasing in the z-coordinate. Hence, at time $t \geq \frac{\pi}{2}$, the image of e_1 ($\in \mathbf{R}^p$, with $p = 1$) intersects the plane $\{(0, y, z); y, z \in \mathbf{R}\} = \mathbf{R}^2$, the vertical foliation.

3 General case

3.1 'Stop and go' technique and preliminaries

The technique of the previous section (i.e., via Marcus equation) does not allow one to generate a p-decomposable flow $\widetilde{\varphi}_t$ of diffeomorphisms close to the original flow φ_t when the vector fields do not commute with each other. In fact, the jumps of Marcus equation occur in the direction of the deterministic flows (they do not reach, for example, the directions of their Lie brackets). In other words, one can not control the proximity of $\widetilde{\varphi}_t$ and φ_t just controlling how close Z_t is from U_t.

The idea of this section is to propose a technique to generate a process $\widetilde{\varphi}_t$, such that, when necessary, it stops at a certain point, then it jumps, at appropriate times, landing exactly at the original flow φ_t. This tool, that we call 'stop and go' technique, allows one to obtain an analogous result of Proposition 2.3 for the general case.

Using the same notation as before, consider the Stratonovich SDE given by:

$$dx_t = X^0(x_t)\,dt + \sum_{i=1}^m X^i(x_t) \circ dB_t^i, \tag{3.1}$$

whose solution is given by the flow of diffeomorphisms φ_t, but here, the vector fields not necessarily commute with each other. Given an arbitrary sequence T of stopping times, i.e.:

$$T: \quad 0 = \overline{T}_0 \le T_1 < \overline{T}_1 \le T_2 < \overline{T}_2 < \dots$$

we define the 'stop and go' equation, as follows:

$$dx_t = \sum_{i=0}^m X^i(x_t) \,\square^T\, dU_t^i, \tag{3.2}$$

where, again $U_t = (t, B_t^1, \dots, B_t^m)$ and whose solution must be interpreted, in terms of action of diffeomorphisms, as the process:

$$\widetilde{\varphi}_t = \begin{cases} \varphi_t, & \text{if } t \in [\overline{T}_j, T_{j+1}); \\ \varphi_{T_j}, & \text{if } t \in [T_j, \overline{T}_j). \end{cases}$$

3.2 Main result

Consider the SDE (3.1) and its corresponding stochastic flow φ_t. Given an initial condition, for each $\omega \in \Omega$, define the set $P(\omega) = \{t \in \mathbf{R} : \det \frac{\partial \varphi_t^2}{\partial y}(\omega) = 0\}$, which is a random set, different from the previous section.

Proposition 3.1. *Given $\varepsilon > 0$, and an open random set $A(\omega) \supseteq P(\omega)$ such that $\mu(A(\omega) \setminus P(\omega)) < \varepsilon$, there exists a sequence T of stopping times such that:*

- *The process $\widetilde{\varphi}_t$, solution of the 'stop and go' equation (3.2), is p-decomposable for all $t \ge 0$;*
- *$\widetilde{\varphi}_t(\omega) = \varphi_t(\omega)$ if $t \notin A(\omega)$.*

Proof. Since $0 \notin P(\omega)$ for all $\omega \in \Omega$, we can assume (reducing the set $A(\omega)$, if necessary), that $0 \notin A(\omega)$. We define a random partition of $\mathbf{R}$ in an analogous way of the previous results. Take, for each ω a continuous function $F_\omega : \mathbf{R} \to [0,1]$ with the property that $F_\omega^{-1}(1) = A^C(\omega)$ and $F_\omega^{-1}(0) = P(\omega)$, and define:

- 'Green zone' $G_\omega = F_\omega^{-1}(1)$;
- 'Yellow zone' $Y_\omega = F_\omega^{-1}\left(\frac{1}{2}, 1\right)$;

- 'Red zone' $R_\omega = F_\omega^{-1}\left[0, \frac{1}{2}\right]$.

Define by induction a sequence of stopping times T, as follows: $\overline{T}_0 = 0$, $T_i(\omega) = \inf\left\{t > \overline{T}_{i-1}(\omega) : t \in R_\omega\right\}$ and $\overline{T}_i(\omega) = \inf\left\{t > T_i(\omega) : t \in G_\omega\right\}$.

The sequence T was constructed in a such way that $\det \frac{\partial \widetilde{\varphi}_t^2}{\partial y}(\omega)$ does not vanish. So $\widetilde{\varphi}_t$ is p-decomposable for all $t \geq 0$. For the second part of the statement, if $t \notin A(\omega)$, then:

$$t \notin \bigcup_{k \in \mathbf{N}} \left(T_k(\omega), \overline{T_k}(\omega)\right),$$

and in this case, $\varphi_t(\omega) = \widetilde{\varphi}_t(\omega)$.

$\square$

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