

Application of an averaging principle on foliated diffusions: topology of the leaves*

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Abstract

We consider an ϵK transversal perturbing vector field in a foliated Brownian motion defined in a foliated tubular neighbourhood of an embedded compact submanifold in \mathbf{R}^3 . We study the effective behaviour of the system under this ϵ perturbation. If the perturbing vector field K is proportional to the Gaussian curvature at the corresponding leaf, we have that the transversal component, after rescaling the time by t/ϵ , approaches a linear increasing behaviour proportional to the Euler characteristic of M , as ϵ goes to zero. An estimate of the rate of convergence is presented.

Keywords: Averaging principle; foliated stochastic flow; Brownian motion on manifolds.

AMS MSC 2010: 60H10 ; 58J65 ; 58J37.

Submitted to ECP on August 8, 2014, final version accepted on March 19, 2015.

1 The set up

The purpose of these notes is to explore a topological application of an averaging principle for foliated stochastic flows, as established in Gargate and Ruffino [3]. Our geometrical setting is a foliation of a tubular neighbourhood of an embedded compact boundaryless submanifold M of \mathbf{R}^3 consisting on diffeomorphic copies of M . Precisely, consider a tubular neighbourhood U of M and a diffeomorphism $\phi : U \rightarrow M \times (-a, a)$ such that each $s \in (-a, a) \in \mathbf{R}$ generates a leaf $\phi^{-1}(M \times \{s\})$ which is diffeomorphic to $M = \phi^{-1}(M \times \{0\})$. The diffeomorphism ϕ is taken such that transversal component $\frac{\partial \phi^{-1}}{\partial s}$ is orthogonal to the leaves, pointing outwards.

The main idea of this application is to consider an unperturbed system whose trajectories stay, each one, in a unique leaf and are Brownian motions on its corresponding leaf. This structure consisting of simultaneous Brownian motions on each leaf is called a *foliated Brownian motion*, see e.g. the seminal article by L. Garnett [4] or more recently Catuogno, Ledesma and Ruffino [1], [2]. We destroy this foliated behaviour of trajectories introducing a perturbing vector field orthogonal to the leaves such that at each point, this vector field is given by the Gaussian curvature of the corresponding leaf, pointing outwards for positive curvature. Putting together the results on the averaging principle, [3], for this particular foliated system and the classical Gauss-Bonnet theorem, we have that, in the average, the transversal behaviour of the perturbed system, with time rescaled by $\frac{t}{\epsilon}$, approaches a deterministic ODE with constant coefficient proportional to

*Partially supported by FAPESP nr. 12/18780-0, 11/50151-0 and CNPq nr. 477861/2013-0.

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the Euler characteristic of the original submanifold M , as ϵ goes to zero. An estimate of the rate of convergence can be obtained.

In the next paragraphs we recall some of the results we are going to use on averaging stochastic flows in foliated spaces.

Previous results on averaging in foliated spaces

We recall the main results in [3] which are relevant here. Let N be a smooth Riemannian manifold with an n -dimensional smooth foliation, i.e. N is endowed with an integrable regular distribution of dimension n , hence N is decomposed in a disjoint union of immersed submanifolds. For a precise definition and further properties of foliated spaces in more general settings see e.g. the initial chapters of Tondeur [9], Walczak [10], among others. We denote by L_x the leaf of the foliation passing through a point $x \in N$. We assume that the leaves are compact and that given an initial condition x_0 , the leaf L_{x_0} has a tubular neighbourhood $U \subset N$ such that there exists a diffeomorphism $\varphi : U \rightarrow L_{x_0} \times V$, where $V \subset \mathbf{R}^d$ is a connected open set containing the origin. For simplicity, the second (vertical) coordinate of a point $x \in U$ is called the vertical projection $p(x) \in V$, i.e. $\varphi(x) = (u, \pi(x))$ for some $u \in L_x$. Hence for any fixed $v \in V$, the inverse image $p^{-1}(v)$ is a compact leaf L_x , where x is any point in U such that the vertical projection $p(x) = v$. In coordinates, we denote

$$p(x) = (p_1(x), \dots, p_d(x)) \in V \subset \mathbf{R}^d$$

for any $x \in U$.

Consider an SDE in M whose associated stochastic flow preserves the foliation, i.e. we consider a Stratonovich equation

$$dx_t = X_0(x_t)dt + \sum_{k=1}^r X_k(x_t) \circ dB_t^k \tag{1.1}$$

where the smooth vector fields X_k are foliated in the sense that $X_k(x) \in T_x L_x$, for $k = 0, 1, \dots, r$. Here $B_t = (B_t^1, \dots, B_t^r)$ is a standard Brownian motion in \mathbf{R}^r with respect to a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbf{P})$. For an initial condition x_0 , the trajectories of the solution x_t in this case lay on the leaf L_{x_0} a.s.. Moreover, there exists a (local) stochastic flow of diffeomorphisms $F_t : N \rightarrow N$ which restricted to the initial leaf is a flow in the compact submanifold L_{x_0} .

We introduce a perturbing smooth vector field K in the system such that this vector field destroys the foliated structure of the trajectories. We denote the perturbed system by x_t^ϵ which satisfies the SDE

$$dx_t^\epsilon = X_0(x_t^\epsilon)dt + \sum_{k=1}^r X_k(x_t^\epsilon) \circ dB_t^k + \epsilon K(x_t^\epsilon) dt, \tag{1.2}$$

with the same initial condition $x_0^\epsilon = x_0$.

For each vertical coordinate $i = 1, \dots, d$, we denote by $Q^i(v)$ the ergodic average of the i -th component of the perturbing vector field K in the corresponding leaf $p^{-1}(v)$. Hence, by the ergodic theorem

$$Q^i(v) := \int_{p^{-1}(v)} Dp_i(K)(x) d\mu_v(x) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Dp_i(K)(F_s(x)) ds$$

for μ_v -almost every point x in the leaf $p^{-1}(v)$; here $D_x p_i$ is the derivative of p_i at the point x and μ_v denotes an invariant measure on $p^{-1}(v)$ which we assume uniquely ergodic.

Denote by $\eta(t)$ an estimate of the rate of convergence of the ergodic limit in L^p for the functions $Q^i(v)$. Hence $\eta(t)$ tends to zero when t goes to infinity. In general, there is no optimal rate of convergence, see e.g. Kakutani-Petersen [6], Krengel [7] and an explicit example of continuous system in the averaging context in [3].

We assume that the averaging functions of the perturbing vector field $Q^i : V \rightarrow \mathbf{R}$ are Lipschitz continuous for $i = 1, \dots, d$. This hypothesis holds naturally if the invariant measures μ_v for the unperturbed foliated system has a sort of weakly continuity on v . For example, for nondegenerate systems on the leaves, this condition is naturally satisfied.

Consider the following ODE on the vertical space V :

$$\frac{dv(t)}{dt} = \left(Q^1(v(t)), \dots, Q^d(v(t)) \right),$$

with initial condition $p(x_0) = 0$. Let T_0 be the time that the solution $v(t)$ hits the boundary ∂V . Let τ^ϵ be the stopping time given by the exit time of the perturbed system x_t^ϵ from the coordinate neighbourhood $U \subset M$.

An averaging principle in this context is established by the theorem below which says that the transversal behaviour of x_t^ϵ can be approximated in the average by an ordinary differential equation in the transversal space whose coefficients are given by the average of the transversal component of the perturbation K with respect to the invariant measure on the leaves for the original dynamics of equation (1.1), when ϵ decreases to zero. The rate of converge is given below:

Theorem 1.1. *Assuming Lipschitz continuity of the averaging functions $Q^i : V \rightarrow \mathbf{R}$ for $i = 1, \dots, d$ we have:*

- (1) *For any $0 < t < T_0$, $\epsilon > 0$, $\beta \in (0, 1/2)$, $\alpha \in (0, 1)$ and $2 \leq q < \infty$, there exist functions $C_1 = C_1(t)$ and $C_2 = C_2(t)$ such that*

$$\left[\mathbb{E} \left(\sup_{s \leq t} \left| p \left(x_{\left(\frac{s}{\epsilon}\right) \wedge \tau^\epsilon}^\epsilon \right) - v(s) \right|^q \right) \right]^{\frac{1}{q}} \leq C_1 \epsilon^\alpha + C_2 \eta \left(t \left| \ln \epsilon \right|^{\frac{2\beta}{q}} \right),$$

where $\eta(t)$ is the rate of convergence in L^q of the ergodic averages of the unperturbed trajectories on the leaves.

- (2) *For $\gamma > 0$, let*

$$T_\gamma = \inf \{ t > 0 \mid \text{dist}(v(t), \partial V) \leq \gamma \}.$$

The exit times of the two systems satisfy the estimates

$$\mathbb{P}(\epsilon \tau^\epsilon < T_\gamma) \leq \gamma^{-q} \left[C_1(T_\gamma) \epsilon^\alpha + C_2(T_\gamma) \eta \left(T_\gamma \left| \ln \epsilon \right|^{\frac{2\beta}{q}} \right) \right]^q.$$

Item (b) of the theorem above guarantees robustness of the result. For the proof of the theorem above see [3]; further extension to Lévy processes has been done in [5].

2 Exploring the topology of the leaves

We consider an unperturbed foliated dynamics in U (degenerate) which is a foliated Brownian motion. That is, for each initial condition $x_0 \in U$, the solution is a Brownian motion on the corresponding leaf L_{x_0} . For more details and construction of this processes, see [1]. In our particular case of embedded manifold we can consider the dynamics generated by gradient vector fields tangent to the leaves. Precisely, at each $x \in U$, let $X^i(x)$ be the orthogonal projection of e_i , the i -th element of the canonical basis onto the

tangent space $T_x L_x$ of the leaf passing through x , for each $i = 1, 2, 3$. The vector fields X^1, X^2 and X^3 determine the following Stratonovich stochastic equation in U :

$$dx_t = \sum_{i=1}^3 X^i(x_t) \circ dB_t^i.$$

The corresponding stochastic flow of this equation generates a foliated Brownian motion in U , i.e. given initial conditions, the solution are simultaneous Brownian motions on each leaf of the foliation, see [2].

We investigate the effective behaviour of a small transversal perturbation of order ϵ :

$$dx_t^\epsilon = \sum_{i=1}^3 X^i(x_t^\epsilon) \circ dB_t^i + \epsilon K(x_t^\epsilon),$$

where the transversal vector field $K(x)$ is orthogonal to the leaves, proportional to the Gaussian curvature of the leaf L_x at the point $x \in U$. We assume that at a point with positive curvature, the corresponding vector field K points outwards. In some sense this equation models an average inertial or centrifugal forces acting on particles moving randomly on the leaves. We have the following averaging result which does not depend on the geometry of the manifold M :

Theorem 2.1. *The transversal dynamics of orthogonal perturbation, given by ϵ times the Gaussian curvature of foliated Brownian motion, behaves according to $v(t) = 2\pi\chi(M) t$, as ϵ goes to zero. Here $\chi(M)$ is the Euler characteristic of the leaves.*

Precisely, up to a stopping time, for $q \geq 2$, $\beta \in (0, 1/2)$, $\epsilon > 0$, for each $t \geq 0$, there exists a constant $C > 0$ such that

$$\left[\mathbb{E} \left(\sup_{s \leq t} \left| p \left(x_{\frac{s}{\epsilon}}^\epsilon \right) - v(s) \right|^q \right) \right]^{\frac{1}{q}} < C |\ln \epsilon|^{-\frac{\beta}{q}}. \tag{2.1}$$

Proof: Most of the proof is straightforward: one just has to put the properties together and apply the classical Gauss-Bonnet theorem. In fact, the invariant measures on the leaves are the normalized Riemannian volume. According to our coordinate systems, the one dimensional transversal component is given by the Gaussian curvature $K(x)$. The average (in the transversal direction) of the perturbing vector field K on each leaf is given by

$$\int_{L_x} K(x) d\mu(x) = 2\pi\chi(M).$$

Hence this average depends only on the topology of the leaf.

The proof finishes as a direct application of Theorem 1.1 with codimension $d = 1$. The rate of ergodic convergence on the leaves $\eta(t)$, in this case, has order $1/\sqrt{t}$ since the system is uniformly elliptic, as in X.-M.-Li [8]. The estimate on the rate of convergence (inequality (2.1)) follows directly .

□

As a simple example, consider an orthogonal perturbation, according to the curvature, of a Brownian motion on any manifold diffeomorphic to the torus. This system presents no transversal behaviour on the average. Indeed, a flat torus illustrates trivially this fact since $K \equiv 0$. On manifolds diffeomorphic to the sphere, the orthogonal behaviour is a linear expansion. For an n -fold torus with genus $n > 1$, the effect is a linear contraction with coefficients proportional to $2 - 2n$.

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