Electron. Commun. Probab. **20** (2015), no. 11, 1–16.
DOI: 10.1214/ECP.v20-3517
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Approximating the Rosenblatt process by multiple Wiener integrals*

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Abstract

Let Z^H be the Rosenblatt process with the representation

$$Z_t^H = \int_0^t \int_0^t L^H(t,s,r) dB_s dB_r,$$

where B is a standard Brownian motion, $\frac{1}{2} < H < 1$ and L^H is a given kernel. By reviewing the kernel L^H we construct its approximation of multiple Wiener integrals of the form

$$\int_0^t \int_0^t \left\{ k_1(sr)^{-\frac{1}{2}H} + k_2(s \vee r)^{\frac{1}{2}H}(s \wedge r)^{-\frac{1}{2}H}|s-r|^{H-1} \right\} dB_s dB_r, \quad k_1, k_2 \geq 0.$$

We find an optimal approximation of Z^H via calculating accurately the values of k_1, k_2.

Keywords: Rosenblatt process; optimal approximation; multiple Wiener integrals.
AMS MSC 2010: 60G18; 60H05.
Submitted to ECP on May 11, 2014, final version accepted on February 4, 2015.

1 Introduction

Hermite process is a special class of self-similar processes with long-range dependence. The processes arise from the *Non Central Limit Theorem* studied by Taqqu [12, 13] and Dobrushin-Majòr [7]. The famous fractional Brownian motion and Rosenblatt process are its special examples. Let us briefly recall the general context.

Let $(\xi_n)_{n\in\mathbb{N}}$ be a stationary centered Gaussian sequence with $E(\xi_n^2) = 1$ such that

$$r(n) := E(\xi_0 \xi_n) = n^{\frac{2H-2}{l}} L(n), \tag{1.1}$$

where $l \geq 1$ is an integer, $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity, and let the Borel function $g: \mathbb{R} \to \mathbb{R}$ satisfy $E(g(\xi_0)) = 0, E(g(\xi_0)^2) < \infty$ and

$$g(x) = \sum_{j=0}^{\infty} c_j H_j(x), \qquad c_j = \frac{1}{j!} E[g(\xi_0) H_j(\xi_0)],$$

*Support: NSFC (11171062) & Innovation Program of Shanghai Municipal Education Commission (12ZZ063)

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where H_j is the Hermite polynomial of order j defined by

$$H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}, \quad j = 1, 2, \dots$$

with $H_0(x) = 1$. Then, the constant

$$l = \min\{j \; ; \; c_j \neq 0\}.$$

is call the Hermite rank of g. Clearly, $l \geq 1$ since $E[g(\xi_0)] = 0$. For a Borel function g with the Hermite rank l, the *Non Central Limit Theorem* implies that the stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j), \qquad n = 1, 2, \dots$$

converges, as $n \to \infty$, in the sense of finite dimensional distributions to the process

$$Z_t(H, l) = \int_{[0,t]^l} L^H(t, s_1, \dots, s_l) dB_{s_1} \cdots dB_{s_l}, \quad t \in [0, 1], \tag{1.2}$$

where $B = \{B_t, t \geq 0\}$ is a standard Brownian motion and

$$L^H(t, s_1, \dots, s_l) = c(H, l) \left(\prod_{j=1}^{l} s_j^{\frac{1}{2} - H'} \right) \int_0^t u^{l(H' - \frac{1}{2})} \prod_{j=1}^{l} (u - s_j)_+^{H' - \frac{3}{2}} du \tag{1.3}$$

with $H' = 1 - \frac{1-H}{l} \in (1 - \frac{1}{2l}, 1)$, $s_1, \dots, s_k \in [0, t]$ and a positive normal constant $c(H, l)$ such that $E\left[(Z_1(H, l))^2\right] = 1$.

Definition 1.1 (Taqqu [13]). *The process $(Z_t(H, l))_{t \geq 0}$ defined by* (1.2) *is called the Hermite process of order l with index H.*

Clearly, when $l = 1$ Hermite process is the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. When $l = 2$ the Hermite process is called the Rosenblatt process (see Taqqu [12]). It is important to note that Hermite process is not Gaussian for $l \geq 2$. The simplest Hermite process is fractional Brownian motion, and the Rosenblatt process is the simplest non-Gaussian Hermite process. Hermite processes are neither a semi-martingale nor a Markov process, and the following properties hold:

(i) they are the long-range dependence in the sense of

$$\sum_{n \geq 1} E[Z_1(H, l)(Z_{n+1}(H, l) - Z_n(H, l))] = \infty;$$

(ii) they are H-selfsimilar;

(iii) they have stationary increments;

(iv) they admit the same covariance functions, i.e.

$$E\left[Z_t(H, l) Z_s(H, l)\right] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t - s|^{2H}\right];$$

(v) they are Hölder continuous of order $\gamma < H$.

These good properties of the Hermite process motivate us to study it. More works for the Hermite process and Rosenblatt process can be found in Bardet *et al* [3], Chen *et al* [4], Chronopoulou *et al* [5, 6], Garzón *et al* [8], Maejima–Tudor [9], Peccati and Taqqu [10], Pipiras–Taqqu [11], Torres–Tudor [14], Tudor [15], Tudor–Viens [16] and the references therein. In this paper we will prove an approximation theorem of Rosenblatt process based on the multiple integrals of form

$$\int_0^t \int_0^t \left\{ k_1(sr)^{-\frac{1}{2}H} + k_2(s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1} \right\} dB_s dB_r, \quad t \geq 0 \tag{1.4}$$

with $k_1, k_2 > 0$. For simplicity we denote $Z_t(H,2) = Z_t^H$. The motivation to consider the approximation arises from the following estimate:

$$L^H(t,s,r) \leq C_{H,T} \left\{ (sr)^{-\frac{1}{2}H} + (s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1} \right\} \tag{1.5}$$

for all $t \in [0,T]$ and $s, r > 0$. In order to prove the above estimate, without loss of generality, we may assume that $s \geq r$ and we have

$$\begin{aligned}\int_s^t \frac{du}{(u-s)^{1-\frac{1}{2}H}(u-r)^{1-\frac{1}{2}H}} &= (s-r)^{H-1} \int_0^{\frac{t-s}{s-r}} \frac{dx}{x^{1-\frac{1}{2}H}(1+x)^{1-\frac{1}{2}H}} \\ &\leq (s-r)^{H-1} \int_0^{\infty} \frac{dx}{x^{1-\frac{1}{2}H}(1+x)^{1-\frac{1}{2}H}}\end{aligned}$$

by making the substitutions $u - s = x(s-r)$. It follows that

$$\begin{aligned}L^H(t,s,r) &= c(H,2)\int_{s\vee r}^t (sr)^{-\frac{1}{2}H}u^H(u-s)^{\frac{1}{2}H-1}(u-r)^{\frac{1}{2}H-1}du \\ &= c(H,2)(sr)^{-\frac{1}{2}H}\int_s^t \frac{(u-s+s)^H du}{(u-s)^{1-\frac{1}{2}H}(u-s+s-r)^{1-\frac{1}{2}H}} \\ &\leq c(H,2)(sr)^{-\frac{1}{2}H}\int_s^t (u-s)^{2H-2}du \\ &\quad + c(H,2)s^{\frac{1}{2}H}r^{-\frac{1}{2}H}\int_s^t \frac{du}{(u-s)^{1-\frac{1}{2}H}(u-r)^{1-\frac{1}{2}H}} \\ &\leq C_{H,T}\left\{(sr)^{-\frac{1}{2}H} + s^{\frac{1}{2}H}r^{-\frac{1}{2}H}(s-r)^{H-1}\right\}\end{aligned}$$

for all $t \in [0,T]$ and $y_1, y_2 > 0$. In general, for every Borel measurable function $\zeta \in L_2([0,T]^2)$ the stochastic integral

$$M_t(\zeta) := \int_0^t \int_0^t \zeta(s,r)dB_s dB_r, \qquad t \in [0,T]$$

is well-defined, and the best approximation problem is to estimate

$$\inf_{\zeta \in L_2([0,T]^2)} \sup_{t\in[0,T]} E(Z_t^H - M_t(\zeta))^2. \tag{1.6}$$

It is important to note that if the above minimum is attained at the function ζ^*, then $\zeta^* > 0$ a.e. In fact, we have

$$\begin{aligned}E\left(Z_t^H - M_t(\zeta)\right)^2 = t^{2H} &+ 2\int_0^t\int_0^t \zeta^2(s,r)dsdr \\ &- 4\int_0^t\int_0^t L^H(t,s,r)\zeta(s,r)dsdr\end{aligned} \tag{1.7}$$

for all $t \geq 0$. If $\zeta^*(y_1,y_2) \not> 0$, then

$$\sup_{t\in[0,T]} E\left(Z_t^H - M_t(\zeta^*)\right)^2 \geq \sup_{t\in[0,T]} E\left(Z_t^H - M_t(|\zeta^*|)\right)^2.$$

This gives the contradiction. Thus, we may assume that $k_1, k_2 > 0$ in (1.4) and study the best approximation problem

$$\inf_{\zeta\in\mathcal{K}} \sup_{t\in[0,T]} E(Z_t^H - M_t(\zeta))^2, \tag{1.8}$$

where

$$\mathcal{K} = \left\{\zeta(s,r) = k_1(sr)^{-\frac{1}{2}H} + k_2(s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1},\ k_1,k_2>0\right\}.$$

For $\zeta \in \mathcal{K}$ we denote

$$f(t,k_1,k_2) := E(Z_t^H - M_t(\zeta))^2$$

with $t \geq 0$.

When $l = 1$, Hermite process is a fractional Brownian Motion with Hurst index H and the similar approximation is first considered by Banna-Mishura [1, 2]. When $l \geq 2$, the question has not been studied and this process is non-Gaussian with non-trivial analysis. In order to state our object, let us consider the kernel K^H of the form

$$K^H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du$$

where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right)^{\frac{1}{2}}$ and $\beta(\cdot,\cdot)$ denotes the classical Beta function. Then we have (see, for example, Tudor [15])

$$L^H(t,s,r) = d(H)\int_{s\vee r}^t \frac{\partial K^{H'}}{\partial u}(u,s)\frac{\partial K^{H'}}{\partial u}(u,r)du,$$

where $d(H) = \frac{1}{H+1}\sqrt{(4H-2)H^{-1}}$ and $H' = \frac{1}{2}(1+H)$. In this short note, our main aim is to find the optimal approximation of Z_t^H by (1.4) via calculating accurately the values of k_1, k_2. In order to end this one can easily check that (see Section 3)

$$\frac{\partial}{\partial t}f(t,k_1,k_2)$$

is a quadratic polynomial in $x = k_1 t^{-2\alpha}$ and its discriminant is also a quadratic polynomial in k_2 with the discriminant

$$\begin{aligned} D_1 = 16&\left[\frac{2C_2(H)}{1-H} - C_1(H)\beta(1-H,H)\right]^2 \\ &- 16\left[\beta^2(1-H,H) - \frac{\beta(1-H,2H-1)}{1-H}\right]\left[C_1(H)^2 - \frac{2H}{1-H}\right] \end{aligned}$$

for $H \in (\frac{1}{2},1)$, where $C_1(H) = d(H)c_{H'}^2\beta^2(1-H,\frac{1}{2}H)$ and

$$C_2(H) = d(H)c_{H'}^2\int_0^1\int_0^s r^{-H}(1-s)^{\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1}(s-r)^{H-1}drds.$$

By using the constant D_1 we give our main result and at the end of this paper we give the numerical simulations of these constants (see Figure 1, 2, 3 and Table 1).

This note is organized as follows. In Section 2, we give the representation of the function $f(t,k_1,k_2) = E(Z_t^H - M_t(\zeta))^2$ for $\zeta \in \mathcal{K}$. In Section 3 and Section 4, we consider the optimal approximation in the two cases $D_1 \leq 0$ and $D_1 > 0$, respectively. In Section 5 we consider two special cases.

2 The representation of $f(t, k_1, k_2)$

In order to give the representation of $f(t,k_1,k_2) = E\left(Z_t^H - M_t(\zeta)\right)^2$ for $\zeta \in \mathcal{K}$, we start with the finiteness of the constant $C_2(H)$.

Lemma 2.1. *For all $\frac{1}{2} < H < 1$ the integral*

$$\int_0^1 \int_0^s r^{-H}(1-s)^{\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1}(s-r)^{H-1} dr ds$$

converges.

Proof. By Young's inequality, we have

$$(1-r)^{\frac{1}{2}H-1} \le (1-s)^{(\frac{1}{2}H-1)\gamma}(s-r)^{(\frac{1}{2}H-1)(1-\gamma)}$$

for all $0 < \gamma < 1$. Notice that $1 - \frac{3}{2}H < \frac{1}{2}H$ for all $\frac{1}{2} < H < 1$. We get

$$\begin{aligned}
&\int_0^1 \int_0^s r^{-H}(1-s)^{\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1}(s-r)^{H-1} dr ds \\
&\quad \le \int_0^1 (1-s)^{(\frac{1}{2}H-1)(1+\gamma)} dy_1 \int_0^s r^{-H}(s-r)^{\frac{3}{2}H-2+(1-\frac{1}{2}H)\gamma} dy_2 \\
&\quad = \int_0^1 s^{-(1-\gamma)(1-\frac{1}{2}H)}(1-s)^{-(1-\frac{1}{2}H)(1+\gamma)} ds \int_0^1 x^{-H}(1-x)^{\frac{3}{2}H-2+(1-\frac{1}{2}H)\gamma} dx < \infty,
\end{aligned}$$

for all $\frac{1-\frac{3}{2}H}{1-\frac{1}{2}H} \vee 0 < \gamma < \frac{\frac{1}{2}H}{1-\frac{1}{2}H}$. This proves $C_2(H) < \infty$. □

Theorem 2.2. *Let $C_1(H)$ and $C_2(H)$ be given in Section 1. Denote*

$$a(k_2) := 1 + H^{-1}2(k_2)^2\beta(1-H, 2H-1) - 4k_2H^{-1}C_2(H)$$

and

$$b(k_2) := C_1(H) - 2k_2\beta(1-H, H)$$

for all $k_1, k_2 \ge 0$ and $\frac{1}{2} < H < 1$. Then we have

$$f(t, k_1, k_2) = a(k_2)t^{2H} - 4k_1b(k_2)t + \frac{2k_1^2}{(1-H)^2}t^{2-2H}, \qquad t \in [0, T].$$

As an immediate result we see that $a(k_2) \ge 0$ and

$$a(k_2) - 2(1-H)^2b^2(k_2) > 0 \tag{2.1}$$

for all k_2 since $f(t, k_1, k_2) \ge 0$ is a quadratic equation in k_1. Notice that $a(k_2)$ is also a quadratic equation in k_2. We get

$$2\left(C_2(H)\right)^2 \le H\beta(1-H, 2H-1)$$

for all $\frac{1}{2} < H < 1$.

Proof of Theorem 2.2. An elementary calculation can show that

$$\begin{aligned}
&\int_0^t \int_0^t L^H(t,s,r)(sr)^{-\frac{1}{2}H} dsdr \\
&\qquad = d(H)c_{H'}^2 \int_0^t \int_0^t \int_{s\vee r}^t (sr)^{-H} u^H (u-s)^{\frac{1}{2}H-1}(u-r)^{\frac{1}{2}H-1} dudsdr \\
&\qquad = \int_0^t \int_0^u \int_o^u d(H)c_{H'}^2 (sr)^{-H} u^H (u-s)^{\frac{1}{2}H-1}(u-r)^{\frac{1}{2}H-1} dsdrdu \\
&\qquad = d(H)c_{H'}^2 \int_0^t \int_0^1 \int_0^1 s^{-H}(1-s)^{\frac{1}{2}H-1} r^{-H}(1-r)^{\frac{1}{2}H-1} dsdrdu \\
&\qquad = d(H)c_{H'}^2 \beta^2(1-H, \frac{1}{2}H)t = C_1(H)t
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \int_0^t L^H(t,s,r)(s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1} dsdr \\
&\qquad = \int_0^t \int_0^u \int_0^u d(H)c_{H'}^2 (sr)^{-\frac{1}{2}H} u^H (u-s)^{\frac{1}{2}H-1}(u-r)^{\frac{1}{2}H-1} \\
&\qquad\qquad \cdot (s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1} dsdrdu \\
&\qquad = d(H)c_{H'}^2 \int_0^t \int_0^1 \int_0^1 u^{2H-1}(sr)^{-\frac{1}{2}H}(1-s)^{\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1} \\
&\qquad\qquad \cdot (s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1} dsdrdu \\
&\qquad = H^{-1} d(H)c_{H'}^2 t^{2H} \int_0^1 \int_0^s r^{-H}(1-s)^{\frac{1}{2}H-1}(1-r)^{\frac{1}{2}H-1}(s-r)^{H-1} drds \\
&\qquad = C_2(H)H^{-1}t^{2H}
\end{aligned}$$

for all $t \in [0,T]$, which give

$$\int_0^t \int_0^t L^H(t,s,r)\zeta(s,r) dsdr = k_1 C_1(H)t + k_2 C_2(H)H^{-1}t^{2H}$$

for all $t \in [0,T]$. On the other hand, it is easy to calculate that

$$\int_0^t \int_0^t \zeta^2(s,r) dsdr = \frac{k_1^2}{(1-H)^2} t^{2-2H} + \frac{k_2^2}{H}\beta(1-H, 2H-1)t^{2H} + 4k_1k_2\beta(1-H,H)t$$

for all $\zeta \in \mathcal{K}$. It follows that

$$\begin{aligned}
f(t,k_1,k_2) &= E\left(Z_t^H - M_t(\zeta)\right)^2 \\
&= t^{2H} + 2\int_0^t \int_0^t \zeta^2(s,r)dsdr - 4\int_0^t \int_0^t L^H(t,s,r)\zeta(s,r)dsdr \qquad (2.2) \\
&= a(k_2)t^{2H} - 4k_1 b(k_2)t + \frac{2k_1^2}{(1-H)^2} t^{2-2H}
\end{aligned}$$

for all $\zeta \in \mathcal{K}$. This completes the proof. □

3 The optimal approximation, case $D_1 \leq 0$

In order to obtain the optimal approximation in the case $D_1 \leq 0$ we need some preliminaries and keep the notation in Section 2. Denote $\alpha = H - \frac{1}{2}$ and define the quadratic function $x \mapsto G(x)$ on $[0,\infty)$ by

$$G(x) := \frac{2}{1-H}x^2 - 2b(k_2)x + Ha(k_2)$$

for $x \geq 0$. Consider the maximum of $t \mapsto f(\cdot,\cdot,t)$. We have for all $t, k_1, k_2 > 0$,

$$\begin{aligned}\frac{\partial}{\partial t} f(t,k_1,k_2) &= 2Ha(k_2)t^{2H-1} - 4k_1 b(k_2) + \frac{4k_1^2}{1-H} t^{1-2H} \\ &= t^{2\alpha}\left(\frac{4k_1^2}{1-H} t^{-4\alpha} - 4b(k_2)k_1 t^{-2\alpha} + 2Ha(k_2)\right) \\ &= 2t^{2\alpha} G(x)\end{aligned} \tag{3.1}$$

with $x = k_1 t^{-2\alpha}$. Clearly, the discriminant D of the quadratic polynomial $G(x)$ satisfies

$$\begin{aligned}\frac{1}{4}D = (b(k_2))^2 - \frac{2H}{1-H}a(k_2) &= 4\left[\beta^2(1-H,H) - \frac{\beta(1-H,2H-1)}{1-H}\right]k_2^2 \\ &\quad + 4\left[\frac{2C_2(H)}{1-H} - C_1(H)\beta(1-H,H)\right]k_2 + C_1(H)^2 - \frac{2H}{1-H}.\end{aligned} \tag{3.2}$$

This gives a quadratic polynomial in k_2 and its discriminant is D_1.

Theorem 3.1. *If $D_1 \leq 0$, then we have*

$$\inf_{\zeta\in\mathcal{K}} \sup_{0\leq t\leq T} E\left(Z_t^H - M_t(\zeta)\right)^2 = a(k_2^*)T^{2H} - 4k_1^* b(k_2^*)T + \frac{2k_1^{*2}}{(1-H)^2}T^{2-2H},$$

where

$$\zeta(s,r) = k_1^*(sr)^{-\alpha'} + k_2^*(s\vee r)^{\alpha'}(s\wedge r)^{-\alpha'}|s-r|^{2\alpha'-1}, \quad s,r>0$$

and (k_1^*, k_2^*) is the stagnation point of the function

$$(k_1,k_2) \mapsto f(T,k_1,k_2).$$

An elementary calculation can obtain

$$k_1^* = \frac{2(1-H)^2\beta(1-H,H)C_2(H) - (1-H)^2\beta(1-H,2H-1)C_1(H)}{4H(1-H)^2\beta^2(1-H,H) - \beta(1-H,2H-1)}T^{2\alpha} \tag{3.3}$$

$$k_2^* = \frac{2H(1-H)^2\beta(1-H,H)C_1(H) - C_2(H)}{4H(1-H)^2\beta^2(1-H,H) - \beta(1-H,2H-1)}. \tag{3.4}$$

Lemma 3.2. *For all $\frac{1}{2} < H < 1$ we have $\beta^2(1-H,H) - \frac{\beta(1-H,2H-1)}{1-H} < 0$.*

Proof. This is a simple exercise. In fact, for all $\frac{1}{2} < H < 1$ we have

$$\begin{aligned}\beta^2(1-H,H) &= \left(\int_0^1 x^{-\frac{H}{2}}(1-x)^{H-1}x^{-\frac{H}{2}}dx\right)^2 \\ &\leq \int_0^1 \left(x^{-\frac{H}{2}}(1-x)^{H-1}\right)^2 dx \int_0^1 x^{-H}dx = \frac{\beta(1-H,2H-1)}{1-H}\end{aligned} \tag{3.5}$$

by Cauchy inequality, and it is easy to check that the inequality above is strict. □

Proof of Theorem 3.1. Let now $D_1 \leq 0$. Then we see that $D \leq 0$ and $\frac{\partial f}{\partial t} \geq 0$ for $H \in (\frac{1}{2}, 1)$ and $k_1, k_2 \geq 0$. It follows that

$$\begin{aligned}\sup_{0\leq t\leq T} E\left(Z_t^H - M_t(\zeta)\right)^2 &= f(T,k_1,k_2) \\ &= a(k_2)T^{2H} - 4k_1 b(k_2)T + \frac{2k_1^2}{(1-H)^2}T^{2-2H}\end{aligned}$$

for all $k_1, k_2 \geq 0$. Let now (k_1^*, k_2^*) be the stagnation point of the function

$$(k_1, k_2) \mapsto f(T, k_1, k_2).$$

Then (k_1^*, k_2^*) can be given by (3.3) and (3.4), and elementary calculations may obtain the Hessian matrix $\mathbf{H}$ on $f(T, k_1, k_2)$ as follows

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f(T,k_1,k_2)}{\partial k_1^2} & \frac{\partial^2 f(T,k_1,k_2)}{\partial k_1 \partial k_2} \\ \frac{\partial^2 f(T,k_1,k_2)}{\partial k_2 \partial k_1} & \frac{\partial^2 f(T,k_1,k_2)}{\partial k_2^2} \end{pmatrix} = \begin{pmatrix} \frac{4T^{2-2H}}{(1-H)^2} & 8\beta(1-H,H)T \\ 8\beta(1-H,H)T & \frac{4}{H}\beta(1-H,2H-1)T^{2H} \end{pmatrix}$$

and $|\mathbf{H}| = 16T^2 \left(\frac{\beta(1-H,2H-1)}{H(1-H)^2} - 4\beta^2(1-H,H) \right)$. Combining this with (3.5), we get $|\mathbf{H}| > 0$ for all $H \in (\frac{1}{2}, 1)$, which means that the minimal value of $(k_1, k_2) \mapsto f(T, k_1, k_2)$ is achieved at the point (k_1^*, k_2^*). Thus, we have proved the theorem. □

4 The optimal approximation, case $D_1 > 0$

In this section we throughout let $D_1 > 0$ and keep the notation in Section 3 and Section 2. When $D_1 > 0$, the equation $D = 0$ admits two real roots as follows

$$k_{2,1} = \frac{-4\left[\frac{2C_2(H)}{1-H} - C_1(H)\beta(1-H,H)\right] + \sqrt{D_1}}{8\left[\beta^2(1-H,H) - \frac{\beta(1-H,2H-1)}{1-H}\right]}$$

and

$$k_{2,2} = \frac{-4\left[\frac{2C_2(H)}{1-H} - C_1(H)\beta(1-H,H)\right] - \sqrt{D_1}}{8\left[\beta^2(1-H,H) - \frac{\beta(1-H,2H-1)}{1-H}\right]}.$$

We get

$$\begin{aligned} \inf_{\zeta \in \mathcal{K}} \sup_{0 \leq t \leq T} E\left(Z_t^H - M_t(\zeta)\right)^2 &= \inf_{\zeta \in \mathcal{K}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \\ &= \min\left\{ \inf_{\substack{k_2 \notin (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \leq t \leq T} f(t, k_1, k_2), \inf_{\substack{k_2 \in (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \right\}. \end{aligned} \tag{4.1}$$

Thus, we can complete the discussion in two cases: $k_2^* \notin (k_{2,2}, k_{2,1})$ and $k_2^* \in (k_{2,2}, k_{2,1})$.

Theorem 4.1. *If $D_1 > 0$ and $k_2^* \notin (k_{2,2}, k_{2,1})$, then we have*

$$\inf_{\zeta \in \mathcal{K}} \sup_{0 \leq t \leq T} E\left(Z_t^H - M_t(\zeta)\right)^2 = a(k_2^*)T^{2H} - 4k_1^* b(k_2^*)T + \frac{2k_1^{*2}}{(1-H)^2} T^{2-2H},$$

where

$$\zeta(y_1, y_2) = k_1^*(y_1 y_2)^{-\alpha'} + k_2^*(y_1 \vee y_2)^{\alpha'}(y_1 \wedge y_2)^{-\alpha'} |y_1 - y_2|^{2\alpha' - 1}, \quad y_1, y_2 > 0.$$

Proof. Let $k_2^* \notin (k_{2,2}, k_{2,1})$. By (4.1) we have that

$$\inf_{\substack{k_2 \notin (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) = f(T, k_1^*, k_2^*),$$

provided $D \leq 0$, and

$$\inf_{\substack{k_2 \in (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \leq t \leq T} f(t, k_1, k_2) \geq \inf_{\substack{k_2 \in (k_{2,2}, k_{2,1}) \\ k_1 > 0}} f(T, k_1, k_2) > f(T, k_1^*, k_2^*),$$

provided $D > 0$, which imply

$$\inf_{\zeta \in \mathcal{K}} \sup_{0 \le t \le T} f(t, k_1, k_2) = f(T, k_1^*, k_2^*),$$

and the theorem follows. □

Next we consider the case $k_2^* \in (k_{2,2}, k_{2,1})$.

Lemma 4.2. *For $k_2^* \in (k_{2,2}, k_{2,1})$, we have*

$$\inf_{\zeta \in \mathcal{K}} \sup_{0 \le t \le T} f(t, k_1, k_2) = \inf_{\substack{k_2 \in (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \le t \le T} f(t, k_1, k_2).$$

Proof. By (4.1) it is enough to show that

$$\inf_{\substack{k_2 \in (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \le t \le T} f(t, k_1, k_2) \le \inf_{\substack{k_2 \notin (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \le t \le T} f(t, k_1, k_2). \tag{4.2}$$

If $k_2 \notin (k_{2,2}, k_{2,1})$, then $D \le 0$ and we have

$$\sup_{0 \le t \le T} f(t, k_1, k_2) = f(T, k_1, k_2).$$

By solving the equation $\frac{\partial f(T, k_1, *)}{k_1} = 0$, we get the stagnation points of the functions $k_1 \mapsto f(T, k_1, k_{2,1})$ and $k_1 \mapsto f(T, k_1, k_{2,2})$ as follow

$$k_{1,1} := \left[C_1(H) - 2k_{2,1}\beta(1-H, H) \right] (1-H)^2 T^{2\alpha}$$

and

$$k_{1,2} := \left[C_1(H) - 2k_{2,2}\beta(1-H, H) \right] (1-H)^2 T^{2\alpha},$$

respectively. It follows that

$$\inf_{\substack{k_2 \notin (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \le t \le T} f(t, k_1, k_2) = \min \left\{ f(T, k_{1,1}, k_{2,1}), f(T, k_{1,2}, k_{2,2}) \right\}.$$

Clearly, if $k_2 = k_{2,2}$ or $k_2 = k_{2,1}$, we have $D = 0$. So $\frac{\partial f}{\partial t} \ge 0$ and

$$\sup_{0 \le t \le T} f(t, k_1, k_{2,1}) = f(T, k_1, k_{2,1}).$$

Hence we have

$$\begin{aligned} \inf_{\substack{k_2 \in (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \le t \le T} f(t, k_1, k_2) &\le \inf_{k_1 > 0} \sup_{0 \le t \le T} f(t, k_1, k_{2,1}) \\ &= \inf_{k_1 > 0} f(T, k_1, k_{2,1}) = f(T, k_{1,1}, k_{2,1}). \end{aligned}$$

On the other hand, we can also get

$$\inf_{\substack{k_2 \in (k_{2,2}, k_{2,1}) \\ k_1 > 0}} \sup_{0 \le t \le T} f(t, k_1, k_2) \le f(T, k_{1,2}, k_{2,2}),$$

and the inequality (4.2) follows. This completes the proof. □

Clearly, $D > 0$ if $k_2 \in (k_{2,2}, k_{2,1})$, and by (3.1) we can see that the equation

$$\frac{\partial f}{\partial t} = 2t^{2\alpha} \left(\frac{2}{1-H} x^2 - 2b(k_2)x + Ha(k_2) \right) = 0 \tag{4.3}$$

has two real roots as follows

$$x_1 := \frac{1-H}{2}\left(b(k_2) + \left(b^2(k_2) - \frac{2Ha(k_2)}{1-H}\right)^{\frac{1}{2}}\right)$$

and

$$x_2 := \frac{1-H}{2}\left(b(k_2) - \left(b^2(k_2) - \frac{2Ha(k_2)}{1-H}\right)^{\frac{1}{2}}\right),$$

which says $t_1 := t_1(k_1,k_2) = k_1^{\frac{1}{2\alpha}} x_1^{-\frac{1}{2\alpha}}$ and $t_2 := t_2(k_1,k_2) = k_1^{\frac{1}{2\alpha}} x_2^{-\frac{1}{2\alpha}}$ are the two stagnation points of the function $t \mapsto f(t,k_1,k_2)$. It follows from the monotonicity of the function $t \mapsto f(t,k_1,k_2)$ that $t_1 := t_1(k_1,k_2)$ and $t_2 := t_2(k_1,k_2)$ are the points of local maximum and minimum, respectively, which implies that

$$\sup_{t\in[0,T]} f(t,k_1,k_2) = \begin{cases} f(T,k_1,k_2), & t_1 \geq T \\ \max\{f(t_1,k_1,k_2), f(T,k_1,k_2)\}, & t_1 < T \end{cases}$$

and

$$\begin{aligned} f(t_1,k_1,k_2) &= a(k_2)\left(\frac{k_1}{x_1}\right)^{\frac{H}{\alpha}} - 4k_1 b(k_2)\left(\frac{k_1}{x_1}\right)^{\frac{1}{2\alpha}} + \frac{2{k_1}^2}{(1-H)^2}\left(\frac{k_1}{x_1}\right)^{\frac{1-H}{\alpha}} \\ &=: k_1^{\frac{H}{\alpha}}\varphi(k_2) = k_1^{\frac{2H}{2H-1}}\varphi(k_2). \end{aligned} \tag{4.4}$$

Lemma 4.3. *If $k_2^* \in (k_{2,2}, k_{2,1})$, we then have*

$$t_1(k_1^*, k_2^*) < T.$$

Proof. Noting that $t_1 = k_1^{\frac{1}{2\alpha}} x_1^{-\frac{1}{2\alpha}}$ and $t_2 = k_1^{\frac{1}{2\alpha}} x_2^{-\frac{1}{2\alpha}}$, we get $t_1^{-2\alpha} = \frac{x_1}{k_1}$, $t_2^{-2\alpha} = \frac{x_2}{k_1}$ and

$$\frac{2}{t_1^{2\alpha}} > \frac{1}{t_1^{2\alpha}} + \frac{1}{t_2^{2\alpha}} = \frac{x_1+x_2}{k_1} = \frac{(1-H)b(k_2)}{k_1}$$

since $t_1 < t_2$. When $k_1 = k_1^*$ and $k_2 = k_2^*$, we have

$$\begin{aligned} \frac{(1-H)b(k_2^*)}{k_1^*} &= \frac{(1-H)\left(C_1(H) - 2k_2^*\beta(1-H,H)\right)}{k_1^*} \\ &= \frac{(1-H)\left(C_1(H) - 2\eta(H)\beta(1-H,H)\right)}{\xi(H)T^{2\alpha}} = \frac{1}{(1-H)T^{2\alpha}} > \frac{2}{T^{2\alpha}} \end{aligned} \tag{4.5}$$

for $H \in (\frac{1}{2}, 1)$, where

$$\eta(H) := \frac{2H(1-H)^2\beta(1-H,H)C_1(H) - C_2(H)}{4H(1-H)^2\beta^2(1-H,H) - \beta(1-H,2H-1)}$$

and

$$\xi(H) := \frac{2(1-H)^2\beta(1-H,H)C_2(H) - (1-H)^2\beta(1-H,2H-1)C_1(H)}{4H(1-H)^2\beta^2(1-H,H) - \beta(1-H,2H-1)}.$$

This proves that $t_1(k_1^*, k_2^*) < T$. □

Lemma 4.4. *If $k_2^* \in (k_{2,2}, k_{2,1})$, we then have $t_2(k_1^*, k_2^*) < T$.*

Proof. From (4.5) it follows that

$$k_1^* = (1-H)^2 T^{2\alpha} b(k_2^*). \tag{4.6}$$

On the other hand, (3.1) implies that

$$\frac{\partial f(t,k_1^*,k_2^*)}{\partial t} = 2t^{2\alpha}\left(\frac{2(k_1^*)^2}{1-H}(t^{-2\alpha})^2 - 2b(k_2^*)k_1^*t^{-2\alpha} + Ha(k_2^*)\right) =: g(t^{-2\alpha}) \tag{4.7}$$

is a quadratic function in $x = t^{-2\alpha}$, and

$$g(t_1^{-2\alpha}(k_1^*,k_2^*)) = g(t_2^{-2\alpha}(k_1^*,k_2^*)) = 0.$$

Noting that

$$\begin{aligned} g(T^{-2\alpha}) &= 2T^{2\alpha}\left(\frac{2(k_1^*)^2}{1-H}(T^{-2\alpha})^2 - 2b(k_2^*)k_1^*T^{-2\alpha} + Ha(k_2^*)\right) \\ &= 2T^{2\alpha}\Big(\frac{2((1-H)^2T^{2\alpha}b(k_2^*))^2}{1-H}(T^{-2\alpha})^2 \\ &\qquad - 2b(k_2^*)(1-H)^2T^{2\alpha}b(k_2^*)T^{-2\alpha} + Ha(k_2^*)\Big) \\ &= 2HT^{2\alpha}\left[a(k_2^*) - 2(1-H)^2b^2(k_2^*)\right] > 0 \end{aligned} \tag{4.8}$$

by (2.1), we get

$$T^{-2\alpha} \notin [t_2^{-2\alpha}(k_1^*,k_2^*), t_1^{-2\alpha}(k_1^*,k_2^*)]$$

and $t_2(k_1^*,k_2^*) < T$ by a simple analysis and Lemma 4.3. □

Lemma 4.5. *Denote*

$$h(k_1,k_2) = f(t_1,k_1,k_2) - f(T,k_1,k_2).$$

For any $k_2 \in (k_{2,2},k_{2,1})$, the equation $h(k_1,k_2) = 0$ (with unknown k_1) has two solutions $\widehat{k}_1$ and $\overline{k}_1$, which satisfy $0 < \widehat{k}_1 < \overline{k}_1$, $\frac{\partial h}{\partial k_1}\mid_{k_1=\widehat{k}_1} > 0$ and $\frac{\partial h}{\partial k_1}\mid_{k_1=\overline{k}_1} = 0$.

Proof. Clearly, $h(k_1,k_2) = 0$ and (4.4) imply that

$$k_1^{\frac{2H}{2H-1}}\varphi(k_2) = a(k_2)T^{2H} - 4k_1b(k_2)T + \frac{2{k_1}^2}{(1-H)^2}T^{2-2H} = f(T,k_1,k_2) \tag{4.9}$$

for all $k_2 \in (k_{2,2},k_{2,1})$. Differentiating (4.9) with respect to k_1 and multiplying by $\frac{2H}{(2H-1)k_1}$ on both sides of the equation (4.9) lead to

$$\frac{2H}{2H-1}k_1^{\frac{1}{2H-1}}\varphi(k_2) = -4b(k_2)T + \frac{4k_1}{(1-H)^2}T^{2-2H} \tag{4.10}$$

and

$$\begin{aligned} \frac{2H}{2H-1}k_1^{\frac{1}{2H-1}}\varphi(k_2) &= \frac{2H}{(2H-1)k_1}a(k_2)T^{2H} \\ &\quad - \frac{8H}{(2H-1)}b(k_2)T + \frac{4Hk_1}{(2H-1)(1-H)^2}T^{2-2H} \end{aligned} \tag{4.11}$$

for all $k_2 \in (k_{2,2},k_{2,1})$. It follows that

$$\begin{aligned} -4b(k_2)T + \frac{4k_1}{(1-H)^2}T^{2-2H} &= \frac{2H}{(2H-1)k_1}a(k_2)T^{2H} \\ &\quad - \frac{8H}{(2H-1)}b(k_2)T + \frac{4Hk_1}{(2H-1)(1-H)^2}T^{2-2H}, \end{aligned}$$

which implies that

$$Ha(k_2)T^{2H} - 2b(k_2)Tk_1 + \frac{2}{1-H}T^{2-2H}k_1^2 = 0 \tag{4.12}$$

for all $k_2 \in (k_{2,2}, k_{2,1})$. This is a quadratic equation in k_1 with the two roots

$$\overline{k}_1 = \frac{1-H}{2}T^{2\alpha}(b(k_2)+\sqrt{D}) = x_1 T^{2\alpha}, \quad \underline{k}_1 = \frac{1-H}{2}T^{2\alpha}(b(k_2)-\sqrt{D}) = x_2 T^{2\alpha}$$

because its discriminant

$$\Delta = 4T^2\left[b^2(k_2) - \frac{2H}{1-H}a(k_2)\right] = T^2 D > 0.$$

It is easily to check that $\overline{k}_1$ is the solution to the equation

$$h(k_1,k_2) = 0 \tag{4.13}$$

and $\frac{\partial h}{\partial k_1}\mid_{k_1=\overline{k}_1} = 0$ for all $k_2 \in (k_{2,2}, k_{2,1})$. In order to see that $\underline{k}_1$ is not the solution to the equation (4.13), we claim that $h(\underline{k}_1, k_2) \neq 0$ for all $k_2 \in (k_{2,2}, k_{2,1})$. We have

$$\begin{aligned}
h(\underline{k}_1,k_2) &= f(t_1,\underline{k}_1,k_2) - f(T,\underline{k}_1,k_2) \\
&= x_2^{\frac{2H}{2H-1}}\varphi(k_2)T^{2H} - a(k_2)T^{2H} + 4x_2 b(k_2)T^{2H} - \frac{2{x_2}^2}{(1-H)^2}T^{2H} \\
&= T^{2H}\left[\left(\frac{x_2}{x_1}\right)^{\frac{2H}{2H-1}}\left(a(k_2) - 4b(k_2)x_1 + \frac{2{x_1}^2}{(1-H)^2}\right)\right. \\
&\qquad \left. - \left(a(k_2) - 4b(k_2)x_2 + \frac{2{x_2}^2}{(1-H)^2}\right)\right].
\end{aligned}$$

Put $u = x_1$ and $z = x_2$, then u and z are the roots of the equation (4.3), and by (3.1) we have

$$\frac{2u^2}{(1-H)^2} = \frac{2b(k_2)u}{1-H} - \frac{Ha(k_2)}{1-H}, \quad \frac{2z^2}{(1-H)^2} = \frac{2b(k_2)z}{1-H} - \frac{Ha(k_2)}{1-H}$$

and

$$u + z = (1-H)b(k_2), \quad uz = \frac{H(1-H)}{2}a(k_2). \tag{4.14}$$

It follows that

$$h(\underline{k}_1,k_2) = \frac{2H-1}{1-H}T^{2H}\left[\left(\frac{z}{u}\right)^{\frac{2H}{2H-1}}(2b(k_2)u - a(k_2)) - 2b(k_2)z + a(k_2)\right]$$

for all $k_2 \in (k_{2,2}, k_{2,1})$. Now, we claim that the inequality

$$\left(\frac{z}{u}\right)^{\frac{2H}{2H-1}}(2b(k_2)u - a(k_2)) - 2b(k_2)z + a(k_2) > 0 \tag{4.15}$$

for all $k_2 \in (k_{2,2}, k_{2,1})$. According to (4.14), the above inequality is equivalent to

$$a(k_2)\left(\left(\frac{z}{u}\right)^{\frac{2H}{2H-1}}\left(\frac{H(u+z)}{z} - 1\right) - \left(\frac{H(u+z)}{u} - 1\right)\right) > 0 \tag{4.16}$$

for all $k_2 \in (k_{2,2}, k_{2,1})$, and it can be simplified as

$$\phi(x) := Hx^{\frac{1}{2H-1}} - (1-H)x^{\frac{2H}{2H-1}} + (1-H) - Hx > 0$$

with $x = \frac{z}{u} \in (0,1)$. This is a simple calculus exercise. In fact, we have $\phi(0) = 1-H$, $\phi(1) = 0$,

$$\phi'(x) = \frac{H}{2H-1}x^{\frac{2-2H}{2H-1}} - \frac{2H(1-H)}{2H-1}x^{\frac{1}{2H-1}} - H$$

and

$$\phi''(x) = \frac{2H(1-H)}{(2H-1)^2} x^{\frac{3-4H}{2H-1}} - \frac{2H(1-H)}{(2H-1)^2} x^{\frac{2-2H}{2H-1}} > 0$$

for all $x \in (0,1)$ since $2H > 1$. This shows that the function ϕ is convex on $(0,1)$ and ϕ' is increasing strictly on $(0,1)$, which gives

$$-H = \phi'(0) < \phi'(x) < \phi'(1) = 0$$

for $x \in (0,1)$. It follows that ϕ is strictly decreasing on $(0,1)$ and

$$\phi(x) > \phi(1) = 0$$

for $x \in (0,1)$. Thus, we have showed that the inequality (4.15) holds and $h(\underline{k}_1, k_2) > 0$ for all $k_2 \in (k_{2,2}, k_{2,1})$.

On the other hand, from $h(0,k_2) = -a(k_2)T^{2H} < 0$ it follows that the equation $h(k_1,k_2) = f(t_1,k_1,k_2) - f(T,k_1,k_2) = 0$ admits a root, denoted by $\widehat{k}_1$, on $(0,\underline{k}_1)$ for all $k_2 \in (k_{2,2}, k_{2,1})$. Noting that the function $k_1 \mapsto f(t_1,k_1,k_2)$ is convex and increasing, we find easily that the equation $h(k_1,k_2) = f(t_1,k_1,k_2) - f(T,k_1,k_2) = 0$ admits two roots at most since the function $k_1 \mapsto f(T,k_1,k_2)$ is a quadratic function. Thus, $\widehat{k}_1$ is unique in $(0,\underline{k}_1)$ and $\frac{\partial h}{\partial k_1} \mid_{k_1=\widehat{k}_1} > 0$, and the lemma follows. □

Now, we can give the solution of the second case.

Theorem 4.6. *Let $D_1 > 0$ and $k_2^* \in (k_{2,2}, k_{2,1})$. Suppose that $\widehat{k}_1$ and t_1 are given as above. Then there exists $\widehat{k}_2 \in [k_{2,2}, k_{2,1}]$ such that the minimal value*

$$\inf_{\zeta\in\mathcal{K}} \sup_{0\le t\le T} f(t,k_1,k_2)$$

is achieved at the point $(T,\widehat{k}_1,\widehat{k}_2)$ and this value equals to $f(T,\widehat{k}_1,\widehat{k}_2)$.

Proof. Let $D_1 > 0$ and $k_2^* \in (k_{2,2}, k_{2,1})$. Then, Lemma 4.3 and Lemma 4.4 imply that

$$\inf_{\zeta\in\mathcal{K}} \sup_{0\le t\le T} f(t,k_1,k_2) = \inf_{\substack{k_2\in(k_{2,2},k_{2,1})\\ k_1>0}} \max\{f(t_1,k_1,k_2), f(T,k_1,k_2)\}.$$

It follows from Lemma 4.5 that

$$\max\{f(t_1,k_1,k_2), f(T,k_1,k_2)\} = f(t_1,k_1,k_2)1_{\{k_1>\widehat{k}_1\}} + f(T,k_1,k_2)1_{\{k_1<\widehat{k}_1\}},$$

which implies that

$$\max\{f(t_1,k_1,k_2), f(T,k_1,k_2)\} = f(T,\widehat{k}_1,k_2)$$

because $k_1 \mapsto f(t_1,k_1,k_2)$ is increasing and $f(T,k_1,k_2)$ is decreasing for $k_1 < \widehat{k}_1$. Combining this with the continuity of $k_2 \mapsto f(T,\widehat{k}_1,k_2)$, we see that there exists $\widehat{k}_2 \in [k_{2,2}, k_{2,1}]$ such that

$$\inf_{\substack{k_2\in(k_{2,2},k_{2,1})\\ k_1>0}} f(T,\widehat{k}_1,k_2) = f(T,\widehat{k}_1,\widehat{k}_2).$$

This completes the proof. □

5 Two special cases

In this section we consider two special classes of the approximation functions $\zeta \in \mathcal{K}$.

Theorem 5.1. *Let* $\mathcal{K}_1 = \left\{\zeta(s,r) = k(sr)^{-\frac{1}{2}H},\ k > 0\right\}$.

(1) *If* $(C_1(H))^2 \le \frac{2H}{1-H}$. *Then*

$$\inf_{\zeta\in\mathcal{K}_1}\sup_{0\le t\le T} E\left(Z_t^H - M_t(\zeta)\right)^2 = [1-2(1-H)^2C_1(H)^2]T^{2H}$$

with $\zeta(s,r) = (1-H)^2C_1(H)T^{2\alpha}(sr)^{-\frac{1}{2}H},\ s,r>0$.

(2) *If* $(C_1(H))^2 > \frac{2H}{1-H}$. *Then*

$$\min_{\zeta\in\mathcal{K}_1}\sup_{0\le t\le T} E\left(Z_t^H - M_t(\zeta)\right)^2 = f(T,k^*,0),$$

where $\zeta(s,r) = \widehat{k}(sr)^{-\frac{1}{2}H}$ $(s,r>0)$ *and* $\widehat{k}$ *is the smallest root of the equation* $f(T,k,0) - f(t_1,k,0) = 0$.

Proof. For $\zeta\in\mathcal{K}_1$ we have

$$E\left(Z_t^H - M_t(\zeta)\right)^2 = f(t,k,0) = t^{2H} - 4kC_1(H)t + \frac{2k^2}{(1-H)^2}t^{2-2H}$$

and $D = 4\left(C_1(H)^2 - \frac{2H}{1-H}\right)$, which complete the proof. □

Finally, denote $\mathcal{K}_2 = \left\{\zeta(s,r) = k(s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1},\ k>0\right\}$. Then, for $\zeta\in\mathcal{K}_2$, by (2.2) and $a(k)>0$ we have

$$\begin{aligned}\inf_{\zeta\in\mathcal{K}_2}\sup_{t\in[0,T]} E\left(Z_t^H - M_t(\zeta)\right)^2 &= \inf_{\zeta\in\mathcal{K}_2} a(k)T^{2H}\\ &= a(k^*)T^{2H} = T^{2H} - \frac{2(C_2(H))^2}{H\beta(1-H,2H-1)}T^{2H}\end{aligned}$$

with $k^* = \frac{C_2(H)}{\beta(1-H,2H-1)}$ and

$$\zeta(s,r) = \frac{C_2(H)}{\beta(1-H,2H-1)}(s\vee r)^{\frac{1}{2}H}(s\wedge r)^{-\frac{1}{2}H}|s-r|^{H-1},\quad s,r>0.$$

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Acknowledgments. The authors would like to express their sincere gratitude to the associate editor and the anonymous referees for their valuable comments.

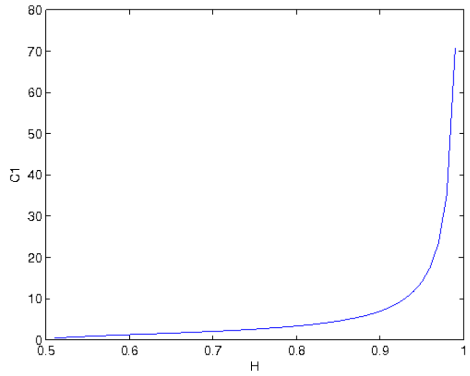

Figure 1: The function $H \mapsto C_1(H)$

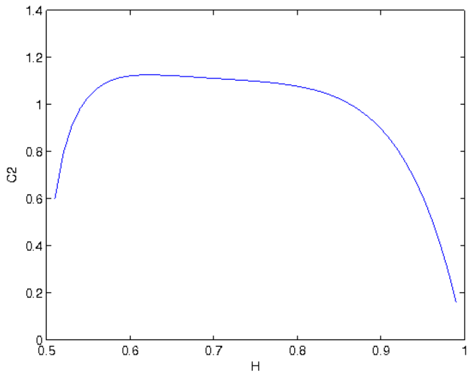

Figure 2: The function $H \mapsto C_2(H)$

Table 1: The enumeration of some constants

H	0.6	0.7	0.8	0.9
$C_1(H)$	1.2522	2.0581	3.3500	6.9514
$C_2(H)$	1.1220	1.1113	1.0777	0.8969
$D_1(H)$	-106.9256	-8.0811	853.7535	$4.463 * 10^4$

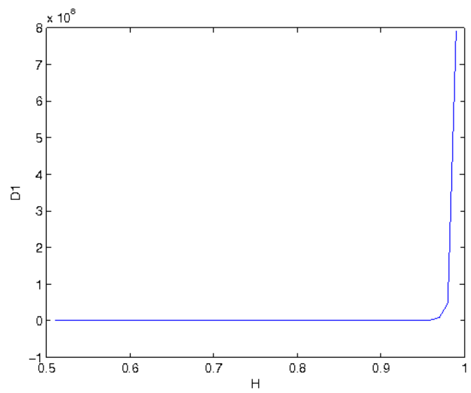

Figure 3: The function $H \mapsto D_1(H)$