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Erratum: Transience and recurrence of rotor-router walks on directed covers of graphs

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Abstract

In the paper [4] *Transience and recurrence of rotor-router walks on directed covers of graphs*, published in ECP volume 17 (2012), no. 41 there is an error in the proof of Corollary 3.8. This corollary is essential for the transient part in the proof of Theorem 3.5(b). We fix this error by constructing a new rotor-router process, which fulfills our needs, and for which the statement of Corollary 3.8. holds.

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Description of the error in the above mentioned paper. In the 5-th line from the end of the proof of Corollary 3.8, the last inequality

$$\left(1-p_k+p_k e^{-1}\right)^{w_{i,k}(h)} e^{\delta_i n} \leq C_k^n e^{\delta_i n},$$

does not hold since the constant $C_k = \left(1-p_k+p_k e^{-1}\right)$ is less than 1. This destroys the subexponential decay of the event $\left[E_n(\mathcal{T}_i,\rho) < \delta_i n\right]$, which is needed in the proof of Theorem 3.5(b). Within this erratum, we fix the error by constructing a new rotor-router process, which we call frontier process.

In the original paper [4] the pages 7 until 9 (not including Lemma 3.9) should be replaced by the following.

1 Fixing the error

For the transience of Theorem 3.5 we can assume that the direct cover $\mathcal{T}_i$ is not isomorphic to single infinite path, since by [1, Theorem 6] we have recurrence in that case for any initial rotor distribution.

The frontier rotor-router process $F_\rho(n)$. For a fixed rotor configuration ρ consider the following process which generates a sequence $F_\rho(n)$ of subsets of vertices of the tree. $F_\rho(n)$ is constructed by a rotor-router process consisting of n rotor-router walks starting at the root r, such that each vertex of $F_\rho(n)$ contains exactly one particle. In the first step put a particle at the root r and set $F_\rho(1)=\{r\}$. Inductively given $F_\rho(n)$ and the rotor configuration that was created by the previous step, we construct the next set $F_\rho(n+1)$ using the following rotor-router procedure. Perform rotor-router walk with a particle starting at the root r, until one of the following stopping conditions occurs:

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(a) The particle reaches the down sink $s_\downarrow$. In this case we set $F_\rho(n+1) = F_\rho(n)$.

(b) The particle first reaches a vertex x, which has never been visited before. In this case we set $F_\rho(n+1) = F_\rho(n) \cup \{x\}$.

(c) The particle reaches an element $y \in F_\rho(n)$. We delete y from $F_\rho(n)$, i.e. set $F'(n) = F_\rho(n) \setminus \{y\}$. At this time there are two particles at y, both of which are restarted until stopping condition (a), (b) or (c) for the set $F'(n)$ applies to them. Note that since we are on a tree at least one particle will stop at a child of y after one step, due to halting condition (b).

We will call the set $F_\rho(n)$ the *frontier* of n particles. In the following we give several properties of the frontier $F_\rho(n)$.

Lemma 1.1. *The process generating $F_\rho(n)$ is always terminating in a finite number of steps and the set of vertices visited by the particles during this process is finite.*

Proof. We prove this by induction. For $n = 1$ the statement is obviously true.

Let $V(n)$ be the set of vertices visited while computing $F_\rho(n)$. Assuming $V(n)$ is finite we know that after a finite number of steps the rotor-router walk first exits the set $V(n)$ at some vertex x. If $x = s_\downarrow$ we are in case (a) and $V(n+1) = V(n)$. If $x \notin F_\rho(n)$ we are in case (b) and we have $V(n+1) = V(n) \cup \{x\}$.

Finally, if $x \in F_\rho(n)$ two particles are restarted at x. If both particles visit children of x in the first step, the process stops and these two vertices are added to $V(n)$. In the case that one of the particles visits the ancestor of x in the first step, this particle continues its walk until of the three stopping conditions occurs. Since the first edge that is traversed in the same direction twice is an edge emanating from the starting vertex (see Angel and Holroyd [2, Lemma 8]) the particle will enter the sink before it will return to x. Hence each vertex of $F_\rho(n)$ can be visited at most once during the formation of $F_\rho(n+1)$. This means in particular that $V(n)$ is expanded by only a finite number of vertices. □

Remark 1.2. Note that whenever a previously unexplored vertex is reached, it is immediately added to the set $F_\rho(n)$. Hence $\max\{|x| : x \in V(n)\} = \max\{|x| : x \in F_\rho(n)\}$.

Definition 1.3. *For each vertex $x \in \mathcal{T}$ denote by $\mathcal{C}(x) = \big\{y \in \mathcal{T} : y \text{ is a descendant of } x\big\} \cup \{x\}$ the* cone *of x.*

Lemma 1.4. *$\mathcal{C}(x) \neq \mathcal{C}(y)$ for all $x, y \in F_\rho(n)$ with $x \neq y$. Let ρ' be the rotor-router configuration at the end of the process generating $F_\rho(n)$. Then for all $x \in F_\rho(n)$ the rotor configuration in the cone of x is unchanged, that is, $\rho_{|\mathcal{C}(x)} \equiv \rho'_{|\mathcal{C}(x)}$.*

Proof. This follows immediately from the procedure generating the frontier. □

Let

$$M(n) = \max_\rho \max\big\{|x| : x \in F_\rho(n)\big\}. \tag{1.1}$$

be the maximal height of the frontier $F_\rho(n)$. We will need an upper bound for $M(n)$. Since whenever stopping condition (c) occurs, the frontier moves one level upwards at the vertex of $F_\rho(n)$ that was hit. We have the trivial upper bound of $M(n) \leq n$. This bound is tight for general trees as shows the example of a single infinite path, where the frontier $F_\rho(n)$ for $n \geq 2$ consists of a single vertex at distance $n-2$ from the root vertex, with the remaining $n-1$ particles in the down sink $s_\downarrow$.

In the case of directed covers with irreducible cone types which are not isomorphic to a single path, $M(n)$ seems to grow logarithmically in n. For our purposes, a weak upper bound of the form $M(n) \leq cn$ for a constant $c < 1$ is sufficient.

Lemma 1.5. *There exists a constant $c < 1$ such that $M(n) < cn$, for all n large enough.*

Proof. Let x be an element of $F_\rho(n)$ with maximal distance $M = |x|$ to the root r. Denote by $p = (r = x_0, x_1, \dots, x_M = x)$ the shortest path between r and x. Since $F_\rho(1) = \{r\}$ and by the iterative construction of $F_\rho(n)$, there exist $1 = n_0 < n_1 < \dots < n_M = n$, such that $x_i \in F_\rho(n_i)$ for all $i \in 0, \dots, M$. Since $\mathcal{T}$ is a directed cover that is not isomorph to a single infinite path, it follows that for all n big enough, there exist a constant $\kappa > 0$, such that $\#\big\{v \in p : \deg(v) \geq 3\big\} \geq \kappa M$.

We want to find a lower bound for $n_{i+2} - n_i$, that is, for the number of steps needed to replace x_i by x_{i+2} in the frontier. At time n_i, the vertex x_i is added to the frontier. The next time after n_i that a particle visits x_i halting condition (c) occurs, thus the rotor at x_i is incremented two times. As long as not all children of x_i are part of the frontier, every particle can visit x_i at most once, since it either stops immediately at a child of x_i on stopping condition (b) or is returned to the ancestor of x_i. Thus at subsequent visits the rotor at x_i is incremented exactly once. In order for x_{i+2} to be added to the frontier, the rotor at x_i has to point at direction x_{i+1} twice. Thus replacing x_i with x_{i+2} in the frontier, needs at least $\deg(x_i)$ particles which visit x_i. Hence, $n_{i+2} - n_i \geq \deg(x_i)$

We have

$$\sum_{i=0}^{M-2} n_{i+2} - n_i = n_M + n_{M-1} - n_1 - n_0 < 2n.$$

On the other hand, denote by

$$\begin{aligned} p_2 &= \#\big\{x_i : i \in \{0, \dots, M-2\} \text{ s.t. } \deg(x_i) = 2\big\} \\ p_3 &= \#\big\{x_i : i \in \{0, \dots, M-2\} \text{ s.t. } \deg(x_i) \geq 3\big\}, \end{aligned}$$

then

$$\begin{aligned} \sum_{i=0}^{M-2} n_{i+2} - n_i \geq \sum_{i=0}^{M-2} \deg(x_i) &\geq 3p_3 + 2p_2 = 3p_3 + 2(M - 1 - p_3) \\ &\geq p_3 + 2M - 2 \geq (\kappa + 2)M - 2\kappa - 2. \end{aligned}$$

Thus $M \leq \frac{2}{\kappa+2} n + 2$, which proves the claim. □

The number of particles on the frontier. For the frontier process $F_\rho(n)$ defined above, when starting n rotor particles at the root, we end up with exactly one particle at each vertex of $F_\rho(n)$ and the rest are in $s_\downarrow = r^{(0)}$ (the ancestor of the root). In order to obtain a lower bound for the cardinality of $F_\rho(n)$, we first get an upper bound for the number of particles stopped at $s_\downarrow$. This will be achieved using Theorem 1 from [3]. Define

$$\ell(n) = \{x \in \mathcal{T}_i : |x| = M(n) \text{ and the path from } r \text{ to } x \text{ contains no vertex of } F_\rho(n)\}, \tag{1.2}$$

where $M(n)$ is defined in (1.1). By construction, the set $F_\rho(n)$ may have "holes": this means that $F_\rho(n)$ is not a cut in the tree. By introducing the set $\ell(n)$ in (1.2), we fill this holes by adding additional vertices on the maximal level $M(n)$. All these additional vertices were not touched by a rotor particle during the formation of $F_\rho(n)$. Fix n and a rotor configuration ρ, and let

$$S = F_\rho(n) \cup \ell(n) \tag{1.3}$$

be the sink determined by the frontier process $F_\rho(n)$. Denote be $\mathcal{T}_i^S$ the finite tree which is obtained by truncating $\mathcal{T}_i$ at S, i.e. $\mathcal{T}_i^S = \{x \in \mathcal{T}_i : \mathcal{C}(x) \cap S \neq \emptyset\}$.

Let (X_t) be the simple random walk on $\mathcal{T}_i$. Let $T_{s_\downarrow} = \min\{t \geq 0 : X_t \in s_\downarrow\}$ and $T_S = \{t \geq 0 : X_t \in S\}$ be the first hitting time of $s_\downarrow$ and S respectively. Consider now the hitting probability

$$h(x) = h^S_{s_\downarrow}(x) = \mathbb{P}_x[T_{s_\downarrow} < T_S], \tag{1.4}$$

that is, the probability to hit $s_\downarrow$ before S, when the random walk starts in x. We have $h(s_\downarrow) = 1$ and $h(x) = 0$, for all $x \in S$ and $h(x) = 0$ for all $x \in \mathcal{T}_i \setminus \mathcal{T}_i^S$.

Start now n rotor particles at the root r, and stop them when they either reach $s_\downarrow$ or S. By the Abelian property of rotor-router walks (see [2, Lemma 24]) and by the construction of the frontier process $F_\rho(n)$ we will have exactly one rotor particle at each vertex of $F_\rho(n)$, no particles at $\ell(n)$, and the rest of the particles are at $s_\downarrow$. In order to estimate the proportion of rotor particles stopped at $s_\downarrow$ we use Theorem 1 from [3], which we state here adapted to our case.

Theorem 1.6 (Theorem 1, [3])**.** *Consider the sinks $s_\downarrow$ and S as above, and let (X_t) be the simple random walk on $\mathcal{T}_i$. Let E be the set of edges of $\mathcal{T}_i$ and suppose that the quantity*

$$K = 1 + \sum_{(x,y)\in E} |h(x) - h(y)| \tag{1.5}$$

is finite. If we start n rotor particles at the root r, then

$$\left| h(r) - \frac{n_{s_\downarrow}}{n} \right| \leq \frac{K}{n}, \tag{1.6}$$

where $n_{s_\downarrow}$ represents the number of particles stopped at $s_\downarrow$.

Lemma 1.7. *The constant K is equal to*

$$K = 1 + \big(M(n) + 1\big)\big(1 - h(r)\big).$$

Proof. The function h is harmonic away from the sink: $h(x) = \frac{1}{\deg(x)} \sum_{y\sim x} h(y)$, if $x \notin s_\downarrow \cup S$, and $h(x) = 0$ for $x \in (\mathcal{T}_i \setminus \mathcal{T}_i^S) \cup S$. Therefore, there are only finitely many non zero summands in (1.5). For a vertex $x \in \mathcal{T}_i^S \setminus S$ and its ancestor $x^{(0)}$, we always have $h\big(x^{(0)}\big) \geq h(x)$, and

$$h\big(x^{(0)}\big) - h(x) = \sum_{i=1}^{\deg(x)-1} \Big(h(x) - h\big(x^{(i)}\big)\Big).$$

Then

$$K = 1 + \Big(h\big(r^{(0)}\big) - h(r)\Big) + \sum_{k=0}^{M(n)-1} \sum_{x\in S^k} \sum_{i=1}^{\deg(x)-1} \Big(h(x) - h\big(x^{(i)}\big)\Big),$$

where $S^k = \{y \in \mathcal{T}_i : |y| = k\}$ represents the k-th level of the tree $\mathcal{T}_i$. For a fixed k

$$\begin{aligned}\sum_{x\in S^k} \sum_{i=1}^{\deg(x)-1} \Big(h(x) - h\big(x^{(i)}\big)\Big) &= \sum_{x\in S^k} h\big(x^{(0)}\big) - h(x) = \sum_{x\in S^{k-1}} \sum_{i=1}^{\deg(x)-1} \Big(h(x) - h\big(x^{(i)}\big)\Big) \\ &= \sum_{x\in S^{k-j}} \sum_{i=1}^{\deg(x)-1} \Big(h(x) - h\big(x^{(i)}\big)\Big), \text{ for } j = 2, \ldots, k-1 \\ &= h\big(r^{(0)}\big) - h(r) = 1 - h(r).\end{aligned}$$

Summing up over all levels the claim follows. ☐

Corollary 1.8. *There is a constant $\kappa \in (0, 1)$, such that $\#F_\rho(n) > \kappa n$, for all n large enough.*

Proof. From (1.6), we have $\frac{n_{s_\downarrow}}{n} \leq \frac{K}{n} + h(r)$, where $n_{s_\downarrow}$ is the number of particles stopping at $s_\downarrow$ in the frontier process $F_\rho(n)$. Putting together Lemma 1.5 and 1.7, we obtain $K < 1 + (cn+1)(1-h(r))$. Putting $\kappa' = h(r)(1-c) + c < 1$, we get $n_{s_\downarrow} < \kappa' n$. Since $\#F_\rho(n) = n - n_{s_\downarrow}$, the claim follows. □

Next, we fix the error in the proof of Corollary 3.8 from [4]. The statement of the result remains unchanged, but the proof is slightly different. Actually, we do not need the constant C_i in the exponential bound.

Corollary 1.9 (Corollary 3.8 from [4])**.** *Let ρ be an initial random rotor configuration with distribution $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_m)$ on the directed cover $\mathcal{T}_i$ with root of type i, of a finite strongly connected graph G with m vertices. Suppose $\mathsf{r}(M(\mathcal{D})) > 1$. Then there exists $\delta_i, c_i > 0$, such that for all n*

$$\mathbb{P}\big[E_n(\mathcal{T}_i, \rho) < \delta_i n\big] \leq e^{-c_i n}, \quad \text{for all } i \in \mathsf{G}.$$

Proof. Consider n rotor walks particles and build the frontier process $F_\rho(n)$. The MBP with probabilities p^i as in equation (3.4) from [4] and $\mathsf{r}(M(\mathcal{D})) > 1$ survives with positive probability p_i. Hence, for each $i \in \mathsf{G}$, with positive probability there exists a live path starting at the root of $\mathcal{T}_i$. Existence of a live path implies that the first particle escapes, hence

$$\mathbb{P}[E_1(\mathcal{T}_i, \rho) = 1] = \mathsf{p}_i > 0, \quad \text{for all } i \in \mathsf{G}.$$

Denote now by X the set of vertices $x \in F_\rho(n)$, for which there is a live path starting at x. Then $\#X = \sum_{x \in F_\rho(n)} Y_x$, where the random variables $Y_x \sim \text{Bernoulli}\big(\mathsf{p}_{\tau(x)}\big)$ are independent Bernoulli random variables. Recall that $\tau(x)$ represents the type of the vertex x. By the construction of $F_\rho(n)$, after starting n rotor walks in the root r, we have exactly one rotor particle in each $x \in F_\rho(n)$. By Corollary 1.8, we have $\#F_\rho(n) > \kappa n$. Hence $\mathbb{E}[\#X] \geq \#F_\rho(n)\mathsf{p} > \kappa \mathsf{p} n$, where $\mathsf{p} = \min_i \mathsf{p}_i > 0$. Let us first prove that

$$E_n(\mathcal{T}_i, \rho) \geq \#X. \tag{1.7}$$

From [3, Lemmas 18,19], it suffices to prove (1.7) for the truncated tree $\mathcal{T}_i^H = \{x \in \mathcal{T}_i : |x| \leq H\}$, with $H > M(n)$, i.e.,

$$E_n(\mathcal{T}_i^H, S^H, \rho^H) \geq \#X. \tag{1.8}$$

$E_n(\mathcal{T}_i^H, S^H, \rho^H)$ represents the number of particles that stop at $S^H = \{x \in \mathcal{T}_i : |x| = H\}$ when we start n rotor-router walks at the root of $\mathcal{T}_i$ and rotor configuration ρ^H (the restriction of ρ on $\mathcal{T}_i^H$). In the tree $\mathcal{T}_i^S$, truncated at the frontier S, start n particles at the root, and stop them when they either reach S or return to $s_\downarrow$. Moreover, the vertices at distance greater than $M(n)$ were not reached, and the rotors there are unchanged. Now for every vertex x in X restart one particle. Since there is a live path a x the particle will reach the level H without leaving the cone of x, at which point the particle is stopped again. Hence if we restart all particles which are located in $F_\rho(n)$ at least $\#X$ of them will reach level H before returning to the root. Because of the abelian property of rotor-router walks, (1.8) follows, therefore also (1.7).

Using the Chernoff bound, there exists $\delta_i \in (0,1)$ such that

$$\begin{aligned}
\mathbb{P}\big[E_n(\mathcal{T}_i, \rho) < \delta_i n\big] &\leq \mathbb{P}\big[\#X < \delta_i n\big] \leq \mathbb{P}\Big[\#X < \frac{\delta_i}{\kappa \mathsf{p}} \mathbb{E}[\#X]\Big] \\
&\leq \exp\bigg\{ - \frac{\big(1 - \frac{\delta_i}{\kappa \mathsf{p}}\big)^2}{2} \mathbb{E}[\#X] \bigg\} \leq \exp\bigg\{ - \frac{\big(1 - \frac{\delta_i}{\kappa \mathsf{p}}\big)^2}{2} \kappa \mathsf{p} n \bigg\}.
\end{aligned}$$

We can then choose $c_i > 0$ such that

$$\mathbb{P}\left[E_n(\mathcal{T}_i, \rho) < \delta_i n\right] \leq e^{-c_i n},$$

which proves the statement. □

The proof of the main result on page 10 [Proof of Theorem 3.5(b)] is unchanged, but it uses the above Corollary.

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