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A note on the $S_2(\delta)$ distribution and the Riemann Xi function

Dmitry Ostrovsky*

Abstract

The theory of $S_2(\delta)$ family of probability distributions is used to give a derivation of the functional equation of the Riemann xi function. The δ deformation of the xi function is formulated in terms of the $S_2(\delta)$ distribution and shown to satisfy Riemann's functional equation. Criteria for simplicity of roots of the xi function and for its simple roots to satisfy the Riemann hypothesis are formulated in terms of a differentiability property of the $S_2(\delta)$ family. For application, the values of the Riemann zeta function at the integers and of the Riemann xi function in the complex plane are represented as integrals involving the Laplace transform of S_2.

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In this paper we contribute to the field of probabilistic studies of values of the Riemann zeta function. This field was pioneered by [4] and [12] and greatly advanced by [1] and [17], which serve as primary references as well as motivation for our results. The field is comprised of roughly two streams of works. The first stream as represented by [2], [9], [10], [13], [14], [15], [16], [22] for example, relates values of the Riemann zeta and other functions of analytic number theory (Riemann xi, Barnes gamma functions, Selberg integral) directly to various probabilistic notions (infinite divisibility, independent product/sum representations, Lévy processes). The second stream as represented by [3], [5], [7], [13] for example, develops random matrix theoretic machinery that is necessary to fully understand the celebrated conjecture of [8] on the moments of the Riemann zeta function on the critical line.

In this paper we continue to study the family of $S_2(\delta)$ probability distributions that we introduced in [16] as a means of approximating the Riemann xi function by a limit of Barnes beta distributions, see [15]. Our contribution is three-fold. First, we give a derivation of the celebrated functional equation of the Riemann xi function using the theory of $S_2(\delta)$ distributions. While this equation has many known proofs, see [20] for example, the novelty of our approach is that our proof is probabilistic in nature and mainly relies on a computation of moments of $S_2(\delta)$ in a way that does not require Jacobi's theta identity or complex integration but rather only uses the Laplace transform of S_2. As an application of our approach, we show that the values of the Riemann zeta function at the integers as well as the values of the Riemann xi function in the complex plane can be represented as simple integrals involving the Laplace transform of S_2. We also show that the functional equation itself is equivalent to a symmetry of a certain integral transform of the Laplace transform. Second, we formulate a functional equation and a generalized

*195 Idlewood Dr., Stamford, CT 06905, USA. E-Mail: dm_ostrov@aya.yale.edu.

xi function that correspond to the $S_2(\delta)$ distribution thereby obtaining a one-parameter deformation of Riemann's xi function that satisfies the functional equation of the xi function. Finally, we show that the behavior of roots of the Mellin transform of $S_2(\delta)$ as a function of δ gives us elementary criteria for the simplicity of roots of the xi function and validity of the Riemann hypothesis for simple roots.

1 Introduction

The theory of the S_2 and related distributions was developed in [1] and [17]. In this section we will review some of the key points of this theory following [1] so as to motivate our generalization of S_2 in the next section.

S_2 is an infinitely divisible, absolutely continuous probability distribution on $(0, \infty)$ that is defined by

$$S_2 \triangleq \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\Gamma_{2,n}}{n^2}, \tag{1.1}$$

and $\{\Gamma_{2,n}\}$ denotes an iid family of gamma distributions on $(0, \infty)$ with the density xe^{-x}. Its Laplace transform is given by[1]

$$\mathbf{E}\big[e^{-qS_2}\big] = \Big[\frac{\sqrt{2q}}{\sinh\sqrt{2q}}\Big]^2, \tag{1.2}$$

$$= \exp\Big(\int\limits_0^\infty (e^{-qt}-1)\big(\theta(\frac{\pi t}{2})-1\big)\frac{dt}{t}\Big),\; q>0, \tag{1.3}$$

where $\theta(t)$ is a special case of Jacobi's θ_3 function

$$\theta(t) \triangleq 1 + 2\sum_{n=1}^{\infty} e^{-\pi t n^2},\; t>0, \tag{1.4}$$

hence the Lévy density of S_2 is $\rho_{S_2}(x) = \big(\theta(\pi x/2)-1\big)/x$. The theta function identity

$$\sqrt{t}\,\theta(t) = \theta(1/t),\, t>0, \tag{1.5}$$

implies that, up to exponentially small terms, $\theta(t) \sim t^{-1/2}$ as $t \to +0$ and $\theta(t) \sim 1$ as $t \to +\infty$ so that $\rho_{S_2}(x)$ is a valid Lévy density, see Theorem 4.3 in Chapter 3 of [19]. The cumulative distribution function of S_2 is[2]

$$\mathbf{P}\big(S_2 < x\big) = \sum_{n\in\mathbb{Z}}(1-n^2\pi^2 x)e^{-n^2\pi^2 x/2}. \tag{1.6}$$

Denote the probability density of S_2 by $f_{S_2}(x)$. Then, it is easy to see from (1.4) and (1.6) that it is related to the Lévy density by

$$f_{S_2}(x) = \frac{d}{dx}\big(1+2x\frac{d}{dx}\big)x\rho_{S_2}(x). \tag{1.7}$$

S_2 satisfies a remarkable functional equation as a corollary of (1.5).

$$\mathbf{E}\Big[g\Big(\frac{4}{\pi^2 S_2}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbf{E}\Big[S_2^{1/2}g(S_2)\Big] \tag{1.8}$$

[1] We mention in passing that S_2 as defined by (1.2) appears also in a model of Anderson localization in the context of statistics of eigenvectors of random banded matrices, see [6].

[2] It is quite non-trivial that the right-hand side of (1.6) is a valid distribution function on $(0, \infty)$. The interested reader is referred to [4], Theorem 7, for a probabilistic proof and to the discussion following it for a direct analytic proof.

that holds for arbitrary test functions g, for which the equation makes sense. It is equivalent to

$$f_{S_2}(x) = \Big(\frac{2}{\pi x}\Big)^{5/2} f_{S_2}\Big(\frac{4}{\pi^2 x}\Big). \tag{1.9}$$

It follows from (1.6) and (1.9) that $f_{S_2}(x)$ is exponentially small in the limits $x \to +0$ and $x \to +\infty$, and we have, up to polynomial prefactors,

$$f_{S_2}(x) \sim e^{-2/x},\ x \to +0, \tag{1.10}$$

$$f_{S_2}(x) \sim e^{-\pi^2 x/2},\ x \to +\infty. \tag{1.11}$$

In particular, the Mellin transform $\mathbf{E}\big[S_2^q\big]$ is entire in q. The relationship between S_2 and the Riemann xi function is equally remarkable.

$$\Big(\frac{2}{\pi}\Big)^q 2\xi(2q) = \mathbf{E}\big[S_2^q\big],\ q \in \mathbb{C}, \tag{1.12}$$

where the entire function $\xi(q)$ is defined in terms of the Riemann zeta function by[3]

$$\xi(q) \triangleq \frac{1}{2} q(q-1)\pi^{-q/2}\Gamma(q/2)\zeta(q),\ \Re(q) > 1, \tag{1.13}$$

and the Riemann zeta function is defined by

$$\zeta(q) \triangleq \sum_{m=0}^{\infty} (m+1)^{-q},\ \Re(q) > 1. \tag{1.14}$$

The Mellin transform in (1.12) is crucial for our purposes so we briefly remind the reader how it can be derived for $\Re(q) > 1/2$ by double integration by parts. One starts with the representation of the density of S_2 in (1.7) and evaluates the resulting Mellin transform by elementary means (boundary terms vanishing by the asymptotics of theta).

$$\begin{aligned}\mathbf{E}\big[S_2^q\big] &= \int\limits_0^\infty x^q \frac{d}{dx}\big(1 + 2x\frac{d}{dx}\big)\big(\theta(\pi x/2) - 1\big)\,dx,\\ &= (2q^2 - q)\int\limits_0^\infty x^{q-1}\big(\theta(\pi x/2) - 1\big)\,dx,\\ &= (2q^2 - q)\Gamma(q)\Big(\frac{2}{\pi^2}\Big)^q 2\zeta(2q),\end{aligned} \tag{1.15}$$

which is equivalent to (1.12) by (1.13). Using (1.12) and the functional equation (1.8) with $g(x) = x^q$, one sees that the xi function satisfies

$$\xi(q) = \xi(1-q) \tag{1.16}$$

for $q \in \mathbb{C}$, which is Riemann's functional equation. We finally note that many important problems in number theory hinge on the location of roots of the xi function, which are known to lie in the critical strip $0 < \Re(q) < 1$. We refer the reader to [20] as a reference on the xi function.

[3] Contrary to the commonly accepted usage, we use q as opposed to s as the generic complex variable to avoid confusion with S_2 and use $\int_0^\infty x^q\, f(x)\,dx$ to define the Mellin transform as it is natural for our purposes.

2 A Review of $S_2(\delta)$

In this section we will remind the reader of our construction of the $S_2(\delta)$ family of probability distributions and their basic properties established in [16].

Definition 2.1. *Let $\delta \geq 0$ and $\{\Gamma_{2,n}\}$ be as in* (1.1).

$$S_2(\delta) \triangleq \sum_{n=1}^{\infty} \frac{\Gamma_{2,n}}{\pi^2 n^2/2+\delta}. \tag{2.1}$$

The main properties of $S_2(\delta)$ are summarized in the following theorem, which we give here with additional details and proof for completeness.

Theorem 2.2 (Properties of $S_2(\delta)$)**.** *$S_2(\delta)$ is infinitely divisible and absolutely continuous. Denote its density by $f_{S_2(\delta)}(x)$. Then, its Laplace transform, density, and Mellin transform satisfy*

$$\mathbf{E}\big[e^{-qS_2(\delta)}\big] = \Big[\frac{\sinh\sqrt{2\delta}}{\sqrt{2\delta}}\Big]^2 \mathbf{E}\big[e^{-(q+\delta)S_2}\big], \tag{2.2}$$

$$= \exp\Big(\int_0^\infty (e^{-qt}-1)e^{-\delta t}\big(\theta(\frac{\pi t}{2})-1\big)\frac{dt}{t}\Big),\ q>0, \tag{2.3}$$

$$f_{S_2(\delta)}(x) = \Big[\frac{\sinh\sqrt{2\delta}}{\sqrt{2\delta}}\Big]^2 e^{-\delta x} f_{S_2}(x),\ x>0, \tag{2.4}$$

$$\mathbf{E}[S_2(\delta)^q] = \Big[\frac{\sinh\sqrt{2\delta}}{\sqrt{2\delta}}\Big]^2 \Big(\frac{2}{\pi}\Big)^q \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{-2\delta}{\pi}\Big)^n 2\xi(2q+2n),\ q\in\mathbb{C},\ \delta<\pi^2/2. \tag{2.5}$$

Given a test function $g(x)$, $S_2(\delta)$ satisfies the general identity

$$\mathbf{E}\Big[\exp\big(-\delta S_2\big)\Big]\mathbf{E}\Big[g\big(S_2(\delta)\big)\Big] = \mathbf{E}\Big[\exp\big(-\delta S_2\big)g(S_2)\Big]. \tag{2.6}$$

Let $\delta>0$ and β be an independent exponential distribution with the density $\delta\exp(-\delta x)$. Define the distribution

$$T(\delta) \triangleq S_2(\delta)+\beta. \tag{2.7}$$

Then, $T(\delta)$ is infinitely divisible and absolutely continuous on $(0,\infty)$ and its density and Laplace transform are

$$f_{T(\delta)}(x) = \delta e^{-\delta x}\Big[\frac{\sinh\sqrt{2\delta}}{\sqrt{2\delta}}\Big]^2 \mathbf{P}(S_2<x),\ x>0, \tag{2.8}$$

$$\mathbf{E}\big[e^{-qT(\delta)}\big] = \exp\Big(\int_0^\infty (e^{-qt}-1)e^{-\delta t}\theta(\frac{\pi t}{2})\frac{dt}{t}\Big),\ q>0. \tag{2.9}$$

The Mellin transforms $\mathbf{E}[S_2(\delta)^q]$ and $\mathbf{E}[T(\delta)^q]$ are entire functions of q.

Proof. The starting point is the formula given in [1], Section 3.2, for the Lévy density $\rho_X(t)$ of the weighted sum of positive, independent, infinitely divisible distributions of the form $X=\sum_n c_n X_n$, where $c_n>0$ and X_n has Lévy density $\rho(t)$ for all n.

$$\rho_X(t) = \sum_n \frac{1}{c_n}\rho(t/c_n). \tag{2.10}$$

The Lévy density of $\Gamma_{2,n}$ is $2e^{-t}/t$ so that the Lévy density $\rho_{S_2(\delta)}(t)$ of $S_2(\delta)$ is

$$\rho_{S_2(\delta)}(t) = \frac{e^{-\delta t}}{t}\big(\theta(\pi t/2)-1\big). \tag{2.11}$$

Then, the Laplace transform of $S_2(\delta)$ can be written as

$$\mathbf{E}\big[e^{-qS_2(\delta)}\big] = \exp\Big(\int\limits_0^\infty (e^{-qt}-1)\rho_{S_2(\delta)}(t)\,dt\Big), \tag{2.12}$$

$$= \Big[\prod_{n=1}^\infty \frac{\delta+\pi^2n^2/2}{q+\delta+\pi^2n^2/2}\Big]^2, \tag{2.13}$$

$$= \Big[\frac{\sinh\sqrt{2\delta}}{\sqrt{2\delta}}\Big]^2\,\mathbf{E}\big[e^{-(q+\delta)S_2}\big], \tag{2.14}$$

where we used Frullani's formula for $\log(x)$ to evaluate the integral in (2.12) and the infinite product representation of $\sinh(x)$ and (1.2) to obtain (2.14). This proves (2.2) and (2.3). The density of $S_2(\delta)$ follows from (2.2) so that the Mellin transform is

$$\mathbf{E}[S_2(\delta)^q] = \Big[\frac{\sinh\sqrt{2\delta}}{\sqrt{2\delta}}\Big]^2 \int\limits_0^\infty x^q e^{-\delta x} f_{S_2}(x)\,dx. \tag{2.15}$$

Expanding the exponential and making use of (1.12), we obtain (2.5), provided that the integral can be computed term by term. The partial sums of $\exp(-\delta x)$ are bounded by $\exp(\delta x)$. If $\delta < \pi^2/2$, then $\exp(\delta x) f_{S_2}(x)$ is exponentially small as $x \to +\infty$, see (1.11), so that the result follows by dominated convergence. The series is absolutely convergent if $\delta < \pi^2/2$ as is clear from (1.13) since $\zeta(q) \to 1$ (uniformly in $\Im(q)$) as $\Re(q) \to +\infty$. (2.6) is immediate from (2.4). The density of $T(\delta)$ in (2.8) is the convolution of the density of $S_2(\delta)$ in (2.4) and the density of β. Since the density and cumulative distribution functions of S_2 are exponentially small as $x \to 0$, the Mellin transforms of $S_2(\delta)$ and $T(\delta)$ are entire in q. $\square$

We mention in passing that our construction of $S_2(\delta)$ in [16] was primarily motivated by $T(\delta)$. There we used Jacobi's triple product to relate $T(\delta)$ to a limit of Barnes beta distributions, which we introduced in a special case in [14] and in general in [15] in the context of the Selberg integral. We will not dwell on this connection here short of pointing out that the Barnes beta distribution approach provides an altogether different way of looking at S_2, see also Corollary 3.5 and Remark 3.6 below.

3 Results

We begin by formulating our result on the functional equation of the xi function, see (1.16). As the equation *per se* is well-known, we must first explain what we assume to be given. Our main assumption is that the relationship of the Mellin transform of S_2 and the xi function in (1.12) is known for all $q \in \mathbb{C}$ (or, equivalently, that the xi function is defined by (1.12), the Mellin transform of S_2 is entire, and (1.13) is known).

Theorem 3.1 (Functional equation of $\xi(q)$)**.** *Let* $q \in \mathbb{C}$. *Then,*

$$\Big(\frac{4}{\pi^2}\Big)^q \mathbf{E}\Big[S_2^{-q}\Big] = \sqrt{\frac{\pi}{2}}\mathbf{E}\Big[S_2^{q+1/2}\Big]. \tag{3.1}$$

Proof. It is sufficient to show that (3.1) holds for any domain of the form $\Re(q) \in (n-1, n)$, $n = 1, 2, 3, \cdots$ because an entire function that is identically zero on such a domain must necessarily be identically zero on the whole complex plane, see Theorem 1.2 in Chapter III of [11]. Let $\Re(p) \in (0,\, 1)$ and $q = n - p$. The starting point and key element of the proof is the identity that is satisfied by the Mellin transform of $S_2(\delta)$.

$$\begin{aligned}\frac{d}{d\delta}\mathbf{E}\big[e^{-\delta S_2}\big]\mathbf{E}\big[S_2(\delta)^q\big] &= \frac{d}{d\delta}\mathbf{E}\big[e^{-\delta S_2}S_2^q\big],\\ &= -\mathbf{E}\big[e^{-\delta S_2}\big]\mathbf{E}\big[S_2(\delta)^{q+1}\big],\ q\in\mathbb{C},\end{aligned} \tag{3.2}$$

which is an elementary corollary of (2.6). It follows by induction that we can write for any $\delta\geq 0$, $q\in\mathbb{C}$, and $n=1,2,3\cdots$

$$\begin{aligned}\mathbf{E}\big[e^{-\delta S_2}\big]\mathbf{E}\big[S_2(\delta)^{-q}\big] &= \int\limits_{\delta}^{\infty} d\delta_1\cdots\int\limits_{\delta_{n-2}}^{\infty} d\delta_{n-1}\int\limits_{\delta_{n-1}}^{\infty} d\delta_n\,\mathbf{E}\big[e^{-\delta_n S_2}\big]\mathbf{E}\big[S_2(\delta_n)^{-q+n}\big],\\ &= \frac{1}{(n-1)!}\int\limits_{\delta}^{\infty}(z-\delta)^{n-1}\mathbf{E}\big[e^{-zS_2}\big]\mathbf{E}\big[S_2(z)^{-q+n}\big]\,dz.\end{aligned} \tag{3.3}$$

Let $q=n-p$, then the expectation on the right-hand side of (3.3) can be computed[4] in terms of the Laplace transform of S_2 using the Cauchy-Saalschütz formula for the gamma function, see Section 12-21 of [21], which holds for $\Re(p)\in(k,\,k+1)$, $k=0,1,2,3\cdots$.

$$x^p = -\frac{1}{\Gamma(-p)}\int\limits_0^{\infty}\frac{du}{u^{1+p}}\Big(\sum_{l=0}^{k}\frac{(-ux)^l}{l!}-e^{-ux}\Big),\ x>0. \tag{3.4}$$

In our case, $\Re(p)\in(0,\,1)$. Hence, by Fubini's theorem and (2.6),

$$\begin{aligned}\mathbf{E}\big[e^{-zS_2}\big]\mathbf{E}\big[S_2(z)^p\big] &= \mathbf{E}\big[e^{-zS_2}S_2^p\big],\\ &= -\frac{1}{\Gamma(-p)}\int\limits_0^{\infty}\frac{du}{u^{1+p}}\Big[\mathbf{E}\big[e^{-zS_2}\big]-\mathbf{E}\big[e^{-(z+u)S_2}\big]\Big].\end{aligned} \tag{3.5}$$

The second element of the proof is the following expansion of the Laplace transform that is immediate from (1.2).

$$\mathbf{E}\big[e^{-zS_2}\big] = 8z\sum_{m=0}^{\infty}(m+1)e^{-\sqrt{8z}(m+1)},\ z>0. \tag{3.6}$$

Unlike the expansion in the moments, it is singular at $z=0$ but is globally convergent. Substituting this expansion into (3.5) and changing variables $u'=u/z$, we obtain

$$\mathbf{E}\big[e^{-zS_2}S_2^p\big] = -\frac{8z^{1-p}}{\Gamma(-p)}\int\limits_0^{\infty}\frac{du}{u^{1+p}}\Big[\sum_{m=0}^{\infty}(m{+}1)e^{-\sqrt{8z}(m+1)}-(1{+}u)\sum_{m=0}^{\infty}(m{+}1)e^{-\sqrt{8z}\sqrt{1+u}(m+1)}\Big]. \tag{3.7}$$

Substituting this equation into (3.3) with $\delta=0$ and applying Fubini's theorem, it is not difficult to evaluate the resulting z integral at any fixed $u>0$ using the definitions of the gamma and Riemann zeta functions as $\Re(n-p+1/2)>1/2$.

$$\mathbf{E}\big[S_2^{-n+p}\big] = -2^{3p-3n+1}\frac{\Gamma(2n-2p+2)}{\Gamma(n)\Gamma(-p)}\zeta(2n-2p+1)\int\limits_0^{\infty}\frac{du}{u^{1+p}}\Big[1-(1+u)^{p-n}\Big]. \tag{3.8}$$

The remaining integral can computed using the identity

$$\int\limits_0^{\infty}\frac{du}{u^{1+p}}\Big[1-(1+u)^{-q}\Big] = -\frac{\Gamma(p+q)\Gamma(-p)}{\Gamma(q)},\ \Re(p)\in(0,1),\ \Re(q)>0, \tag{3.9}$$

[4] We note that (2.5) cannot be used here as we need the Mellin transform for arbitrary $z>0$.

which easily follows from the standard properties of the gamma and beta functions and integration by parts. Thus, we have shown

$$\mathbf{E}\big[S_2^{-n+p}\big] = 2^{3p-3n+1}\frac{\Gamma(2n-2p+2)}{\Gamma(n-p)}\zeta(2n-2p+1). \tag{3.10}$$

On the other hand, the right-hand side of (3.1) can be computed by (1.12) and (1.13) as $\Re(n-p+1/2) > 1/2$. Using the doubling formula of the gamma function in the form

$$\Gamma(2n-2p) = \Gamma(n-p)\Gamma(n-p+1/2)(2\pi)^{-1/2}2^{2n-2p-1/2}, \tag{3.11}$$

we obtain after several lines of straightforward algebra

$$\Big(\frac{4}{\pi^2}\Big)^{p-n}\sqrt{\frac{\pi}{2}}\mathbf{E}\Big[S_2^{n-p+1/2}\Big] = 2^{3p-3n+1}\frac{\Gamma(2n-2p+2)}{\Gamma(n-p)}\zeta(2n-2p+1). \tag{3.12}$$

Recalling (1.12), we have checked that $\xi(-2q) = \xi(1+2q)$ for $\Re(q) \in (n, n-1)$. As $\xi(q)$ is entire, this must be true for all $q \in \mathbb{C}$. □

Corollary 3.2 (Some explicit formulas)**.** *Let* $n = 1, 2, 3, \cdots$.

$$\frac{2^{3n-1}}{\Gamma(2n+2)}\int_0^\infty u^{n-1}\Big[\frac{\sqrt{2u}}{\sinh\sqrt{2u}}\Big]^2 du = \zeta(2n+1), \tag{3.13}$$

$$\frac{2^{3n-5/2}}{\Gamma(2n+1)}\int_0^\infty u^{n-3/2}\Big[\frac{\sqrt{2u}}{\sinh\sqrt{2u}}\Big]^2 du = \zeta(2n). \tag{3.14}$$

Let $\Re(p) \in (0,\, 1/2)$*. Then,* (3.1) *in the critical strip is equivalent to*

$$\begin{aligned} 2\xi(2p) = 2\xi(1-2p) &= \Big(\frac{2}{\pi}\Big)^{-p}\frac{1}{\Gamma(-p)}\int_0^\infty \frac{du}{u^{1+p}}\left[\Big[\frac{\sqrt{2u}}{\sinh\sqrt{2u}}\Big]^2 - 1\right], \\ &= \Big(\frac{2}{\pi}\Big)^{p-1/2}\frac{1}{\Gamma(p-1/2)}\int_0^\infty \frac{du}{u^{3/2-p}}\left[\Big[\frac{\sqrt{2u}}{\sinh\sqrt{2u}}\Big]^2 - 1\right]. \end{aligned} \tag{3.15}$$

Let $\Re(p) \in (k, k+1)$, $k = 0, 1, 2, 3\cdots$ *(and* $\Re(p) > 1/2$ *in the case of* $k = 0$*). Then,* (3.1) *outside of the critical strip is equivalent to*

$$\begin{aligned} 2\xi(2p) = 2\xi(1-2p) &= \Big(\frac{2}{\pi}\Big)^{-p}\frac{1}{\Gamma(-p)}\int_0^\infty \frac{du}{u^{1+p}}\left[\Big[\frac{\sqrt{2u}}{\sinh\sqrt{2u}}\Big]^2 - \sum_{l=0}^{k}\frac{u^l}{l!}\frac{d^l}{d\delta^l}|_{\delta=0}\Big[\frac{\sqrt{2\delta}}{\sinh\sqrt{2\delta}}\Big]^2\right], \\ &= \Big(\frac{2}{\pi}\Big)^{p-1/2}\frac{1}{\Gamma(p-1/2)}\int_0^\infty \frac{du}{u^{3/2-p}}\Big[\frac{\sqrt{2u}}{\sinh\sqrt{2u}}\Big]^2. \end{aligned} \tag{3.16}$$

Proof. The formulas for the values of the Riemann zeta at the integers in (3.13) and (3.14) are special cases of (3.16) ((3.13) also follows by letting $p \to 0$ in (3.10) and then using (3.3)). If $\Re(p) \in (0,\, 1/2)$, then $\Re(1/2-p) \in (0,\, 1/2)$ so that both the left- and right-hand sides of (3.1) with $q = -p$ can be computed by means of (3.4). In the remaining cases, we use the standard definition of the gamma function to compute $\mathbf{E}\big[S_2^{1/2-p}\big]$ and (3.4) to compute $\mathbf{E}\big[S_2^p\big]$. □

We now proceed to our result on the δ deformation of the Riemann xi function.

Theorem 3.3 (Functional equation of $S_2(\delta)$)**.** *Let* $\delta \geq 0$.

$$\mathbf{E}\Big[\exp\Big(-\frac{4\delta}{\pi^2 S_2(\delta)}\Big) g\Big(\frac{4}{\pi^2 S_2(\delta)}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbf{E}\Big[S_2(\delta)^{1/2}\exp\Big(-\frac{4\delta}{\pi^2 S_2(\delta)}\Big) g\big(S_2(\delta)\big)\Big]. \tag{3.17}$$

The generalized xi function defined by

$$\Big(\frac{2}{\pi}\Big)^q 2\xi_\delta(2q) \triangleq \mathbf{E}\Big[\exp\big(-\delta S_2\big)\Big]\mathbf{E}\Big[\exp\Big(-\frac{4\delta}{\pi^2 S_2(\delta)}\Big) S_2(\delta)^q\Big],\ q \in \mathbb{C}, \tag{3.18}$$

is entire in q, $\xi_{\delta=0}(q) = \xi(q)$, *and*

$$\xi_\delta(q) = \xi_\delta(1-q). \tag{3.19}$$

Proof. This result is a corollary of (1.8) and (2.6). The function $x \to \exp\big(-\delta(x + 4/\pi^2 x)\big)$ is symmetric under $x \to 4/\pi^2 x$ so that we have by (1.8)

$$\mathbf{E}\Big[\exp\Big(-\delta S_2 - \frac{4\delta}{\pi^2 S_2}\Big) g\Big(\frac{4}{\pi^2 S_2}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbf{E}\Big[S_2^{1/2}\exp\Big(-\delta S_2 - \frac{4\delta}{\pi^2 S_2}\Big) g\big(S_2\big)\Big]. \tag{3.20}$$

By (2.6) this is equivalent to (3.17). (3.19) follows by letting $g(x) = x^q$. □

Remark 3.4. It is not difficult to see that the same approach gives us also a two-parameter deformation of the xi function by defining for $\kappa, \delta \geq 0$ the entire function

$$\Big(\frac{2}{\pi}\Big)^q 2\xi_{\delta,\kappa}(2q) \triangleq \mathbf{E}\Big[\exp\big(-\delta S_2\big)\Big]\mathbf{E}\Big[\exp\Big(-\frac{4\kappa}{\pi^2 S_2(\delta)}\Big) S_2(\delta)^q\Big], \tag{3.21}$$

Clearly, $\xi_{\delta,\delta}(q) = \xi_\delta(q)$ and, moreover,

$$\xi_{\delta,\kappa}(q) = \xi_{\kappa,\delta}(1-q), \tag{3.22}$$

which follows from the more general identity

$$\mathbf{E}\Big[\exp\Big(-\delta S_2 - \frac{4\kappa}{\pi^2 S_2}\Big) g\Big(\frac{4}{\pi^2 S_2}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbf{E}\Big[S_2^{1/2}\exp\Big(-\kappa S_2 - \frac{4\delta}{\pi^2 S_2}\Big) g\big(S_2\big)\Big]. \tag{3.23}$$

In particular, by letting $g(x) = 1$ and using (3.4), we obtain from (3.23)

$$\mathbf{E}\Big[e^{-\delta S_2 - 4\kappa/\pi^2 S_2}\Big] = \frac{1}{2\sqrt{2}}\int\limits_0^\infty \frac{dz}{z^{3/2}}\Big[\mathbf{E}\big[e^{-\kappa S_2 - 4\delta/\pi^2 S_2}\big] - \mathbf{E}\big[e^{-(\kappa+z)S_2 - 4\delta/\pi^2 S_2}\big]\Big]. \tag{3.24}$$

This shows that the functional equation of S_2 is equivalent to a functional equation for the joint Laplace transform of $(S_2,\, 4/\pi^2 S_2)$.

Corollary 3.5 (Functional equation of $T(\delta)$)**.** *Let* $\delta > 0$ *and define the entire function*

$$\chi_\delta(q) \triangleq \mathbf{E}\Big[\exp\big(-\delta S_2\big)\Big]\mathbf{E}\Big[\exp\Big(-\frac{4\delta}{\pi^2 T(\delta)}\Big) T(\delta)^q\Big],\ q \in \mathbb{C}. \tag{3.25}$$

Then, (3.19) *is equivalent to*

$$\Big(\frac{4}{\pi^2}\Big)^{-q}\Big(\chi_\delta(q) - \frac{q}{\delta}\chi_\delta(q-1) - \frac{4}{\pi^2}\chi_\delta(q-2)\Big) = \sqrt{\frac{\pi}{2}}\Big(\chi_\delta\big(\frac{1}{2}-q\big) - \frac{\frac{1}{2}-q}{\delta}\chi_\delta\big(-\frac{1}{2}-q\big) - \frac{4}{\pi^2}\chi_\delta\big(-\frac{3}{2}-q\big)\Big). \tag{3.26}$$

Proof. It is easy to see from (2.4) and (2.8) that the density of $T(\delta)$ satisfies

$$f_{S_2(\delta)}(x) = f_{T(\delta)}(x) + \frac{1}{\delta} f'_{T(\delta)}(x). \tag{3.27}$$

Hence,

$$\mathbf{E}\Big[\exp\big(-\delta S_2\big)\Big]\mathbf{E}\Big[\exp\Big(-\frac{4\delta}{\pi^2 S_2(\delta)}\Big) S_2(\delta)^q\Big] = \chi_\delta(q) - \frac{q}{\delta}\chi_\delta(q-1) - \frac{4}{\pi^2}\chi_\delta(q-2), \tag{3.28}$$

and the result is equivalent to (3.19). □

Remark 3.6. We note that it is also possible to re-formulate (3.17) in terms of $T(\delta)$ using

$$\mathbf{E}\Big[g\big(S_2(\delta)\big)\Big] = \mathbf{E}\Big[g\big(T(\delta)\big)\Big] - \frac{1}{\delta}\mathbf{E}\Big[g'\big(T(\delta)\big)\Big]. \tag{3.29}$$

The interest in (3.26) is that it gives us an equivalent formulation of the functional equation directly in terms of the $T(\delta)$ distribution. We showed in [16] that $T(\delta)$ can be obtained as a limit of Barnes beta distributions. This leads to the interesting problem of deriving the functional equation by the Barnes beta distribution route, which, however, is beyond the scope of this paper.

Finally, we will consider the roots of the Mellin transform of S_2, *i.e.* of $\xi(2q)$, see (1.12). Before we can state our result, we need an auxiliary lemma.

Lemma 3.7. *Let $\delta \in \mathbb{C}$, $|\delta| < \pi^2/2$, and $q \in \mathbb{C}$. Define the functions*

$$M_1(\delta,\, q) \triangleq \mathbf{E}\Big[e^{-\delta S_2} S_2^q\Big], \tag{3.30}$$

$$M_2(\delta,\, q) \triangleq \mathbf{E}\Big[e^{-4\delta/\pi^2 S_2} S_2^q\Big]. \tag{3.31}$$

They satisfy the identities

$$M_1(\delta,\, q) = \Big(\frac{2}{\pi}\Big)^q \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{-2\delta}{\pi}\Big)^n 2\xi(2q+2n), \tag{3.32}$$

$$M_2(\delta,\, q) = \Big(\frac{2}{\pi}\Big)^q \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{-2\delta}{\pi}\Big)^n 2\xi(1+2n-2q). \tag{3.33}$$

$M_1(\delta,\, q)$ and $M_2(\delta,\, q)$ are holomorphic in δ over the domain $|\delta| < \pi^2/2$ for any fixed q and are entire functions of q of order 1 with infinitely many zeroes for any fixed $|\delta| < \pi^2/2$. $M_1(\delta,\, q)$ and $M_2(\delta,\, q)$ are related to each other by

$$\Big(\frac{4}{\pi^2}\Big)^q M_2(\delta, -q) = \sqrt{\frac{\pi}{2}} M_1(\delta, q+1/2). \tag{3.34}$$

Proof. The first equation is a slight extension of (2.5). Both (3.32) and (3.33) are verified in the same way as (2.5) by expanding the functional in the moments of S_2 (and using the functional equation in the case of (3.33)). The tail behavior of the series at any fixed q is easily estimated by Stirling's formula and the fact that $\zeta(q) \to 1$ (uniformly in $\Im(q)$) as $\Re(q) \to +\infty$. The stated restriction on the domain of δ is immediate from the asymptotics of $f_{S_2}(x)$ given in (1.10) and (1.11). Hence, $M_1(\delta,\, q)$ and $M_2(\delta,\, q)$ are holomorphic in δ and entire in q. The identity in (3.34) follows from (1.8). To prove that $M_1(\delta,\, q)$ is an entire function of order 1 in q and has infinitely many roots, we use the theory of entire functions of finite order and classical estimate of the growth of $\xi(q)$ at infinity. It is not difficult to show that $M_1(\delta,\, q)$ has the same asymptotic bound as $\xi(2q)$,

$$\log|M_1(\delta,\, q)| = \mathcal{O}\big(|q|\log|q|\big),\ |q| \to \infty, \tag{3.35}$$

see (2.12.3) in [20], so that $M_1(\delta, q)$ is of at most order 1. It is exactly of order 1 due to its behavior along the positive real axis

$$M_1(\delta, q) \sim e^{q \log q}, \; q \to +\infty, \tag{3.36}$$

which follows by Stirling's formula, and, therefore, has infinitely many roots by the Hadamard product formula, see Theorem 3.5 in Section XIII.3 of [11]. □

We will now study the roots of $M_i(\delta, q)$ $i = 1, 2$ as a deformation of those of the xi function. Specifically, given a δ, $M_i(\delta, q)$ has infinitely many roots as a function of q by Lemma 3.7. We are interested in how these roots depend on δ. For simplicity, we will restrict ourselves to $\delta \in (-\pi^2/2, \pi^2/2)$. Let q_0 be a root of the Mellin transform of S_2 so that $\xi(2q_0) = 0$, necessarily $0 < \Re(2q_0) < 1$, and $\xi(1 - 2q_0) = 0$ by the functional equation. Define $q_1(\delta \,|\, q_0)$ and $q_2(\delta \,|\, q_0)$ to be functions of δ having value q_0 at $\delta = 0$ that are defined implicitly as curves of roots of $M_1(\delta, q)$ and $M_2(\delta, q)$.

Definition 3.8. *Let* $\xi(2q_0) = 0$, $\delta \in (-\pi^2/2, \pi^2/2)$, *and* $q_1(0 \,|\, q_0) = q_2(0 \,|\, q_0) = q_0$.

$$M_1\big(\delta, q_1(\delta \,|\, q_0)\big) = 0, \tag{3.37}$$

$$M_2\big(\delta, q_2(\delta \,|\, q_0)\big) = 0. \tag{3.38}$$

Clearly,

$$q_2(\delta \,|\, q_0) = 1/2 - q_1(\delta \,|\, 1/2 - q_0) \tag{3.39}$$

by (3.34), and (3.37) is equivalent to $\mathbf{E}\big[S_2(\delta)^{q_1(\delta \,|\, q_0)}\big] = 0$ for $\delta \geq 0$ by (2.6). If q_0 is a simple root of $\xi(2q)$, then $q_1(\delta \,|\, q_0)$ and $q_2(\delta \,|\, q_0)$ are differentiable at $\delta = 0$ by the implicit function theorem. The following result establishes the converse.

Theorem 3.9 (Criterion for simplicity of roots of $\xi(q)$)**.** *If the function* $q_i(\delta \,|\, q_0)$, $i = 1, 2$ *is differentiable at* $\delta = 0$, *then* q_0 *is a simple root of* $\xi(2q)$ *and*

$$\xi'(2q_0) \frac{dq_1}{d\delta}(\delta \,|\, q_0)|_{\delta=0} = \frac{1}{\pi} \xi(2q_0 + 2), \tag{3.40}$$

$$\xi'(2q_0) \frac{dq_2}{d\delta}(\delta \,|\, q_0)|_{\delta=0} = \frac{1}{\pi} \xi(2q_0 - 2). \tag{3.41}$$

Proof. We will give proof for $q_1(\delta \,|\, q_0)$, the proof for $q_2(\delta \,|\, q_0)$ goes through verbatim. Assume that $q_1(\delta \,|\, q_0)$ is differentiable at $\delta = 0$. Consider the composite function $\delta \to \mathbf{E}\big[e^{-\delta S_2} S_2^{q_1(\delta \,|\, q_0)}\big]$

$$\mathbf{E}\big[e^{-\delta S_2} S_2^{q_1(\delta \,|\, q_0)}\big] = 0, \tag{3.42}$$

which is identically zero by construction, hence

$$\frac{d}{d\delta}|_{\delta=0} \mathbf{E}\big[e^{-\delta S_2} S_2^{q_1(\delta \,|\, q_0)}\big] = 0. \tag{3.43}$$

Since $q_1(\delta \,|\, q_0)$ is assumed to be differentiable, by the chain rule we have

$$\frac{d}{d\delta}|_{\delta=0} \mathbf{E}\big[e^{-\delta S_2} S_2^{q_1(\delta \,|\, q_0)}\big] = \frac{\partial}{\partial\delta}|_{\delta=0} \mathbf{E}\big[e^{-\delta S_2} S_2^{q_1(\delta \,|\, q_0)}\big] + \frac{\partial}{\partial q} \mathbf{E}\big[S_2^q\big]|_{q=q_0} \frac{dq_1}{d\delta}(\delta \,|\, q_0)|_{\delta=0}. \tag{3.44}$$

The calculation of the partial derivatives is elementary. Using that $\xi(2q_0) = 0$ by construction, we have by (1.12) and (3.2), respectively,

$$\frac{\partial}{\partial q} \mathbf{E}\big[S_2^q\big]|_{q=q_0} = \big(\frac{2}{\pi}\big)^{q_0} 4\xi'(2q_0), \tag{3.45}$$

$$\frac{\partial}{\partial\delta}|_{\delta=0} \mathbf{E}\big[e^{-\delta S_2} S_2^{q_1(\delta \,|\, q_0)}\big] = -\mathbf{E}\big[S_2^{q_0+1}\big] = \big(\frac{2}{\pi}\big)^{q_0+1} 2\xi(2q_0 + 2). \tag{3.46}$$

Thus, we have proved (3.40). It remains to notice that $\xi(2q_0+2) \neq 0$ as $\Re(2q_0+2) > 2$ so that

$$\xi'(2q_0)\frac{dq_1}{d\delta}(\delta\,|\,q_0)|_{\delta=0} \neq 0. \tag{3.47}$$

□

Corollary 3.10 (Criterion for simple roots of $\xi(q)$ to satisfy the Riemann hypothesis)**.** *Assume q_0 to be a simple root of $\xi(2q)$. Then, $\Re(2q_0) = 1/2$ iff*

$$\xi'(2q_0)\frac{dq_1}{d\delta}(\delta\,|\,q_0)|_{\delta=0} = \xi'(2\overline{q_0})\frac{d\overline{q_2}}{d\delta}(\delta\,|\,q_0)|_{\delta=0}, \tag{3.48}$$

$$= \xi'(1-2\overline{q_0})\frac{dq_1}{d\delta}(\delta\,|\,1/2-\overline{q_0})|_{\delta=0}. \tag{3.49}$$

The proof requires the following auxiliary result.

Lemma 3.11. *Let $p > 1$ and $0 < Re(s) < 1$. Then,*

$$\xi(s+p) = \xi(\overline{s}-p) \Leftrightarrow \Re(s) = 1/2. \tag{3.50}$$

Proof. If $\xi(s+p) = \xi(\overline{s}-p)$, then

$$\xi(s+p) = \xi(1+p-\overline{s}) \tag{3.51}$$

by the functional equation. Obviously,

$$\Im(s+p) = \Im(1+p-\overline{s}) \tag{3.52}$$

and

$$\Re(s+p),\, \Re(1+p-\overline{s}) > 1 \tag{3.53}$$

by construction. By Theorem 1 of [18], the modulus of $\xi(q)$ is strictly increasing along any horizontal half-line that is located to the right of the critical strip. Hence,

$$s+p = 1+p-\overline{s} \tag{3.54}$$

so that $\Re(s) = 1/2$. Conversely, if $\Re(s) = 1/2$, the result is immediate. □

We can now complete the proof of Corollary 3.10.

Proof. By Theorem 3.9, (3.48) is equivalent to

$$\xi(2q_0+2) = \xi(2\overline{q_0}-2), \tag{3.55}$$

which is equivalent to $\Re(2q_0) = 1/2$ by Lemma 3.11. To verify (3.49), it is sufficient to note the identities

$$\xi'(2q_0) = -\xi'(1-2q_0), \tag{3.56}$$

$$\overline{q_1}(\delta\,|\,q_0) = q_1(\delta\,|\,\overline{q_0}), \tag{3.57}$$

and recall (3.39). □

Remark 3.12. We believe that the differentiability condition in Theorem 3.9 is quite natural as $M_1(\delta,\, q)$ and $M_2(\delta,\, q)$ are "smooth" deformations of the Mellin transform of S_2, which suggests that their roots should also generate a "smooth" deformation of the roots of the Mellin transform. In this sense, Theorem 3.9 "explains" why the roots of the xi function might be expected to be simple.

4 Conclusions

We have given a derivation of the functional equation of the Riemann xi function that is based on the theory of $S_2(\delta)$ probability distributions. Using this theory, we have reduced the functional equation to a simple integral relation involving the Laplace transform of S_2 and then verified it using elementary means. Our approach has shown that the Laplace transform of S_2 is fundamental to the structure of the xi function, for in addition to the functional equation itself, we have given a probabilistic derivation of explicit formulas for the values of the xi function in the complex plane and of the Riemann zeta at the integers in terms of simple integrals involving the Laplace transform of S_2.

We have shown that a particular transform of the $S_2(\delta)$ distribution gives rise to a one-parameter family $\xi_\delta(q)$ of entire functions, which extend the Riemann xi function and satisfy its functional equation. In particular, this construction opens up a possibility of approaching the functional equation from the viewpoint of the theory of Barnes beta distributions.

We have introduced a class of transforms of the S_2 distribution that naturally extend the Mellin transform of $S_2(\delta)$ to holomorphic functions $M_1(\delta, q)$ and $M_2(\delta, q)$ of two variables. We have noted that the differentiability of their roots as functions of q with respect to δ is equivalent to the simplicity of roots of the xi function and, assuming simplicity, we have formulated a criterion for the validity of the Riemann hypothesis.

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