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Large deviation bounds for the volume of the largest cluster in 2D critical percolation

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Abstract

Let M_n denote the number of sites in the largest cluster in critical site percolation on the triangular lattice inside a box of side length n. We give lower and upper bounds on the probability that $M_n/\mathbb{E}M_n > x$ of the form $\exp(-Cx^{2/\alpha_1})$ for $x \geq 1$ and large n with $\alpha_1 = 5/48$ and $C > 0$. Our results extend to other two dimensional lattices and strengthen the previously known exponential upper bound derived by Borgs, Chayes, Kesten and Spencer [3]. Furthermore, under some general assumptions similar to those in [3], we derive a similar upper bound in dimensions $d > 2$.

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1 Introduction and statement of the main results

For a general introduction to the percolation model we refer to [10], [7], and [2]. Consider the critical bond percolation model on the lattice $\mathbb{Z}^d$ for $d \geq 2$. For $n \in \mathbb{N}$ let

$$\Lambda_n := \{-n, -n+1, \ldots, n\}^d$$

denote the hypercube (ball) centred at the origin with radius n. For $v \in V(\mathbb{T})$ we write $\Lambda_n(v) := v + \Lambda_n$. Further let ∂A denote the (outer) boundary of $A \subseteq \mathbb{Z}^d$, that is

$$\partial A := \left\{ v \in \mathbb{Z}^d \setminus A : \exists u \in A \text{ such that } u \sim v \right\}.$$

We say that two sites v, w are connected by an open path and denote it by $v \leftrightarrow w$ if there is a sequence of open edges which starts at v, ends at w, and the consecutive edges share a vertex. Let $v \overset{S}{\leftrightarrow} w$ denote the event where there is an open path connecting v to w which only uses vertices in $S \subseteq \mathbb{Z}^d$. For $A, B \subseteq \mathbb{Z}^d$, $A \overset{S}{\leftrightarrow} B$ denotes the event where there are vertices $v \in A, w \in B$ such that $v \overset{S}{\leftrightarrow} w$. When S is omitted, it is assumed to be equal to $\mathbb{Z}^d$.

The open cluster of the vertex v in Λ_n is denoted by

$$\mathcal{C}_n(v) := \left\{ w \in \Lambda_n \,|\, w \overset{\Lambda_n}{\longleftrightarrow} v \right\}.$$

Herein the size of a cluster is measured by its number of vertices. Further, let $\mathcal{C}_n^{(i)}$ denote the ith largest cluster in Λ_n. If there are clusters with the same size, we order

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them in some arbitrary but deterministic way. For $m \le n$ we write $\pi(m,n)$ for the probability $\mathbb{P}_{p_c}(\partial\Lambda_m \leftrightarrow \partial\Lambda_n)$. We set $\pi(n) := \pi(1,n)$. We will work under the following assumptions.

Assumption 1.1 (Quasi-multiplicativity)**.** *There exists a constant C_1 such that for all $0 \le k \le l \le m$ we have*

$$\pi(k,l)\pi(l,m) \le C_1\pi(k,m). \tag{1.1}$$

Assumption 1.2. *There exist constants $C_2 > 0$ and $\alpha < d$ such that for all $n \ge m \ge 1$*

$$\frac{\pi(n)}{\pi(m)} \ge C_2\left(\frac{n}{m}\right)^{-\alpha}. \tag{1.2}$$

Assumption 1.1 and 1.2 hold for $d = 2$, as proved in [7] and [14]. Furthermore, Assumption 1.2 holds in high ($d \ge 19$) dimensions, however, we do not expect Assumption 1.1 to hold in this case. See Remark ix) below for more details on this case. To our knowledge, it is an open question whether any of Assumption 1.1 or 1.2 is satisfied in dimensions $3 \le d \le 18$.

In [3] the following bound was given:

Theorem 1.3 (Proposition 6.3 of [3])**.** *Suppose that Assumption 1.2 holds. Then there exist positive constants c_1, c_2 such that for all $x, n \ge 0$,*

$$\mathbb{P}_{p_c}\left(|\mathcal{C}_n^{(1)}| \ge xn^d\pi(n)\right) \le c_1\exp(-c_2x). \tag{1.3}$$

We strengthen this result when both of Assumption 1.1 and 1.2 are satisfied:

Theorem 1.4. *Let $d \ge 2$, and suppose that Assumptions 1.1 and 1.2 hold. There exist positive constants c_1, c_2 depending only on d and the constants appearing in the assumptions, such that for all $n, u > 1$,*

$$\mathbb{P}_{p_c}\left(|\mathcal{C}_n^{(1)}| \ge n^d\pi(n/u)\right) \le c_1\exp(-c_2u^d). \tag{1.4}$$

Furthermore, for $d = 2$ there are constants $c_3, c_4 > 0$ such that the lower bound

$$\mathbb{P}_{p_c}\left(|\mathcal{C}_n^{(1)}| \ge n^d\pi(n/u)\right) \ge c_3\exp(-c_4u^d) \tag{1.5}$$

holds for all $1 \le u \le n$.

The lower bound in Theorem 1.4 follows from standard RSW methods, nevertheless, for completeness we include its proof in Section 3.2. The upper bound above relies on Theorem 1.5 below, which is our main contribution. Let

$$\mathcal{V}_n := \{v \in \Lambda_n \mid v \leftrightarrow \partial\Lambda_{2n}\} \tag{1.6}$$

denote the set of vertices in Λ_n which are connected to $\partial\Lambda_{2n}$.

Theorem 1.5. *Let $d \ge 2$, and suppose that Assumptions 1.1 and 1.2 hold. There is a constant c_1 such that for all $n, u > 0$ and $k \in \mathbb{N}$*

$$\mathbb{E}_{p_c}\binom{|\mathcal{V}_n|}{k} \le (c_1n^d\pi(n/\sqrt[d]{k})/k)^k. \tag{1.7}$$

Consequently, for some positive constants c_2, c_3, we have

$$\mathbb{P}_{p_c}\left(|\mathcal{V}_n| \ge n^d\pi\left(n/u\right)\right) \le c_2\exp(-c_3u^d). \tag{1.8}$$

The constants c_1, c_2, c_3 above only depend on d and the constants appearing in Assumptions 1.1 and 1.2.

A weaker version of Theorem 1.5 is proved in [3] as Lemma 6.1. Theorem 1.4 follows from Theorem 1.5 by arguments analogous to those in [3] which lead from [3, Lemma 6.1] to Theorem 1.3. Thus we only prove Theorem 1.5 and the lower bound in Theorem 1.4 here.

Remarks. i) We believe that a lower bound matching (1.7) with a constant smaller than c_1 holds. Such lower bound would immediately imply (1.8). Nevertheless, we chose to prove (1.8) directly, since the construction is rather simple, but it gives an example when the rare event $|\mathcal{C}_n^{(1)}| \geq n^d \pi(n/u)$ happens.

ii) Our motivation for studying the size of large critical clusters comes from the forest-fire processes described as follows. Let λ be some small positive number. At time 0 all the vertices of $\mathbb{Z}^d$ are empty. As time goes on, empty vertices get occupied by a tree at rate 1, independently from each other. Vertices with trees get struck by lightning at rate λ independently from each other. When a tree gets struck by lightning, its forest (its connected component in $\mathbb{Z}^d$ of vertices with trees) is ignited, that is, all of the trees are removed in this forest. Then trees occupy empty vertices with rate 1, and lightnings strike and so on. We are particularly interested in the case where $\lambda > 0$ is small.

As we can see, a forest burns down at rate proportional to its size, thus a precise control of the size of critical clusters can be useful for the study of the processes above.

iii) [3, Proposition 6.3] also treats the case where the percolation parameter p is different from p_c. Our results extend to this case in an analogous way as in [3]. Furthermore, Assumptions 1.1 and 1.2, our results, as well as those in [3], in the case $d = 2$ hold for site/bond percolation on other lattices: As long as the lattice is invariant under a translation, a rotation around the origin with some angle and a reflection on one of the coordinate axes, the results above follow. Furthermore, these results remain valid for some inhomogeneous percolation models. See [7] for more details.

iv) The proof of Theorem 1.5 relies on the method presented in [11]. However, the computation there only considers the case $d = 2$. As we will see below, the arguments in [11] extend to the case $d \geq 3$ in a straightforward way.

v) Recall a ratio limit theorem, Proposition 4.9 of [6] for the one arm events. Combining it with Theorem 1.4 we get, for site percolation on the triangular lattice,

$$\begin{aligned} \mathbb{P}_{p_c}\left(\left|\mathcal{C}_n^{(1)}\right| \geq xn^2\pi(n)\right) &\leq c_1 \exp(-c_2 x^{96/5}), \\ &\geq c_3 \exp(-c_4 x^{96/5}) \end{aligned}$$

with some universal constants c_i for all $x > 0$ and $n \geq n_0(x)$.

vi) The upper bound in Theorem 1.4 trivially extends to $|\mathcal{C}_n^{(l)}|$ the volume of the lth largest cluster. Furthermore, in dimension 2 the same lower bound with different constants also holds. Its derivation is analogous to that for the largest cluster, hence we omit it.

vii) Theorem 1.5 gives upper bounds on the moments and on the tail probability of $\mathcal{V}_n/n^2\pi(n)$, where, roughly speaking, $\mathcal{V}_n$ counts the points in Λ_n with one long open arm. Similar upper bounds can be achieved for the number of points with multiple disjoint arms.

Let $l \in \mathbb{N}$ and $\sigma \in \{0,1\}^l$. Let $\pi_\sigma(m,n)$ denote the probability that $\partial\Lambda_m$ and $\partial\Lambda_n$ are connected by l disjoint arms, where in a counter-clockwise order of these arms the

ith arm is open when $\sigma_i = 1$ and dual closed otherwise. Suppose that Assumption 1.1 and 1.2 are satisfied when π is replaced by π_σ with some constants C_1, C_2 and for some $\alpha_\sigma > 0$ not necessarily smaller than d. We have two cases: when $\alpha_\sigma < d$, Lemma 3.1 applies with π replaced by π_σ, and we get results analogous to Theorem 1.5. However, when $\alpha_\sigma > d$, Lemma 3.1 fails and we see that $\sum_{k=1}^{\infty} k^{d-1}\pi_\sigma(k) < \infty$. By slightly modifying the computations in the proof of Theorem 1.5 in the case $\alpha_\sigma > 2$ we get

$$\mathbb{E}_{p_c}\binom{|\mathcal{V}_n^\sigma|}{k} \le c_1^k n^d \pi_\sigma(n)$$

for some constant c_1 where $\mathcal{V}_n^\sigma$ denotes the multi-arm analogue of $\mathcal{V}_n$.

We believe that a lower bound matching (1.8) holds in two dimensions when σ switches colours at most four times and $\alpha_\sigma < 2$. However, in this case the construction in the lower bound is more delicate and rather technical hence we omit it.

viii) In the case of the critical site percolation triangular lattice, Morrow and Zhang [13] gave upper and lower bounds for the moments for quantities similar to $|\mathcal{V}_\eta^\sigma|$. More precisely, they considered L_n, the set of vertices in the lowest crossing of Λ_n, the pioneering and pivotal vertices of L_n, denoted by F_n and Q_n, respectively. From each site in L_n, F_n, and Q_n arms with colour sequence $\sigma(L) = (1,0,1)$, $\sigma(F) = (1,0)$ and $\sigma(Q) = (1,0,1,0)$ start and extend till $\partial\Lambda_n$, respectively. For a precise definition see [13]. It was showed that $\mathbb{E}_{p_c}(|X_n|^k) = n^{(2-\alpha_X)k+o(1)}$ for $X = L, F, Q$. Here $\alpha_L, \alpha_F, \alpha_Q$ coincide with the multi-chromatic 3, 2 and 4 arm exponents for critical site percolation on the triangular lattice [17], and the results in [13] are similar to the muli-arm analogues of Theorem 1.5 noted in the previous remark.

In view of Remark i), v) and vii), we believe that the arguments herein can be applied to improve the results of [13] to $\mathbb{E}_{p_c}(|X_n|^k) = (O(1)n^2\pi_{\sigma(X)}(n/\sqrt{k}))^k$ for $X = L, F, Q$.

ix) Let us turn to the case $d \ge 19$. Kozma and Nachmias [12, Theorem 1] proved that $\pi(n) = O(n^{-2})$ building on the results in [8]. This combined with [1, Theorem 5] gives that $|\mathcal{C}_n^{(1)}|$ is of order $n^{4+o(1)}$. Hence the bounds in Theorem 1.3 and 1.4 are much weaker than those in [1, Theorem 5]. Nevertheless, we get some new conditional results which are interesting in dimensions below 19.

x) We note some results on the distribution of $|\mathcal{C}_n^{(l)}|$ for $l \ge 1$. We already mentioned the results of [3] which are the most relevant for our purposes. The same authors in [4] describe the connection between the volume and the diameter of the largest critical and near-critical clusters. Járai [9] showed, among other things, that the microscopic scale behaviour of the largest critical clusters can be described by that of the incipient infinite cluster. Finally, van den Berg and Conijn [18] proved that the probability of $|\mathcal{C}_n^{(1)}|/n^2\pi(n) \in (a,b)$ is positive for all $0 < a < b$ for sufficiently large n. While in [19] they showed, roughly speaking, that the distribution of $|\mathcal{C}_n^{(1)}|/n^2\pi(n)$ has no atoms for large n and that $|\mathcal{C}_n^{(l)}| - |\mathcal{C}_n^{(l+1)}| = O(n^2\pi(n))$ for $l \ge 1$.

Organization of the paper

In Section 2 we provide some more notation. We sketch the arguments of [11] which are essential for the proofs of our results in Section 2.1. Building on these results, we prove Theorem 1.5 in Section 3.1. We conclude in Section 3.2 where we deduce the lower bound in Theorem 1.4.

2 Notation and preliminaries

The space of configurations is $\Omega := \{0,1\}^{E(\mathbb{Z}^d)}$. For $\omega \in \Omega$ let $\omega(e) \in \{0,1\}$ denote its value at $e \in E(\mathbb{Z}^d)$. We say that $e \in E(\mathbb{Z}^d)$ is open, if $\omega(e) = 1$, otherwise e is closed. For $p \in [0,1]$ let $\mathbb{P}_p$ denote the product measure on Ω where $\mathbb{P}_p(\omega(e) = 1) = p$. Let $p_c = p_c(d)$ denote the critical percolation parameter. That is, $p_c = \sup\{p \,|\, \mathbb{P}_p(0 \leftrightarrow \infty) = 0\}$.

2.1 The counting argument of [11]

The proof of Theorem 1.5 is based on a counting argument found in [11]. This argument strengthens the proof of [3, Lemma 6.1] and it counts certain passage points, which, roughly speaking, are the starting points of six disjoint open and closed arms. Herein we give a sketch of the argument in the one arm case.

Let $k \in \mathbb{N}$ and

$$X = \{x_1, x_2, \dots, x_k\} \subseteq \Lambda_n.$$

We give a bound on the probability of the event $\{\mathcal{V}_n \supseteq X\}$, but first some definitions.

Let T_0 denote the empty graph on the vertex set X. Let us start blowing a ball at each point of X at unit speed. That is, at time $t \geq 0$, we have the balls $\Lambda_t(x)$, $x \in X$.

For small values of t these balls are pairwise disjoint. As t increases, more and more of these balls intersect each other. Let r_1, denote the smallest t when the first pair of balls touch. We pick one such pair balls in some deterministic way, with centres $u_1, v_1 \in X$. We draw an edge e_1 between u_1 and v_1 and label it with $l(e_1) := r_1$, and get the graph T_1. Note that $||u_1 - v_1||_\infty = 2r_1$. Then we continue with the growth process, and stop at time r_2 if we find a pair of vertices $u_2, v_2 \in X$ such that u_2, v_2 are in different connected components of T_1 and $\Lambda_{r_2}(u_2)$ and $\Lambda_{r_2}(v_2)$ touch. Then we draw an edge e_2 between one such deterministically chosen pair with the label $l(e_2) := r_2$ and get T_2. Note that it can happen that $r_1 = r_2$. We continue with this procedure till we arrive to the tree T_{k-1}. Let $\mathcal{R}(X)$ denote the multiset containing r_i for $i = 1, 2, \dots, k-1$.

As we saw above, $r_1 = \frac{1}{2}\min_{u,v \in X, u \neq v} ||u - v||$. Furthermore, it is easy to see that for $i = 1, 2, \dots, k-1$ there are at least $k + 1 - i$ vertices of X such that any pair of them is at least $2r_i$ distance from other. This combined with the pigeon-hole principle provides the following observation:

Observation 2.1. *For all $i \in [0, \sqrt[d]{k-1}] \cap \mathbb{Z}$ we have $r_{k-i^d} < \frac{n}{i}$. Equivalently, $\overline{r}_j := r_{k-j} \leq n/\lfloor \sqrt[d]{j} \rfloor$ for $j = 1, \dots, k-1$.*

We say that B is a blob, if B is a non-empty connected component of T_i for some i. In the growth process above blobs merge with other blobs and form bigger ones over time. Let

$$b(B) := \min\{r_i \,:\, B \text{ is a connected component of } T_i\},$$
$$d(B) := \max\{r_i \,:\, B \text{ is a connected component of } T_i\}$$

denote the birth time, and the death time of a blob B. It is easy to see that the sets

$$G(B) := \begin{cases} \bigcup_{x \in B} \Lambda_{d(B)}(x) \setminus \bigcup_{x \in B} \Lambda_{b(B)}(x) & B \neq X, \\ \Lambda_{2n} \setminus \bigcup_{x \in B} \Lambda_{d(B)}(x) & B = X \end{cases}$$

are pairwise disjoint. See Figure 1. Let

$$ib(B) := \partial \left(\bigcup_{x \in B} \Lambda_{b(B)}(x) \right), \qquad ob(B) := \begin{cases} \partial \left(\bigcup_{x \in B} \Lambda_{d(B)}(x) \right) & B \neq X, \\ \partial \Lambda_{2n} & B = X \end{cases}$$

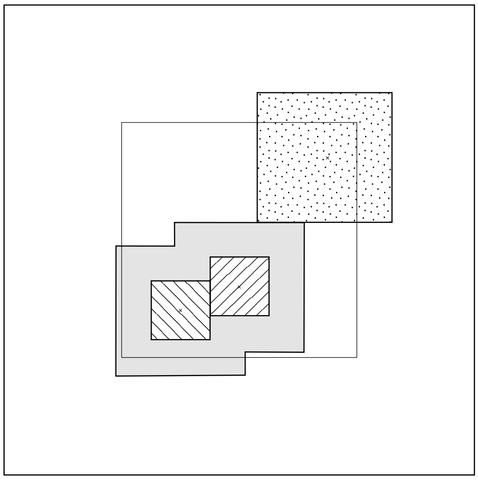

Figure 1: The areas with different patterns correspond the sets $G(B)$.

denote the boundary of the inner and outer faces of the sets $G(B)$, respectively. Now we are ready to make a bound on the probability $\mathbb{P}(\mathcal{V}_n \supseteq X)$. Recall the definition of $\mathcal{V}_n$ from (1.6). For all $x \in V(B)$ we have

$$\{\mathcal{V}_n \supseteq X\} \subseteq \{x \leftrightarrow \partial\Lambda_{2n}\} \subseteq \{ib(B) \leftrightarrow ob(B)\}.$$

The events $\{ib(B) \leftrightarrow ob(B)\}$ are independent since they depend only on the state of the edges in $G(B)$, which are pairwise disjoint subsets of Λ_{2n}. Hence

$$\begin{aligned} \mathbb{P}_{p_c}(\mathcal{V}_n \supseteq X) &\leq \mathbb{P}_{p_c}\left(\bigcap_{B \text{blob}} \{ib(B) \leftrightarrow ob(B)\}\right) \\ &\leq \prod_{B \text{ blob}} \mathbb{P}_{p_c}(ib(B) \leftrightarrow ob(B)). \end{aligned}$$

Then, as in the proof of [11, Proposition 14], an induction on the blobs leads to the following bound.

Proposition 2.2. *Suppose that Assumption (I) and (II) holds. Then there is a constant $C_3 = C_3(c_1, c_2, \alpha, d)$ such that*

$$\mathbb{P}_{p_c}(\mathcal{V}_n \supseteq X) \leq C_3\pi(n) \prod_{r \in \mathcal{R}(X)} C_3\pi(r)$$

for all $X \subseteq \Lambda_n$

Proposition 2.2 provides an upper bound on $\mathbb{P}_{p_c}(\mathcal{V}_n \supseteq X)$ as a function of $\mathcal{R}(X)$. To give a bound on $\mathbb{E}_{p_c}\binom{|\mathcal{V}_n|}{k}$, we bound the number of sets X such that $\mathcal{R}(X) = R$ for fixed R. By arguments analogous to the proof of [11, Proposition 15] we get the following.

Proposition 2.3. *There is a universal constant C_4 such that for all multisets R with $k-1$ elements we have*

$$\#\{X \subseteq \Lambda_n : |X| = k, \mathcal{R}(X) = R\} \leq C_4\mathcal{O}(R)n^d \prod_{r \in R} dC_4 r^{d-1}, \qquad (2.1)$$

where $\mathcal{O}(R)$ denotes the number of different ways the elements of R can be ordered.

3 Proof of Theorem 1.4 and 1.5

We start with the following consequence of Assumption (II).

Lemma 3.1 (Lemma 4.4 of [3]). *If Assumption (II) holds, then there is a constant* $C_5 = C_5(C_2, \alpha, d)$ *such that for all* $n \geq 0$ *we have*

$$\sum_{k=1}^{n} k^{d-1}\pi(k) \leq C_5 n^d \pi(n). \tag{3.1}$$

3.1 Proof of Theorem 1.5

Combining Proposition 2.2 and 2.3 with $C_6 = dC_3C_4$ we get:

$$\begin{aligned}
\mathbb{E}\binom{|\mathcal{V}_n|}{k} &= \sum_{X \subseteq \Lambda_n} \mathbb{P}_{p_c}\left(\mathcal{V}_n \supseteq X\right) \\
&\leq d \sum_{R} C_3C_4\mathcal{O}(R)n^d\pi(n) \prod_{r\in R} dC_3C_4 r^{d-1}\pi(r) &(3.2)\\
&= C_6^k n^d \pi(n) \sum_{\tilde{R}} \prod_{\tilde{r}\in\tilde{R}} \tilde{r}^{d-1}\pi(\tilde{r}) = C_6^k n^d\pi(n)\left(\sum_{r=1}^{n} r^{d-1}\pi(r)\right)^{k-1} &(3.3)
\end{aligned}$$

where the first summation in (3.2) runs over the $k-1$ element mulitsets of $\{1,2,\ldots,n\}$, while in (3.3) $\tilde{R}$ runs through the $k-1$ long sequences in $\{1,2,\ldots,n\}$. Note that by Observation 2.1, many terms in (3.3) are redundant. We exploit this in the following.

Let $\bar{r}_i$ denote the ith largest element of $\tilde{R}$. Observation 2.1 provides an upper bound on $\mathbb{E}\binom{|\mathcal{V}_n|}{k}$ where in the sum in (3.3) we restrict to the terms such that $\bar{r}_i \leq n/2^l$ for all i with $2^{dl} \leq i < 2^{d(l+1)}$. We indicate this restriction by an additional tilde above the sum. Let $j := \lfloor \log_{2^d}(k) \rfloor$ and $m = k-1-2^{dj}$. We arrive to the following bound:

$$\begin{aligned}
\mathbb{E}\binom{|\mathcal{V}_n|}{k} &\leq C_6^k n^d\pi(n) \widetilde{\sum_{\tilde{R}}} \prod_{\tilde{r}\in\tilde{R}} \tilde{r}^{d-1}\pi(\tilde{r}) \\
&\leq C_6^k n^d \pi(n) \binom{k-1}{2^d-1, (2^d-1)2^d, \ldots, (2^d-1)2^{d(j-1)}, m} &(3.4)\\
&\quad \prod_{i=1}^{j-1}\left(\sum_{r=1}^{n/2^i} r^{d-1}\pi(r)\right)^{(2^d-1)2^{di}} \left(\sum_{r=1}^{n/2^{j-1}} r^{d-1}\pi(r)\right)^{m}.
\end{aligned}$$

The multinomial term in (3.4) bounds the number of ways we can order $k-1$ (not necessarily different) numbers when we do not distinguish between the largest 2^d-1, the next $(2^d-1)2^d$ largest,..., and the next $(2^d-1)^{d(j-1)}$ largest of them. The product terms in (3.4) apply the above bounds on the range of $\bar{r}_i$. Hence by Lemma 3.1, we have that

$$\begin{aligned}
\mathbb{E}\binom{|\mathcal{V}_n|}{k} &\leq (C_5C_6)^k n^{dk} \binom{k-1}{2^d-1, (2^d-1)2^d, \ldots, (2^d-1)2^{d(j-1)}, m} \\
&\quad 2^{-m(j-1)d} \prod_{i=1}^{j-1} 2^{-di(2^d-1)2^{id}} \cdot \pi(n)\pi(n/2^{j-1})^m \prod_{i=1}^{j-1} \pi(n/2^i)^{(2^d-1)2^{di}}. &(3.5)
\end{aligned}$$

We estimate the multinomial, and the two product terms separately. It is a simple computation to show that there is a constant $C_7 = C_7(d)$ such that

$$\binom{k-1}{2^d-1, (2^d-1)2^d, \ldots, (2^d-1)2^{d(j-1)}, m} \leq C_7^{k-1}, \tag{3.6}$$

and that

$$2^{-m(j-1)d} \prod_{i=1}^{j-1} 2^{-di(2^d-1)2^{id}} \leq C_7^k k^{-k} \tag{3.7}$$

for all $k \geq 1$. We combine (3.5), (3.6), and (3.7) with the trivial bound $\pi(n/\sqrt[d]{k})^k$ for the product of π's, and get

$$\mathbb{E}\binom{|\mathcal{V}_n|}{k} \leq C_8^k n^{kd} k^{-k} \pi(n/\sqrt[d]{k})^k \tag{3.8}$$

with $C_8 = C_5 C_6 C_7^2$. This finishes the proof of the first part of Theorem 1.5.

Let us proceed to the proof of the second part. The statement is trivial for $u > n$, hence we assume $u \in [1, n]$ in the following. For $t \geq 1$ by (3.8) we get

$$\begin{aligned} \mathbb{E}\left(t^{|\mathcal{V}_n|}\right) &= \sum_{k=1}^{\infty} (t-1)^k \mathbb{E}\binom{|\mathcal{V}_n|}{k} \\ &\leq \sum_{k=0}^{\infty} \left((t-1) C_8 n^d \pi(n/\sqrt[d]{k})/k\right)^k . \end{aligned}$$

Take $t = 1 + \frac{u^d}{C_2 C_8 n^d \pi(n/u)}$ where $u \in [1, n]$. With Assumption (II) we get

$$\begin{aligned} \mathbb{E}\left(t^{|\mathcal{V}_n|}\right) &\leq \sum_{k=0}^{\infty} \left(\frac{u^d \pi(n/\sqrt[d]{k})}{C_2 k \pi(n/u)}\right)^k \\ &\leq \sum_{k=0}^{C_2^{-1} u^d} \left(\frac{u^d}{C_2 k}\right)^k + \sum_{k=C_2^{-1} u^d + 1}^{\infty} \left(\frac{u^d}{k}\right)^{(1-\alpha/d)k} \\ &\leq \sum_{k=0}^{\infty} \frac{u^{dk}}{C_2^k k!} + C_2^{-1} u^d \sum_{l=1}^{\infty} \left(l^{1-\alpha/d}\right)^{-C_2^{-1} u^d l} \\ &\leq \exp(C_2^{-1} u^d) + C_2^{-1} u^d \sum_{l=1}^{\infty} l^{-(1-\alpha/d) l} \\ &\leq C_9 \exp(C_2^{-1} u^d) \end{aligned} \tag{3.9}$$

for some constant $C_9 = C_9(\alpha, d)$. Note that the function $x \to (1+x)^{1/x}$ is decreasing, and that $\frac{u^d}{n^d \pi(n/u)} \leq C_2^{-1} (u/n)^{d-\alpha} \leq C_2^{-1}$ since $u \in [1, n]$. Hence there is a constant C_{10} such that for all $K > 0$

$$t^{K n^d \pi(n/u)} = \left(1 + \frac{u^d}{C_2 C_8 n^d \pi(n/u)}\right)^{K n^d \pi(n/u)} \geq \exp\left(C_{10} K u^d\right). \tag{3.10}$$

Then the Markov inequality, (3.9) and (3.10) with $K = 2/(C_2 C_{10})$ gives that

$$\mathbb{P}_{p_c}\left(|\mathcal{V}_n| \geq \frac{2}{C_2 C_{10}} n^d \pi\left(n/u\right)\right) \leq C_9 \exp\left(-u^d/C_8\right), \tag{3.11}$$

From (3.11) by Assumption 1.2 the second part of Theorem 1.5 follows. This finishes the proof of Theorem 1.5. □

3.2 Proof of the lower bound of Theorem 1.4

In this section we consider the case $d = 2$.

For $n, m \geq 1$ let $B(n, m)$ denote the rectangle $B(n, m) := [0, n] \times [0, m] \cap \mathbb{Z}^2$. Further, let $\mathcal{H}(B(n, m))$ denote the event that there is an open path connecting $\{0\} \times [0, m]$ to $\{n\} \times [0, m]$. The notation extends to translates of $B(n, m)$ in the usual way. Furthermore, we define the event $\mathcal{V}(B(n, m))$ that there is a vertical crossing of $B(n, m)$. The following well-known statement first appeared in [16], see also [15].

Lemma 3.2 (RSW)**.** *There is a positive constant $C_{11} > 0$ such that for all $n \geq 1$*

$$\mathbb{P}_{p_c}(\mathcal{H}(B(n, 2n))) \geq e^{-C_{11}}.$$

We say that an event $\mathcal{A}$ is increasing, if $\omega \in \mathcal{A}$ then $\omega' \in \mathcal{A}$ for all $\omega' \in \Omega$ with $\omega' \geq \omega$, where $\geq$ is understood coordinate-wise. We recall the FKG -inequality [5]:

Lemma 3.3. *(FKG) Let $\mathcal{A}, \mathcal{B}$ be increasing events, then*

$$\mathbb{P}_{p_c}(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_{p_c}(\mathcal{A})\mathbb{P}_{p_c}(\mathcal{B}).$$

We start with the following lemma.

Lemma 3.4. *There are positive constants C_{12}, C_{13} such that for all $n \geq 1$*

$$\mathbb{P}_{p_c}(|\mathcal{V}_n| \geq C_{12} n^2 \pi(n)) \geq e^{-C_{13}}.$$

Proof of Lemma 3.4. Simple computation gives that

$$\mathbb{E}_{p_c}(|\mathcal{V}_n|) \geq n^2 \pi(3n) \geq C_2 3^{-\alpha} n^2 \pi(n).$$

This combined with Theorem 1.5 provides the desired constants C_{12} and C_{13}. □

Now we proceed to the proof of the lower bound in Theorem 1.4.

Proof of the lower bound in Theorem 1.4. For $v \in \mathbb{Z}^2$, we set $B(v; n, m) := B(n, m) + v$, and

$$\mathcal{V}_n(v) := \{w \in \Lambda_n(v) \,|\, w \leftrightarrow \partial\Lambda_{2n}(v)\}$$

Note that it is enough to prove (1.5) when u is an integer in $[2, n]$. We set $n' = \lfloor n/u \rfloor$. Let $\mathcal{D}_n(u)$ denote the event

$$\mathcal{D}_n(u) := \bigcap_{v \in \Lambda_u} \mathcal{H}\left(B\left(n'v; n', 2n'\right)\right) \cap \mathcal{V}\left(B\left(n'v; 2n', n'\right)\right).$$

It is easy to check that on the event $\mathcal{D}_n(u)$, all the vertices $w \in \Lambda_{n-n'}$ with $w \leftrightarrow \partial\Lambda_{2n'}(w)$ belong to the same cluster. In particular, on $\mathcal{D}_n(u)$ we have

$$\sum_{v \in \Lambda_{u-1}} |\mathcal{V}_{n'}\left(n'v\right)| \leq |\mathcal{C}_n^{(1)}|. \tag{3.12}$$

Lemma 3.2 and 3.3 gives that

$$\mathbb{P}_{p_c}(\mathcal{D}_n(u)) \geq e^{-C_{11} 2u^2}. \tag{3.13}$$

Combination of (3.12), (3.13) and Lemma 3.3 gives that for $C_{12} > 0$ as in Lemma 3.4 we have

$$\begin{aligned}
\mathbb{P}_{p_c}\left(|\mathcal{C}_n^{(1)}| \geq \frac{C_{12}}{2} n^2 \pi(n/u)\right) \\
&\geq \mathbb{P}_{p_c}\left(\mathcal{D}_n(u), \sum_{v\in\Lambda_{u-1}} |\mathcal{V}_{n'}(n'v)| \geq \frac{C_{12}}{2} n^2 \pi(n/u)\right) \\
&\geq e^{-2C_{10}u^2}\mathbb{P}_{p_c}\left(\sum_{v\in\Lambda_{u-1}} |\mathcal{V}_{n'}(n'v)| \geq \frac{C_{12}}{2} n^2 \pi(n/u)\right) \qquad (3.14) \\
&\geq e^{-2C_{11}u^2}\mathbb{P}_{p_c}\left(\mathcal{V}_{n'} \geq C_{12} n'^2 \pi(n')\right)^{u^2} \\
&\geq e^{-(2C_{11}+C_{13})u^2}. \qquad (3.15)
\end{aligned}$$

Above we used Lemma 3.4 in (3.14) and in (3.15). Simple application of Assumption 1.2 finishes the proof of the lower bound of Theorem 1.4. □

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