

Large gaps asymptotics for the 1-dimensional random Schrödinger operator

Stéphanie Jacquot*

Abstract

We show that in the Schrödinger point process, Sch_τ , $\tau > 0$, the probability of having no eigenvalue in a fixed interval of size λ is given by

$$\exp\left(-\frac{\lambda^2}{4\tau} + \left(\frac{2}{\tau} - \frac{1}{4}\right)\lambda + o(\lambda)\right),$$

as $\lambda \rightarrow \infty$. It is a slightly more precise version of the formula given in a previous work.

Keywords: Point process ; stationary distribution ; Girsanov transformation ; diffusion.

AMS MSC 2010: 60G10; 60G55; 60H20; 60H30.

Submitted to EJP on April 5, 2013, final version accepted on August 30, 2014.

1 Introduction

For $\tau > 0$, we consider the Schrödinger point process, Sch_τ as introduced in [3]. For $a, b \in \mathbb{R}$, let $\text{Sch}_\tau[a, b]$ be the number of points of Sch_τ in the interval $[a, b]$. We define

$$E_\tau(0, \lambda) = \mathbb{P}(\text{Sch}_\tau[0, \lambda] = 0).$$

We want to compute the asymptotics of $E_\tau(0, \lambda)$ as $\lambda \rightarrow \infty$. An asymptotic of this probability was computed in [3] and is given by:

$$E_\tau(0, \lambda) = \exp\left(-\frac{\lambda^2}{4\tau} + o(\lambda^2)\right),$$

as $\lambda \rightarrow \infty$. Our aim in this paper is to compute the asymptotics of $E_\tau(0, \lambda)$ to the next order. We will be proving the following theorem:

Theorem 1.1. As $\lambda \rightarrow \infty$,

$$E_\tau(0, \lambda) = \exp\left(-\frac{\lambda^2}{4\tau} + \left(\frac{2}{\tau} - \frac{1}{4}\right)\lambda + o(\lambda)\right). \quad (1.1)$$

*Statistical Laboratory, Center for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK.

For $\tau > 0$, let us first introduce the Sch_τ point process. We follow the description of [3]. Consider the family of SDE's,

$$d\phi^\lambda(t) = \lambda dt + dB(t) + \Re\left(e^{-i\phi^\lambda} dW(t)\right), \quad \phi^\lambda(0) = 0,$$

coupled together for all values of $\lambda \in \mathbb{R}$, where B and W are standard real and complex Brownian motions respectively. By Corollary 4, [3], this SDE has a unique strong solution and for each time t , the function $\lambda \rightarrow \phi^\lambda(t)$ is strictly increasing and real-analytic with probability one. We define Sch_τ as follows:

$$\text{Sch}_\tau := \left\{ \lambda : \phi^{\lambda/\tau}(\tau) \in 2\pi\mathbb{Z} \right\}.$$

Let $\alpha^\lambda(t) = \phi^\lambda(t) - \phi^0(t)$. Then α^λ satisfies

$$d\alpha^\lambda(t) = \lambda dt + \Re\left[\left(e^{-i\alpha^\lambda} - 1\right) dZ(t)\right], \quad \alpha^\lambda(0) = 0 \tag{1.2}$$

where Z is a complex Brownian motion with $dZ(t) = e^{-i\phi^0} dW(t)$. For a fixed $\lambda \in \mathbb{R}$, this can be rewritten as,

$$d\alpha^\lambda(t) = \lambda dt + \sqrt{2} \sin\left(\frac{\alpha^\lambda}{2}\right) dB(t), \quad \alpha^\lambda(0) = 0. \tag{1.3}$$

Then the following events are the same $\{\text{Sch}_\tau[0, \lambda] = 0\} = \{\alpha^{\lambda/\tau}(\tau) < 2\pi\}$. We will find it convenient to remove the dependance in α^λ in front of the Brownian motion. For this, we make the change of variables $\tilde{X}^\lambda = \log[\tan(\alpha^\lambda/4)]$, which is well-defined for $\alpha^\lambda \in (0, 2\pi)$. It satisfies the following SDE,

$$d\tilde{X}^\lambda(t) = \frac{\lambda}{2} \cosh(\tilde{X}^\lambda) dt + \frac{1}{4} \tanh(\tilde{X}^\lambda) dt + \frac{1}{\sqrt{2}} dB(t), \quad \tilde{X}^\lambda(0) = -\infty. \tag{1.4}$$

Now we make a change of time variable by setting $X^\lambda(t) = \tilde{X}^\lambda(2t)$. It satisfies the following SDE,

$$dX^\lambda(t) = \lambda \cosh(X^\lambda) dt + \frac{1}{2} \tanh(X^\lambda) dt + dB(t), \quad X^\lambda(0) = -\infty. \tag{1.5}$$

Then,

$$\{\text{Sch}_\tau[0, \lambda] = 0\} = \left\{ X^{\lambda/\tau}(\tau/2) < \infty \right\}.$$

For convenience, we will simply write X when we refer to $X^{\lambda/\tau}$. Thus,

$$E_\tau(0, \lambda) = \mathbb{E}\left[1(X(t) \text{ finite on } [0, \tau/2])\right]$$

In order to prove Theorem 1.1, we will follow a similar method to [5]. In [5], the authors compute the large gaps asymptotics of the Sine_β process, $\beta > 0$, the point process limit of the bulk eigenvalue of β -ensemble of random matrices. Their result is that

$$\mathbb{P}(\text{Sine}_\beta[0, \lambda] = 0) = (\kappa_\beta + o(1)) \exp\left(-\frac{\beta}{64} \lambda^2 + \left(\frac{\beta}{8} - \frac{1}{4}\right) \lambda + \gamma_\beta \log(\lambda)\right)$$

as $\lambda \rightarrow \infty$, where

$$\gamma_\beta = \frac{1}{4} \left(\frac{\beta}{2} + \frac{2}{\beta} - 3\right), \quad \kappa_\beta > 0.$$

For $\beta > 0$, the Sine_β process (or Brownian Carousel) can be introduced as follows. We consider the strong solution of the following coupled one-parameter family of SDE

$$d\alpha_\lambda(t) = \lambda f(t) dt + \Re\left(\left(e^{i\alpha_\lambda} - 1\right) d\tilde{Z}(t)\right), \quad \alpha_\lambda(0) = 0$$

where \tilde{Z} is a complex Brownian motion and $f(t) = (\beta/4) \exp(-(\beta/4)t)$. For a single λ , this reduces to

$$d\alpha_\lambda(t) = \lambda f(t)dt + 2 \sin\left(\frac{\alpha_\lambda(t)}{2}\right) dW(t), \quad \alpha_\lambda(0) = 0. \tag{1.6}$$

In [4], it is shown that the quantity

$$\lim_{t \rightarrow \infty} \frac{\alpha_\lambda(t)}{2\pi},$$

exists for every λ and is equal to an integer a.s. We define $N(\lambda)$ as the unique random right-continuous function which agrees with for every λ a.s. For $\lambda_1 < \lambda_2$, we have $N(\lambda_1) < N(\lambda_2)$ a.s. $N(\lambda)$ is just the counting function of the Sine_β process, giving the number of point of the process in an interval of length λ .

We set $Z = \log[\tan(\alpha_\lambda/4)]$. Then Z satisfies

$$dZ(t) = \lambda f(t) \cosh(Z)dt + \frac{1}{2} \tanh(Z)dt + dB(t), \quad Z(0) = -\infty. \tag{1.7}$$

Observe that our SDE (1.5) and (1.7) are very similar, except that in our case, we have $f(t) = (2/\tau)1_{[0, \tau/2]}(t)$.

Our main tool to prove Theorem 1.1 will be the Cameron-Martin-Girsanov formula, which allows one to compare the measure on paths given by two diffusions. If we knew the conditional distribution of the diffusion X under the event that it does not blow up before time $\tau/2$, then we could use the Cameron-Martin-Girsanov formula to compute $E_\tau(0, \lambda)$ explicitly. While we cannot do this, the next best option is to find a new diffusion Y which approximates this conditional distribution. The next section gives the statement of the Girsanov theorem we will be using.

2 The Cameron-Martin-Girsanov formula

We will use the same version of the Girsanov formula than the one introduced in [5].

Proposition 2.1. *Consider the following stochastic differential equations:*

$$dX(t) = g(t, X)dt + dB(t), \quad \lim_{t \rightarrow 0} X(t) = -\infty, \tag{2.1}$$

$$dY(t) = h(t, Y)dt + d\tilde{B}(t), \quad \lim_{t \rightarrow 0} Y(t) = -\infty \tag{2.2}$$

on the interval $(0, T]$ where B, \tilde{B} are standard Brownian motions. Assume that (8) has a unique solution X in law taking values in $(-\infty, \infty]$. Let

$$G_s = G_s(X) = \int_0^s (h(t, X) - g(t, X)) dX - \frac{1}{2} \int_0^s (h(t, X)^2 - g(t, X)^2) dt \tag{2.3}$$

and assume that:

(A) $g^2 - h^2$ and $g - h$ are bounded when x is bounded above. (Then G_s is almost surely well defined when X_s is finite.)

(B) G_s is bounded above by a deterministic constant.

(C) $G_s \rightarrow \infty$ when $s \uparrow S$ if X hits $+\infty$ at time S . In this case, we define $G_s := -\infty$ for $s \geq S$. Consider the process \tilde{Y} whose density with respect to the distribution of the process X is given by e^{G_τ} . Then, \tilde{Y} satisfies the second SDE (9) and never blows-up to $+\infty$ almost surely. Moreover, for any nonnegative function F of the path of X that vanishes when X blows up we have

$$\mathbb{E}[F(X)] = \mathbb{E}\left[F(Y)e^{-G_\tau(Y)}\right]. \tag{2.4}$$

The proof of Proposition 2.1 is given in [5].

3 Construction of the diffusion Y

In this section, we construct a diffusion Y which approximates the conditional distribution of X under the event that it does not explode before time $\tau/2$. We will construct a drift function $h(x)$ for which the diffusion Y ,

$$dY(t) = h(Y)dt + d\tilde{B}(t), \quad Y(0) = -\infty \tag{3.1}$$

is well defined, a.s. finite for $t \leq \tau/2$ and with Radon-Nikodym derivative $e^{G_{\tau/2}}$ with $G_{\tau/2}$ as in Proposition 2.1.

Lemma 3.1. *For the diffusion $X^{\lambda/\tau}$ (which we simply denote X), and $T = \tau/2$, there exists a function $h(x)$ so that conditions (A)-(C) of Proposition 2.1 hold, and $G_{\tau/2}$ has the following form:*

$$\begin{aligned} -G_{\tau/2} = & -\frac{\lambda^2}{4\tau} - \frac{\lambda}{4} + \frac{\tau}{16} + \frac{3}{2} \log(2) + 2X(\tau/2)^+ + \frac{\lambda}{\tau} e^{X(\tau/2)} \\ & + \omega_1(X(\tau/2)) + \omega_2(X(\tau/2)) + \int_0^{\tau/2} \Phi(X(s))ds, \end{aligned}$$

where ω_1, ω_2, Φ are continuous with

$$\omega_1(x) = \frac{1}{2} \log(\cosh(x)) + \log(\cosh(x/2)) - |x|,$$

and there exists $C_1, C_2 > 0$ such that

$$\|\omega_2\|_\infty \leq \frac{C_1}{\lambda}, \quad \|\Phi\|_\infty \leq \frac{C_2}{\lambda}.$$

Also,

$$h(x) = -\frac{\lambda}{\tau} \sinh(x) - 1 - \frac{1}{2} \tanh\left(\frac{x}{2}\right) + h_4(x)$$

with h_4 continuous and such that there exists $c > 0$, with $\|h_4\|_\infty < c/\lambda$.

Proof. In order to construct the function h , we follow [5]. We will use

$$-G_s(X) = \int_0^s (g(t, X) - h(t, X)) dX + \frac{1}{2} \int_0^s (h(t, X)^2 - g(t, X)^2) dt$$

where

$$g = g_1 + g_2, \quad g_1(x) = \frac{\lambda}{\tau} \cosh(x), \quad g_2(x) = \frac{1}{2} \tanh(x).$$

Our goal is to find the appropriate drift term h in a way that the diffusion Y will approximate the conditional distribution of X given that it does not blow up in the interval $[0, \tau/2]$. We will do this term by term starting with the highest order; we write $h = h_1 + h_2 + h_3 + h_4$. We set

$$h_1(x) = -\frac{\lambda}{\tau} \sinh(x),$$

as it yields the nice cancellation,

$$h_1^2 - g_1^2 = \left(\frac{\lambda}{\tau}\right)^2 \sinh^2(x) - \left(\frac{\lambda}{\tau}\right)^2 \cosh^2(x) = -\left(\frac{\lambda}{\tau}\right)^2.$$

The contribution to the drift terms h_1 and g_1 to the stochastic integral part of $-G_s$ is given by,

$$\int_0^s (g_1 - h_1) dX = \frac{\lambda}{\tau} \int_0^s e^X dX.$$

We will use Ito's formula to evaluate integrals with respect to dX . Let b be a continuously differentiable function and denote by \tilde{b} its antiderivative. We have,

$$b(X)dX = d\left(\tilde{b}(X)\right) - \frac{1}{2}b'(X)dt \tag{3.2}$$

Thus, by applying (3.2), we have

$$\frac{\lambda}{\tau} \int_0^s e^X dX = \frac{\lambda}{\tau} e^{X(s)} - \frac{\lambda}{2\tau} \int_0^s e^{X(t)} dt. \tag{3.3}$$

We choose h_2 so that the integral term in the right-hand side of (3.3) simplifies, that is h_2 such that $h_1 h_2 = -\frac{\lambda}{2\tau}(1 - e^x)$. This gives,

$$h_2(x) = -\frac{1}{2} \left(1 + \tanh\left(\frac{x}{2}\right)\right)$$

We now choose h_3 so that $h_1 h_3 = g_1 g_2$, that is,

$$h_1 h_3 = g_1 g_2 = \frac{\lambda}{\tau} \cosh(x) \frac{1}{2} \tanh(x),$$

which leads to $h_3 = -\frac{1}{2}$. Thus,

$$\begin{aligned} -G_s &= \frac{\lambda}{\tau} e^{X(s)} - \frac{1}{2} \left(\frac{\lambda}{\tau}\right)^2 s - \frac{\lambda}{2\tau} s \\ &\quad + \frac{1}{2} \int_0^s \left\{ 2h_1 h_4 + (h_2 + h_3 + h_4)^2 - g_2^2 \right\} dt \\ &\quad - \int_0^s h_4 dX + \int_0^s (g_2 - h_2 - h_3) dX. \end{aligned}$$

Let $u = g_2 - h_2 - h_3 = 1 + \frac{1}{2} \tanh(x) + \frac{1}{2} \tanh(x/2)$. Then by Ito's formula,

$$u(X)dX = d(\tilde{u}(X)) - \frac{1}{2}u'(X)dt$$

where

$$u'(x) = \frac{1}{2} \frac{1}{\cosh^2(x)} + \frac{1}{4} \frac{1}{\cosh^2(\frac{x}{2})}, \quad \tilde{u}(x) = x + \frac{1}{2} \log(\cosh(x)) + \log\left(\cosh\left(\frac{x}{2}\right)\right).$$

Also,

$$\lim_{x \rightarrow -\infty} \tilde{u}(x) = -\frac{3}{2} \log(2).$$

Thus,

$$\begin{aligned} \int_0^s u(X)dX &= \tilde{u}(X(s)) + \frac{3}{2} \log(2) - \frac{1}{4} \int_0^s \left(\frac{1}{\cosh^2(X(s))} + \frac{1}{2} \frac{1}{\cosh^2(X(s)/2)} \right) dt \\ &= \tilde{u}(X(s)) + \frac{3}{2} \log(2) - \frac{3}{8} s + \frac{1}{4} \int_0^s \left(\tanh^2(X(s)) + \frac{1}{2} \tanh^2(X(s)/2) \right) dt \end{aligned} \tag{3.4}$$

Then,

$$\begin{aligned} -G_s &= -\frac{1}{2} \left(\frac{\lambda}{\tau}\right)^2 s - \frac{\lambda}{2\tau} s + \frac{1}{8} s \\ &\quad + \frac{\lambda}{\tau} e^{X(s)} + \tilde{u}(X(s)) + \frac{3}{2} \log(2) \\ &\quad + \frac{1}{2} \int_0^s \left\{ 2h_1 h_4 + 2(h_2 + h_3) h_4 + h_4^2 \right\} dt \\ &\quad - \int_0^s h_4 dX + \int_0^s \eta(X(t)) dt. \end{aligned}$$

The coefficient $1/8$ in front of s in the first line in the expression above come from the $-3/8$ in front of s in (3.4) and the constant term in $(h_2 + h_3)^2/2$. The term η collects the integrand in (3.4), the terms $(h_2 + h_3)^2/2$ with the constant term removed and $-g_2^2/2$. Explicitly, we have

$$\eta(x) = \frac{1}{4} \tanh(x/2)^2 + \frac{1}{8} \tanh(x)^2 + \frac{1}{2} \tanh(x/2).$$

We choose h_4 so that $h_1 h_4 = -\eta$, that is

$$h_4 = \frac{\tau}{\lambda} \frac{\eta}{\sinh(x)}.$$

By Ito's formula,

$$-\int_0^s h_4 dX = -\tilde{h}_4(X(s)) + \frac{1}{2} \int_0^s \partial_x h_4 dt.$$

where $-\tilde{h}_4$ is the antiderivative of h_4 . Thus,

$$\begin{aligned} -G_s &= -\frac{1}{2} \left(\frac{\lambda}{\tau}\right)^2 s - \frac{\lambda}{2\tau} s + \frac{1}{8} s \\ &\quad + \frac{\lambda}{\tau} e^{X(s)} + \tilde{u}(X(s)) + \frac{3}{2} \log(2) \\ &\quad + \frac{1}{2} \int_0^s \{2(h_2 + h_3) h_4 + h_4^2 + \partial_x h_4\} dt - \tilde{h}_4(X(s)). \end{aligned}$$

Using the fact that $\log \cosh(x) - |x|$ is bounded, one can rewrite the second line of $-G_s$ as follows:

$$2X(s)^+ + \frac{\lambda}{\tau} e^{X(s)} + \omega_1(X(s)) + \frac{3}{2} \log(2)$$

where $\omega_1(x) = \frac{1}{2} \log(\cosh(x)) + \log(\cosh(x/2)) - |x|$ and so is bounded.

We now plug $s = \tau/2$ to get,

$$\begin{aligned} -G_{\tau/2} &= -\frac{\lambda^2}{4\tau} - \frac{\lambda}{4} + \frac{\tau}{16} \\ &\quad + 2X(\tau/2)^+ + \frac{\lambda}{\tau} e^{X(\tau/2)} + \omega_1(X(\tau/2)) + \frac{3}{2} \log(2) \\ &\quad + \frac{1}{2} \int_0^{\tau/2} \{2(h_2 + h_3) h_4 + h_4^2 + \partial_x h_4\} dt - \tilde{h}_4(X(\tau/2)). \end{aligned}$$

We now need to check that that the proposed choice for h satisfies conditions (A)-(C) from Proposition 2.1. First, observe that $h_2, h_3, h_4, \tilde{h}_4, \partial_x h_4$ are all bounded above by an absolute constant and in particular there is a $1/\lambda$ coefficient in front of $h_4, \tilde{h}_4, \partial_x h_4$. This implies that we can write the third line of G_s

$$\omega_2(X(\tau/2)) + \int_0^{\tau/2} \Phi(X(s)) ds$$

with $\omega_2 = -\tilde{h}_4$ and

$$\int_0^{\tau/2} \Phi(X(s)) ds = \frac{1}{2} \int_0^{\tau/2} \{2(h_2 + h_3) h_4 + h_4^2 + \partial_x h_4\} dt$$

and that there exists $C_1, C_2 > 0$ such that

$$\|\omega_2\|_\infty \leq \frac{C_1}{\lambda}, \quad \|\Phi\|_\infty \leq \frac{C_1}{\lambda}.$$

We now need to check that the conditions (A)-(C) of Proposition 2.1 are satisfied. We proceed as in [5].

Condition (A): As $x \rightarrow -\infty$, one can write

$$g(x) = \frac{\lambda}{2\tau}e^{-x} - \frac{1}{2} + \hat{g}(x), \quad h(x) = \frac{\lambda}{2\tau}e^{-x} - \frac{1}{2} + \hat{h}(x)$$

where $|\hat{g}| \leq Ce^x$ and $|\hat{h}| \leq C'e^x$ for some positive constants C and C' . Thus, $g - h$ and $g^2 - h^2$ are both bounded if x is bounded above.

Condition (B): The first and third line in $-G_{\tau/2}$ are both bounded as well as ω_1 in the second line. Thus, we only need to show that

$$2X(s)^+ + \frac{\lambda}{\tau}e^{X(s)}$$

is bounded from below by a constant only depending on λ and τ , which is obvious. Thus G_s is bounded from above by a constant (depending only on λ and τ).

Condition (C): As $s \uparrow S$, the time when X reaches ∞ , we have that

$$2X(s)^+ + \frac{\lambda}{\tau}e^{X(s)}$$

goes to ∞ while the other terms in G_s are bounded. Thus, $G_s \rightarrow -\infty$.

4 Proof of Theorem 1.1

Recall that

$$\mathbb{P}(\text{Sch}_\tau[0, \lambda] = 0) = \mathbb{E}[1(X(t) \text{ finite on } [0, \tau/2])].$$

We consider the diffusion Y given by the SDE (3.1) with a drift function $h(x)$ as in Lemma 3.1. That is Y satisfies the following SDE:

$$dY(t) = \left(-\frac{\lambda}{\tau} \sinh(Y) - 1 - \frac{1}{2} \tanh\left(\frac{Y}{2}\right) + h_4(Y) \right) dt + dB(t), \quad Y(0) = -\infty$$

where there exists $c > 0$ such that $\|h_4\|_\infty \leq c/\lambda$. We apply Proposition 2.1 with $F(X) = 1(X \text{ finite on } [0, \tau/2])$. Then,

$$\mathbb{P}(\text{Sch}_\tau[0, \lambda] = 0) = \exp\left(-\frac{\lambda^2}{4\tau} - \frac{\lambda}{4} + \frac{\tau}{16} + \frac{3}{2} \log(2)\right) \times \mathbb{E}[\exp(\Psi(Y))] \tag{4.1}$$

where

$$\Psi(Y) = 2Y(\tau/2)^+ + \frac{\lambda}{\tau}e^{Y(\tau/2)} + \omega_1(Y(\tau/2)) + \omega_2(Y(\tau/2)) + \int_0^{\tau/2} \Phi(Y(s)) ds.$$

Since there exists $C_1, C_2 > 0$ such that $\|\omega_2\|_\infty < C_1/\lambda$, and $\|\Phi\|_\infty < C_2/\lambda$, we can find $C > 0$ such that

$$\left| \omega_2(Y(\tau/2)) + \int_0^{\tau/2} \Phi(Y(s)) ds \right| < \frac{C}{\lambda}. \tag{4.2}$$

Our aim is to evaluate $\mathbb{E}[\exp(\Psi(Y))]$ as $\lambda \rightarrow \infty$. We will prove the following lemma.

Lemma 4.1. As $\lambda \rightarrow \infty$,

$$\mathbb{E}[\exp(\Psi(Y))] \sim \exp\left(\frac{2}{\tau}\lambda + o(\lambda)\right).$$

This lemma, combined with (4.1) proves Theorem 1.1. In order to prove the lemma we will bound Y above and below in order to get upper and lower bounds on $\mathbb{E}[\exp(\Psi(Y))]$. For the upper bound, we will bound Y above by the stationary solution of an SDE whose drift is very close to the one of Y . This will give a very precise upper bound which we think is actually the full asymptotics (up to the constant term) for $\mathbb{P}(\text{Sch}_\tau[0, \lambda] = 0)$. Unfortunately, this method cannot be applied for the lower bound. For the lower bound we will bound Y below by the (random) solution of an ODE. This will give a less precise asymptotics but good enough to get the $\exp(2\tau/\lambda)$ that we need. Observe that we could have used a similar method for the upper bound but we chose to proceed differently as the bound obtained is more precise and likely to be the full asymptotics up to the constant term.

Proof of Lemma 4.1:

Upper Bound Consider,

$$dY_1(t) = \left(-\frac{\lambda}{\tau} \sinh(Y_1) - 1 - \frac{1}{2} \tanh\left(\frac{Y_1}{2}\right) + \frac{c}{\lambda} \right) dt + dB(t).$$

If we drive Y_1 and Y with the same Brownian motion and consider Y_1 started from its stationary distribution, then, since $Y(0) = -\infty < Y_1(0)$ and the drift of Y_1 is greater than the drift of Y we have that

$$Y(t) \leq Y_1(t), \quad \text{for all } t \geq 0.$$

Now, let us compute the stationary distribution for Y_1 . The adjoint of the infinitesimal generator of Y_1 is given by

$$A^* p_\lambda(t, y) = -\frac{\partial}{\partial y} \left[\left(-\frac{\lambda}{\tau} \sinh(y) - 1 - \frac{1}{2} \tanh(y/2) + \frac{c}{\lambda} \right) p_\lambda(t, y) \right] + \frac{1}{2} \frac{\partial^2}{\partial y^2} p_\lambda(t, y).$$

The stationary distribution of Y_1 satisfies $\partial/\partial t(p_\lambda(t, y)) = 0$ and $A^* p_\lambda(t, y) = 0$. So,

$$p_\lambda(t, y) = p_\lambda(y) = K(\lambda) \exp\left(-\frac{2\lambda}{\tau} \cosh(y) - 2y - 2 \log(\cosh(y/2)) + \frac{2c}{\lambda} y \right)$$

where $K(\lambda)$ is a constant such that

$$\int_{-\infty}^{\infty} p_\lambda(y) dy = 1.$$

We would like to asymptotically evaluate $K(\lambda)$. Now,

$$\frac{1}{K(\lambda)} = \int_{-\infty}^{\infty} \exp\left(-\frac{2\lambda}{\tau} \cosh(y) - 2y - 2 \log(\cosh(y/2)) + \frac{2c}{\lambda} y \right) dy.$$

we will use Lemma 5.1 from Appendix A. We can neglect the term Cy/λ in the exponential. We call $I(\lambda)$ the integral above (after the term cy/λ has been removed). We set $g(y) = (2/\tau) \cosh(y)$ and $h(y) = \exp(-2 \log(\cosh(y/2)) - 2y)$. The minimum of g is attained at 0 and $g(0) = 2/\tau$, $g'(0) = 0$, $g''(0) = 2/\tau > 0$ and $h(0) = 1 \neq 0$. Thus, by Lemma 5.1

$$I(\lambda) = \exp\left(-\frac{2}{\tau} \lambda \right) \sqrt{\frac{\pi\tau}{\lambda}} (1 + o(1)), \tag{4.3}$$

that is,

$$K(\lambda) = \exp\left(\frac{2}{\tau} \lambda \right) \sqrt{\frac{\lambda}{\pi\tau}} (1 + o(1)),$$

Using (4.2) and the fact that $x \rightarrow 2x^+ + (\lambda/\tau)e^x + \omega_1(y)$ is strictly increasing, we have that

$$\mathbb{E}[\exp(\Psi(Y))] \leq \mathbb{E}\left[\exp\left(2Y_1(\tau/2)^+ + \frac{\lambda}{\tau}e^{Y_1(\tau/2)} + \omega_1(Y_1(\tau/2))\right)\right] \exp\left(\frac{C}{\lambda}\right). \quad (4.4)$$

Now,

$$\begin{aligned} &\mathbb{E}\left[\exp\left(2Y_1(\tau/2)^+ + \frac{\lambda}{\tau}e^{Y_1(\tau/2)} + \omega_1(Y_1(\tau/2))\right)\right] = \\ &K(\lambda) \int_{-\infty}^{\infty} \exp\left(2y^+ + \frac{\lambda}{\tau}\exp(y) + \frac{1}{2}\log(\cosh(y)) + \log\left(\cosh\left(\frac{y}{2}\right)\right) - |y|\right) \\ &\quad \times \exp\left(-\frac{2\lambda}{\tau}\cosh(y) - 2y - 2\log\left(\cosh\left(\frac{y}{2}\right)\right) + \frac{2c}{\lambda}y\right) dy \\ &= K(\lambda) \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda}{\tau}\exp(-y) + \frac{1}{2}\log(\cosh(y)) - \log\left(\cosh\left(\frac{y}{2}\right)\right) - y + \frac{2c}{\lambda}y\right) dy \end{aligned}$$

since $2y^+ - |y| - 2y = -y$. Now, we can neglect the term in cy/λ in the exponential above and so we write

$$\mathbb{E}\left[\exp\left(2Y_1(\tau/2)^+ + \frac{\lambda}{\tau}e^{Y_1(\tau/2)} + \omega_1(Y_1(\tau/2))\right)\right] = K(\lambda)J(\lambda) \quad (4.5)$$

with

$$J(\lambda) = \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda}{\tau}\exp(-y) + \frac{1}{2}\log(\cosh(y)) - \log\left(\cosh\left(\frac{y}{2}\right)\right) - y\right) dy.$$

We make the following change of variables: $u = \exp(-y)$. Then,

$$J(\lambda) = \exp\left(\frac{1}{2}\log(2)\right) \int_0^{\infty} \exp\left(-\frac{\lambda}{\tau}u\right) \frac{\sqrt{u^2+1}}{u+1} du.$$

We set $g(u) = -u/\tau$ and $h(u) = \sqrt{u^2+1}/(u+1)$. We have $g'(u) = -1/\tau \neq 0$. The maximum is attained in 0 with $g(0) = 0, h(0) = 1$. Thus, by Lemma 5.2,

$$J(\lambda) \sim \exp\left(\frac{1}{2}\log(2)\right) \frac{\tau}{\lambda} (1 + o(1)).$$

Thus,

$$\mathbb{E}\left[\exp\left(2Y_1(\tau/2)^+ + \frac{\lambda}{\tau}e^{Y_1(\tau/2)} + \omega_1(Y_1(\tau/2))\right)\right] \leq \exp\left(\frac{2}{\tau}\lambda\right) \exp\left(\frac{1}{2}\log(2)\right) \sqrt{\frac{\tau}{\pi\lambda}} (1 + o(1)) \quad (4.6)$$

Lower Bound For the lower bound, we are going to bound Y stochastically below by the (random) solution of an ODE. We know that $\|h_4\|_{\infty} \leq c/\lambda$ for some $c > 0$. Thus, $-1 - (1/2)\tanh(y/2) + h_4(y) \geq -(3/2) - c/\lambda$. Since, eventually we will let $\lambda \rightarrow \infty$, one can choose λ large enough so that $-(3/2) - c/\lambda \geq -2$. Then $Z_2 \leq Y$ where

$$dZ_2(t) = \left(-\frac{\lambda}{\tau}\sinh(Z_2) - 2\right) dt + dB(t), \quad Z_2(0) = -\infty.$$

Let $U_2 = Z_2 - B$. Then,

$$dU_2(t) = \left(-\frac{\lambda}{\tau}\sinh(U_2 + B) - 2\right) dt, \quad U_2(0) = -\infty.$$

Thus $Y \geq U_2 + B$. By definition, for $t \in [0, \tau/2]$, the process $Y - B$ lies above the (random) solution of the differential equation

$$y'(t) = \left(-\frac{\lambda}{\tau} \sinh(y(t) + M) - 2 \right) dt, \quad y(0) = -\infty$$

where $M = \sup_{s \in [0, \tau/2]} B(s)$. This differential equation admits almost surely a unique solution which satisfies:

$$\left| \frac{e^{y(t)+M} - u_1}{e^{y(t)+M} - u_2} \right| = \left| \frac{u_1}{u_2} \right| \exp\left(\frac{-\lambda\sqrt{\Delta}}{\tau} t \right) \tag{4.7}$$

where

$$u_1 = -\frac{4\tau}{\lambda} + \sqrt{\Delta}, \quad u_2 = -\frac{4\tau}{\lambda} - \sqrt{\Delta}, \quad \Delta = \left(\frac{4\tau}{\lambda} \right)^2 + 4.$$

Thus, for all $t \in [0, \tau/2]$, we have (after Taylor expansion for large λ)

$$y(t) = \begin{cases} \ln(2) - M + 2e^{-2\lambda t/\tau} + o(e^{-2\lambda t/\tau}) & \text{if } y(t) + M \geq \ln(u_1), \\ \ln(2) - M - 2e^{-2\lambda t/\tau} + o(e^{-2\lambda t/\tau}) & \text{if } y(t) + M \leq \ln(u_1). \end{cases}$$

By definition, $Y(\tau/2) \geq B(\tau/2) + y(\tau/2)$. Thus, we obtain that

$$Y\left(\frac{\tau}{2}\right) \geq \ln(2) + B\left(\frac{\tau}{2}\right) - M - \epsilon(\lambda)$$

where $\epsilon(\lambda) > 0$ and $\epsilon(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Now, we know that the probability density function of the random variable $U = M - B(\tau/2)$ is given by

$$f_U(x) = \frac{2}{\sqrt{\pi\tau}} \exp\left(-\frac{x^2}{\tau}\right) 1_{\{x \geq 0\}}.$$

Using (4.2) and the fact that $x \rightarrow 2x^+ + (\lambda/\tau)e^x + \omega_1(y)$ is strictly increasing, we have

$$\mathbb{E} \left[\exp\left(\frac{\lambda}{\tau} e^{\ln(2)-U-\epsilon(\lambda)} + \omega_1(\ln(2) - U - \epsilon(\lambda)) \right) \right] \exp\left(-\frac{C}{\lambda}\right) \leq \mathbb{E} [\exp(\Psi(Y))]. \tag{4.8}$$

Moreover, $\omega_1(x) = (1/2) \log(\cosh(x)) + \log(\cosh(\frac{x}{2})) - |x| \geq -|x|$. Thus,

$$\omega_1(\ln(2) - U - \epsilon(\lambda)) \geq -|\ln(2) - U - \epsilon(\lambda)| \geq -\ln(2) - |U| - \epsilon(\lambda).$$

Thus,

$$\mathbb{E} \left[\exp\left(\frac{2}{\tau} \lambda e^{-U} e^{-\epsilon(\lambda)} \right) \right] \exp\left(-\frac{C}{\lambda} - \ln(2) - \epsilon(\lambda)\right) \leq \mathbb{E} [\exp(\Psi(Y))].$$

Now,

$$\mathbb{E} \left[\exp\left(\frac{2}{\tau} \lambda e^{-U} e^{-\epsilon(\lambda)} \right) \right] = \int_{-\infty}^{\infty} \exp\left(\frac{2}{\tau} \lambda e^{-x} e^{-\epsilon(\lambda)} \right) f_U(x) dx.$$

One can neglect the term $e^{-\epsilon(\lambda)} \sim 1$ as λ is large, in the integral above. Thus, let

$$L(\lambda) = \int_{-\infty}^{\infty} \exp\left(\frac{2}{\tau} \lambda e^{-x} \right) f_U(x) dx = \int_0^{\infty} \frac{2}{\sqrt{\pi\tau}} \exp\left(\frac{2}{\tau} \lambda e^{-x} \right) \exp\left(-\frac{x^2}{\tau}\right) dx$$

Then,

$$L(\lambda) \geq \int_0^{10} \frac{2}{\sqrt{\pi\tau}} \exp\left(\frac{2}{\tau} \lambda e^{-x} \right) \exp\left(-\frac{x^2}{\tau}\right) dx = N(\lambda).$$

Let $g(x) = \exp(-x)$. On the interval $[0, 10]$, g attains its maximum at 0 with $g(0) = 1$ and $g' \neq 0$. Let $h(x) = (2/\sqrt{\pi\tau}) \exp\left(-\frac{x^2}{\tau}\right)$. Then $h(0) = \frac{2}{\sqrt{\pi\tau}}$. Thus by Lemma 5.2,

$$N(\lambda) \sim \frac{2}{\lambda\sqrt{\pi\tau}} \exp\left(\frac{2}{\tau}\lambda\right) (1 + o(1))$$

Thus, combining this and (4.6) we obtain that

$$\mathbb{E} [\exp(\Psi(Y))] \sim \exp\left(\frac{2}{\tau}\lambda + o(\lambda)\right),$$

as $\lambda \rightarrow \infty$, as required. This completes the proof.

Remark 4.2. We believe that the RHS of the expression (4.6) combined with (4.1) is the full asymptotics up to the constant term of $\mathbb{P}(\text{Sch}_\tau[0, \lambda] = 0)$ and that

$$\mathbb{P}(\text{Sch}_\tau[0, \lambda] = 0) = (\kappa_\tau + o(1)) \exp\left(-\frac{\lambda^2}{4\tau} + \left(\frac{2}{\tau} - \frac{1}{4}\right)\lambda - \frac{1}{2}\log(\lambda)\right), \quad (4.9)$$

as $\lambda \rightarrow \infty$, where $\kappa_\tau = 2^2 \exp(\tau/16)\sqrt{\tau/\pi}$.

Remark 4.3. Comparison of Sine_β with Sch_τ .

Recall that, for $\beta > 0$,

$$\mathbb{P}(\text{Sine}_\beta[0, \lambda] = 0) = (\kappa_\beta + o(1)) \exp\left(-\frac{\beta}{64}\lambda^2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)\lambda + \gamma_\beta \log(\lambda)\right)$$

as $\lambda \rightarrow \infty$, where

$$\gamma_\beta = \frac{1}{4} \left(\frac{\beta}{2} + \frac{2}{\beta} - 3\right), \quad \kappa_\beta > 0,$$

and that for $\tau > 0$,

$$\mathbb{P}(\text{Sch}_\tau[0, \lambda] = 0) = \exp\left(-\frac{\lambda^2}{4\tau} + \left(\frac{2}{\tau} - \frac{1}{4}\right)\lambda + o(\lambda)\right),$$

as $\lambda \rightarrow \infty$. If we take $\beta = 16/\tau$, then we get the same leading terms in front of λ^2 and λ for both Sine_β with Sch_τ . We believe however (from the heuristic expression given by (4.9) that the term in front of $\ln(\lambda)$ would differ in both expressions which is probably due to how strongly the function $f(t)$ decays. We recall that in the case of Sine_β , $f(t) = (\beta/4) \exp(-(\beta/4)t)$ and for the Sch_τ process, $f(t) = (2/\tau)1_{[0, \tau/2]}(t)$.

5 Appendix A: The Laplace Method for approximating integrals

Lemma 5.1. Consider an integral of the form

$$I(\lambda) = \int_a^b h(x)e^{-\lambda g(x)} dx, \quad a, b \in \mathbb{R} \cup \{-\infty, \infty\}$$

where $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are smooth functions.

- If g has strict minimum over $[a, b]$ at an interior critical point c such that $g'(c) = 0, g''(c) > 0$ and $h(c) \neq 0$ then

$$I(\lambda) \sim e^{-\lambda g(c)} h(c) \sqrt{\frac{2\pi}{\lambda g''(c)}} (1 + o(1)), \quad \text{as } \lambda \rightarrow \infty.$$

- If g has its minimum over $[a, b]$ at an end point (say $x=a$) with $g'(a) = 0, g''(a) > 0$ and $h(a) \neq 0$ then

$$I(\lambda) \sim e^{-\lambda g(a)} h(a) \sqrt{\frac{\pi}{2\lambda g''(a)}} (1 + o(1)), \text{ as } \lambda \rightarrow \infty.$$

Lemma 5.2. Consider an integral of the form

$$I(\lambda) = \int_a^b h(x) e^{\lambda g(x)} dx, \quad a, b \in \mathbb{R} \cup \{-\infty, \infty\}$$

where $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are smooth functions. Suppose that $g'(x) \neq 0$ in $[a, b]$. Then g has no local maximums in $[a, b]$ so the absolute maximum must occur at an endpoint.

- If the maximum is attained at $x = a$, then, provided $h(a) \neq 0$,

$$I(\lambda) \sim -\frac{h(a)}{\lambda g'(a)} e^{\lambda g(a)} (1 + o(1)), \text{ as } \lambda \rightarrow \infty.$$

- If the maximum is attained at $x = b$, then, provided $h(b) \neq 0$,

$$I(\lambda) \sim \frac{h(b)}{\lambda g'(b)} e^{\lambda g(b)} (1 + o(1)), \text{ as } \lambda \rightarrow \infty.$$

References

- [1] Pablo Groisman and Julio D. Rossi *The explosion time in stochastic differential equations with small diffusion*
- [2] Stéphanie Jacquot and Benedek Valkó. *Bulk scaling limit for the laguerre ensemble*. Electronic Journal of Probability Vol. 16 (2011), Paper no. 11, pages 314-346. MR-2771139
- [3] Eugene Kritchanski, Benedek Valkó and Bálint Virág. *The scaling limit of the critical one-dimensional random Schrödinger operator*. <http://arxiv.org/abs/1107.3058>
- [4] Benedek Valkó and Bálint Virág. *Continuum limits of random matrices and the Brownian carousel*. Invent. Math., 177(3):463–508, 2009. MR2534097. MR-2534097
- [5] Benedek Valkó and Bálint Virág. *Large gaps between random eigenvalues* The Annals of Probability Vol.38 (2010), Paper no.3,1263-1279 MR-2674999