Electron. Commun. Probab. **18** (2013), no. 85, 1–7.
DOI: 10.1214/ECP.v18-2918
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The quenched limiting distributions of a one-dimensional random walk in random scenery*

Nadine Guillotin-Plantard† Yueyun Hu‡ Bruno Schapira§

Abstract

For a one-dimensional random walk in random scenery (RWRS) on $\mathbb{Z}$, we determine its quenched weak limits by applying Strassen [13]'s functional law of the iterated logarithm. As a consequence, conditioned on the random scenery, the one-dimensional RWRS does not converge in law, in contrast with the multi-dimensional case.

Keywords: Random walk in random scenery; Weak limit theorem; Law of the iterated logarithm; Brownian motion in Brownian Scenery; Strong approximation.
AMS MSC 2010: 60F05; 60G52.
Submitted to ECP on July 8, 2013, final version accepted on October 25, 2013.

1 Introduction

Random walks in random sceneries were introduced independently by Kesten and Spitzer [9] and by Borodin [3, 4]. Let $S = (S_n)_{n\geq 0}$ be a random walk in $\mathbb{Z}^d$ starting at 0, i.e., $S_0 = 0$ and $(S_n - S_{n-1})_{n\geq 1}$ is a sequence of i.i.d. $\mathbb{Z}^d$-valued random variables. Let $\xi = (\xi_x)_{x\in\mathbb{Z}^d}$ be a field of i.i.d. real random variables independent of S. The field ξ is called the random scenery. The random walk in random scenery (RWRS) $K := (K_n)_{n\geq 0}$ is defined by setting $K_0 := 0$ and, for $n \in \mathbb{N}^*$,

$$K_n := \sum_{i=1}^{n} \xi_{S_i}. \tag{1.1}$$

We will denote by $\mathbb{P}$ the joint law of S and ξ. The law $\mathbb{P}$ is called the *annealed* law, while the conditional law $\mathbb{P}(\cdot|\xi)$ is called the *quenched* law.

Limit theorems for RWRS have a long history, we refer to [7] or [8] for a complete review. Distributional limit theorems for *quenched* sceneries (i.e. under the quenched law) are however quite recent. The first result in this direction that we are aware of was obtained by Ben Arous and Černý [1], in the case of a heavy-tailed scenery and planar random walk. In [7], quenched central limit theorems (with the usual $\sqrt{n}$-scaling and Gaussian law in the limit) were proved for a large class of transient random walks. More

*Research partially supported by ANR (MEMEMOII) 2010 BLAN 0125.
†Institut Camille Jordan, Université Lyon 1, France.
E-mail: nadine.guillotin@univ-lyon1.fr
‡Département de Mathématiques (LAGA CNRS-UMR 7539) Université Paris 13, France.
E-mail: yueyun@math.univ-paris13.fr
§Centre de Mathématiques et Informatique, Aix-Marseille Université, France.
E-mail: bruno.schapira@latp.univ-mrs.fr

recently, in [8], the case of the planar random walk was studied, the authors proved a quenched version of the annealed central limit theorem obtained by Bolthausen in [2].

In this note we consider the case of the simple symmetric random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}$, the random scenery $(\xi_x)_{x\in\mathbb{Z}}$ is assumed to be centered with finite variance equal to one and there exists some $\delta > 0$ such that $\mathbb{E}(|\xi_0|^{2+\delta}) < \infty$. We prove that under these assumptions, there is no quenched distributional limit theorem for K. In the sequel, for $-\infty \leq a < b \leq \infty$, we will denote by $\mathcal{AC}([a,b] \to \mathbb{R})$ the set of absolutely continuous functions defined on the interval $[a,b]$ with values in $\mathbb{R}$. Recall that if $f \in \mathcal{AC}([a,b] \to \mathbb{R})$, then the derivative of f (denoted by $\dot{f}$) exists almost everywhere and is Lebesgue integrable on $[a,b]$. Define

$$\mathcal{K}^* := \Big\{ f \in \mathcal{AC}(\mathbb{R} \to \mathbb{R}) : f(0) = 0, \int_{-\infty}^{\infty} (\dot{f}(x))^2 dx \leq 1 \Big\}. \tag{1.2}$$

Theorem 1. *For $\mathbb{P}$-a.e. ξ, under the quenched probability $\mathbb{P}\left(. \mid \xi\right)$, the process*

$$\tilde{K}_n := \frac{K_n}{(2n^{3/2}\log\log n)^{1/2}}, \qquad n > e^e,$$

does not converge in law. More precisely, for $\mathbb{P}$-a.e. ξ, under the quenched probability $\mathbb{P}\left(. \mid \xi\right)$, the limit points of the law of $\tilde{K}_n$, as $n \to \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B, with

$$\Theta_B := \Big\{ \int_{-\infty}^{\infty} f(x) dL_1(x) : f \in \mathcal{K}^* \Big\}, \tag{1.3}$$

where $(L_1(x), x \in \mathbb{R})$ denotes the family of local times at time 1 of a one-dimensional Brownian motion B starting from 0.

The set Θ_B is closed for the topology of weak convergence of measures, and is a compact subset of $L^2((B_t)_{t\in[0,1]})$.

Let us mention that the set $\mathcal{K}^*$ directly comes from Strassen [13]'s limiting set. The precise meaning of $\int_{-\infty}^{\infty} f(x)dL_1(x)$ can be given by the integration by parts and the occupation times formula:

$$\int_{-\infty}^{\infty} f(x)dL_1(x) = -\int_{-\infty}^{\infty} L_1(x)\dot{f}(x)dx = -\int_0^1 \dot{f}(B_s)ds, \tag{1.4}$$

where as before, $\dot{f}$ denotes the almost everywhere derivative of f.

Instead of Theorem 1, we shall prove that there is no quenched limit theorem for the continuous analogue of K introduced by Kesten and Spitzer [9] and deduce Theorem 1 by using a strong approximation for the one-dimensional RWRS. Let us define this continuous analogue: Assume that $B := (B(t))_{t\geq 0}$, $W := (W(t))_{t\geq 0}$, $\tilde{W} := (\tilde{W}(t))_{t\geq 0}$ are three real Brownian motions starting from 0, defined on the same probability space and independent of each other. For brevity, we shall write $W(x) := W(x)$ if $x \geq 0$ and $\tilde{W}(-x)$ if $x < 0$ and say that W is a two-sided Brownian motion. We denote by $\mathbb{P}_B$, $\mathbb{P}_W$ the law of these processes. We will also denote by $(L_t(x))_{t\geq 0, x\in\mathbb{R}}$ a continuous version with compact support of the local time of the process B. We define the continuous version of the RWRS, also called *Brownian motion in Brownian scenery*, as

$$Z_t := \int_0^{+\infty} L_t(x)dW(x) + \int_0^{+\infty} L_t(-x)d\tilde{W}(x) \equiv \int_{-\infty}^{+\infty} L_t(x)dW(x).$$

In dimension one, under the annealed measure, Kesten and Spitzer [9] proved that the process $(n^{-3/4}K([nt]))_{t\geq 0}$ weakly converges in the space of continuous functions to the continuous process $Z = (Z(t))_{t\geq 0}$. Zhang [14] (see also [6, 10]) gave a stronger version of this result in the special case when the scenery has a finite moment of order $2+\delta$ for some $\delta > 0$, more precisely, there is a coupling of ξ, S, B and W such that (ξ, W) is independent of (S, B) and for any $\varepsilon > 0$, almost surely,

$$\max_{0\leq m\leq n} |K(m) - Z(m)| = o(n^{\frac{1}{2}+\frac{1}{2(2+\delta)}+\varepsilon}), \quad n \to +\infty. \tag{1.5}$$

Theorem 1 will follow from this strong approximation and the following result.

Theorem 2. *$\mathbb{P}_W$-almost surely, under the quenched probability $\mathbb{P}(\cdot|W)$, the limit points of the law of*

$$\tilde{Z}_t := \frac{Z_t}{(2t^{3/2}\log\log t)^{1/2}}, \qquad t \to \infty,$$

under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B defined in Theorem 1. Consequently under $\mathbb{P}(\cdot|W)$, as $t \to \infty$, $\tilde{Z}_t$ does not converge in law.

To prove Theorem 2, we shall apply Strassen [13]'s functional law of the iterated logarithm applied to the two-sided Brownian motion W; we shall also need to estimate the stochastic integral $\int g(x)dL_1(x)$ for a Borel function g, see Section 2 for the details.

2 Proofs

For a two-sided one-dimensional Brownian motion $(W(t), t \in \mathbb{R})$ starting from 0, let us define for any $\lambda > e^e$,

$$W_\lambda(t) := \frac{W(\lambda t)}{(2\lambda \log\log\lambda)^{1/2}}, \qquad t \in \mathbb{R}.$$

Lemma 3. *(i) Almost surely, for any $s < 0 < r$ rational numbers, $(W_\lambda(t), s \leq t \leq r)$ is relatively compact in the uniform topology and the set of its limit points is $\mathcal{K}_{s,r}$, with*

$$\mathcal{K}_{s,r} := \Big\{ f \in \mathcal{AC}([s,r] \to \mathbb{R}) : f(0) = 0, \int_s^r (\dot{f}(x))^2 dx \leq 1 \Big\}.$$

(ii) There exists some finite random variable $\mathcal{A}_W$ only depending on $(W(x), x \in \mathbb{R})$ such that for all $\lambda \geq e^{36}$,

$$\sup_{t\in\mathbb{R}, t\neq 0} \frac{|W_\lambda(t)|}{\sqrt{|t|\log\log(|t| + \frac{1}{|t|} + 36)}} \leq \mathcal{A}_W < \infty.$$

Remark 4. *The statement (i) is a reformulation of Strassen's theorem and holds in fact for all real numbers s and r. Moreover, using the notation $\mathcal{K}^*$ in (1.2), we remark that $\mathcal{K}_{s,r}$ coincides with the restriction of $\mathcal{K}^*$ on $[s,r]$: for any $s < 0 < r$,*

$$\mathcal{K}_{s,r} = \Big\{ f\big|_{[s,r]} : f \in \mathcal{K}^* \Big\}.$$

Proof: (i) For any fixed $s < 0 < r$, by applying Strassen's theorem ([13]) to the two-dimensional rescaled Brownian motion: $(\frac{W(\lambda r u)}{\sqrt{2\lambda r \log\log\lambda}}, \frac{W(\lambda s u)}{\sqrt{2\lambda |s| \log\log\lambda}})_{0\leq u\leq 1}$, we get that a.s., $(W_\lambda(t), s \leq t \leq r)$ is relatively compact in the uniform topology with $\mathcal{K}_{s,r}$ as the set of limit points. By inverting a.s. and s, r, we obtain (i).

(ii) By the classical law of the iterated logarithm for the Brownian motion W (both at 0 and at ∞), we get that

$$\widetilde{\mathcal{A}}_W := \sup_{x\in\mathbb{R}, x\neq 0} \frac{|W(x)|}{\sqrt{|x|\log\log(|x| + \frac{1}{|x|} + 36)}}$$

is a finite variable. Observe that for any $t > 0$ and $\lambda > e^{36}$,

$$\begin{aligned} \log\log\Big(\lambda t + \frac{1}{\lambda t} + 36\Big) &= \log\Big(\log\lambda + \log\Big(t + \frac{1}{\lambda^2 t} + \frac{36}{\lambda}\Big)\Big) \\ &\le \log\Big(\log\lambda + \log\Big(t + \frac{1}{t} + 36\Big)\Big) \\ &\le \log\log\lambda + \log\log\Big(t + \frac{1}{t} + 36\Big) \end{aligned}$$

using that for every $a, b \ge 2$, $\log(a+b) \le \log(a) + \log(b)$. The Lemma follows if we take for e.g. $\mathcal{A}_W := 2\widetilde{\mathcal{A}}_W$. □

Next, we recall some properties of Brownian local times: The process $x \mapsto L_1(x)$ is a (continuous) semimartingale (by Perkins [11]), moreover, the quadratic variation of $x \mapsto L_1(x)$ equals $4\int_{-\infty}^x L_1(z)dz$. By Revuz and Yor ([12], Exercise VI (1.28)), for any locally bounded Borel function f,

$$\frac{1}{2}\int_{-\infty}^{\infty} f(x)dL_1(x) = -\int_0^{B_1} f(u)du + \int_0^1 f(B_u)dB_u. \tag{2.1}$$

Let us define for all $\lambda > e^e$ and $n \ge 0$,

$$H_\lambda := \int_{-\infty}^{\infty} W_\lambda(x)dL_1(x), \qquad H_\lambda^{(n)} := \int_{-n}^{n} W_\lambda(x)dL_1(x),$$

with $H_\lambda^{(0)} = 0$. Denote by $\mathbb{E}_B$ the expectation with respect to the law of B.

Lemma 5. *There exists some positive constant c_1 such that for any $\lambda > e^{36}$ and $n \ge 0$, we have*

$$\mathbb{E}_B\Big|H_\lambda - H_\lambda^{(n)}\Big| \le c_1\, e^{-\frac{n^2}{4}}\, \mathcal{A}_W, \tag{2.2}$$

$$\mathbb{E}_B\Big(\int_{-\infty}^{\infty} f(x)dL_1(x)\Big)^2 \le 16\, s(f), \tag{2.3}$$

$$\mathbb{E}_B\Big|\int_{-\infty}^{\infty} f(x)dL_1(x) - \int_{-n}^{n} f(x)dL_1(x)\Big| \le 4\sqrt{2s(f)}\, e^{-\frac{n^2}{4}}, \tag{2.4}$$

for any Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $s(f) := \sup_{0\le u\le 1} \mathbb{E}_B\Big[f^2(B_u)\Big] < \infty$.

Remark that if f is bounded, then $s(f) \le \sup_{x\in\mathbb{R}} f^2(x)$.

Proof: We first prove that there exists some positive constant c_2 such that for all $n \ge 0$ and $\lambda > e^{36}$,

$$\mathbb{E}_B\Big[(H_\lambda - H_\lambda^{(n)})^2\Big] \le c_2\, \mathcal{A}_W^2. \tag{2.5}$$

In fact, by applying (2.1) and using the Brownian isometry for $f(x) = W_\lambda(x)1_{(|x|>n)}$, we get that

$$\mathbb{E}_B\Big[(H_\lambda - H_\lambda^{(n)})^2\Big] \le 8\mathbb{E}_B\Big[F_{n,\lambda}(B_1)^2\Big] + 8\mathbb{E}_B\Big[\int_0^1 (W_\lambda(B_u))^2 1_{(|B_u|>n)}du\Big],$$

with $F_{n,\lambda}(x) := \int_0^x W_\lambda(y)1_{(|y|>n)}dy$ for any $x \in \mathbb{R}$. By Lemma 3 (ii),

$$|F_{n,\lambda}(x)| \le \mathcal{A}_W\,\Big|\int_0^x (|y|\log\log(|y| + \frac{1}{|y|} + 36))^{1/2}dy\Big| \le c_3\,\mathcal{A}_W\,(1+x^2), \qquad \forall x \in \mathbb{R},$$

with some constant $c_3 > 0$. (Here we used that $\log x < x$ for $x > 0$ and that for any $a, b > 0$, $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$). Hence $\mathbb{E}_B\Big[F_{n,\lambda}(B_1)^2\Big] \le 6\, c_3^2\, \mathcal{A}_W^2$. In the same way,

$\mathbb{E}_B\big[(W_\lambda(B_u))^2\big] \le \mathcal{A}_W^2\, \mathbb{E}_B\big[|B_u|\log\log(|B_u| + \frac{1}{|B_u|} + 36)\big]$ which is integrable for $u \in (0,1]$. Then (2.5) follows.

To check (2.2), we remark that $H_\lambda - H_\lambda^{(n)} = 0$ if $\sup_{0\le u\le 1}|B_u| \le n$. Then by Cauchy-Schwarz' inequality and (2.5), we have that

$$\begin{aligned}\mathbb{E}_B\Big|H_\lambda - H_\lambda^{(n)}\Big| &= \mathbb{E}_B\Big[\big|H_\lambda - H_\lambda^{(n)}\big|\, 1_{(\sup_{0\le u\le 1}|B_u|>n)}\Big] \\ &\le \sqrt{\mathbb{E}_B\Big[(H_\lambda - H_\lambda^{(n)})^2\Big]}\sqrt{\mathbb{P}_B\Big(\sup_{0\le u\le 1}|B_u| > n\Big)} \\ &\le \sqrt{c_2}\,\mathcal{A}_W\,\sqrt{2}\,e^{-\frac{n^2}{4}},\end{aligned}$$

by the standard Gaussian tail: $\mathbb{P}_B\big(\sup_{0\le u\le 1}|B_u| > x\big) \le 2e^{-x^2/2}$ for any $x>0$. Then we get (2.2).

To prove (2.3), we use again (2.1) and the Brownian isometry to arrive at

$$\mathbb{E}_B\Big(\int_{-\infty}^{\infty} f(x)dL_1(x)\Big)^2 \le 8\mathbb{E}_B\Big[G^2(B_1)\Big] + 8\int_0^1 \mathbb{E}_B\Big[f^2(B_u)\Big]du \le 8\mathbb{E}_B\Big[G^2(B_1)\Big] + 8s(f),$$

with $G(x) := \int_0^x f(y)dy$ for any $x \in \mathbb{R}$. By Cauchy-Schwarz' inequality, $(G(x))^2 \le \Big|x\int_0^x f^2(y)dy\Big|$ for any $x\in\mathbb{R}$, from which we use the integration by parts for the density of B_1 and deduce that $\mathbb{E}_B\Big[G^2(B_1)\Big] \le \mathbb{E}_B\Big[f^2(B_1)\Big]$. Then (2.3) follows.

Finally for (2.4), we use (2.3) to see that

$$\mathbb{E}_B\Big(\int_{-\infty}^{\infty} f(x)dL_1(x) - \int_{-n}^{n} f(x)dL_1(x)\Big)^2 = \mathbb{E}_B\Big(\int_{-\infty}^{\infty} f(x)1_{(|x|>n)}dL_1(x)\Big)^2 \le 16s(f),$$

for any n. Then (2.4) follows from the Cauchy-Schwarz inequality and the Gaussian tail, exactly in the same way as (2.2). □

Recalling (1.3) for the definition of Θ_B. For any $p>0$, it is easy to see that $\Theta_B \subset L^p(B)$, since from Cauchy-Schwarz' inequality, using the relation (1.4), we deduce that

$$\Big(\int_{-\infty}^{\infty} f(x)dL_1(x)\Big)^2 \le \Big(\int_{-\infty}^{\infty}(L_1(x))^2dx\Big)\Big(\int_{-\infty}^{\infty}(\dot f(x))^2dx\Big) \le \sup_x L_1(x) \in L^p(B),$$

see Csáki [5], Lemma 1 for the tail of $\sup_x L_1(x)$. Write $d_{L^1(B)}(\xi,\eta)$ for the distance in $L^1(B)$ for any $\xi,\eta\in L^1(B)$.

Lemma 6. $\mathbb{P}_W$*-almost surely,*

$$d_{L^1(B)}(H_\lambda, \Theta_B) \to 0, \qquad \text{as } \lambda\to\infty,$$

where Θ_B *is defined in* (1.3)*. Moreover,* $\mathbb{P}_W$*-almost surely for any* $\xi\in\Theta_B$*,*

$$\liminf_{\lambda\to\infty} d_{L^1(B)}(H_\lambda,\xi) = 0.$$

Proof: Let $\varepsilon>0$. Choose a large $n=n(\varepsilon)$ such that $c_1e^{-n^2/4} \le \varepsilon$. By Lemma 3 (i), for all large $\lambda \ge \lambda_0(W,\varepsilon,n)$, there exists some function $g = g_{\lambda,W,\varepsilon,n} \in \mathcal{K}_{-n,n}$ such that $\sup_{|x|\le n}|W_\lambda(x) - g(x)| \le \varepsilon$. Applying (2.3) to $f(x) = (W_\lambda(x) - g(x))1_{(|x|\le n)}$ which is bounded by ε, we get that

$$\mathbb{E}_B\Big|H_\lambda^{(n)} - \int_{-n}^{n} g(x)dL_1(x)\Big| \le 4\sqrt{s(f)} \le 4\varepsilon.$$

We extend g to $\mathbb{R}$ by letting $g(x) = g(n)$ if $x \geq n$ and $g(x) = g(-n)$ if $x \leq -n$, then $g \in \mathcal{K}^*$ and $\int_{-\infty}^{\infty} g(x)dL_1(x) = \int_{-n}^{n} g(x)dL_1(x)$. By the triangular inequality and (2.2),

$$\mathbb{E}_B\Big|H_\lambda - \int_{-\infty}^{\infty} g(x)dL_1(x)\Big| \leq 4\varepsilon + \mathbb{E}_B\Big|H_\lambda - H_\lambda^{(n)}\Big| \leq (4 + c_1\mathcal{A}_W)\varepsilon.$$

It follows that $d_{L^1(B)}(H_\lambda, \Theta_B) \leq (4 + c_1\mathcal{A}_W)\varepsilon$. Hence $\mathbb{P}_W$-a.s.,

$$\limsup_{\lambda\to\infty} d_{L^1(B)}(H_\lambda, \Theta_B) \leq (4 + c_1\mathcal{A}_W)\varepsilon,$$

showing the first part in the Lemma.

For the other part of the Lemma, let $h \in \mathcal{K}^*$ such that $\xi = \int_{-\infty}^{\infty} h(x)dL_1(x)$. Observe that $|h(x)| \leq \sqrt{\left|x\int_0^x (\dot{h}(y))^2 dy\right|} \leq \sqrt{|x|}$ for all $x \in \mathbb{R}$, $s(h) = \sup_{0\leq u\leq 1} \mathbb{E}_B[h^2(B_u)] \leq \mathbb{E}_B[|B_1|]$, then for any $\varepsilon > 0$, we may use (2.4) and choose an integer $n = n(\varepsilon)$ such that $(c_1 + 4\sqrt{2})e^{-n^2/4} \leq \varepsilon$ and

$$d_{L^1(B)}(\xi, \xi_n) \leq \varepsilon,$$

where $\xi_n := \int_{-n}^{n} h(x)dL_1(x)$. Applying Lemma 3 (i) to the restriction of h on $[-n, n]$, we may find a sequence $\lambda_j = \lambda_j(\varepsilon, W, n) \to \infty$ such that $\sup_{|x|\leq n} |W_{\lambda_j}(x) - h(x)| \leq \varepsilon$. By applying (2.3) to $f(x) = (W_{\lambda_j}(x) - h(x))1_{(|x|\leq n)}$, we have that

$$d_{L^1(B)}(H_{\lambda_j}^{(n)},\, \xi_n) \leq 4\varepsilon.$$

By (2.2) and the choice of n, $d_{L^1(B)}(H_{\lambda_j}^{(n)}, H_{\lambda_j}) \leq \varepsilon\mathcal{A}_W$ for all large λ_j, it follows from the triangular inequality that

$$d_{L^1(B)}(\xi, H_{\lambda_j}) \leq (5 + \mathcal{A}_W)\varepsilon,$$

implying that $\mathbb{P}_W$-a.s., $\liminf_{\lambda\to\infty} d_{L^1(B)}(H_\lambda, \xi) \leq (5 + \mathcal{A}_W)\varepsilon \to 0$ as $\varepsilon \to 0$. □

We now are ready to give the proof of Theorems 2 and 1.

Proof of Theorem 2. Firstly, we remark that by Brownian scaling, $\mathbb{P}_W$-a.s.,

$$\frac{Z_t}{t^{3/4}} \overset{(d)}{=} -\int_{m_1}^{M_1} \frac{1}{t^{1/4}} W(\sqrt{t}y)dL_1(y). \tag{2.6}$$

In fact, by the change of variables $x = y\sqrt{t}$, we get

$$\int_{-\infty}^{+\infty} L_t(x)dW(x) = \sqrt{t}\int_{-\infty}^{+\infty} \left(\frac{L_t(y\sqrt{t})}{\sqrt{t}}\right) dW(y\sqrt{t})$$

which has the same distribution as

$$\sqrt{t}\int_{-\infty}^{+\infty} L_1(y)dW(y\sqrt{t})$$

from the scaling property of the local time of the Brownian motion. Since $(L_1(x))_{x\in\mathbb{R}}$ is a continuous semi-martingale, independent from the process W, from the formula of integration by parts, we get that $\mathbb{P}_W$ -a.s.,

$$\sqrt{t}\int_{-\infty}^{+\infty} L_1(y)dW(y\sqrt{t}) = -t^{3/4}\int_{m_1}^{M_1} \left(\frac{W(\sqrt{t}y)}{t^{1/4}}\right) dL_1(y),$$

yielding (2.6). Theorem 2 follows from Lemma 6. □

Proof of Theorem 1. We use the strong approximation of Zhang [14] : there exists on a suitably enlarged probability space, a coupling of ξ, S, B and W such that (ξ, W) is independent of (S, B) and for any $\varepsilon > 0$, almost surely,

$$\max_{0\le m\le n} |K(m) - Z(m)| = o(n^{\frac{1}{2}+\frac{1}{2(2+\delta)}+\varepsilon}), \quad n \to +\infty.$$

From the independence of (ξ, W) and (S, B), we deduce that for $\mathbb{P}$-a.e. (ξ, W), under the quenched probability $\mathbb{P}(.|\xi, W)$, the limit points of the laws of $\tilde{K}_n$ and $\tilde{Z}_n$ are the same ones. Now, by adapting the proof of Theorem 2, we have that for $\mathbb{P}$-a.e. (ξ, W), under the quenched probability $\mathbb{P}(.|\xi, W)$, the limit points of the laws of $\tilde{Z}_n$, as $n \to \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B. It gives that for $\mathbb{P}$-a.e. (ξ, W), under the quenched probability $\mathbb{P}(.|\xi, W)$, the limit points of the laws of $\tilde{K}_n$, as $n \to \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B and the first part of Theorem 1 follows.

Let $(\zeta_n)_n$ be a sequence of random variables in Θ_B, each ζ_n being associated to a function $f_n \in \mathcal{K}^*$. The sequence of the (almost everywhere) derivatives of f_n is then a bounded sequence in the Hilbert space $L^2(\mathbb{R})$, so we can extract a subsequence which weakly converges. Using the definition of the weak convergence and the relation (1.4), $(\zeta_n)_n$ converges almost surely and the closure of Θ_B follows. Since the sequence $(\zeta_n)_n$ is bounded in $L^p(B)$ for any $p \ge 1$, the convergence also holds in $L^2(B)$. Therefore Θ_B is a compact set of $L^2(B)$ as closed and bounded subset. □

References

[1] Ben Arous, G. and Černý, J., (2007) Scaling limit for trap models on $\mathbb{Z}^d$, *Ann. Probab.* 35 (6), 2356 – 2384. MR-2353391

[2] Bolthausen, E. (1989) A central limit theorem for two-dimensional random walks in random sceneries, *Ann. Probab.* 17, 108–115. MR-0972774

[3] Borodin, A.N. (1979) A limit theorem for sums of independent random variables defined on a recurrent random walk, *Dokl. Akad. nauk SSSR* 246 (4), 786 – 787. MR-0543530

[4] Borodin, A.N. (1979) Limit theorems for sums of independent random variables defined in a transient random walk, in Investigations in the Theory of Probability Distributions, IV, Zap, Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 85, 17-29. 237 244. MR-0535455

[5] Csáki, E. (1989) An integral test for the supremum of Wiener local time. *Probab. Th. Rel. Fields* 83, 207–217. MR-1012499

[6] Csáki, E., König, W. and Shi, Z. (1999) An embedding for the Kesten-Spitzer random walk in random scenery, *Stochastic Process. Appl.* 82 (2), 283-292. MR-1700010

[7] Guillotin-Plantard, N. and Poisat, J. (2013) Quenched central limit theorems for random walks in random scenery, *Stochastic Process. Appl.* 123 (4), 1348 – 1367. MR-3016226

[8] Guillotin-Plantard, N., Poisat, J. and Dos Santos, R.S. (2013) A quenched central limit theorem for planar random walks in random sceneries, *Submitted.*

[9] Kesten, H. and Spitzer, F. (1979) A limit theorem related to a new class of self-similar processes, *Z. Wahrsch. Verw. Gebiete* 50 (1), 5–25. MR-0550121

[10] Khoshnevisan, D. and Lewis, T.M. (1998) A law of the iterated logarithm for stable processes in random scenery, *Stochastic Process. Appl.* 74 (1), 89–121. MR-1624017

[11] Perkins, E. (1982). Local time is a semimartingale. *Z. Wahrsch. Verw. Gebiete* **60** no. 1, 79 – 117. MR-0661760

[12] Revuz, D. and Yor, M. (1999). *Continuous martingales and Brownian motion.* Third edition. Springer-Verlag, Berlin. MR-1725357

[13] Strassen, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. Verw. Gebiete* **3**, Issue 3, 211 – 226. MR-0175194

[14] Zhang, L. (2001) The strong approximation for the Kesten-Spitzer random walk. *Statistics & Probability Letters* **53**, 21 – 26. MR-1843337

Acknowledgments. We are grateful to Mikhail Lifshits for interesting discussions. The authors thank the referee for recommending various improvements in exposition.