Containing internal diffusion limited aggregation

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Abstract

Internal Diffusion Limited Aggregation (IDLA) is a model that describes the growth of a random aggregate of particles from the inside out. Shellef proved that IDLA processes on supercritical percolation clusters of integer-lattices fill Euclidean balls, with high probability. In this article, we complete the picture and prove a limit-shape theorem for IDLA on such percolation clusters, by providing the corresponding upper bound.

The technique to prove upper bounds is new and robust: it only requires the existence of a "good" lower bound. Specifically, this way of proving upper bounds on IDLA clusters is more suitable for random environments than previous ways, since it does not harness harmonic measure estimates.

1 Introduction

The Internal Diffusion Limited Aggregation (IDLA) model was introduced by Diaconis and Fulton in [7], and gives a protocol for recursively building a random aggregate of particles. At each step, the first vertex visited outside the current aggregate by a random walk started at the origin is added to the aggregate. This simulates the growth of an aggregate of particles from the inside out.

In a number of settings, this model is known to have a deterministic limit-shape, meaning that a random aggregate with a large number of particles has a typical shape. On $\mathbb{Z}^d$, Lawler, Bramson and Griffeath [13] were the first to identify this limit-shape, in the case of simple random walks, as an Euclidean ball. Their result was later sharpened by Lawler [12], and was recently drastically improved with the simultaneous works of Asselah and Gaudillère [2, 3] and Jerison, Levine and Sheffield [10, 11, 9]. On other graphs, current knowledge is less precise. On groups with polynomial growth, the existence of the limiting shape is unknown, although Blachère gave bounds on the cluster [5]. On finitely generated groups with exponential growth, with a suitable metric, a limit-shape result was proved by Blachère and Brofferio [6]. Huss also studied IDLA for a large class of random walks on general non-amenable graphs in [8].

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Another interesting question about IDLA is whether the limit-shape is robust to small perturbations of the underlying graph; For example, on the infinite cluster of supercritical percolation cluster of $\mathbb{Z}^d$. Shellef proved a sharp inner bound for the IDLA model on the infinite cluster [14]. Figure 1 presents the IDLA aggregate built on the supercritical bond percolation cluster.
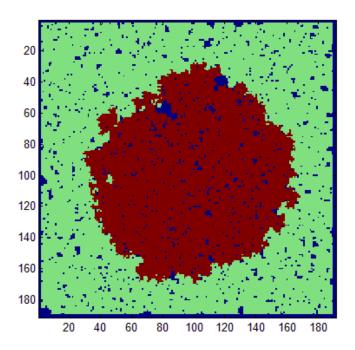

Figure 1: IDLA *aggregate with 6,300 particles on the supercritical percolation cluster on $\mathbb{Z}^2$ (edges are deleted w.p. 0.4). Red points are in the aggregate, green points are in the cluster, and blue points are outside the cluster.*

To prove the existence of a limit-shape on the supercritical percolation cluster on $\mathbb{Z}^d$, it thus remains to show a sharp outer bound on IDLA. We provide such an outer bound, relying on Shellef's inner bound.

Theorem 1.1. *Fix $d > 0$ and $p > p_c(\mathbb{Z}^d)$ and let ω be the infinite cluster of percolation on $\mathbb{Z}^d$ with parameter p, conditioned on the origin o belonging to ω. Let $B_o(n)$ be the Euclidean ball of radius n centered at the origin. Let $b_o(n) = |\omega \cap B_o(n)|$, the size of the part of the cluster that is in $B_o(n)$. Let $A_{b_o(n)}(o)$ be* IDLA *generated by $b_o(n)$ particles started at the origin. Then, for every $\varepsilon > 0$ and a.s. every ω, the* IDLA *on ω a.s. satisfies:*

$$B_o((1-\varepsilon)n) \subset A_{b_o(n)}(o) \subset B_o((1+\varepsilon)n) \qquad \forall\, n \text{ large enough.}$$

More generally, Theorem 1.2 below states that given a sharp inner bound on IDLA for a graph (with some regularity assumptions), a sharp outer bound exists as well. The inner and outer bounds together prove the almost sure convergence to a limit-shape. The regularity assumptions are quite mild, and the theorem should be useful in great generality. Since the percolation cluster satisfies the necessary regularity assumptions, Theorem 1.2 implies Theorem 1.1. We also mention other environments that satisfy the regularity assumptions, like Cayley graphs with polynomial growth (more details follow).

Let us mention that, despite the fact that the upper bound on $\mathbb{Z}^d$ is not intrinsically harder than the lower bound, they invoke different ingredients. The lower bound usually harnesses estimates on the Green function, while the upper bound requires the use of upper bounds on harmonic measures. The Green function is fairly well understood in several (random) environments such as supercritical percolation, random conductances with elliptic condition, and so forth. This is not the case for harmonic measures. The techniques developed in this article allow to bypass this difficulty. Our main result relates an upper bound on IDLA directly to a lower bound. Roughly speaking, we show that if one knows that balls of a certain metric are contained in the aggregate, and not many particles are left over, then one can deduce an upper bound and shape theorem.

1.1 Definition of IDLA

Let G be a graph. Let $S \subset G$ be a finite subset of G. In order to define IDLA, first define adding one particle started at x to an existing aggregate S. For $x \in G$, denote by $A(S; x)$ the IDLA aggregate obtained as follows: Let $\xi = (\xi(0), \xi(1), \ldots)$ be a random walk on G started at $\xi(0) = x$ and let t_S be the first time this walk is not in S. Define

$$A(S; x) := S \cup \{\xi(t_S)\}.$$

It is standard to consider a slightly more general process, where the growth of the aggregate is stopped at certain stopping times, *e.g.*, upon exiting a set T. Denote by $A(S; x \mapsto T)$ the aggregate obtained by letting a particle randomly walk from x, but pausing it if it exits T, which is defined as follows. Let ξ be a random walk on G started at x. Let t_S be the first time this walk is not in S as above, and let t_T be the first time ξ exits T. Define

$$A(S; x \mapsto T) := S \cup \{\xi(t_S \wedge (t_T - 1))\}.$$

To keep track of the position of a paused particle, define

$$P(S; x \mapsto T) = \begin{cases} \xi(t_T) & \text{if } t_T \leq t_S, \\ \bot & \text{otherwise,} \end{cases}$$

so $\bot$ means that the particle has settled before exiting T and will never reach its complement.

Given vertices $x_1, \ldots, x_k$ in G and a set T, define $A(S; x_1, \ldots, x_k \mapsto T)$ to be the IDLA aggregate formed from an existing aggregate S by k particles, started at $x_1, \ldots, x_k$, and paused upon exiting T. That is, define inductively: $S_0 = S$, $S_j = A(S_{j-1}, x_j \mapsto T)$ for $j \in \{1, \ldots, k\}$, and $A(S; x_1, \ldots, x_k \mapsto T) = S_k$. Again, to keep track of paused particles, define $P(S; x_1, \ldots, x_k \mapsto T)$ to be the sequence of particles paused in this process; Formally, if $p_j = P(S_{j-1}; x_j \mapsto T)$ for $j \in \{1, \ldots, k\}$, then $P(S; x_1, \ldots, x_k \mapsto T)$ is the sequence $(p_j : p_j \neq \bot)$. When particles are not stopped, we define the aggregate similarly and we denote it by $A(S; x_1, \ldots, x_k)$.

One reason to keep track of these paused particles is the so called *Abelian property* of IDLA:

$A(S; x_1, \ldots, x_k)$ has the same distribution as
$$A\big(A(S; x_1, \ldots, x_k \mapsto T); P(S; x_1, \ldots, x_k \mapsto T)\big). \tag{1.1}$$

Equation (1.1) says that in order to sample $A(S, x_1, \ldots, x_k)$, one can sample $A(S; x_1, \ldots, x_k \mapsto T)$ while pausing particles upon exiting T and keeping track of them via $P(S; x_1, \ldots, x_k \mapsto T)$, and then restart the paused particles on the obtained aggregate $A(S; x_1, \ldots, x_k \mapsto T)$. For more details, see [7, 13].

We are mostly interested in n particles starting at just one point. For an integer $n \geq 0$, by $A_n(x)$ we denote the IDLA aggregate with n particles started at x, that is,

$$A_n(x) = A(\emptyset; \underbrace{x, \ldots, x}_{n \text{ times}}).$$

We also focus on pausing particles according to a metric ρ. Define the *ball* of radius r around x to be $B_x(r) = \{y : \rho(x, y) < r\}$, and denote its size by $b_x(r) = |B_x(r)|$. As above, set

$$A_n(x \mapsto r) = A(\emptyset; x, \ldots, x \mapsto B_x(r))$$

and

$$P_n(x \mapsto r) = P(\emptyset; x, \ldots, x \mapsto B_x(r)).$$

1.2 Assumptions

We now make a few assumptions. The first two assumptions are independent of the IDLA process. We will always compare the IDLA cluster to a ball for a certain metric ρ on G, on which we now make a few hypotheses. Assume that ρ satisfies the following assumptions:

Continuity (C)**:** the metric is dominated by the graph distance d_G on G: there exists $c > 0$ such that

$$\rho(x, y) \leq c \cdot d_G(x, y) \quad \forall \, x, y \in G.$$

Regular volume growth (VG)**:** there exist $c, d > 0$ such that for every $n > 0$,

$$\frac{1}{c} r^d \leq |B_x(r)| \leq c r^d \qquad \forall \, x \in B_o(n) \text{ and } n^{1/d^3} \leq r \leq n.$$

The conditions on r and n allow to consider inhomogeneous environments. Keep in mind that the percolation cluster contains arbitrary finite graphs infinitely often.

We will also make the following assumption on IDLA:

Weaker lower bound (wLB)**:** there exists $\alpha > 0$ such that for every $n > 0$,

$$\mathbb{P}[B_x(r) \subset A_{b_x(r/\alpha)}(x \mapsto r)] \geq \alpha \qquad \forall \, x \in B_o(n + r) \text{ and } n^{1/d^3} \leq r \leq n.$$

In words, with noticeable probability, when releasing order $b_x(r)$ particles at x the aggregate contains $B_x(r)$. This assumption is easy to verify for many IDLA processes. In order to prove the existence of a limiting shape, we will assume a stronger lower bound for the aggregate grown around the origin, as we will see in the next section.

1.3 Main results

The first theorem relates a lower bound to an upper bound in well-behaved environments.

Theorem 1.2. *Let G be a graph and ρ be a metric on G satisfying conditions (C), (VG) and (wLB). Then, there exists a constant $c_1 > 0$ such that for any $\varepsilon > 0$,*

$$\mathbb{P}\big[A_{b_o(n)}(o) \not\subset B_o((1 + \varepsilon)n) \ i.o.\big] \leq \mathbb{P}\big[|A_{b_o(n)}(o \mapsto n)| < b_o(n)(1 - c_1 \varepsilon^d) \ i.o.\big].$$

The previous theorem is especially useful when a lower bound is known. Indeed, although we need a statement stronger than a simple lower bound, the usual proofs of lower bounds always yield the fact that the IDLA process (almost) fills all large enough balls when stopped on their boundary:

Lower bound (LB)**:** $|A_{b_o(n)}(o \mapsto n)|/b_o(n)$ converges almost surely to 1.

Note that (LB) implies not only that the aggregate eventually contains a large ball, but also that the aggregate contains a large ball, even when stopped upon exiting a (slightly bigger) ball.

Corollary 1.3. *Let G be a graph and ρ a metric on G satisfying conditions (C), (VG), (wLB), and (LB), then for every $\varepsilon > 0$, a.s.*

$$A_{b_o(n)}(o) \subset B_o((1+\varepsilon)n) \qquad \forall \, n \text{ large enough}.$$

Even though the main application of the theorem above and its corollary will be in the case of supercritical percolation, they apply in great generality and we believe that they will be useful in future works on IDLA.

2 An upper bound on IDLA on percolation clusters

Let us prove that Theorem 1.2 and its corollary imply Theorem 1.1. In this case, ρ will be the Euclidean distance. By Theorem 1.2, we only need to check that (C), (VG), (wLB) and (LB) are satisfied:

(C) Since $\rho \leq d_\omega$ deterministically, property (C) is satisfied for every ω with the constant 1.

(VG) Barlow proved in [4] that (VG) is satisfied for the distance d_ω for almost every environment. A classical result of [1] easily implies that for almost every environment ω, there exists $c_1 = c_1(\omega) > 0$ such that for every $n > 0$,

$$\rho(x,y) \leq d_\omega(x,y) \leq c_1 \rho(x,y) \quad \forall \, x,y \in B_o(n) : c_1 \log n \leq \rho(x,y) \leq n \tag{2.1}$$

so that (VG) is also satisfied for ρ with a possibly different constant (the result also follows from [4]).

(wLB) In [14], Shellef proved that for any $\varepsilon > 0$, there exists $\eta > 0$ such that the following holds for almost every environment ω: there exists $c_2 = c_2(\omega) > 0$ such that

$$\mathbb{P} \left[B_o((1-\varepsilon)n) \subset A_{b_o(n)}(o \mapsto n) \right] \geq 1 - \frac{c_2}{n^{d+2}}$$

and $\mathbb{P}_p[c_2 \geq \lambda] \leq e^{-\lambda^\eta}$ for all $\lambda > 0$. All together, this implies that for almost every environment ω, there exists $c_3 = c_3(\omega) > 0$ such that for every $n > 0$,

$$\mathbb{P} \left[B_x((1-\varepsilon)r) \subset A_{b_x(r)}(x \mapsto r) \quad \forall x \in B_o(n) \text{ and } n^{1/d^3} \leq r \leq n \right] \geq 1 - \frac{c_3}{n}. \tag{2.2}$$

The condition (wLB) follows readily. The result in Shellef deals with the event $B_o((1-\varepsilon)n) \subset A_{b_o(n)}(o)$ (particles are not stopped at distance n), but the proof actually implies this stronger result.

(LB) Finally, note that the comparison between distances (2.1) and (VG) imply that

$$b_o(n) - b_o((1-\varepsilon)n) \ \leq \ c_4 \varepsilon^d b_o(n)$$

for some constant $c_4 = c_4(\omega) > 0$ depending on the environment, so that (2.2) implies (LB) for almost every environment.

3 Proof of Theorem 1.2

From now on, we fix a graph G satisfying (C), (VG) and (wLB). Constants in the proof always depend only on the constants involved in (C), (VG) and (wLB), *i.e.* c, d and α.

The following lemma shows that a lower bound on the aggregate implies a lower bound on hitting probabilities. It is a general statement not invoking any of the conditions (C), (VG) or (wLB). In the following, we make a slight abuse of notations: ξ will denote a random walk as well as its trace.

Lemma 3.1. *Let $Q \subset B \subset G$ and $x \in B$. Let ξ be a random walk started at x and stopped on exiting B. For any $t > 0$,*

$$\mathbb{P}[\xi \cap Q \neq \emptyset] \geq \mathbb{P}[B \subset A_t(x \mapsto B)] \cdot |Q|/t.$$

The above lemma is most useful when t is chosen so that $\mathbb{P}[B \subset A_t(x \mapsto B)]$ is of order 1.

Proof. Let $\xi_1, \ldots, \xi_t$ be the t independent random walks started at x and stopped on exiting B that generate the aggregate $A_t(x \mapsto B)$. Let $J \in \{1, \ldots, t\}$ be a uniformly chosen index independent of the random walks. Consider the set Γ of $j \in \{1, \ldots, t\}$ so that ξ_j hits Q before exiting B. Since $Q \subset B$, the inclusion $B \subset A_t(x \mapsto B)$ implies $|\Gamma| \geq |Q|$. Since J is independent of $\xi_1, \ldots, \xi_t$,

$$\mathbb{P}[\xi_J \cap Q \neq \emptyset \mid B \subset A_t(x \mapsto B)] \geq \mathbb{P}[J \in \Gamma \mid B \subset A_t(x \mapsto B)] \geq |Q|/t.$$

The lemma follows since the distribution of ξ_J is that of a random walk started at x and stopped when exiting B. $\qquad\square$

By assumption on G, we can hence get the following hitting probability estimate, which states that a random walk hits a set, whose complement has size at most ϵr^d, with a probability that is bounded away from zero.

Lemma 3.2. *There exist $\epsilon, \eta > 0$ such that for large enough n and $n^{1/(d(d+1))} < r < n$, the following holds. Let $x \in B_o(n)$ and let $S \subset B_o(n+r)$ be so that $|S \setminus B_o(n)| \leq \epsilon r^d$. Let ξ be a random walk started at x and stopped upon exiting $B_o(n+r)$. Then,*

$$\mathbb{P}\left[\xi \cap \big(B_o(n+r) \setminus (S \cup B_o(n))\big) \neq \emptyset\right] \geq \eta.$$

Proof. For every path γ from inside $B_o(n)$ to outside $B_o(n+r)$, let $y(\gamma)$ be the first vertex on γ so that $\rho(y(\gamma), B_o(n)) \geq r/2$. Denote by Y the set of all $y(\gamma)$ for such paths γ. Every path from x to outside $B_o(n+r)$ must hit Y. By Markov's property, it thus suffices to prove the theorem for starting points $y \in Y$. Fix $y \in Y$.

Let $B = B_y(r/3)$ and $Q = B \setminus S$. By (VG) and by assumption on S,

$$|Q| \geq \frac{1}{c}(r/3)^d - \epsilon r^d \geq \frac{1}{c}(r/4)^d,$$

with $\epsilon = 4^{-d}/c$. By (wLB) with $t = b_y(r/(3\alpha))$,

$$\mathbb{P}[B \subset A_t(y \mapsto r/3)] \geq \alpha.$$

Let ξ be a random walk started at y and stopped on exiting B. Lemma 3.1 and (VG) imply that

$$\mathbb{P}[\xi \cap Q \neq \emptyset] \geq \alpha \frac{(r/4)^d}{cb_y(r/(3\alpha))} \geq \alpha(\alpha/4)^d/c^2 =: \eta.$$

Note that $r/2 \leq \rho(y, B_o(n)) \leq r/2 + c$ thanks to the definition of Y and (C). Therefore, $\rho(y, G \setminus B_o(n+r)) \geq r - r/2 - c > r/3$ for n large, and so $B \subset B_o(n+r) \setminus B_o(n)$. We deduce

$$\mathbb{P}\left[\xi \cap \left(B_o(n+r) \setminus (S \cup B_o(n))\right) \neq \emptyset\right] \geq \mathbb{P}[\xi \cap Q \neq \emptyset] \geq \eta.$$

$\square$

After analyzing the behavior of a single particle, we can analyze the behavior of the whole aggregate. The following lemma says that, with high probability, a constant fraction of the aggregate is absorbed in a wide enough (yet still very fine) annulus.

Lemma 3.3. *There exist $\delta > 0$ and $p < 1$ such that for all n large enough, for all $n^{1/(d+1)} < k < n$ and $x_1, \ldots, x_k \in B_o(n)$, and for all $S \subset B_o(n)$,*

$$\mathbb{P}\left[|A(S; x_1, \ldots, x_k \mapsto B_o(n + k^{1/d})) \setminus S| \leq \delta k\right] \leq p^k.$$

Proof. Let $r = k^{1/d}$. Fix ϵ, η as in Lemma 3.2. Let $\xi_1, \ldots, \xi_k$ be the random walks started at $x_1, \ldots, x_k$ that generates the aggregate. Let $k' = \lfloor \epsilon k \rfloor \leq \epsilon r^d$. For $j \in \{1, \ldots, k'\}$, denote

$$A_j = A(S; x_1, \ldots, x_j \mapsto B_o(n+r)).$$

Since $|A_j \setminus B_o(n)| \leq j \leq \varepsilon r^d$, Lemma 3.2 implies that for all $j \in \{1, \ldots, k'\}$,

$$\mathbb{P}\left[\xi_{j+1} \cap \left(B_o(n+r) \setminus A_j\right) \neq \emptyset \mid A_j\right] \geq \eta.$$

Therefore, $|A(S; x_1, \ldots, x_k \mapsto B_o(n+r)) \setminus S|$ dominates a (k', η)-binomial random variable. Thus, there exist $\delta > 0$ and $p < 1$ depending only on ϵ, η such that

$$\mathbb{P}\left[|A(S; x_1, \ldots, x_k \mapsto B_o(n+r)) \setminus S| \leq \delta k\right] \leq p^k.$$

$\square$

We now turn to the proof of Theorem 1.2. The proof consists of inductively constructing a sequence of aggregates A_j by pausing the particles at different distances n_j from the origin. If k_j is the number of paused particles, we choose the next distance n_{j+1}, at which we pause the particles again, in terms of n_j and k_j. We iterate this procedure until there are less than $n^{1/(d+1)}$ paused particles. At this point, there are too few particles to matter.

Proof of Theorem 1.2. Fix $n > 0$. Define A_j, n_j, P_j, k_j as follows:

- Let $n_0 = n$ and $A_0 = A_{b_o(n)}(o \mapsto n)$. Let $P_0 = P_{b_o(n)}(o \mapsto n)$ and let $k_0 = |P_0|$.
- For $j \geq 0$, define
$$n_{j+1} = \begin{cases} n_j + k_j^{1/d} & \text{if } k_j > n^{1/(d+1)}, \\ \infty & \text{otherwise.} \end{cases}$$
  Let $A_{j+1} = A(A_j; P_j \mapsto B_o(n_{j+1}))$. Let $P_{j+1} = P(A_j; P_j \mapsto B_o(n_{j+1}))$ and let $k_{j+1} = |P_{j+1}|$.

Let J be the (random) first time at which $k_J \leq n^{1/(d+1)}$. By construction, $A_j = A_{J+1}$ for any $j \geq J+1$. The Abelian property (1.1) guarantees that A_{J+1} and $A_{b_o(n)}(o)$ have the same law.

By construction, $A_J \subset B_o(n_J)$. Since $k_J \leq n^{1/(d+1)}$ and ρ is continuous (C), the k_J last particles cannot grow long arms. Formally, $A_{J+1} \subset B_o(n_J + cn^{1/(d+1)})$.

Since $J \leq n$, by Lemma 3.3, for some $\delta = \delta(\alpha, c, d) < 1$,

$$\mathbb{P}[\exists \, 1 \leq j \leq J : k_j > (1-\delta)^j k_0] \leq \mathbb{P}[\exists \, 1 \leq j \leq J : k_j > (1-\delta)k_{j-1}] \leq np^{n^{1/(d+1)}}.$$

This implies that with probability at least $1 - np^{n^{1/(d+1)}}$,

$$n_J = n + k_0^{1/d} + \cdots + k_{J-1}^{1/d} \leq n + k_0^{1/d} \cdot \frac{1}{1 - (1-\delta)^{1/d}},$$

and if $n_J + Cn^{1/(d+1)} > (1+\varepsilon)n$, then $k_0^{1/d} > (\varepsilon n - Cn^{1/(d+1)})(1 - (1-\delta)^{1/d})$. So, using (VG), for any $\varepsilon > 0$,

$$\left\{ A_{b_o(n)}(o) \not\subset B_o((1+\varepsilon)n) \right\} \subseteq \left\{ k_0 > c_1 \varepsilon^d b_o(n) \right\} \cup \left\{ \exists\, 1 \leq j \leq J \,:\, k_j > (1-\delta)^j k_0 \right\}.$$

Hence, applying the Borel-Cantelli Lemma to the events

$$\left\{ A_{b_o(n)}(o) \not\subset B_o((1+\varepsilon)n) \right\} \setminus \left\{ k_0 > c_1 \varepsilon^d b_o(n) \right\} \subset \left\{ \exists\, 1 \leq j \leq J \,:\, k_j > (1-\delta)^j k_0 \right\},$$

we obtain that the event described on the left hand side happens finitely often almost surely, which concludes the proof of Theorem 1.2. □

References

[1] Peter Antal and Agoston Pisztora. On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.*, 24(2):1036–1048, 1996. MR-1404543

[2] A. Asselah and A. Gaudilliere. From logarithmic to subdiffusive polynomial fluctuations for internal DLA and related growth models. arXiv:1009.2838, 2010.

[3] A. Asselah and A. Gaudilliere. Sub-logarithmic fluctuations for internal DLA. arXiv:1011.4592, 2010.

[4] M.T. Barlow. Random walks on supercritical percolation clusters. *The Annals of Probability*, 32(4):3024–3084, 2004. MR-2094438

[5] S. Blachère. Internal diffusion limited aggregation on discrete groups of polynomial growth. In *Random walks and geometry: proceedings of a workshop at the Erwin Schrödinger Institute, Vienna, June 18-July 13, 2001*, page 377. De Gruyter, 2004. MR-2087790

[6] S. Blachère and S. Brofferio. Internal diffusion limited aggregation on discrete groups having exponential growth. *Probability Theory and Related Fields*, 137(3):323–343, 2007. MR-2278460

[7] P. Diaconis and W. Fulton. A growth model, a game, an algebra, lagrange inversion, and characteristic classes. *Rend. Sem. Mat. Univ. Pol. Torino*, 49(1):95–119, 1991. MR-1218674

[8] W. Huss. Internal diffusion-limited aggregation on non-amenable graphs. *Electronic Communications in Probability*, 13:272–279, 2008. MR-2415135

[9] D. Jerison, L. Levine, and S. Sheffield. Internal DLA in higher dimensions. arXiv:1012.3453, 2010.

[10] D. Jerison, L. Levine, and S. Sheffield. Logarithmic fluctuations for internal DLA. arXiv:1010.2483, 2010.

[11] D. Jerison, L. Levine, and S. Sheffield. Internal DLA and the Gaussian free field. arXiv:1101.0596, 2011.

[12] Gregory F. Lawler. Subdiffusive fluctuations for internal diffusion limited aggregation. *Ann. Probab.*, 23(1):71–86, 1995. MR-1330761

[13] Gregory F. Lawler, Maury Bramson, and David Griffeath. Internal diffusion limited aggregation. *Ann. Probab.*, 20(4):2117–2140, 1992. MR-1188055

[14] E. Shellef. Idla on the supercritical percolation cluster. *Electronic Journal of Probability*, 15:723–740, 2010. MR-2650780