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Illustration of various methods for solving partly Skorokhod's embedding problem*

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Abstract

We show that excursion theory and Azéma's exponential result allow to solve partly Skorokhod's embedding problem.

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1 Some particular Brownian stopping times

Throughout the paper, (B_t) denotes one-dimensional standard Brownian motion, and (L_t) is its local time. In the sequel, we look at some variant of the Azéma-Yor algorithm [2] for solving Skorokhod's embedding problem with the help of stopping times depending only on Brownian motion and its supremum.

1.1

In this paper, we wish to identity the law of B_{θ_F}, for the stopping time:

$$\theta_F = \inf\{t : F(L_t; |B_t|) \geq a\}.$$

The function $F : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function with the following property: denoting $F(\sigma, x) \equiv F_\sigma(x)$, we assume that F_σ is strictly increasing from 0 to ∞, and we denote by $F_\sigma^{-1}(\cdot)$ the inverse of F_σ:

$$F_\sigma^{(-1)}(y) = \inf\{x : F(\sigma, x) = y\}.$$

Thus, we may rewrite:

$$\begin{aligned} \theta_F &= \inf\{t : |B_t| \geq F_{L_t}^{-1}(a)\} \\ &= \inf\{t : h(L_t)|B_t| \geq 1\} \ := \ \theta^{(h)}, \end{aligned}$$

where:

$$h(l) = \frac{1}{F_l^{-1}(a)}.$$

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The main result in this paper is that:

$$|B_{\theta^{(h)}}| \sim \frac{1}{h\big(H^{-1}(\mathbf{e})\big)} \tag{1.1}$$

where $\mathbf{e}$ is a standard exponential variable, $H(x) = \int_0^x dy\; h(y)$.

As an illustration, we note that for $h_1(l) = l$,

$$|B_{\theta^{(h_1)}}| \sim \frac{1}{\sqrt{2\mathbf{e}}}.$$

It would be interesting to know which class of distributions is obtained from the RHS of (1.1). In fact, if $h\big(H^{-1}(u)\big) = \varphi(u)$ is Lipschitz, then H is the only solution of the ordinary differential equation $H'(t) = \varphi(H(t))$; $H(0) = 0$. Thus, the family of laws obtained from (1.1) is quite rich; for example take for h a positive power of l.

1.2

The remainder of this paper consists in three sections:

- in Section 2, we use Azéma exponential result to obtain (1.1);
- in Section 3, we use an excursion argument for the same purpose;
- in Section 4, we mention two points to be looked at carefully;
- in Section 5, we sketch how the previous arguments allow to recover the Azéma-Yor algorithm for solving Skorokhod's embedding problem.

2 A proof of (1.1) via Azéma's exponential result

2.1

We first state Azéma's exponential result:

Proposition 2.1 (Azéma [1]). *Let $(A_t^{\mathcal{L}}, t \geq 0)$ be the predictable compensator of $1_{(\mathcal{L} \leq t)}$, where $\mathcal{L}$ stands for the end of a predictable set on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, i.e:*

$$\mathcal{L} = \sup\{t : (t, \omega) \in \Gamma\},$$

for Γ a predictable set.

Then, under the hypothesis (CA): all martingales are continuous, and $\mathcal{L}$ avoids all $(\mathcal{F}_t)$ stopping times T, i.e.: $P(\mathcal{L} = T) = 0$, the variable $A_\infty^{\mathcal{L}}$ is a standard exponential variable with mean 1.

2.2

We compute $\big(A_t^{\mathcal{L}^{(h)}}\big)$ where $\mathcal{L}^{(h)} = \sup\{t \leq \theta^{(h)} : B_t = 0\}$.

Proposition 2.2.

$$A_t^{\mathcal{L}^{(h)}} = H(L_{\theta^{(h)} \wedge t}).$$

Proof. We use the balayage formula (see, e.g. [3], Chapter VI) to assert that, for any bounded predictable process (K_s), one has:

$$K_{g_t}|B_t| = \int_0^t K_{g_s} d|B_s|.$$

where $g_t = \sup\{s < t : B_s = 0\}$. In fact, we shall use the following variant:

$$K_{g_t} h(L_t)|B_t| = \int_0^t K_{g_s} h(L_s) d|B_s|.$$

Thus, applying the optional stopping theorem, we get:

$$E\Big[K_{g_{\theta^{(h)}}}\Big] = E\Big[\int_0^{\theta^{(h)}} K_s h(L_s) dL_s\Big].$$

which yields the desired result. □

2.3

As a consequence of the definition of $\theta^{(h)}$, we get

$$|B_{\theta^{(h)}}| = \frac{1}{h(L_{\theta^{(h)}})} \tag{2.1}$$

from Proposition 2.2, we deduce:

$$L_{\theta^{(h)}} \overset{\text{(law)}}{=} H^{-1}(\mathbf{e}) \tag{2.2}$$

which proves, together with (2.1) that (1.1) is satisfied.

3 The excursion theory argument

3.1

Call $(\tau_l, l \geq 0)$ the inverse local time. The excursion theory argument runs in the following equalities between the random sets:

$$\begin{aligned}
&(L_{\theta^{(h)}} \geq l) = (\theta^{(h)} \geq \tau_l) \\
&= (\forall \lambda \leq l,\ \text{for } t \in (\tau_{\lambda -}, \tau_\lambda),\ \text{one has: } h(\lambda)|B_t| < 1) \\
&= \Big(\sum_{\lambda \leq l} 1_{\{h(\lambda) \sup\limits_{\tau_{\lambda-} \leq t \leq \tau_\lambda} |B_t| \geq 1\}} = 0\Big).
\end{aligned}$$

From excursion theory, we now deduce:

Proposition 3.1.

$$L_{\theta^{(h)}} \overset{\text{(law)}}{=} H^{-1}(\mathbf{e}),$$

hence (1.1) holds.

Proof. By excursion theory, the process

$$N_l^{(h)} = \sum_{\lambda \leq l} 1_{\{h(\lambda) \sup\limits_{\tau_{\lambda-} \leq t \leq \tau_\lambda} |B_t| \geq 1\}}$$

is an inhomogeneous Poisson process, whose intensity measure may be expressed simply in terms of the Itô measure $\mathbf{n}$; precisely, we have:

$$(L_{\theta^{(h)}} \geq l) = (N_l^{(h)} = 0)$$

hence, denoting by ε the generic excursion, and $V(\varepsilon)$ its life time:

$$\begin{aligned}
P(L_{\theta^{(h)}} \geq l) &= \exp\Big(- \int_0^l d\lambda\ \mathbf{n}\big(h(\lambda) \sup_{t \leq V(\varepsilon)} |\varepsilon_t| \geq 1\big)\Big) \\
&= \exp\Big(- \int_0^l d\lambda\ \frac{1}{1/h(\lambda)}\Big) \\
&= \exp(-H(l)) \\
&= P(\mathbf{e} \geq H(l)),
\end{aligned}$$

hence the result. For the second equality, we have used:

$$\mathbf{n}\big(\sup_{t\le V(\varepsilon)} |\varepsilon_t| > a\big) = \frac{1}{a}.$$

□

4 Taking some care

In our discussion, two points need to be looked at carefully.

(i) First, we want $\theta^{(h)} < \infty$ a.s. This may be ensured as follows: there is the representation:

$$h(L_t)|B_t| \quad = \quad \beta\big(\int_0^t h^2(L_s)ds\big),$$

as a consequence of Dubins-Schwarz and the balayage formula, where $(\beta(u), u \ge 0)$ is a reflecting Brownian motion. Thus, if $\int_0^\infty h^2(L_u)du = \infty$ a.s., it follows that $\theta^{(h)} < \infty$ a.s. Now, it is easily shown that $\int^\infty h^2(L_u)du = \infty$ iff $\int^\infty h(l)dl = \infty$, a condition we assume in the paper. Indeed, that these two integrals are infinite simultaneously follows from the general fact that the bracket of a martingale at infinity is infinite if and only if its local time at infinity is infinite. Here, this martingale is $M_t = h(L_t)B_t$, whose local time is $H(L_t)$.

(ii) Some care also has to be taken in the application of the optional stopping theorem in our proof of Proposition 2.2. But, in fact, replacing $\theta^{(h)}$ by $\theta^{(h)} \wedge n$, and using dominated convergence, and monotone convergence, we justify the use of the optional stopping theorem.

5 A relation with the Azéma-Yor algorithm for solving Skorokhod's problem

We note that the arguments in Section 2 and Section 3 allow (almost) to recover the Azéma-Yor result for Skorokhod embedding. Azéma-Yor [2] have obtained an explicit solution to Skorokhod's embedding problem, as follows: given a probability $\mu(dx)$ on $\mathbb{R}$, with first moment, and centered, if:

$$T_\mu = \inf\{t : S_t \ge \psi_\mu(B_t)\},$$

then

$$B_{T_\mu} \sim \mu,$$

where $S_t = \sup_{s\le t} B_s$, and $\psi_\mu(x) = \frac{1}{\mu[x,\infty)} \int_{[x,\infty)} t d\mu(t)$ is the Hardy-Littlewood function attached to μ.

Indeed, similar calculation as above show that, if

$$G_\mu = \sup\{t \le T_\mu : S_t - B_t = 0\},$$

then the increasing process associated to G_μ is: $\Sigma(S_{t\wedge T_\mu})$, where $\Sigma(x) = \int_0^x \frac{dy}{y-\phi(y)}$, with ϕ the inverse of ψ_μ. Thus,

$$\Sigma(S_{T_\mu}) \stackrel{\text{(law)}}{=} \mathbf{e},$$

that is:

$$P(S_{T_\mu} \geq x) = \exp(-\Sigma(x)).$$

But, it is easily shown, from this result, that:

$$B_{T_\mu} = \phi(S_{T_\mu}) \sim \mu.$$

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