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Erratum: Convergence in law in the second Wiener/Wigner chaos

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Abstract

We correct an error in our paper [1].

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1 Introduction

We use the same notation as in [1] and we assume that the reader is familiar with it. We are indebted to Giovanni Peccati for pointing out, in the most constructive and gentle way, an error in [1, Theorem 3.4] and for providing an explicit counterexample supporting his claim.

2 A correct version of Lemma 3.5

Unfortunately, Lemma 3.5 in [1] is not correct. Our mistake comes from an improper calculation involving a Vandermonde determinant at the end of its proof. To fix the error is not a big deal though: it suffices to replace *different* by *consecutive* in the statement of Lemma 3.5, see below for a correct version together with its proof. As a direct consequence of this new version, we should also replace *different* by *consecutive* in the assumption (ii-c) of both Theorems 3.4 and 4.3 in [1]. We restate these latter results correctly in Section 2 for convenience.

Lemma 3.5. *Let $\mu_0 \in \mathbb{R}$, let $a \in \mathbb{N}^*$, let $\mu_1, \ldots, \mu_a \neq 0$ be pairwise distinct real numbers, and let $m_1, \ldots, m_a \in \mathbb{N}^*$. Set*

$$Q(x) = x^{2(1+\mathbf{1}_{\{\mu_0 \neq 0\}})} \prod_{i=1}^{a} (x - \mu_i)^2.$$

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Assume that $\{\lambda_j\}_{j\geqslant 0}$ *is a square-integrable sequence of real numbers satisfying*

$$\lambda_0^2+\sum_{j=1}^{\infty}\lambda_j^2=\mu_0^2+\sum_{i=1}^{a}m_i\,\mu_i^2 \tag{2.1}$$

$$\sum_{r=3}^{2(1+\mathbf{1}_{\{\mu_0\neq 0\}}+a)}\frac{Q^{(r)}(0)}{r!}\sum_{j=1}^{\infty}\lambda_j^r=\sum_{r=3}^{2(1+\mathbf{1}_{\{\mu_0\neq 0\}}+a)}\frac{Q^{(r)}(0)}{r!}\sum_{i=1}^{a}m_i\,\mu_i^r \tag{2.2}$$

$$\sum_{j=1}^{\infty}\lambda_j^r=\sum_{i=1}^{a}m_i\,\mu_i^r,\text{ for `}a\text{' consecutive values of } r\geqslant 2(1+\mathbf{1}_{\{\mu_0\neq 0\}}). \tag{2.3}$$

Then:

- *(i)* $|\lambda_0|=|\mu_0|$.
- *(ii) The cardinality of the set* $S=\{j\geqslant 1:\ \lambda_j\neq 0\}$ *is finite.*
- *(iii)* $\{\lambda_j\}_{j\in S}=\{\mu_i\}_{1\leqslant i\leqslant a}$.
- *(iv) for any* $i=1,\ldots,a$*, one has* $m_i=\#\{j\in S:\ \lambda_j=\mu_i\}$.

Proof. As in the original proof of [1, Lemma 3.5], we divide the proof according to the nullity of μ_0.

First case: $\mu_0=0$. We have $Q(x)=x^2\prod_{i=1}^{a}(x-\mu_i)^2$. Since the polynomial Q can be rewritten as

$$Q(x)=\sum_{r=2}^{2(1+a)}\frac{Q^{(r)}(0)}{r!}x^r,$$

assumptions (2.1) and (2.2) together ensure that

$$\lambda_0^2\prod_{i=1}^{a}\mu_i^2+\sum_{j=1}^{\infty}Q(\lambda_j)=\sum_{i=1}^{a}m_iQ(\mu_i)=0.$$

Because Q is positive and $\prod_{i=1}^{a}\mu_i^2\neq 0$, we deduce that $\lambda_0=0$ and $Q(\lambda_j)=0$ for all $j\geqslant 1$, that is, $\lambda_j\in\{0,\mu_1,\ldots,\mu_a\}$ for all $j\geqslant 1$. This shows claims (i) as well as:

$$\{\lambda_j\}_{j\in S}\subset\{\mu_i\}_{1\leqslant i\leqslant a}. \tag{2.4}$$

Moreover, since the sequence $\{\lambda_j\}_{j\geqslant 1}$ is square-integrable, claim (ii) holds true as well. It remains to show (iii) and (iv). For any $i=1,\ldots,a$, let $n_i=\#\{j\in S:\ \lambda_j=\mu_i\}$. Also, let $r\geqslant 2$ be such that $r,r+1,\ldots,r+a-1$ are 'a' consecutive values satisfying (2.3). We then have

$$\begin{pmatrix}\mu_1^r & \mu_2^r & \cdots & \mu_a^r\\ \mu_1^{r+1} & \mu_2^{r+1} & \cdots & \mu_a^{r+1}\\ \vdots & \vdots & \ddots & \vdots\\ \mu_1^{r+a-1} & \mu_2^{r+a-1} & \cdots & \mu_a^{r+a-1}\end{pmatrix}\begin{pmatrix}n_1-m_1\\ n_2-m_2\\ \vdots\\ n_a-m_a\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}.$$

Since $\mu_1,\ldots,\mu_a\neq 0$ are pairwise distinct, one has (Vandermonde matrix)

$$\det\begin{pmatrix}\mu_1^r & \mu_2^r & \cdots & \mu_a^r\\ \mu_1^{r+1} & \mu_2^{r+1} & \cdots & \mu_a^{r+1}\\ \vdots & \vdots & \ddots & \vdots\\ \mu_1^{r+a-1} & \mu_2^{r+a-1} & \cdots & \mu_a^{r+a-1}\end{pmatrix}$$
$$=\quad\prod_{i=1}^{a}\mu_i^r\times\det\begin{pmatrix}1 & 1 & \cdots & 1\\ \mu_1 & \mu_2 & \cdots & \mu_a\\ \vdots & \vdots & \ddots & \vdots\\ \mu_1^{a-1} & \mu_2^{a-1} & \cdots & \mu_a^{a-1}\end{pmatrix}\neq 0,$$

from which (iv) follows. Finally, recalling the inclusion (2.4) we deduce (iii).

Second case: $\mu_0 \neq 0$. In this case, one has $Q(x) = x^4 \prod_{i=1}^{a}(x-\mu_i)^2$ and claims (ii), (iii) and (iv) may be shown by following the same line of reasoning as above. We then deduce claim (i) by looking at (2.1). $\square$

3 Correct versions of Theorems 3.4 and 4.3

For convenience, we restate Theorems 3.4 and 4.3 correctly here. Their proofs are unchanged.

Theorem 3.4. *Let $f \in L_s^2(\mathbb{R}_+^2)$ with $0 \leqslant \mathrm{rank}(f) < \infty$, let $\mu_0 \in \mathbb{R}$ and let $N \sim \mathcal{N}(0, \mu_0^2)$ be independent of the underlying Brownian motion W. Assume that $|\mu_0| + \|f\|_{L^2(\mathbb{R}_+)} > 0$ and set*

$$Q(x) = x^{2(1+\mathbf{1}_{\{\mu_0 \neq 0\}})} \prod_{i=1}^{a(f)} (x - \lambda_i(f))^2.$$

Let $\{F_n\}_{n \geqslant 1}$ be a sequence of double Wiener-Itô integrals. Then, as $n \to \infty$, we have

(i) $F_n \overset{\mathrm{law}}{\to} N + I_2^W(f)$

if and only if all the following are satisfied:

(ii-a) $\kappa_2(F_n) \to \kappa_2(N + I_2^W(f)) = \mu_0^2 + 2\|f\|^2_{L^2(\mathbb{R}_+^2)}$;

(ii-b) $\sum_{r=3}^{\deg Q} \frac{Q^{(r)}(0)}{r!} \frac{\kappa_r(F_n)}{(r-1)!2^{r-1}} \to \sum_{r=3}^{\deg Q} \frac{Q^{(r)}(0)}{r!} \frac{\kappa_r(I_2^W(f))}{(r-1)!2^{r-1}}$;

(ii-c) $\kappa_r(F_n) \to \kappa_r(I_2^W(f))$ *for $a(f)$ consecutive values of r, with $r \geqslant 2(1+\mathbf{1}_{\{\mu_0 \neq 0\}})$.*

Theorem 4.3. *Let $f \in L_s^2(\mathbb{R}_+^2)$ with $0 \leqslant \mathrm{rank}(f) < \infty$, let $\mu_0 \in \mathbb{R}$ and let $A \sim \mathcal{S}(0, \mu_0^2)$ be independent of the underlying free Brownian motion S. Assume that $|\mu_0| + \|f\|_{L^2(\mathbb{R}_+)} > 0$ and set*

$$Q(x) = x^{2(1+\mathbf{1}_{\{\mu_0 \neq 0\}})} \prod_{i=1}^{a(f)} (x - \lambda_i(f))^2.$$

Let $\{F_n\}_{n \geqslant 1}$ be a sequence of double Wigner integrals. Then, as $n \to \infty$, we have

(i) $F_n \overset{\mathrm{law}}{\to} A + I_2^S(f)$

if and only if all the following are satisfied:

(ii-a) $\widehat{\kappa}_2(F_n) \to \widehat{\kappa}_2(A + I_2^S(f)) = \mu_0^2 + \|f\|^2_{L^2(\mathbb{R}_+^2)}$;

(ii-b) $\sum_{r=3}^{\deg Q} \frac{Q^{(r)}(0)}{r!} \widehat{\kappa}_r(F_n) \to \sum_{r=3}^{\deg Q} \frac{Q^{(r)}(0)}{r!} \widehat{\kappa}_r(I_2^S(f))$;

(ii-c) $\widehat{\kappa}_r(F_n) \to \widehat{\kappa}_r(I_2^W(f))$ *for $a(f)$ consecutive values of r, with $r \geqslant 2(1+\mathbf{1}_{\{\mu_0 \neq 0\}})$.*

References

[1] I. Nourdin and G. Poly (2012): Convergence in law in the second Wiener/Wigner chaos. *Electron. Comm. Probab.* **17**, no. 36. DOI: 10.1214/ECP.v17-2023