

## Convergence of integral functionals of one-dimensional diffusions\*

Aleksandar Mijatović<sup>†</sup>      Mikhail Urusov<sup>‡</sup>

### Abstract

In this paper we describe the pathwise behaviour of the integral functional  $\int_0^t f(Y_u) du$  for any  $t \in [0, \zeta]$ , where  $\zeta$  is (a possibly infinite) exit time of a one-dimensional diffusion process  $Y$  from its state space,  $f$  is a nonnegative Borel measurable function and the coefficients of the SDE solved by  $Y$  are only required to satisfy weak local integrability conditions. Two proofs of the deterministic characterisation of the convergence of such functionals are given: the problem is reduced in two different ways to certain path properties of Brownian motion where either the Williams theorem and the theory of Bessel processes or the first Ray-Knight theorem can be applied to prove the characterisation. As a simple application of the main results we give a short proof of Feller's test for explosion.

**Keywords:** Integral functional; one-dimensional diffusion; local time; Bessel process; Ray-Knight theorem; Williams theorem.

**AMS MSC 2010:** 60H10; 60J60.

Submitted to ECP on February 20, 2012, final version accepted on October 30, 2012.

## 1 Introduction

In this paper we investigate a problem of the convergence of the integral functional

$$\int_0^t f(Y_u) du, \quad t \in [0, \zeta], \quad (1.1)$$

of a one-dimensional  $J$ -valued diffusion  $Y$  that solves an SDE up to an exit time  $\zeta$ . The coefficients of the SDE are required to satisfy only some weak local integrability conditions on the open interval  $J = (l, r)$  (see (2.2) and (2.3)) and the function  $f: J \rightarrow [0, \infty]$  is assumed to be Borel measurable. The main results in this paper (see Theorems 2.6, 2.10 and 2.11) study the integral functional as a process, identify the stopping time after which the integral explodes, and give a deterministic criterion for the convergence of the integral functional at this stopping time. It turns out that this stopping time is the first time the process  $Y$  hits the set where a local integrability condition fails. Questions of such type appear naturally in stochastic analysis in connection with Girsanov's measure change (see e.g. the discussion in Section 2.7 below) or in insurance mathematics, where (1.1) is interpreted as the present value of a continuous stream of perpetuities (see [7]).

---

\*Supported by the LMS grant R.STAA.3037 and by the Mathematisches Forschungsinstitut Oberwolfach.

<sup>†</sup>Department of Mathematics, Imperial College, London, UK. E-mail: a.mijatovic@imperial.ac.uk

<sup>‡</sup>Faculty of Mathematics, University of Duisburg-Essen, Essen, Germany.  
E-mail: mikhaail.urusov@uni-due.de

The proof of the results consists of two steps. The first step, which uses only basic properties of diffusion processes and their local times, reduces the original problem to a question of the convergence of an integral functional of Brownian motion. The answer to this question is given in Lemma 4.1, which is not new. This result or its generalizations appeared in [8, p. 225–226], [1, Lem. 1.4.1], [3, Lem. 2], [2, Prop. A1.8], [11, Prop. 3.2]. In the second step, we present two short proofs of Lemma 4.1: (i) Williams' theorem (see [18, Ch. VII, Cor. 4.6]) and the result on integral functionals of Bessel processes from [4] are applied; (ii) a direct approach based on the first Ray-Knight theorem (see [18, Ch. XI, Th. 2.2]) is followed. The second proof uses the same ideas as the proofs in the references mentioned above (with a single technical point worked out differently – see the discussion following the statement of Lemma 4.1). The two proofs are presented for their simplicity and in order to make the paper self-contained.

In [11] the convergence of the integral functionals of the form  $\int_0^t f(X_s) ds$  is investigated, where  $f$  is a nonnegative Borel function and  $X$  a strong Markov continuous local martingale. The analytic condition that characterises the convergence of the integral functionals is given in terms of the speed measure of  $X$ . The diffusion  $Y$  considered in this paper is not necessarily a local martingale, but the process  $s(Y)$ , where  $s$  is the scale function of  $Y$ , is a strong Markov continuous local martingale, thus our characterisation theorems can be deduced from the ones in [11]. In our paper the behaviour of integral functional (1.1) is described in a more compact way than in [11]. We do not deduce our characterisation results from the ones in [11], but give a direct argument, based on the path properties of the underlying diffusion process, which turns out to be shorter. We would however like to emphasise that many ideas applied in this paper are present in [11] and in the literature preceding [11] (for an extensive literature review, see the discussion after Lemma 4.1).

In [14] a similar question is treated under different assumptions with the answer given in different terms. Namely, in [14] the diffusion is assumed to be transient (e.g.  $\lim_{t \nearrow \zeta} Y_t = r$  a.s.) and the function  $f$  to be locally bounded on  $J = (l, r)$ , while it is proved that convergence of a perpetual integral functional is equivalent to that  $g(r)$  is an exit boundary for some auxiliary diffusion with the state space  $(g(l), g(r))$ , where  $g$  is a certain increasing function. Further related papers are [19], where the conditions are found which imply that a perpetual integral functional is identical in law with the first hitting time of a point for some other diffusion, and [20], where the emphasis is on perpetual integral functionals of a Brownian motion with drift, and (i) their finiteness, (ii) moments, (iii) exponential moments and (iv) boundedness of potential are discussed.

The rest of the paper is organized as follows. Section 2 describes the setting, states the main results deduces Feller's test for explosions and presents other applications. In Section 3 we show how to reduce the main theorems to a problem for Brownian motion. Section 4 gives the characterisation for the convergence of integral functionals of Brownian motion. Sections 5 and 6 give the two proofs of Lemma 4.1.

## 2 The Setting and Main Results

**2.1.** First we introduce some common notations used in the sequel. Let us consider an open interval  $J = (l, r) \subseteq \mathbb{R}$ .

- By  $\bar{J}$  we denote  $[l, r]$ .
- By  $L_{\text{loc}}^1(J)$  we denote the set of Borel functions  $J \rightarrow [-\infty, \infty]$ , which are locally integrable on  $J$ , i.e. integrable on compact subsets of  $J$ .
- For  $x \in J$ ,  $L_{\text{loc}}^1(x)$  denotes the set of Borel functions  $f: J \rightarrow [-\infty, \infty]$  such that  $\int_{x-\varepsilon}^{x+\varepsilon} |f(y)| dy < \infty$  for some  $\varepsilon > 0$ .

- Let  $\alpha \in [l, r)$ ,  $\beta \in (l, r]$ . By  $L_{\text{loc}}^1(\alpha+)$  we denote the set of Borel functions  $f: J \rightarrow [-\infty, \infty]$  such that  $\int_{\alpha}^z |f(y)| dy < \infty$  for some  $z \in J$ ,  $z > \alpha$ . The notation  $L_{\text{loc}}^1(\beta-)$  is introduced similarly.

We will need the following statement. Its proof is straightforward.

**Lemma 2.1.**  $L_{\text{loc}}^1(J) = \bigcap_{x \in J} L_{\text{loc}}^1(x)$ .

**2.2.** Let the state space be  $J = (l, r)$ ,  $-\infty \leq l < r \leq \infty$ , and  $Y = (Y_t)_{t \in [0, \infty)}$  be a  $J$ -valued solution of the one-dimensional SDE

$$dY_t = \mu(Y_t) dt + \sigma(Y_t) dW_t, \quad Y_0 = x_0, \tag{2.1}$$

on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ , where  $x_0 \in J$  and  $W$  is an  $(\mathcal{F}_t, \mathbb{P})$ -Brownian motion. We allow  $Y$  to exit its state space  $J$  at a finite time in a continuous way. The exit time is denoted by  $\zeta$ . That is to say,  $\mathbb{P}$ -a.s. on  $\{\zeta = \infty\}$  the trajectories of  $Y$  do not exit  $J$ , while  $\mathbb{P}$ -a.s. on  $\{\zeta < \infty\}$  we have: either  $\lim_{t \uparrow \zeta} Y_t = r$  or  $\lim_{t \uparrow \zeta} Y_t = l$ . Then we need to specify the behaviour of  $Y$  after  $\zeta$  on  $\{\zeta < \infty\}$ . In what follows we assume that on  $\{\zeta < \infty\}$  the process  $Y$  stays at the endpoint of  $J$  where it exits after  $\zeta$ , i.e.  $l$  and  $r$  are by convention absorbing boundaries.

Throughout the paper it is assumed that the coefficients  $\mu$  and  $\sigma$  in (2.1) satisfy the Engelbert–Schmidt conditions

$$\sigma(x) \neq 0 \quad \forall x \in J, \tag{2.2}$$

$$\frac{1}{\sigma^2}, \frac{\mu}{\sigma^2} \in L_{\text{loc}}^1(J). \tag{2.3}$$

Under (2.2) and (2.3) SDE (2.1) has a weak solution, unique in law, which possibly exits  $J$  (see [10] or Th. 5.15 and the discussion in the beginning of Sec. 5.C, in particular, conditions  $(ND)'$  and  $(LI)'$  in [13, Ch. 5]). Conditions (2.2)–(2.3) are reasonable weak assumptions: any locally bounded Borel function  $\mu$  and locally bounded away from 0 Borel function  $\sigma$  on  $J$  satisfy (2.2) and (2.3).

Finally, the reason for considering an arbitrary interval  $J \subseteq \mathbb{R}$  as a state space, and not just  $\mathbb{R}$  itself, is that there are natural examples, where (2.2)–(2.3) hold only on a subset of  $\mathbb{R}$ . Even for geometric Brownian motion conditions (2.2)–(2.3) with  $J = \mathbb{R}$  are not satisfied, hence our results would not be applicable even to a geometric Brownian motion if we had assumed (2.2)–(2.3) only with  $J = \mathbb{R}$ . To view a geometric Brownian motion as a special case of (2.1) under assumptions (2.2)–(2.3), we need to take  $J = (0, \infty)$ .

**2.3.** Now we state some well-known results about the behaviour of one-dimensional diffusions with the coefficients satisfying (2.2)–(2.3) that will be extensively used in the sequel. Let us also note that these results do not hold beyond (2.2)–(2.3).

Let  $s$  denote the scale function of  $Y$  and  $\rho$  the derivative of  $s$ , i.e.

$$\rho(x) = \exp \left\{ - \int_c^x \frac{2\mu}{\sigma^2}(y) dy \right\}, \quad x \in J, \tag{2.4}$$

$$s(x) = \int_c^x \rho(y) dy, \quad x \in \bar{J}, \tag{2.5}$$

for some  $c \in J$ . In particular,  $s$  is an increasing  $C^1$ -function  $J \rightarrow \mathbb{R}$  with a strictly positive absolutely continuous derivative, while  $s(r)$  (resp.  $s(l)$ ) may take value  $\infty$  (resp.  $-\infty$ ).

For  $a \in \bar{J}$  let us define the stopping time

$$\tau_a^Y = \inf\{t \in [0, \infty): Y_t = a\} \quad (\inf \emptyset := \infty). \tag{2.6}$$

**Proposition 2.2.** For any  $a \in J$  we have  $\mathbb{P}(\tau_a^Y < \infty) > 0$ .

Even though it is assumed in Proposition 2.2 that  $a \in J$ , we stress that  $\tau_a^Y$  is defined for any  $a \in \bar{J}$ , which will be needed in Remark 2.7 below.

Further let us consider the sets

$$\begin{aligned} A &= \left\{ \zeta = \infty, \limsup_{t \rightarrow \infty} Y_t = r, \liminf_{t \rightarrow \infty} Y_t = l \right\}, \\ B_r &= \left\{ \zeta = \infty, \lim_{t \rightarrow \infty} Y_t = r \right\}, \\ C_r &= \left\{ \zeta < \infty, \lim_{t \uparrow \zeta} Y_t = r \right\}, \\ B_l &= \left\{ \zeta = \infty, \lim_{t \rightarrow \infty} Y_t = l \right\}, \\ C_l &= \left\{ \zeta < \infty, \lim_{t \uparrow \zeta} Y_t = l \right\}. \end{aligned}$$

**Proposition 2.3.** Either  $\mathbb{P}(A) = 1$  or  $\mathbb{P}(B_r \cup B_l \cup C_r \cup C_l) = 1$ .

**Proposition 2.4.** (i)  $\mathbb{P}(B_r \cup C_r) = 0$  holds if and only if  $s(r) = \infty$ .

(ii)  $\mathbb{P}(B_l \cup C_l) = 0$  holds if and only if  $s(l) = -\infty$ .

In particular, we get that  $\mathbb{P}(A) = 1$  holds if and only if  $s(r) = \infty, s(l) = -\infty$ .

**Proposition 2.5.** Assume that  $s(r) < \infty$ . Then either  $\mathbb{P}(B_r) > 0, \mathbb{P}(C_r) = 0$  or  $\mathbb{P}(B_r) = 0, \mathbb{P}(C_r) > 0$ . Furthermore, we have

$$\mathbb{P} \left( \lim_{t \uparrow \zeta} Y_t = r, Y_t > a \forall t \in [0, \zeta) \right) > 0$$

for any  $a < x_0$ .

Propositions 2.2–2.5 are well-known and follow from the construction of solutions (see e.g. [10] or [13, Ch. 5.5]) or can be deduced from the results in [9, Sec. 1.5]. Clearly, Proposition 2.5, which contains statements about the behaviour of one-dimensional diffusions at the endpoint  $r$ , has its analogue for the behaviour at  $l$ .

**2.4.** In this paper we study convergence of the integral functional

$$\int_0^t f(Y_u) du, \quad t \in [0, \zeta], \tag{2.7}$$

where  $f: J \rightarrow [0, \infty]$  is a nonnegative Borel function. In this subsection we reduce the study of convergence of (2.7) in general to that of convergence of the integral

$$\int_0^\zeta f(Y_u) du \tag{2.8}$$

for a nonnegative Borel function  $f: J \rightarrow [0, \infty]$  such that  $\frac{f}{\sigma^2} \in L^1_{\text{loc}}(J)$ . In the next subsection we formulate the answer to the latter problem.

Let us consider the set

$$D = \left\{ x \in J: \frac{f}{\sigma^2} \notin L^1_{\text{loc}}(x) \right\} \tag{2.9}$$

and note that  $D$  is a closed subset in  $J$ . Let us further define the stopping time

$$\eta_D = \zeta \wedge \inf \{ t \in [0, \infty): Y_t \in D \} \quad (\inf \emptyset := \infty). \tag{2.10}$$

**Theorem 2.6.** P-a.s. we have:

$$\int_0^t f(Y_u) du < \infty, \quad t \in [0, \eta_D], \tag{2.11}$$

$$\int_0^t f(Y_u) du = \infty, \quad t \in (\eta_D, \zeta]. \tag{2.12}$$

**Remark 2.7.** After Theorem 2.6 it remains only to study the convergence of the integral

$$\int_0^{\eta_D} f(Y_u) du.$$

If  $x_0 \in D$ , then  $\eta_D \equiv 0$ , and the integral is clearly zero. Let us assume that  $x_0 \notin D$  and set

$$\begin{aligned} \alpha &= \sup(l, x_0) \cap D && (\sup \emptyset := l), \\ \beta &= \inf(x_0, r) \cap D && (\inf \emptyset := r). \end{aligned}$$

It is easy to see that  $\eta_D = \tau_\alpha^Y \wedge \tau_\beta^Y$ . Now if we consider  $I := (\alpha, \beta)$  as a new state space for  $Y$ , then  $\tau_\alpha^Y \wedge \tau_\beta^Y$  will be the new exit time, and we will have  $\frac{f}{\sigma^2} \in L_{loc}^1(I)$  by Lemma 2.1. This concludes the reduction of the study of the convergence of (2.7) to that of the convergence of (2.8).

In order to prove Theorem 2.6 we need some additional notation. Since  $Y$  is a continuous semimartingale up to the exit time  $\zeta$ , one can define its local time  $\{L_t^y(Y); y \in J, t \in [0, \zeta]\}$  on the stochastic interval  $[0, \zeta]$  for any  $y \in J$  in the usual way (e.g. via the obvious generalization of [18, Ch. VI, Th. 1.2]). It follows from Theorem VI.1.7 in [18] that the random field  $\{L_t^y(Y); y \in J, t \in [0, \zeta]\}$  admits a modification such that the map  $(y, t) \mapsto L_t^y(Y)$  is a.s. continuous in  $t$  and cadlag in  $y$ .<sup>1</sup> As usual we always work with such a modification. Let us further recall that a.s. on  $\{t < \zeta\}$  the function  $y \mapsto L_t^y(Y)$  has a compact support in  $J$  and hence is bounded as a cadlag function with a compact support.

We will need the following result.

**Lemma 2.8** (Theorem 2.7 in [5]). *Let  $a \in J$ . Then*

$$L_t^a(Y) > 0 \quad \text{and} \quad L_t^{a-}(Y) > 0 \quad \text{P-a.s. on } \{\tau_a^Y < t < \zeta\}.$$

**Remark 2.9.** (i) *By Proposition 2.2 we have  $P(\tau_a^Y < \zeta) > 0$ . Hence, there exists  $t \in (0, \infty)$  such that  $P(\tau_a^Y < t < \zeta) > 0$ .*

(ii) *Let us note that the result of Lemma 2.8 no longer holds if the coefficients  $\mu$  and  $\sigma$  of (2.1) fail to satisfy (2.2)–(2.3) (see Theorem 2.6 in [5]).*

*Proof of Theorem 2.6.* By the occupation times formula, P-a.s. we have

$$\int_0^t f(Y_u) du = \int_0^t \frac{f}{\sigma^2}(Y_u) d\langle Y, Y \rangle_u = \int_J \frac{f}{\sigma^2}(y) L_t^y(Y) dy, \quad t \in [0, \zeta]. \tag{2.13}$$

Then (2.11) follows from the fact that P-a.s. on  $\{t < \zeta\}$  the function  $y \mapsto L_t^y(Y)$  is a cadlag function with a compact support in  $J$ .

As for (2.12), it immediately follows from (2.13) and Lemma 2.8 in the case  $x_0 \in D$ . If  $x_0 \notin D$ , we first observe that  $\eta_D = \tau_\alpha^Y \wedge \tau_\beta^Y$  (see Remark 2.7), and hence  $\{\eta_D < \zeta\} = \{\tau_\alpha^Y < \zeta\} \cup \{\tau_\beta^Y < \zeta\}$ . If  $P(\tau_\alpha^Y < \zeta) > 0$ , then, since  $D$  is closed we have  $\alpha \in D$  (note that  $(l, x_0) \cap D \neq \emptyset$  in this case because otherwise  $\alpha = l$  and  $P(\tau_\alpha^Y < \zeta) = 0$ ). Thus, (2.12) on  $\{\tau_\alpha^Y < \zeta\}$  follows from (2.13) and Lemma 2.8 applied with  $a = \alpha \in D$ . Similarly we get (2.12) on  $\{\tau_\beta^Y < \zeta\}$ . This concludes the proof.  $\square$

<sup>1</sup>Moreover, it can be proved that for a diffusion  $Y$  driven by (2.1) under conditions (2.2) and (2.3), any such modification is, in fact, a.s. jointly continuous in  $(t, y)$ ; see [17, Proposition A.1].

**2.5.** As pointed out in the previous subsection, it remains to study the convergence of the integral

$$\int_0^\zeta f(Y_u) du \tag{2.14}$$

for a nonnegative Borel function  $f: J \rightarrow [0, \infty]$  satisfying

$$\frac{f}{\sigma^2} \in L^1_{\text{loc}}(J). \tag{2.15}$$

This study is performed in the following two theorems, where we separately treat the cases  $P(A) = 1$  and  $P(B_r \cup B_l \cup C_r \cup C_l) = 1$  (see Propositions 2.3 and 2.4). Below  $\nu_L$  denotes the Lebesgue measure on  $J$ .

**Theorem 2.10.** Assume that the function  $f: J \rightarrow [0, \infty]$  satisfies (2.15). Let  $s(r) = \infty$  and  $s(l) = -\infty$ .

(i) If  $\nu_L(f > 0) = 0$ , then

$$\int_0^\zeta f(Y_u) du = 0 \quad \text{P-a.s.}$$

(ii) If  $\nu_L(f > 0) > 0$ , then

$$\int_0^\zeta f(Y_u) du = \infty \quad \text{P-a.s.}$$

Let us also note that  $\zeta = \infty$  P-a.s. in the case  $s(r) = \infty, s(l) = -\infty$ .

In the remaining case  $s(l) > -\infty$  or  $s(r) < \infty$  we have

$$\Omega = \left\{ \lim_{t \uparrow \zeta} Y_t = r \right\} \cup \left\{ \lim_{t \uparrow \zeta} Y_t = l \right\} \quad \text{P-a.s.}$$

In the following theorem we investigate the convergence of (2.14) on  $\{\lim_{t \uparrow \zeta} Y_t = r\}$ . To this end we need to assume  $s(r) < \infty$  because otherwise  $P(\lim_{t \uparrow \zeta} Y_t = r) = 0$  by Proposition 2.4.

**Theorem 2.11.** Assume that the function  $f: J \rightarrow [0, \infty]$  satisfies (2.15). Let  $s(r) < \infty$ .

(i) If

$$\frac{(s(r) - s)f}{\rho\sigma^2} \in L^1_{\text{loc}}(r-),$$

then

$$\int_0^\zeta f(Y_u) du < \infty \quad \text{P-a.s. on } \left\{ \lim_{t \uparrow \zeta} Y_t = r \right\}.$$

(ii) If

$$\frac{(s(r) - s)f}{\rho\sigma^2} \notin L^1_{\text{loc}}(r-),$$

then

$$\int_0^\zeta f(Y_u) du = \infty \quad \text{P-a.s. on } \left\{ \lim_{t \uparrow \zeta} Y_t = r \right\}.$$

Clearly, Theorem 2.11 has its analogue that describes the convergence of (2.14) on  $\{\lim_{t \uparrow \zeta} Y_t = l\}$ .

**2.6.** As an application let us deduce a deterministic distinction between the cases  $P(B_r) > 0, P(C_r) = 0$  and  $P(B_r) = 0, P(C_r) > 0$  of Proposition 2.5.

**Proposition 2.12** (Feller’s test for explosions). *We have  $P(B_r) = 0$ ,  $P(C_r) > 0$  if and only if*

$$s(r) < \infty \quad \text{and} \quad \frac{s(r) - s}{\rho\sigma^2} \in L^1_{\text{loc}}(r-).$$

Clearly, Proposition 2.12 has its analogue for the exit at  $l$ . Feller’s test for explosions in this form can be found in [5, Sec. 4.1]. For a different (but equivalent) form see e.g. [13, Ch. 5, Th. 5.29].

*Proof.* By Propositions 2.4 and 2.5, we need to have  $s(r) < \infty$ . Now the result follows from Theorem 2.11 applied to  $f \equiv 1$ .  $\square$

We note that Proposition 2.12 is not used in the proofs of our main results. Thus, this is a valid method of obtaining Feller’s test for explosions.

**2.7.** The following discussion is inspired by the treatment in [15, Ch. 6]. Liptser and Shiryaev consider a nonnegative local martingale (hence, a supermartingale)  $\mathfrak{z} = (\mathfrak{z}_t)_{t \in [0, \infty)}$  of the form

$$\mathfrak{z}_t = 1 + \int_0^t \gamma_u dW_u, \quad t \in [0, \infty), \tag{2.16}$$

where  $\gamma$  is a progressively measurable process satisfying

$$\int_0^t \gamma_u^2 du < \infty, \quad t \in [0, \infty),$$

and prove that it can always be represented as a “generalized” stochastic exponential

$$\mathfrak{z}_t = \exp \left\{ \int_0^{t \wedge \xi} \beta_u dW_u - \frac{1}{2} \int_0^{t \wedge \xi} \beta_u^2 du \right\}, \quad t \in [0, \infty), \tag{2.17}$$

where

$$\begin{aligned} \xi &= \inf \{ t \in [0, \infty) : \mathfrak{z}_t = 0 \} \quad (\inf \emptyset := \infty), \\ \beta_u &= \frac{\gamma_u}{\mathfrak{z}_u}, \quad u \in [0, \xi), \end{aligned}$$

and we necessarily have

$$\int_0^\xi \beta_u^2 du = \infty \quad \text{a.s. on } \{ \xi < \infty \} \tag{2.18}$$

(see [15, Lem. 6.2]).

Let us now define another “generalized” stochastic exponential  $Z$  in our diffusion setting as follows. Let  $b: J \rightarrow \mathbb{R}$  be an arbitrary Borel function. We define

$$Z_t^- := \begin{cases} \exp \left\{ \int_0^{t \wedge \zeta} b(Y_u) dW_u - \frac{1}{2} \int_0^{t \wedge \zeta} b^2(Y_u) du \right\} & \text{if } \int_0^{t \wedge \zeta} b^2(Y_u) du < \infty, \\ 0 & \text{if } \int_0^{t \wedge \zeta} b^2(Y_u) du = \infty, \end{cases} \tag{2.19}$$

$$Z_t := \lim_{\varepsilon \searrow 0} Z_{t+\varepsilon}^-, \quad t \in [0, \infty) \tag{2.20}$$

(cf. with (2.17) and (2.18)). Note that  $t \mapsto \int_0^{t \wedge \zeta} b^2(Y_u) du$  is a left-continuous increasing process that may go to  $\infty$  by a jump (see the discussion below), so we need (2.20) to get a càdlàg process  $Z$ .

Clearly, Theorems 2.6 and 2.11 applied to  $f := b^2$  yield explicit expressions for when  $\int_0^{t \wedge \zeta} b^2(Y_u) du$  is finite, and hence provide us with an explicit description of (2.19), (2.20).

We omit the full description here as it is straightforward but lengthy. Instead, let us make a comment about the relation between “generalized” stochastic exponentials (2.16), (2.17) and (2.19), (2.20). We will now use the set  $D$  of (2.9) with  $f := b^2$  and the stopping time  $\eta_D$  of (2.10). If  $x_0 \in D$ , then  $Z \equiv 0$ . Below we assume that  $x_0 \notin D$  and consider two following cases.

*Case 1:*  $D = \emptyset$ . In this case  $Z$  is always a continuous local martingale, which is strictly positive on  $[0, \zeta)$ . Note that we do not need (2.20) in this case, that is  $Z \equiv Z^-$ . At time  $\zeta$  the process  $Z$  may (but not necessarily will) attain zero. Here (2.19) is a special case of (2.16), (2.17) (just take  $\gamma_u := Z_u b(Y_u) I(u < \zeta)$ ). In the case  $D = \emptyset$  the local martingale  $Z$  is studied in [17], where deterministic necessary and sufficient conditions for  $Z$  to be a martingale and to be a uniformly integrable martingale are established.

*Case 2:*  $D \neq \emptyset, x_0 \notin D$ . Here  $Z_t \equiv 0$  for  $t \geq \eta_D$  on the set  $\{\eta_D < \zeta\}$  (note that  $P(\eta_D < \zeta) > 0$ ). It may happen that  $Z_{\eta_D-} > 0$  with positive probability on  $\{\eta_D < \zeta\}$ . In this situation  $Z$  is a supermartingale but not a local martingale, hence (2.19), (2.20) is not a special case of (2.16), (2.17). On the other hand,  $Z$  is a local martingale (and (2.19), (2.20) is a special case of (2.16), (2.17)) whenever  $Z_{\eta_D-} = 0$  a.s. on  $\{\eta_D < \zeta\}$ . Such a process  $Z$  is studied in [16], where deterministic criteria for it to be a local martingale, martingale, and uniformly integrable martingale are provided.

### 3 Proofs of Theorems 2.10 and 2.11

In this section we prove Theorems 2.10 and 2.11. In the latter proof we apply Lemma 4.1 below, which will be proved in the next sections.

Let us set

$$\tilde{Y}_t = s(Y_t), \quad t \in [0, \zeta). \tag{3.1}$$

Then

$$d\tilde{Y}_t = \tilde{\sigma}(\tilde{Y}_t) dW_t, \quad t \in [0, \zeta), \tag{3.2}$$

where

$$\tilde{\sigma}(x) = (\rho\sigma) \circ s^{-1}(x), \quad x \in (s(l), s(r)).$$

In particular,  $\tilde{Y}$  is a continuous local martingale on the stochastic interval  $[0, \zeta)$ . By the Dambis–Dubins–Schwarz theorem, there exists a Brownian motion  $B$  starting from  $s(x_0)$  (possibly on an enlargement of the initial probability space) such that

$$\tilde{Y}_t = B_{\langle \tilde{Y}, \tilde{Y} \rangle_t} \quad \text{P-a.s.}, \quad t \in [0, \zeta). \tag{3.3}$$

Let us also introduce the function

$$\tilde{f}(x) = f \circ s^{-1}(x), \quad x \in (s(l), s(r)).$$

*Proof of Theorem 2.10.* Here  $s(r) = \infty$  and  $s(l) = -\infty$ . Hence  $\zeta = \infty$  P-a.s. and, moreover,  $P(A) = 1$  (see Propositions 2.3 and 2.4). Then (3.1) implies that

$$P \left( \limsup_{t \rightarrow \infty} \tilde{Y}_t = \infty, \liminf_{t \rightarrow \infty} \tilde{Y}_t = -\infty \right) = 1.$$

Now it follows from (3.3) that  $\langle \tilde{Y}, \tilde{Y} \rangle_\infty = \infty$  P-a.s. We have

$$\begin{aligned} \int_0^\infty f(Y_u) du &= \int_0^\infty \tilde{f}(\tilde{Y}_u) du = \int_0^\infty \frac{\tilde{f}}{\tilde{\sigma}^2} \left( B_{\langle \tilde{Y}, \tilde{Y} \rangle_u} \right) d\langle \tilde{Y}, \tilde{Y} \rangle_u \\ &= \int_0^\infty \frac{\tilde{f}}{\tilde{\sigma}^2}(B_v) dv = \int_{\mathbb{R}} \frac{\tilde{f}}{\tilde{\sigma}^2}(x) L_\infty^x(B) dx \quad \text{P-a.s.} \end{aligned}$$



The first equality above is clear (we used  $\zeta = \infty$  P-a.s.), the second follows from (3.2) and (3.3), the third is due to the continuity of  $\langle \tilde{Y}, \tilde{Y} \rangle$  and the fact that  $\langle \tilde{Y}, \tilde{Y} \rangle_\infty = \infty$  P-a.s., and the last one follows from the occupation times formula ( $L_t^x(B)$  denotes the local time of the Brownian motion  $B$  at time  $t$  and at level  $x$ ). It remains to note that  $\nu_L(f > 0) > 0$  is equivalent to  $\nu_L(\tilde{f} > 0) > 0$  and that for a Brownian local time, P-a.s. it holds  $L_\infty^x(B) \equiv \infty \forall x \in \mathbb{R}$  (see e.g. [18, Ch. VI, § 2]). The proof is completed.  $\square$

*Proof of Theorem 2.11.* Here  $s(r) < \infty$ , i.e.  $\mathbb{P}(\lim_{t \uparrow \zeta} Y_t = r) > 0$ . Let us set  $R := \{\lim_{t \uparrow \zeta} Y_t = r\}$  and observe that (3.1) and (3.3) imply

$$R \equiv \left\{ \lim_{t \uparrow \zeta} Y_t = r \right\} = \left\{ \lim_{t \uparrow \zeta} \tilde{Y}_t = s(r) \right\} = \left\{ \lim_{t \uparrow \zeta} B_{\langle \tilde{Y}, \tilde{Y} \rangle_t} = s(r) \right\}.$$

In particular,

$$\langle \tilde{Y}, \tilde{Y} \rangle_\zeta = \tau_{s(r)}^B \quad \text{P-a.s. on } R, \tag{3.4}$$

where  $\tau_{s(r)}^B$  denotes the hitting time of the level  $s(r)$  by the Brownian motion  $B$ . Let us note that  $\zeta$  may be finite or infinite on  $R$ , but it follows from (3.4) that  $\langle \tilde{Y}, \tilde{Y} \rangle_\zeta$  is in either case finite on  $R$ . Similarly to the previous proof we get

$$\int_0^\zeta f(Y_u) du = \int_0^{\tau_{s(r)}^B} \frac{\tilde{f}}{\tilde{\sigma}^2}(B_v) dv \quad \text{P-a.s. on } R. \tag{3.5}$$

The question of convergence of the integral in the right-hand side of (3.5) is studied in Lemma 4.1 below. It is easy to obtain from (2.15) that  $\frac{\tilde{f}}{\tilde{\sigma}^2} \in L_{\text{loc}}^1(s(J))$ , which means that Lemma 4.1 can be applied (see (4.1)). Thus, to study the convergence of the integral in the right-hand side of (3.5) we need to check whether

$$(s(r) - x) \frac{\tilde{f}}{\tilde{\sigma}^2}(x) \in L_{\text{loc}}^1(s(r)-)$$

(the notation “ $f(x) \in \mathfrak{M}$ ” for a function  $f$  and a class of functions  $\mathfrak{M}$  is understood to be synonymous to “ $f \in \mathfrak{M}$ ”). We have

$$\int_{s(\cdot)}^{s(r)} \frac{(s(r) - x) f(s^{-1}(x))}{\rho^2(s^{-1}(x)) \sigma^2(s^{-1}(x))} dx = \int_{\cdot}^r \frac{(s(r) - s(y)) f(y)}{\rho(y) \sigma^2(y)} dy$$

(recall that  $s' = \rho$ ). Now the statement of Theorem 2.11 follows from (3.5) and Lemma 4.1.  $\square$

## 4 The Setting and Notation in the Brownian Case

It remains to prove Lemma 4.1 below. From now on let us consider a Brownian motion  $B$  starting from  $x_0 \in \mathbb{R}$ . We will extensively use the notation  $\tau_a^B$  ( $a \in \mathbb{R}$ ) for the stopping time defined as in (2.6). Below we use the notation “ $f(x) \in \mathfrak{M}$ ” for a function  $f$  and a class of functions  $\mathfrak{M}$  as a synonym for “ $f \in \mathfrak{M}$ ”.

**Lemma 4.1.** *Let  $B$  be a Brownian motion starting from  $x_0 \in \mathbb{R}$  and  $x_0 < r < \infty$ . Assume that the function  $f: I \rightarrow [0, \infty]$  with  $I := (-\infty, r)$  satisfies*

$$f \in L_{\text{loc}}^1(I). \tag{4.1}$$

(i) *If  $(r - x)f(x) \in L_{\text{loc}}^1(r-)$ , then*

$$\int_0^{\tau_r^B} f(B_u) du < \infty \quad \text{P-a.s.}$$

(ii) If  $(r - x)f(x) \notin L^1_{\text{loc}}(r-)$ , then

$$\int_0^{\tau_r^B} f(B_u) du = \infty \quad \text{P-a.s.}$$

As it was explained in the introduction, Lemma 4.1 is in no way new. First it appeared implicitly in [8, p. 225–226] and was stated explicitly in [3, Lem. 2]. The proofs in [8] and [3] rest on an application of the Ray-Knight theorem and Shepp’s [21] dichotomy result for Gaussian processes. This dichotomy argument was at some point replaced by Jeulin’s [12] lemma (e.g. see [1, Lem. 1.4.2], [2, Lem. A1.7] or [11, Lem. 3.1]). We also refer to the statements [1, Lem. 1.4.1], [2, Prop. A1.8], [11, Prop. 3.2] about Brownian local time integrated in the space variable against a positive measure, which contain Lemma 4.1 as a special case and which are proved in a similar way (the Ray-Knight theorem and Jeulin’s lemma are used).

Thus, one can get a proof of Lemma 4.1 from any of the references in the previous paragraph. In order to make the paper self-contained, we also present two short proofs of Lemma 4.1 below. The first one is very short and is completely different from the ones mentioned in the previous paragraph. It is based on Williams’ theorem (see [18, Ch. VII, Cor. 4.6]) and Cherny’s investigation of convergence of integral functionals of Bessel processes (see [4, Th. 2.1]). The second proof follows the same lines as the ones mentioned in the previous paragraph, only the application of Shepp’s dichotomy result (resp. of Jeulin’s lemma) in [8], [3] (resp. in [1], [2], [11]) is replaced by a direct argument similar to the one in the proof in [6, Th. 1.4].

### 5 First Proof of Lemma 4.1

By the occupation times formula and (4.1), P-a.s. we get

$$\int_0^t f(B_u) du < \infty, \quad t \in [0, \tau_r^B]. \tag{5.1}$$

By  $\rho = (\rho_t)_{t \in [0, \infty)}$  we denote a three-dimensional Bessel process starting from 0. Let us set

$$\xi = \sup\{t \in [0, \infty) : \rho_t = r - x_0\}$$

(note that  $\xi$  is a finite random variable because  $\rho_t \rightarrow \infty$  a.s.). By Williams’ theorem,

$$\text{Law}(r - B_{\tau_r^B - t}; t \in [0, \tau_r^B)) = \text{Law}(\rho_t; t \in [0, \xi)), \tag{5.2}$$

where “Law” means distribution. It follows from Theorem 2.1 in [4] that, for a nonnegative function  $g$ ,

(A)  $xg(x) \in L^1_{\text{loc}}(0+)$  implies that a.s. it holds  $\exists \varepsilon > 0 \int_0^\varepsilon g(\rho_u) du < \infty$ ;

(B)  $xg(x) \notin L^1_{\text{loc}}(0+)$  implies that a.s. it holds  $\forall \varepsilon > 0 \int_0^\varepsilon g(\rho_u) du = \infty$ .

By (5.1), (5.2) and (A), (B), the question reduces to whether  $xf(r - x) \in L^1_{\text{loc}}(0+)$ , or, equivalently, to whether  $(r - x)f(x) \in L^1_{\text{loc}}(r-)$ . This concludes the proof.

### 6 Second Proof of Lemma 4.1

By the occupation times formula,

$$\int_0^{\tau_r^B} f(B_u) du = \int_{-\infty}^r f(x)L_{\tau_r^B}^x(B) dx \quad \text{P-a.s.} \tag{6.1}$$

Since P-a.s. the mapping  $x \mapsto L_{\tau_r^B}^x(B)$  is a continuous function with a compact support in  $\mathbb{R}$ , we get from (4.1) that

$$\int_{-\infty}^{x_0} f(x)L_{\tau_r^B}^x(B) dx < \infty \quad \text{P-a.s.}$$

By (6.1), the question of whether  $\int_0^{\tau_r^B} f(B_u) du$  is finite reduces to the question of whether  $\int_{x_0}^r f(x)L_{\tau_r^B}^x(B) dx$  is finite, or, equivalently, to

$$\text{whether } \int_0^{r-x_0} f(r-u)L_{\tau_r^B}^{r-u}(B) du \text{ is finite.} \tag{6.2}$$

Let  $\overline{W}$  and  $\widetilde{W}$  be independent Brownian motions starting from 0. Let us set

$$\eta_t = \overline{W}_t^2 + \widetilde{W}_t^2, \tag{6.3}$$

i.e.  $\eta = (\eta_t)_{t \in [0, \infty)}$  is a squared two-dimensional Bessel process starting from 0. It follows from the first Ray-Knight theorem that

$$\text{Law} \left( L_{\tau_r^B}^{r-u}; u \in [0, r-x_0] \right) = \text{Law} (\eta_u; u \in [0, r-x_0]). \tag{6.4}$$

In what follows we prove that, for a Brownian motion  $W$  starting from 0,

(A)  $xf(r-x) \in L_{\text{loc}}^1(0+)$  implies that  $\int_0^{r-x_0} f(r-u)W_u^2 du < \infty$  a.s.;

(B)  $xf(r-x) \notin L_{\text{loc}}^1(0+)$  implies that  $\int_0^{r-x_0} f(r-u)W_u^2 du = \infty$  a.s.

Together with (6.2)–(6.4) this will complete the proof of Lemma 4.1.

By Fubini’s theorem,

$$\mathbb{E} \int_0^{r-x_0} f(r-u)W_u^2 du = \int_0^{r-x_0} f(r-u)u du,$$

so (A) is immediate (recall that (4.1) holds).

In order to prove (B) we assume that

$$\mathbb{P} \left( \int_0^{r-x_0} f(r-u)W_u^2 du < \infty \right) > 0. \tag{6.5}$$

Then there exists a sufficiently large  $M < \infty$  such that

$$\gamma := \mathbb{P}(R) > 0, \quad \text{where } R := \left\{ \int_0^{r-x_0} f(r-u)W_u^2 du \leq M \right\}.$$

Let us note that, for any positive  $\delta$  and  $u$ , the probability

$$\mathbb{P}(W_u^2 \geq \delta^2 u) = \mathbb{P}(|W_u/\sqrt{u}| \geq \delta) = \mathbb{P}(|N(0, 1)| \geq \delta)$$

does not depend on  $u$ . We take a sufficiently small  $\delta > 0$  such that

$$\mathbb{P}(|N(0, 1)| \geq \delta) \geq 1 - \frac{\gamma}{2}.$$

Then, for any  $u$ ,

$$\mathbb{E}(W_u^2 I_R) \geq \delta^2 u \mathbb{P}(R \cap \{W_u^2 \geq \delta^2 u\}) \geq \frac{\gamma}{2} \delta^2 u.$$

By Fubini’s theorem,

$$\mathbb{E} \left[ I_R \int_0^{r-x_0} f(r-u)W_u^2 du \right] = \int_0^{r-x_0} f(r-u) \mathbb{E}(W_u^2 I_R) du \geq \frac{\gamma}{2} \delta^2 \int_0^{r-x_0} f(r-u)u du.$$

The left-hand side is finite as on the event  $R$  the integral is not greater than  $M$ . Thus, (6.5) implies  $uf(r-u) \in L_{\text{loc}}^1(0+)$ , which proves (B) and completes the proof of Lemma 4.1.

## References

- [1] S. Assing. *Homogene stochastische Differentialgleichungen mit gewöhnlicher Drift*. Promotionschrift. Friedrich-Schiller-Universität, Jena, 1994.
- [2] S. Assing and W. Schmidt. *Continuous strong Markov processes in dimension one: A stochastic calculus approach*, volume 1688 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. MR-1724313
- [3] S. Assing and T. Senf. On stochastic differential equations without drift. *Stochastics and Stochastics Reports*, 36:21–39, 1997. MR-1117203
- [4] A. Cherny. Convergence of some integrals associated with Bessel processes. *Theory Probab. Appl.*, 45(2):195–209, 2001. MR-1967756
- [5] A. Cherny and H.-J. Engelbert. *Singular Stochastic Differential Equations*, volume 1858 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005. MR-2112227
- [6] F. Delbaen and H. Shirakawa. No arbitrage condition for positive diffusion price processes. *Asia-Pacific Financial Markets*, 9:159–168, 2002.
- [7] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990. MR-1129194
- [8] H.-J. Engelbert and W. Schmidt. On the behaviour of certain Bessel functionals. An application to a class of stochastic differential equations. *Math. Nachr.*, 131:219–234, 1987. MR-0908813
- [9] H.-J. Engelbert and W. Schmidt. Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. I. *Math. Nachr.*, 143:167–184, 1989. MR-1018242
- [10] H.-J. Engelbert and W. Schmidt. Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. III. *Math. Nachr.*, 151:149–197, 1991. MR-1121203
- [11] H.-J. Engelbert and G. Tittel. Integral functionals of strong Markov continuous local martingales. In R. Buckdahn, H.-J. Engelbert, and M. Yor, editors, *Stochastic Processes and Related Topics*, Proceedings of the 12th Winter School, Siegmundsburg (Germany), February 27-March 4, 2000, pages 111–132. CRC Press, 2002. MR-1987312
- [12] T. Jeulin. *Semi-martingales et grossissement d'une Filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1980. MR-0604176
- [13] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. MR-1121940
- [14] D. Khoshnevisan, P. Salminen, and M. Yor. A note on a.s. finiteness of perpetual integral functionals of diffusions. *Electron. Comm. Probab.*, 11:108–117, 2006. MR-2231738
- [15] R. S. Liptser and A. N. Shiryaev. *Statistics of Random Processes. I, II*, volume 5, 6 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2001. MR-1800858
- [16] A. Mijatović, N. Novak, and M. Urusov. Martingale property of generalized stochastic exponentials. In C. Donati-Martin, M. Émery, A. Rouault, and C. Stricker, editors, *Séminaire de Probabilités XLIV*, volume 2046 of *Lecture Notes in Math.*, pages 41–59. Springer, Berlin, 2012. MR-2933932
- [17] A. Mijatović and M. Urusov. On the martingale property of certain local martingales. *Probability Theory and Related Fields*, 152(1):1–30, 2012. MR-2875751
- [18] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition, 1999. MR-1725357
- [19] P. Salminen and M. Yor. Perpetual integral functionals as hitting and occupation times. *Electron. J. Probab.*, 10:371–419, 2005. MR-2147313
- [20] P. Salminen and M. Yor. Properties of perpetual integral functionals of Brownian motion with drift. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(3):335–347, 2005. MR-2139023

- [21] L. A. Shepp. Radon-Nikodym derivatives of Gaussian measures. *Ann. Math. Stat.*, 37:321–354, 1966. MR-0190999

**Acknowledgments.** We thank Hans-Jürgen Engelbert for comments on our work and Albert Shiryaev for an advice on this topic. MU would like to thank the Statistics Department at Warwick, where a part of this work was carried out.