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THE GROWTH CONSTANTS OF LATTICE TREES AND LATTICE ANIMALS IN HIGH DIMENSIONS

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Abstract
We prove that the growth constants for nearest-neighbour lattice trees and lattice (bond) animals on the integer lattice $\mathbb{Z}^d$ are asymptotic to $2de$ as the dimension goes to infinity, and that their critical one-point functions converge to e. Similar results are obtained in dimensions $d > 8$ in the limit of increasingly spread-out models; in this case the result for the growth constant is a special case of previous results of M. Penrose. The proof is elementary, once we apply previous results of T. Hara and G. Slade obtained using the lace expansion.

1 The main result

We define two different regular graphs with vertex set $\mathbb{Z}^d$, as follows. The *nearest-neighbour* graph has edge set consisting of pairs $\{x,y\}$ with $\|x-y\|_1 = 1$. The *spread-out* graph has edge set consisting of pairs $\{x,y\}$ with $0 < \|x-y\|_\infty \le L$, with $L \ge 1$ fixed. These graphs have degrees $2d$ and $(2L+1)^d - 1$, respectively. Often we discuss both graphs simultaneously, and use K to denote the degree in either case. Also, we will write $\lim_{K\to\infty}$ to simultaneously denote the limit as $d \to \infty$ for the nearest-neighbour case, and the limit as $L \to \infty$ for the spread-out case.

On either graph, a *lattice animal* is a finite connected subgraph, and a *lattice tree* is a finite connected subgraph without cycles. These very natural combinatorial objects are also fundamental in polymer science [13]. We denote the number of lattice animals containing n bonds and containing the origin of $\mathbb{Z}^d$ by a_n, and the number of lattice trees containing n bonds and containing the origin of $\mathbb{Z}^d$ by t_n. Standard subadditivity arguments [14, 15] provide the existence of the *growth*

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constants

$$\tau = \lim_{n\to\infty} t_n^{1/n}, \qquad \alpha = \lim_{n\to\infty} a_n^{1/n}. \tag{1.1}$$

The growth constants of course depend on d, and for the spread-out model, also on L. The *one-point functions*

$$g(z) = \sum_{n=0}^{\infty} t_n z^n \quad \text{and} \quad g^{(a)}(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.2}$$

have radii of convergence $z_c = \tau^{-1}$ and $z_c^{(a)} = \alpha^{-1}$, respectively.

We will rely on a result obtained by Hara and Slade [7] using the lace expansion, but we will not need any details about the lace expansion in this paper. It is shown in [7] that $g(z_c)$ and $g^{(a)}(z_c^{(a)})$ are finite (in fact, at most 4) for the nearest-neighbour model in sufficiently high dimensions, and for the spread-out model in dimensions $d > 8$ if L is sufficiently large, and that, in these two limits, z_c and $z_c^{(a)}$ obey the equations

$$\lim_{K\to\infty} K z_c g(z_c) = \lim_{K\to\infty} K z_c^{(a)} g^{(a)}(z_c^{(a)}) = 1. \tag{1.3}$$

This is discussed for the nearest-neighbour model in [6] (see, in particular, [6, (1.31)]), and the same considerations apply for the spread-out model. In fact, much more is known [19].

Our main result is the following theorem. The asymptotic relation in its statement means that the limit of the ratio of left- and right-hand sides is equal to 1.

Theorem 1. *For the nearest neighbour model as $d \to \infty$, and for the spread-out model in dimensions $d > 8$ as $L \to \infty$,*

$$\tau \sim K\mathrm{e} \quad \text{and} \quad \alpha \sim K\mathrm{e}, \tag{1.4}$$

and, in these same limits,

$$\lim_{K\to\infty} g(z_c) = \lim_{K\to\infty} g^{(a)}(z_c^{(a)}) = \mathrm{e}. \tag{1.5}$$

To our knowledge, Theorem 1 is new for the nearest-neighbour model. The proof of Theorem 1 is the same for both the nearest-neighbour and spread-out models. No bound on the rate of convergence is obtained here for either (1.4) or (1.5). Given (1.3), the statements $\tau \sim K\mathrm{e}$ and $g(z_c) \to \mathrm{e}$ are equivalent, as are the statements $\alpha \sim K\mathrm{e}$ and $g^{(a)}(z_c^{(a)}) \to \mathrm{e}$.

Stronger results than (1.4) have been obtained by Penrose [18] for the spread-out model using a completely different method of proof, without restriction to $d > 8$ and with the error estimate

$$\frac{K^K}{(K-1)^{K-1}} - O(K^{5/7}\log K) \le \tau \le \alpha \le \frac{K^K}{(K-1)^{K-1}} \tag{1.6}$$

in all dimensions $d \ge 1$. Both the right- and left-hand sides of (1.6) are of course asymptotic to $K\mathrm{e}$ as $K \to \infty$. When combined with (1.3), (1.5) then follows from (1.6) for the spread-out model in dimensions $d > 8$.

Much stronger results than (1.4) have been obtained for the closely related models of self-avoiding walks and percolation. Let c_n denote the number of n-step self-avoiding walks starting at the origin. For nearest-neighbour self-avoiding walks, it was proved in [8] that the connective constant $\mu = \lim_{n\to\infty} c_n^{1/n}$ has an asymptotic expansion $\mu \sim \sum_{i=-1}^{\infty} m_i (2d)^{-i}$ (as $d \to \infty$), with $m_i \in \mathbb{Z}$ for all i. The first thirteen coefficients in this expansion are now known [2], and it appears likely that the series $\sum_i m_i x^i$ has radius of convergence equal to zero. It may however be Borel summable, and a partial result in this direction is given in [5]. Some related results for nearest-neighbour bond

percolation are obtained in [8, 11], and for spread-out models of percolation and self-avoiding walks in [10, 17, 18].
The behaviour of τ and α for the nearest-neighbour model, as $d \to \infty$, has been extensively studied in the physics literature. For τ, the expansion

$$\tau = \sigma \mathrm{e} \exp\left(-\frac{1}{2}\frac{1}{\sigma} - \frac{8}{3}\frac{1}{\sigma^2} - \frac{85}{12}\frac{1}{\sigma^3} - \frac{931}{20}\frac{1}{\sigma^4} - \frac{2777}{10}\frac{1}{\sigma^5} + \cdots \right) \quad \text{where} \quad \sigma = 2d-1 \tag{1.7}$$

is reported in [4], but without a rigorous estimate for the error term. This raises the question of whether there exists an asymptotic expansion for τ of the form $\mathrm{e}\sum_{i=-1}^{\infty} r_i (2d)^{-i}$, with $r_i \in \mathbb{Q}$. For α, the series

$$\begin{aligned}\alpha = \sigma \mathrm{e} \exp\Big(&-\frac{1}{2}\frac{1}{\sigma} - \Big(\frac{8}{3} - \frac{1}{2\mathrm{e}}\Big)\frac{1}{\sigma^2} - \Big(\frac{85}{12} - \frac{1}{4\mathrm{e}}\Big)\frac{1}{\sigma^3} - \Big(\frac{931}{20} - \frac{139}{48\mathrm{e}} - \frac{1}{8\mathrm{e}^2}\Big)\frac{1}{\sigma^4} \\ &-\Big(\frac{2777}{10} + \frac{177}{32\mathrm{e}} - \frac{29}{12\mathrm{e}^2}\Big)\frac{1}{\sigma^5} + \cdots \Big)\end{aligned} \tag{1.8}$$

was derived in [9, 16], again without a rigorous error estimate; here the role of the transcendental number e is more delicate. Theorem 1 provides a rigorous confirmation of the leading terms in (1.7)–(1.8).

2 Main steps in the proof

We define

$$z_0 = \frac{1}{K\mathrm{e}}. \tag{2.1}$$

Since $\tau \le \alpha$ by definition, the critical points obey

$$z_c \ge z_c^{(a)}. \tag{2.2}$$

In fact, the strict inequality $\tau < \alpha$ is known [13]. The proof of Theorem 1 uses the following two ingredients. The content of the first is that $z_c^{(a)} \ge z_0$, or, equivalently, that $\alpha \le K\mathrm{e}$. This fact is presumably well-known, though we did not find an explicit proof in the literature. Klarner [14] proves that for 2-dimensional nearest-neighbour site animals the growth constant is at most $27/4 = 3^3/2^2$ and Penrose [18] states that this can be generalised to the upper bound $\alpha \le K^K/(K-1)^{K-1} \sim K\mathrm{e}$ for bond animals on an arbitrary regular graph. We will provide an elementary proof that $\alpha \le K\mathrm{e}$ in Lemma 2 below, both to keep self-contained and because elements of the proof are also useful elsewhere in our approach.

Lemma 2. *In all dimensions $d \ge 1$, and for the nearest-neighbour or spread-out models,*

$$z_c \ge z_c^{(a)} \ge z_0 = \frac{1}{K\mathrm{e}}. \tag{2.3}$$

Proposition 3. *For the nearest-neighbour model, or for the spread-out model in dimensions $d \ge 1$,*

$$\lim_{K\to\infty} g(z_0) = \mathrm{e}. \tag{2.4}$$

Proof of Theorem 1. We will prove that, under the hypotheses of Theorem 1,

$$\lim_{K\to\infty} g(z_c) = \mathrm{e}. \tag{2.5}$$

It then follows from (1.3) that $z_c \sim z_0$. Lemma 2 then implies that $z_c^{(a)} \sim z_0$, and finally (1.3) implies that $\lim_{K\to\infty} g(z_c^{(a)}) = \mathrm{e}$. Thus Theorem 1 will follow, once we prove (2.5). By Proposition 3 and (1.3),

$$\lim_{K\to\infty} (Kz_c g(z_c) - \mathrm{e}^{-1} g(z_0)) = 0. \tag{2.6}$$

This can be rewritten as

$$\lim_{K\to\infty} [K(z_c - z_0)g(z_c) + \mathrm{e}^{-1}(g(z_c) - g(z_0))] = 0. \tag{2.7}$$

By Lemma 2 and the monotonicity of g, both terms in the limit are non-negative, and therefore

$$\lim_{K\to\infty} (g(z_c) - g(z_0)) = 0. \tag{2.8}$$

With Proposition 3, this gives (2.5) and completes the proof. □

It remains to prove Lemma 2 and Proposition 3.

3 The proof completed

We will make use of the following mean-field model (see [1, 19]), which is related to the Galton–Watson branching process with critical Poisson offspring distribution. In other developments, the connection with the mean-field model is reflected by the super-Brownian scaling limits proved for lattice trees in high dimensions [3, 12].

Let $\mathscr{T}_n$ denote the set of n-edge rooted plane trees [20], and let $\mathscr{T} = \cup_{n=0}^{\infty} \mathscr{T}_n$. Given $T \in \mathscr{T}$, we consider mappings $\varphi : T \to \mathbb{Z}^d$ with the property that φ maps the root to the origin, and maps each other vertex of T to a neighbour of its parent (nearest-neighbour or spread-out, depending on the setting); the set of such mappings is denoted $\Phi(T)$. There is no self-avoidance constraint. By definition, for $T \in \mathscr{T}_n$, the cardinality of $\Phi(T)$ is K^n. We may interpret the image of T under φ as a multigraph without self-lines, and we refer to the pair (T, φ) as a *mean-field* configuration. The set of all mean-field configurations (T, φ) with $T \in \mathscr{T}_n$ is denoted $\mathscr{M}_n$.

Let ξ_i denote the forward degree of a vertex $i \in T$; this is the degree of the root when i is the root of T and otherwise it is the degree of i minus 1. For $n \geq 0$, let

$$f_n = \sum_{(T,\varphi)\in\mathscr{M}_n} \prod_{i\in T} \frac{1}{\xi_i!} = K^n \sum_{T\in\mathscr{T}_n} \prod_{i\in T} \frac{1}{\xi_i!}, \tag{3.1}$$

where the second equality follows from the fact that $\Phi(T)$ has K^n elements. Let $|T|$ denote the number of edges in T. Then

$$\sum_{n=0}^{\infty} f_n z^n = (Kz)^{-1} \sum_{T\in\mathscr{T}} (K\mathrm{e}z)^{|T|+1} \prod_{i\in T} \frac{1}{\mathrm{e}\xi_i!}. \tag{3.2}$$

Moreover, since $\mathbb{P}(T) = \prod_{i\in T}(\mathrm{e}\xi_i!)^{-1}$ is the probability that T arises as the family tree of a Galton–Watson branching process with critical Poisson offspring distribution, it follows from (3.2) that

$$\sum_{n=0}^{\infty} f_n z_0^n = \mathrm{e} \sum_{T\in\mathscr{T}} \mathbb{P}(T) = \mathrm{e}. \tag{3.3}$$

The relation with the critical Poisson branching process can easily be exploited further (see, e.g., [1, Theorem 2.1]) to obtain

$$\sum_{n=0}^{\infty} f_n z^n = (Kz)^{-1} \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(Kz)^n. \tag{3.4}$$

The series on the right-hand side converges if and only if $|K\mathrm{e}z| \le 1$, by Stirling's formula, and hence

$$\lim_{n\to\infty} f_n^{1/n} = K\mathrm{e} = \frac{1}{z_0}. \tag{3.5}$$

Let $\mathscr{L}_n$ denote the set of n-bond lattice trees containing the origin; its cardinality is t_n. We will use the fact, proved in [1, (5.5)], that for every $L \in \mathscr{L}_n$,

$$\sum_{(T,\varphi)\in\mathscr{M}_n:\varphi(T)=L} \prod_{i\in T} \frac{1}{\xi_i!} = 1. \tag{3.6}$$

The proof of (3.6) in [1] is given for the nearest-neighbour model, but it applies without change also to the spread-out model. By summing (3.6) over $L \in \mathscr{L}_n$, we obtain

$$t_n \le f_n, \tag{3.7}$$

and hence $\tau \le \lim_{n\to\infty} f_n^{1/n} = K\mathrm{e}$. This gives the inequality $z_c \ge z_0$, which is weaker than the inequality $z_c^{(a)} \ge z_0$ that we seek in Lemma 2.

Proof of Lemma 2. The inequality $z_c \ge z_c^{(a)}$ follows from $t_n \le a_n$, and the equality $z_0 = (K\mathrm{e})^{-1}$ holds by definition, so it suffices to prove that $z_c^{(a)} \ge z_0$. By (3.5), for this it suffices to prove that

$$a_n \le f_n. \tag{3.8}$$

To prove this, we adapt the proof of (3.6) from [1].

The first step involves a unique determination of a tree structure within a lattice animal. For this, we order all bonds in the infinite lattice lexicographically. Also, we regard a bond as an arc joining the vertices of its endpoints, and we order the two halves of this arc as minimal and maximal. These orderings are fixed once and for all. Given a lattice animal A, suppose that it contains c cycles. Choose the minimal bond whose removal would break a cycle, and remove its minimal half from the animal. Repeat this until no cycles remain. The result is a kind of lattice tree, which we will call the *cut-tree* A^*, in which c leaves are endpoints of half edges. See Figure 1. Let $\mathscr{A}_n$ denote the set of n-bond lattice animals that contain the origin. Let $\mathscr{A}_n^*$ denote the set of n-bond cut-trees that can be produced from a lattice animal in $\mathscr{A}_n$ by this procedure. By construction, lattice animals and cut-trees are in one-to-one correspondence, so $\mathscr{A}_n^*$ has cardinality a_n.

We may regard the edges of $T \in \mathscr{T}$ as directed away from the root, and we write a directed edge as (i,i'). Given $A^* \in \mathscr{A}_n^*$ and $(T,\varphi) \in \mathscr{M}_n$, we say that $\varphi(T) = A^*$ if (i) each bond in A is the image of a unique edge in T under φ, and if, in addition, (ii) if (b^+, b^-) is a directed bond in A from which the half-bond containing b^- is removed in A^*, and if the edge of T that is mapped by φ to (b^+, b^-) is (i,i'), then i' is a leaf of T. Roughly speaking, the condition $\varphi(T) = A^*$ means that the mapping φ "folds" T over A^* in such a way that the tree structure of T is preserved in A^*. We claim that for every $A^* \in \mathscr{A}_n^*$,

$$\sum_{(T,\varphi)\in\mathscr{M}_n:\varphi(T)=A^*} \prod_{i\in T} \frac{1}{\xi_i!} = 1. \tag{3.9}$$

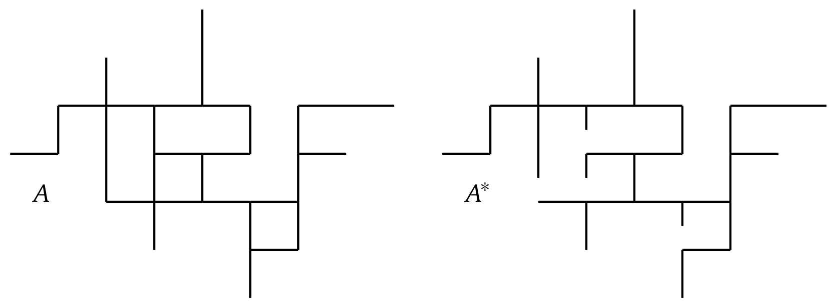

Figure 1: A lattice animal A and its associated cut-tree A^*.

This implies that

$$a_n = \sum_{(T,\varphi)\in\mathcal{M}_n:\varphi(T)\in\mathcal{A}_n^*} \prod_{i\in T} \frac{1}{\xi_i!} \le \sum_{(T,\varphi)\in\mathcal{M}_n} \prod_{i\in T} \frac{1}{\xi_i!} = f_n, \tag{3.10}$$

which is the required inequality (3.8). Thus it suffices to prove (3.9).
To prove (3.9), we adapt the proof of (3.6) from [1], as follows. Let b_0 be the degree of 0 in A^*, and given a nonzero vertex $x \in A^*$, let b_x be the degree of x in A^* minus 1 (the forward degree of x). Then the set $\{b_x : x \in A^*\}$ (with repetitions) must be equal to the set $\{\xi_i : i \in T\}$ (with repetitions) for any T such that $\varphi(T) = A^*$. Defining $\nu(A^*)$ to be the cardinality of $\{(T,\varphi) : \varphi(T) = A^*\}$, (3.9) is therefore equivalent to

$$\nu(A^*) = \prod_{x\in A^*} b_x!. \tag{3.11}$$

We prove (3.11) by induction on the number N of generations of A^*, i.e., the number of bonds or half-bonds in the longest self-avoiding path in A^* starting from the origin. The identity (3.11) clearly holds if $N = 0$. Our induction hypothesis is that (3.11) holds if there are $N-1$ or fewer generations. Suppose A^* has N generations, and let $A_1^*, \ldots, A_{b_0}^*$ denote the cut-trees resulting by deleting from A^* the origin and all bonds and half-bonds incident on the origin. We regard each A_a^* as rooted at the non-zero vertex in the corresponding deleted bond. It suffices to show that $\nu(A^*) = b_0! \prod_{a=1}^{b_0} \nu(A_a^*)$, since each A_a^* has fewer than N generations.
To prove this, we note that each pair (T,φ) with $\varphi(T) = A^*$ induces a set of (T_a,φ_a) such that $\varphi_a(T_a) = A_a^*$. This correspondence is $b_0!$ to 1, since (T,φ) is determined by the set of (T_a,φ_a), up to permutation of the branches of T at its root. This proves $\nu(A^*) = b_0! \prod_{a=1}^{b_0} \nu(A_a^*)$, and completes the proof of the lemma. □

Lemma 4. *For the nearest-neighbour or spread-out models (the latter in all dimensions $d \ge 1$), for each fixed $n \ge 0$,*

$$\lim_{K\to\infty} \frac{t_n}{K^n} = \sum_{T\in\mathcal{T}_n} \prod_{i\in T} \frac{1}{\xi_i!}. \tag{3.12}$$

Proof. By (3.6),

$$t_n = \sum_{(T,\varphi)\in\mathcal{M}_n:\varphi(T)\in\mathcal{L}_n} \prod_{i\in T} \frac{1}{\xi_i!} = \sum_{T\in\mathcal{T}_n} \prod_{i\in T} \frac{1}{\xi_i!} \sum_{\varphi\in\Phi(T):\varphi(T)\in\mathcal{L}_n} 1. \tag{3.13}$$

Given $T \in \mathscr{T}_n$, the cardinality of $\Phi(T)$ is K^n, so there are at most K^n nonzero terms in the above sum over φ. On the other hand, there are at least $K(K-1)\cdots(K-n+1)$ nonzero terms. To see this, consider the mapping φ of T to proceed in a connected fashion to map the edges of T one by one to bonds in $\mathbb{Z}^d$, starting from the root. The first edge of T can be mapped to any one of K possible bonds. The second edge of T includes one of the vertices of the first edge, and to avoid the image of the other vertex of the first edge, it can be mapped to any one of $K-1$ possible edges. In this way, as φ proceeds from the root to map vertices of T into $\mathbb{Z}^d$, the restriction that the image contain $n+1$ distinct vertices allows K choices for the first bond, $K-1$ choices for the second bond, at least $K-2$ for the third, at least $K-3$ for the fourth, and so on. This implies that

$$K(K-1)\cdots(K-n+1)\sum_{T\in\mathscr{T}_n}\prod_{i\in T}\frac{1}{\xi_i!} \le t_n \le K^n \sum_{T\in\mathscr{T}_n}\prod_{i\in T}\frac{1}{\xi_i!}, \tag{3.14}$$

and the desired conclusion follows. □

Proof of Proposition 3. By (3.7), (3.1) and (2.1),

$$t_n z_0^n \le f_n z_0^n = \mathrm{e}^{-n}\sum_{T\in\mathscr{T}_n}\prod_{i\in T}\frac{1}{\xi_i!}, \tag{3.15}$$

which is independent of K. Also, by (3.3), $\sum_{n=0}^{\infty} f_n z_0^n = \mathrm{e}$. Hence, by Lemma 4 and the dominated convergence theorem, we have

$$\lim_{K\to\infty} g(z_0) = \sum_{n=0}^{\infty}\lim_{K\to\infty} t_n z_0^n = \sum_{n=0}^{\infty}\left(\sum_{T\in\mathscr{T}_n}\prod_{i\in T}\frac{1}{\xi_i!}\right)\mathrm{e}^{-n} = \sum_{n=0}^{\infty} f_n z_0^n = \mathrm{e}. \tag{3.16}$$

□

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