A REMARK ON LOCALIZATION FOR BRANCHING RANDOM WALKS IN RANDOM ENVIRONMENT

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Abstract

We prove a localization-result for branching random walks in random environment, namely that if the process does not die out, the most populated site will infinitely often contain more than a fixed percentage of the population. This had been proven already before by Hu and Yoshida, but it is possible to drop their assumption that particles may not die.

1 Branching Random Walks in Random Environment

1.1 Informal descripion

Branching Random Walks in Random Environment (BRWRE) are a model for the spread of particles on an inhomogeneous media, such as bacteria that move around and encounter food supply or environmental conditions variable in time and space. These environmental conditions have an impact on the reproduction rate of the particles.

The randomness of the model occurs in two steps. The first step is the setting of the environment, which determines the offspring distribution at different times and places. In our case, these offspring distributions are to be i.i.d..

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The second step is the development of the population given the environment randomly generated in the first step. Starting with one particle at the origin, each particle generates offspring according to the offspring distribution associated with the time-space-location where it is born. It carries this offspring to adjacent sites in the manner of a simple random walk, and dies, leaving the new particles to start over, independently of each other.

As it is possible that particles die without leaving any offspring, the whole population might die out. This phenomenon is described in the event of "extinction". In the present article, however, we are more interested in the long-term-behaviour of the population, and usually work on the complementary event, called "survival". All the notions will be thoroughly defined in Subsection 1.3.

1.2   Brief history

Branching random walks in random environment have been introduced in [Birk], and Birkner, Geiger and Kersting [BGK05] revealed a phase change of the model which was subsequently characterized as a dichotomy: [Nak11] revealed that this model exhibits a phase transition beween what is called slow and regular growth, respectively.

The question of localization in this model, that is whether or not it is possible that in the long term, many particles may become concentrated on few sites, was answered positively for the slow growth phase by Hu and Yoshida [HY09] for environments that do not allow for extinction. A similar answer is given for the more general model of Linear Stochastic Evolution (LSE) in [Yos10]. BRWRE's survival, together with growth rates for the population, are studied by Comets and Yoshida [CY].

Uniting tools from the last three articles is what allows us to prove a localization result in a setting where extinction is possible.

A central limit theorem for BRWRE in the regular growth phase is proved in [HNY]. In that article, a more complete outline of the history of CLTs for BRW, BRWRE and related models can be found, and pictures of the BRWRE are given.

1.3   Thorough definition of the model

We define the random environment as i.i.d. offspring distributions $(q_{t,x})_{t\in\mathbb{N}_0,x\in\mathbb{Z}^d}$ under some (product-)measure Q on $\Omega_q := \mathscr{P}(\mathbb{N}_0)^{\mathbb{N}_0\times\mathbb{Z}^d}$, where $\mathscr{P}(\mathbb{N}_0)$ is the set of probability measures on $\mathbb{N}_0$, and may be equipped with the natural Borel-σ-field induced from that of $[0,1]^{\mathbb{N}_0}$. We call this product-σ-field $\mathscr{F}_q$.

$$q_{t,x} = (q_{t,x}(k))_{k\in\mathbb{N}_0} \in [0,1]^{\mathbb{N}_0}, \ \sum_{k\in\mathbb{N}_0} q_{t,x}(k) = 1.$$

On a measurable space $(\Omega_K, \mathscr{F}_K)$, to each fixed environment $q = (q_{t,x})_{t\in\mathbb{N}_0,x\in\mathbb{Z}^d}$ we associate a probability measure P_K^q such that the random variables $K := (K_{t,x}^v)_{t\in\mathbb{N}_0,x\in\mathbb{Z}^d,v\in\mathbb{N}}$ are independent in the number v of the particle and the space-time point (t,x) while being distributed according to $q_{t,x}$:

$$P_K^q(K_{t,x}^v = k) = q_{t,x}(k), \ k \in \mathbb{N}_0. \tag{1.1}$$

These random variables $K_{t,x}^v$ describe the number of children born to the v-th particle at time-space-location (t,x).

The v-th particle ($v \in \mathbb{N}$) at time $t \in \mathbb{N}_0 := \{0,1,\dots\} =: \mathbb{N} \cup \{0\}$ and site $x \in \mathbb{Z}^d$ moves (together with all of his offspring) to some site adjacent to his birthplace, determined by the $\mathbb{Z}^d$-valued

random variable $X_{t,x}^\nu$. The $X := (X_{t,x}^\nu)_{t \in \mathbb{N}_0, x \in \mathbb{Z}^d, \nu \in \mathbb{N}}$, defined on a probability space $(\Omega_X, \mathscr{F}_X, P_X)$, are defined to be the one-step transitions of a simple random walk, and i.i.d. in all three time, space, and particles:

$$P_X(X_{t,x}^\nu = y) = p(x,y) := \begin{cases} 1/2d & \text{if } |x-y| = 1 \\ 0 & \text{if } |x-y| \neq 1; \end{cases} \tag{1.2}$$

$|\cdot|$ designates the one-norm.

At its time-space destination $(t+1, X_{t,x}^\nu)$, the said ν-th particle from (t,x) dies and leaves place to its children, and the procedure starts over for every child.

Of course, we can combine the realization of X and K on one probability space

$$(\Omega_X \times \Omega_K, \mathscr{F}_X \otimes \mathscr{F}_K, P^q), \text{ where } P^q := P_X \otimes P_K^q \tag{1.3}$$

and finally merge all our construction to

$$\Omega := \Omega_X \times \Omega_K \times \Omega_q, \quad \mathscr{F} := \mathscr{F}_X \otimes \mathscr{F}_K \otimes \mathscr{F}_q,$$
$$P(A) := \int_A Q(dq) P^q(d\omega), \quad A \in \mathscr{F}. \tag{1.4}$$

P^q can be seen as the quenched measure and P as the annealed one of the model.

Now we come to the population at time t and site x. We start at time 0 with one particle at the origin, and define inductively

$$N_{0,x} := \mathbb{1}_{x=0}, \quad N_{t,x} = \sum_{y \in \mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} \mathbb{1}_{X_{t,y}^\nu = x} K_{t-1,y}^\nu, \ t \geq 1. \tag{1.5}$$

The filtration

$$\mathscr{F}_0 := \{\emptyset, \Omega\}, \quad \mathscr{F}_t := \sigma\left(X_{s,\cdot}^\cdot, K_{s,\cdot}^\cdot, q_{s,\cdot}; s \leq t-1\right), \quad t \geq 1, \tag{1.6}$$

makes the process $t \mapsto (N_{t,x})_{x \in \mathbb{Z}^d}$ adapted. The total population at time t can now be obtained by summation over all sites:

$$N_t := \sum_{y \in \mathbb{Z}^d} N_{t,y} = \sum_{y \in \mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} K_{t-1,y}^\nu, \ t \geq 1. \tag{1.7}$$

Important quantities of this model are the averaged and local moments of the offpring distributions

$$m^{(p)} := Q\left(m_{t,x}^{(p)}\right), \ m_{t,x}^{(p)} := \sum_{k \in \mathbb{N}_0} k^p q_{t,x}(k), \ p \in \mathbb{N}. \tag{1.8}$$

We also write $\mathbf{m} := m^{(1)}$.

1.4 The phase transition of the normalized population

It has been proven in [Nak11] that the total population exhibits a phase transition, where the one phase amounts to population growing as fast as its expectation, while the other phase means slower-than-the-expectation growth.

Proposition 1.4.1. *The normalized population $\overline{N}_t := N_t/\mathbf{m}^t$ is a martingale, and hence its limit exists:*

$$\overline{N}_\infty := \lim_{t\to\infty} \frac{N_t}{\mathbf{m}^t}, \; P\text{-a.s.}. \tag{1.9}$$

Further,

$$P(\overline{N}_\infty) = \begin{cases} 1 & \text{"regular growth phase", or} \\ 0 & \text{"slow growth phase".} \end{cases}$$

Sufficient conditions for both phases are given by the following two Propositions 1.4.3 and 1.4.4, which necessitate a bit of

Notation 1.4.2. Given the simple symmetric random walk S_t on $\mathbb{Z}^d$, we call π_d the probability of the return event $\bigcup_{t\geq1}\{S_t = 0\}$. Furthermore, we write

$$\alpha := \frac{Q(m_{t,x}^2)}{\mathbf{m}^2}.$$

Proposition 1.4.3. *There exists a constant $\alpha^* > 1/\pi_d$ such that, if*

$$\mathbf{m} > 1, \; m^{(2)} < \infty, \; d \geq 3, \text{ and } \alpha < \alpha^*, \tag{1.10}$$

then $P(\overline{N}_\infty > 0) > 0$.

Proposition 1.4.4. *On the other hand, $P(\overline{N}_\infty = 0) = 1$ is provided by any of the following three conditions:*

> **(a1)** $d = 1$; $Q(m_{t,x} = \mathbf{m}) \neq 1$.
>
> **(a2)** $d = 2$; $Q(m_{t,x} = \mathbf{m}) \neq 1$.
>
> **(a3)** $d \geq 3$; $Q\left(\dfrac{m_{t,x}}{\mathbf{m}} \ln \dfrac{m_{t,x}}{\mathbf{m}}\right) > \ln(2d)$.

Propositions 1.4.3 and 1.4.4 were obtained first in [BGK05, Theorem 4]. Proposition 1.4.3 plays a crucial role in our proof as it allows us in the slow growth phase to conclude $\alpha > \alpha^* > 1/\pi_d$.

Remark 1.4.5. We would at this point recall the non-random environment case [AN72, Theorem 1, page 24], where

$$P(\overline{N}_\infty = 0) = 1 \text{ if and only if } P(K_{t,x}^\nu \ln K_{t,x}^\nu) = \infty \text{ or } \mathbf{m} \leq 1. \tag{1.11}$$

In our case here, with the additional randomness of the environment, $P(\overline{N}_\infty = 0) = 1$ can happen even if the $K_{t,x}^\nu$ are bounded (see Remark 1.6.3 b) below).

1.5   Survival and the global growth estimate

Another dichotomy of this model is the one of survival and extinction. We define

$$\{survival\} := \{\forall t \in \mathbb{N}_0, N_t > 0\}. \tag{1.12}$$

The event of extinction is defined as the complement.
The following global growth estimate obtained in [CY, Theorem 2.1.1] characterizes the event of survival:

Lemma 1.5.1. *Suppose $Q(m_{t,x} + m_{t,x}^{-1}) < \infty$ and let $\varepsilon > 0$. Then, for large t,*

$$N_t \leq e^{(\Psi + \varepsilon)t}, \ P\text{-a.s.},$$

where the limit

$$\Psi := \lim_{t \to \infty} \frac{1}{t} Q\big(\ln P^q(N_{t,0}) \big)$$

exists.
If $\Psi > 0$ and $m^{(2)} < \infty$, then

$$\{survival\} = \{N_t \geq e^{(\Psi - \varepsilon)t} \text{ for all large } t\}, \ P\text{-a.s..} \tag{1.13}$$

Remark 1.5.2. a) Actually, the hypotheses given in [CY] are somewhat weaker, and are implied by our assumption $m^{(2)} < \infty$. See the Remark 2) right after [CY, Theorem 2.1.1].
b) It is proved in [CY] as well that "$\Psi > 0$" is implied by

$$Q(m_{t,x} = \mathbf{m}) \neq 1, \ Q(\ln m_{t,x}) \geq 0. \tag{1.14}$$

The object we investigate is the population density

$$\rho_{t,x} = \rho_t(x) := \frac{N_{t,x}}{N_t} \mathbb{1}_{N_t > 0}, \ t \in \mathbb{N}_0, \ x \in \mathbb{Z}^d. \tag{1.15}$$

It describes the distribution of the population in space.
Related important objects are

$$\rho_t^* := \max_{x \in \mathbb{Z}^d} \rho_{t,x} \quad \text{and} \quad \mathscr{R}_t := \sum_{x \in \mathbb{Z}^d} \rho_{t,x}^2. \tag{1.16}$$

They are, respectively, the density at the most populated site and the probability that two particles picked randomly from the total population are at the same site at time t. We will call this latter value the "replica overlap".
It is possible to relate the event of survival to this replica overlap.

Theorem 1.5.3. *Suppose $m^{(2)} < \infty$. Then, if $P(\overline{N}_\infty = 0) = 1$,*

$$\{survival\} \subseteq \Big\{ \sum_{t=1}^{\infty} \mathscr{R}_t = \infty \Big\}. \tag{1.17}$$

The proof of this Theorem can be found in Section 2.2. While it is true that the opposite inclusion does hold under the stronger assumption $m^{(3)} < \infty$, we do not state this formally here. The proof can be found in [HNY].

1.6 The main result

Hu and Yoshida, using the assumption that particles may not die, proved in [HY09, Theorem 1.3.2] the following

Theorem 1.6.1. *Suppose $P(\overline{N}_\infty = 0) = 1$ and*

$$m^{(3)} < \infty, \ Q(m_{t,x} = \mathbf{m}) \neq 1, \ Q\big(q_{t,x}(0) = 0\big) = 1. \qquad ([\text{HY09, (1.18)}])$$

Then, there exists a non-random number $c \in (0,1)$ such that,

$$\limsup_{t \to \infty} \rho_t^* \geq \limsup_{t \to \infty} \mathscr{R}_t \geq c, \ P\text{-a.s..} \tag{6.19}$$

In this setting, extinction (i.e. the event that at some time, the total population becomes 0) cannot occur. However, it is possible to drop this assumption with the help of a few additional tools. Our main result is indeed that the last two hypotheses can be replaced by weaker ones.

Theorem 1.6.2. *Suppose* $P(\overline{N}_\infty = 0) = 1$ *and*

$$m^{(3)} < \infty, \ \Psi > 0, \ Q(m_{t,x}^{-1}) < \infty. \tag{1.20}$$

Then, there exists a non-random number $c \in (0,1)$ *such that*

$$\limsup_{t \to \infty} \rho_t^* \geq \limsup_{t \to \infty} \mathscr{R}_t \geq c, \ P\text{-a.s. on the event of survival.} \tag{1.21}$$

The proof of the Theorem is postponed to its own Section 2.4.

Remark 1.6.3. a) The fact that Theorem 1.6.1 does not allow for dying particles has two implications, namely $\Psi > 0$ (rather trivially by (1.14)) and $Q(m_{t,x}^{-1}) < \infty$. Our theorem shows that we can indeed content ourselves with these two weaker conditions themselves.
b) The hypotheses $P(\overline{N}_\infty = 0) = 1$ and $\Psi > 0$ are difficult to check in practice. Yet, it is possible to give an example that satisfies the easier **(a1)** − **(a3)** of Proposition 1.4.4 and (1.14), but not the hypotheses of Theorem 1.6.1. It is given by the following class of environments constituted only of two states: for $n \in \mathbb{N}$,

$$q_{\cdot,\cdot}(0) = q_{\cdot,\cdot}(n^2) = \frac{1}{2} \text{ with probability } \frac{1}{n}, \tag{1.22}$$

$$q_{\cdot,\cdot}(1) = 1 \text{ with probability } 1 - \frac{1}{n}. \tag{1.23}$$

In this case, $Q\left(\frac{m_{t,x}}{\mathbf{m}} \ln \frac{m_{t,x}}{\mathbf{m}}\right) \sim \ln n$, and hence any dimension can be covered by n large enough.

2 Proofs

2.1 Tools for the proof of Theorem 1.5.3

The following Definition will be useful at several points. It provides notation for the thorough calculus of the fluctuation of the normalized population.

Definition 2.1.1. *Let*

$$U_{s+1,x} := \frac{\mathbb{1}_{N_s>0}}{\mathbf{m}N_s} \sum_{v=1}^{N_{s,x}} K_{s,x}^v \geq 0, \quad U_{s+1} := \sum_{x \in \mathbb{Z}^d} U_{s+1,x} = \frac{N_{s+1}}{\mathbf{m}N_s}\mathbb{1}_{N_s>0} = \frac{\overline{N}_{s+1}}{\overline{N}_s}\mathbb{1}_{\overline{N}_s>0}.$$

The $(U_{s+1,x})_{x\in\mathbb{Z}^d}$ *are independent under* $P(\cdot|\mathscr{F}_s)$. *It is not difficult to see that, on the event* $\{N_s > 0\}$,

$$P(U_{s+1,x}|\mathscr{F}_s) = \rho_s(x), \text{ and hence } P(U_{s+1}|\mathscr{F}_s) = 1.$$

Also, with $\widetilde{c}_i = \frac{m^{(i)}}{\mathbf{m}^i}$, $i = 2,3$,

$$\alpha\rho(x)^2 = \frac{1}{\mathbf{m}^2 N_s^2} N_{s,x}^2 Q(m_{s,x}^2) \leq P\big(U_{s+1,x}^2 \big| \mathscr{F}_s\big) \tag{2.1}$$

$$= \frac{1}{\mathbf{m}^2 N_s^2} P\Big(\big(\sum_{\nu=1}^{N_{s,x}} K_{s,x}^\nu\big)^2 \Big| \mathscr{F}_s\Big) \leq \frac{N_{s,x}^2 m^{(2)}}{\mathbf{m}^2 N_s^2} = \widetilde{c}_2 \rho_s(x)^2, \tag{2.2}$$

$$P\big(U_{s+1,x}^3 \big| \mathscr{F}_s\big) \leq \frac{m^{(3)}}{\mathbf{m}^3} \rho_s(x)^3 = \widetilde{c}_3 \rho_s(x)^3, \textit{ again on the event } \{N_s > 0\}. \tag{2.3}$$

Theorem 1.5.3 is a consequence of the following Proposition. It can be found in [Yos10, Proposition 2.1.2] and relates survival and boundedness of the predictable quadratic variation for some abstract martingale.

Proposition 2.1.2. *Let $(Y_t)_{t \in \mathbb{N}_0}$ be a mean-zero square-integrable martingale on a probability space with measure $\mathbb{E}$ and filtration $(\mathscr{F}_t)_{t \in \mathbb{N}_0}$. Suppose $-1 \leq \Delta Y_t := Y_t - Y_{t-1}$ for all $t \in \mathbb{N}$, and let*

$$X_t := \prod_{s=1}^{t} (1 + \Delta Y_s). \tag{2.4}$$

If $P\big((\Delta Y_t)^2 \big| \mathscr{F}_{t-1}\big)$ is uniformly bounded in t, then

$$\{X_\infty = 0\} \subseteq \{Extinction\} \cup \Big\{ \sum_{s=1}^{\infty} P\big((\Delta Y_s)^2 \big| \mathscr{F}_{s-1}\big) = \infty \Big\}, \tag{2.5}$$

where $\{Extinction\} := \{\exists t > 0 : X_t = 0\}$.

2.2 Proof of Theorem 1.5.3

We want to apply the abstract result that is Proposition 2.1.2 to our setting. To get the notation right, we take $X_t := \overline{N}_t$, and remark that the definition

$$\Delta Y_t := \frac{\overline{N}_t}{\overline{N}_{t-1}} \mathbb{1}_{N_{t-1}>0} - \mathbb{1}_{N_{t-1}>0} = \sum_x \big[U_{t,x} - \rho_{t,x} \big] \geq -1 \tag{2.6}$$

verifies (2.4); the $U_{t,x}$ are taken from Definition 2.1.1. As for the other hypothesis of the Proposition, we need not even to check it in order to find $\sum_{s=1}^{\infty} P\big((\Delta Y_s)^2 \big| \mathscr{F}_{s-1}\big) = \infty$: if uniform boundedness does not hold, it is true anyway, and if uniform boundedness holds, we derive it from Proposition 2.1.2 on the event $\{survival\} \cap \{\overline{N}_\infty = 0\}$.

Now, with (2.1), we see that $\sum_{s=1}^{t} P\big((\Delta Y_s)^2 \big| \mathscr{F}_{s-1}\big)$ shares its asymptotic behaviour with $\sum_{s=1}^{t} \mathscr{R}_s$, so we conclude (1.17). $\qquad\square$

2.3 Tools for the proof of Theorem 1.6.2

One result that has not been taken into account in [HY09] and that helps us making the slight improvement of the hypotheses is the following improved version of the Borel-Cantelli-lemma, stated in [Yos10, Lemma 2.2.1]:

Lemma 2.3.1. *Let $(R_t)_{t\in\mathbb{N}}$ be an integrable, adapted process defined on a probability space with measure $\mathbb{E}$ and a filtration $(\mathscr{F}_t)_{t\in\mathbb{N}_0}$. Define $V_0 := 0 =: T_0$ and*

$$V_t := \sum_{s=1}^{t} R_s, \qquad T_t := \sum_{s=1}^{t} \mathbb{E}[R_s|\mathscr{F}_{s-1}], \ t \in \mathbb{N}.$$

a) Suppose there is a constant $C_1 \in (0,\infty)$ such that

$$R_t - \mathbb{E}[R_t|\mathscr{F}_{t-1}] \geq -C_1, \ \mathbb{E}\text{-a.s. for all } t \in \mathbb{N}. \tag{2.7}$$

Then,

$$\left\{ \lim_{t\to\infty} V_t = \infty \right\} = \left\{ \lim_{t\to\infty} V_t = \infty, \ \limsup_{t\to\infty} \frac{T_t}{V_t} \geq 1 \right\} \subseteq \left\{ \sup_{t\geq 1} T_t = \infty \right\}.$$

b) Suppose that $(R_t)_{t\in\mathbb{N}}$ is in $L^2(\mathbb{E})$, and that there exists a constant $C_2 \in (0,\infty)$ such that

$$\mathbb{V}\mathrm{ar}[R_t|\mathscr{F}_{t-1}] \leq C_2 \mathbb{E}[R_t|\mathscr{F}_{t-1}] \ \mathbb{E}\text{-a.s. for all } t \in \mathbb{N},$$

where $\mathbb{V}\mathrm{ar}[R_t|\mathscr{F}_{t-1}] := \mathbb{E}[R_t^2|\mathscr{F}_{t-1}] - \mathbb{E}[R_t|\mathscr{F}_{t-1}]^2$. Then, $\mathbb{E}$-a.s.,

$$\left\{ \lim_{t\to\infty} T_t = \infty \right\} = \left\{ \lim_{t\to\infty} T_t = \infty, \ \limsup_{t\to\infty} \frac{V_t}{T_t} = 1 \right\} \subseteq \left\{ \sup_{t\geq 1} V_t = \infty \right\}.$$

This Lemma admits in our setting, with a slight abuse of notation, for the following

Corollary 2.3.2. *On the event $\{\lim_{t\to\infty} V_t = \infty\}$, there exists a constant $c_0 \in [1,\infty)$ such that*

$$T_t := \sum_{s=1}^{t} P(\mathscr{R}_s|\mathscr{F}_{s-1}) \leq c_0 \sum_{s=1}^{t} \mathscr{R}_s =: c_0 V_t \tag{2.8}$$

holds for large t.

Proof. In fact, the hypotheses of both **a)** and **b)** of Lemma 2.3.1 are satisfied. Indeed, $0 \leq \mathscr{R}_t = \sum_x \rho_{t,x}^2 \leq 1$ is square-integrable and adapted, and (2.7) is satisfied with $C_1 = 2$. Also,

$$\mathbb{V}\mathrm{ar}(\mathscr{R}_t|\mathscr{F}_{t-1}) \leq P(\mathscr{R}_t^2|\mathscr{F}_{t-1}) \leq P(\mathscr{R}_t|\mathscr{F}_{t-1}).$$

Hence, with **a)**, $\{\lim_{t\to\infty} V_t = \infty\}$ implies $\{\sup_t T_t = \infty\}$. But T_t is a sum over positive terms, so its supremum is equal to its limes, and we can readily apply part **b)**. The statement is then trivial. $\square$

The following Lemma is an extension to [Yos10, Lemma 3.2.1] and replaces [HY09, Lemma 3.1.1].

Lemma 2.3.3. *Let $(U_i)_{1\leq i\leq n}$, $n \geq 2$, be non-negative, independent and cube-integrable random variables on our general probability space with probability measure $\mathbb{E}$ such that for*

$$U = \sum_{i=1}^{n} U_i, \qquad \mathbb{E}[U] = 1. \tag{2.9}$$

Let furthermore X be a random variable such that $0 \leq X \leq U_1^2$ a.s.. Then,

$$\mathbb{E}\left[\frac{U_1 U_2}{U^2} : U > 0\right] \geq \mathbb{E}[U_1]\mathbb{E}[U_2] - 2\mathbb{E}[U_2]\mathbb{V}\mathrm{ar}[U_1] - 2\mathbb{E}[U_1]\mathbb{V}\mathrm{ar}[U_2], \tag{2.10}$$

$$\mathbb{E}\left[\frac{X}{U^2} : U > 0\right] \geq \mathbb{E}[U_1^2](1 + 2\mathbb{E}[U_1]) - 2\mathbb{E}[U_1^3] - 3\mathbb{E}[U_1^2 - X]. \tag{2.11}$$

Proof. The first inequality is proved in [Yos10]. We will prove the second one.
Note that $u^{-2} \geq 3 - 2u$ for $u \in (0, \infty)$. Thus, we have that

$$
\begin{aligned}
\mathbb{E}\left[\frac{X}{U^2} : U > 0\right] &\geq \mathbb{E}[X(3 - 2U) : U > 0] = \mathbb{E}[X(3 - 2U)] \\
&= \mathbb{E}[(X - U_1^2)(3 - 2U)] + \mathbb{E}[U_1^2(3 - 2U)] \\
&= \mathbb{E}[(X - U_1^2)(3 - 2U)] + 3\mathbb{E}[U_1^2] - 2\mathbb{E}[U_1^3] - 2\mathbb{E}[U_1^2]\mathbb{E}[\textstyle\sum_{i \neq 1} U_i] \\
&= 2\mathbb{E}[(U_1^2 - X)U] - 3\mathbb{E}[U_1^2 - X] + 3\mathbb{E}[U_1^2] - 2\mathbb{E}[U_1^3] - 2\mathbb{E}[U_1^2](1 - \mathbb{E}[U_1]) \\
&\geq -3\mathbb{E}[U_1^2 - X] + \mathbb{E}[U_1^2](1 + 2\mathbb{E}[U_1]) - 2\mathbb{E}[U_1^3]. \qquad\qquad \square
\end{aligned}
$$

At this point, we need some further notations. We denote by $\mathscr{P}_s(x, y)$ the probability that the simple random walk starting in $x \in Z^d$ goes to $y \in Z^d$ in exactly $s \in \mathbb{N}$ steps. We write $r_j := \mathscr{P}_{2j}(x, x)$. Also, we can define the semigroup of the simple random walk by $\mathscr{P}_s f(x) := \sum_y \mathscr{P}_s(x, y) f(y)$. We write $\mathscr{P} := \mathscr{P}_1$.

Remark 2.3.4. With the Cauchy-Schwarz-inequality, we have

$$
\max_x \left(\mathscr{P}_j \rho_t(x)\right)^2 \leq \sum_x \left(\mathscr{P}_j \rho_t(x)\right)^2 \leq \sum_x \mathscr{P}_j \rho_t^2(x) = \mathscr{R}_t = \sum_x \rho_t^2(x) \leq 1. \tag{2.12}
$$

We now start estimates on the population density. The following result corresponds to the inequality (3.7) in [HY09, Lemma 3.1.4].

Lemma 2.3.5. *Suppose* (1.20). *On the event of survival up to time* $s \in \mathbb{N}$, *for any* $y_1, y_2 \in \mathbb{Z}^d$, *we have*

$$
\begin{aligned}
P\left(\rho_{s+1}(y_1)\rho_{s+1}(y_2)\big|\mathscr{F}_s\right) \\
&\geq \mathscr{P}\rho_s(y_1)\mathscr{P}\rho_s(y_2) + (\alpha - 1)\sum_z \rho_s(z)^2 p(z, y_1) p(z, y_2) \tag{2.13} \\
&\quad - 2\widetilde{c}_2\left[\mathscr{P}\rho_s(y_2)\mathscr{P}(\rho_s^2)(y_1) + \mathscr{P}\rho_s(y_1)\mathscr{P}(\rho_s^2)(y_2)\right] \\
&\quad - 2\widetilde{c}_3\sum_z \rho_s(z)^3 p(z, y_1) p(z, y_2) - 3\widetilde{c}_2\frac{1}{N_s}\sum_z \rho_s(z) p(z, y_1) p(z, y_2).
\end{aligned}
$$

where the $\widetilde{c}_\cdot$ *are the same as in Definition 2.1.1.*

Proof. We have

$$
\begin{aligned}
P\left(\rho_{s+1}(y_1)\rho_{s+1}(y_2)|\mathscr{F}_s\right) &= \sum_{z_1,z_2} \sum_{\nu_1=1}^{N_{s,z_1}} \sum_{\nu_2=1}^{N_{s,z_2}} P\left(\frac{\mathbb{1}_{X_{s,z_1}^{\nu_1}=y_1}\mathbb{1}_{X_{s,z_2}^{\nu_2}=y_2} K_{s,z_1}^{\nu_1} K_{s,z_2}^{\nu_2}}{N_{s+1}^2}\mathbb{1}_{N_{s+1}>0}\bigg|\mathscr{F}_s\right) \\
&\geq \sum_{z_1 \neq z_2} p(z_1, y_1) p(z_2, y_2) P\left(\frac{\sum_{\nu_1=1}^{N_{s,z_1}} K_{s,z_1}^{\nu_1} \sum_{\nu_2=1}^{N_{s,z_2}} K_{s,z_2}^{\nu_2}}{N_{s+1}^2}\mathbb{1}_{N_{s+1}>0}\bigg|\mathscr{F}_s\right) \tag{2.14} \\
&\quad + \sum_z p(z, y_1) p(z, y_2) P\left(\frac{\sum_{\nu_1 \neq \nu_2=1}^{N_{s,z}} K_{s,z}^{\nu_1} K_{s,z}^{\nu_2}}{N_{s+1}^2}\mathbb{1}_{N_{s+1}>0}\bigg|\mathscr{F}_s\right). \tag{2.15}
\end{aligned}
$$

Now, we would like to estimate (2.14) and (2.15). We can rewrite these lines with the processes from Definition 2.1.1. These verify the hypotheses of Lemma 2.3.3. The estimates obtained by the

application of this Lemma comprise second and third moments which we cannot provide explicitly. We therefore replace them by the estimates obtained in Definition 2.1.1; note that survival up to time $s+1$ implies survival up to time s.

Since $\{N_{s+1} > 0\} \subseteq \{U_{s+1} > 0\}$, by (2.10), we have

$$
\begin{aligned}
(2.14) &= \sum_{z_1 \neq z_2} p(z_1, y_1) p(z_2, y_2) P\left(\frac{U_{s+1, z_1} U_{s+1, z_2}}{U_{s+1}^2} \mathbb{1}_{U_{s+1} > 0} \Big| \mathscr{F}_s \right) \\
&\geq \sum_{z_1 \neq z_2} p(z_1, y_1) p(z_2, y_2) \left(\rho_s(z_1) \rho_s(z_2) - 2\widetilde{c}_2 \left[\rho_s(z_2) \rho_s(z_1)^2 + \rho_s(z_1) \rho_s(z_2)^2 \right] \right).
\end{aligned}
$$

Also, with $X(z) = \left(\left(\sum_{v=1}^{N_{s,z}} K_{s,z}^v \right)^2 - \sum_{v=1}^{N_{s,z}} (K_{s,z}^v)^2 \right) / \mathbf{m}^2 N_s^2$ and (2.11),

$$
\begin{aligned}
(2.15) &= \sum_z p(z, y_1) p(z, y_2) P\left(\frac{X(z)}{U_{s+1}^2} \mathbb{1}_{U_{s+1} > 0} \Big| \mathscr{F}_s \right) \\
&\geq \sum_z p(z, y_1) p(z, y_2) \\
&\qquad \left[P\left(U_{s+1,z}^2 \big| \mathscr{F}_s \right) \left[1 + 2\rho_s(z) \right] - 2P\left(U_{s+1,z}^3 \big| \mathscr{F}_s \right) - 3 \sum_{v=1}^{N_{s,z}} P\left(\left(\frac{K_{s,z}^v}{\mathbf{m} N_s} \right)^2 \Big| \mathscr{F}_s \right) \right] \\
&\geq \sum_z p(z, y_1) p(z, y_2) \left[\alpha \rho_s(z)^2 - 2\widetilde{c}_3 \rho_s(z)^3 - 3\widetilde{c}_2 \frac{\rho_s(z)}{N_s} \right].
\end{aligned}
$$

These estimates imply the statement. □

Lemma 2.3.6. *Suppose* (1.20). *For all* $1 \leq j \leq t - 1$,

$$
\begin{aligned}
P\left(\sum_x (\mathscr{P}_{j-1} \rho_{t-j+1}(x))^2 \Big| \mathscr{F}_{t-j} \right) &\geq \sum_x (\mathscr{P}_j \rho_{t-j}(x))^2 + (\alpha - 1) r_j \mathscr{R}_{t-j} \\
&\qquad - (4\widetilde{c}_2 + 2\widetilde{c}_3) \mathscr{R}_{t-j}^{3/2} - \frac{3\widetilde{c}_2}{N_{t-j}}.
\end{aligned}
$$

Proof. If we apply the definition of the semigroup operator $\mathscr{P}$, we get

$$
\sum_x (\mathscr{P}_{j-1} \rho_{t-j+1}(x))^2 = \sum_x \sum_{y_1, y_2} \mathscr{P}_{j-1}(x, y_1) \mathscr{P}_{j-1}(x, y_2) \rho_{t-j+1}(y_1) \rho_{t-j+1}(y_2).
$$

Applying (2.13) gives

$$
P\left(\sum_x (\mathscr{P}_{j-1} \rho_{t-j+1}(x))^2 \Big| \mathscr{F}_{t-j} \right) \geq \left[I + (\alpha - 1) II - 2\widetilde{c}_2 III - 2\widetilde{c}_3 IV - 3\widetilde{c}_2 \frac{1}{N_{t-j}} V \right],
$$

where

$$I := \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\mathscr{P}\rho_{t-j}(y_1)\mathscr{P}\rho_{t-j}(y_2)$$

$$= \sum_x \left(\mathscr{P}_j\rho_{t-j}(x)\right)^2 \text{ by definition of the semigroup-operator;}$$

$$II := \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_z \rho_{t-j}(z)^2 p(z,y_1)p(z,y_2)$$

$$= \sum_x \sum_z \left(\mathscr{P}_j(x,z)\right)^2 \rho_{t-j}^2(z) = r_j \sum_z \rho_{t-j}^2(z) \text{ because } \sum_x \left(\mathscr{P}_j(x,z)\right)^2 = r_j;$$

$$III := \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\left(\mathscr{P}\rho_{t-j}(y_2)\mathscr{P}\rho_{t-j}^2(y_1) + \mathscr{P}\rho_{t-j}(y_1)\mathscr{P}\rho_{t-j}^2(y_2)\right)$$

$$= 2\sum_x \mathscr{P}_j\rho_{t-j}(x)\mathscr{P}_j(\rho_{t-j}^2)(x)$$

$$\leq 2\max_x \mathscr{P}_j\rho_{t-j}(x)\sum_x \mathscr{P}_j(\rho_{t-j}^2)(x) \leq 2\mathscr{R}_{t-j}^{1/2}\mathscr{R}_{t-j} \text{ by Remark 2.3.4;}$$

$$IV := \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_z \rho_{t-j}^3(z)p(z,y_1)p(z,y_2)$$

$$\leq \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_{z_1,z_2} \rho_{t-j}(z_1)p(y_1,z_1)\rho_{t-j}^2(z_2)p(y_2,z_2)$$

$$= \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\mathscr{P}\rho_{t-j}(y_1)\mathscr{P}\rho_{t-j}^2(y_2) \leq III;$$

$$V := \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_z \rho_{t-j}(z)p(z,y_1)p(z,y_2)$$

$$= \sum_x \sum_z \left(\mathscr{P}_j(x,z)\right)^2 \rho_{t-j}(z) = \sum_z \rho_{t-j}(z)r_j = r_j.$$

In these computations, the symmetry of $p(\cdot,\cdot)$ has been used at appropriate places. If we put together the pieces, we obtain the statement of the Lemma. □

Later, in the proof of the main theorem, we are going to perform a division by $\sum_{s=1}^t \mathscr{R}_s$ at some point. The following Lemma helps showing that a certain term then vanishes asymptotically. We recall the definition of $V_t := \sum_{s=1}^t \mathscr{R}_s$ from Corollary 2.3.2 and write $V_\infty := \lim_{t\to\infty} V_t$.

Lemma 2.3.7. *Assume* (1.20)*, and fix some $j \geq 1$. The martingale $Z_j(\cdot)$ defined by*

$$Z_j(t) := \sum_{s=1}^t \sum_x \left[\left(\mathscr{P}_j\rho_s(x)\right)^2 - P\left((\mathscr{P}_j\rho_s(x))^2 \big| \mathscr{F}_{s-1}\right) \right], \ t \geq 1,$$

satisfies the following law of large numbers:

$$\{V_\infty = \infty\} \subseteq \left\{ \frac{Z_j(t)}{V_t} \xrightarrow[t\to\infty]{} 0 \right\}, \ P\text{-a.s..}$$

Remark 2.3.8. The increments of $Z_j(t)$ will be used later, and in squared form in the proof of the Lemma. They are given by

$$Z_j(t+1) - Z_j(t) = \sum_x \left[\left(\mathscr{P}_j\rho_{t+1}(x)\right)^2 - P\left((\mathscr{P}_j\rho_{t+1}(x))^2 \big| \mathscr{F}_t\right) \right];$$

recalling Remark 2.3.4, we can further estimate

$$\left(Z_j(t+1) - Z_j(t)\right)^2 \leq \left(\sum_x \left(\mathscr{P}_j \rho_{t+1}(x)\right)^2\right)^2 + \left(\sum_x P\left((\mathscr{P}_j \rho_{t+1}(x))^2 \big| \mathscr{F}_t\right)\right)^2$$

$$\leq \mathscr{R}_{t+1}^2 + P\left(\mathscr{R}_{t+1} \big| \mathscr{F}_t\right)^2 \leq \mathscr{R}_{t+1} + P\left(\mathscr{R}_{t+1} \big| \mathscr{F}_t\right).$$

Proof of Lemma 2.3.7. The idea of the proof is to make use of the increasing process $\langle Z_j \rangle_t$ associated with $Z_j(t)$ in order to monitor the growth of $Z_j(t)$ itself.

With the previous Remark, it is indeed possible to estimate $\langle Z_j \rangle_t$ by the sum of the conditional replica-overlap:

$$\langle Z_j \rangle_t = \sum_{s=0}^{t-1} \langle Z_j \rangle_{s+1} - \langle Z_j \rangle_s = \sum_{s=0}^{t-1} P\left(\left(Z_j(s+1) - Z_j(s)\right)^2 \big| \mathscr{F}_s\right) \tag{2.16}$$

$$\leq 2 \sum_{s=0}^{t-1} P\left(\mathscr{R}_s \big| \mathscr{F}_{s-1}\right) = \dots,$$

but this, by Corollary 2.3.2, is in turn related to the replica overlap itself:

$$\dots \leq 2c_0 V_t, \ t \geq 1. \tag{2.17}$$

The rest is easy. Either $\langle Z_j \rangle_\infty < \infty$, in which case $Z_j(t)$ converges and the statement is trivial anyway, or $\langle Z_j \rangle_\infty = \infty$, in which case we can apply the law of large numbers for square-integrable martingales, see [Dur91, p. 253], which gives us

$$\left|\frac{Z_j(t)}{V_t}\right| \leq \frac{1}{2c_0} \left|\frac{Z_j(t)}{\langle Z_j \rangle_t}\right| \xrightarrow[t \to \infty]{} 0. \qquad \square$$

As a final ingredient, we give a statement that compares parameters of the simple random walk with ones of the BRWRE-model.

Lemma 2.3.9. *Suppose* (1.20) *and* $P(\overline{N}_\infty = 0) = 1$. *There exist* $\varepsilon > 0$ *and* $t_0 \in \mathbb{N}$ *such that*

$$\sum_{s=1}^{t_0} r_s \geq \frac{1+\varepsilon}{\alpha - 1}. \tag{2.18}$$

Furthermore, with $T > t_0$ *and* $c_4 := (\alpha - 1) t_0 \sum_{j=1}^{t_0} r_j$,

$$\sum_{t=t_0+1}^{T} \left[(\alpha - 1) \sum_{j=1}^{t_0} r_j \mathscr{R}_{t-j} - \mathscr{R}_t\right] \geq \varepsilon V_T - c_4. \tag{2.19}$$

Proof. In dimensions $d = 1, 2$, the first statement is trivial as $\sum_{s=1}^{\infty} r_s = \infty$.

In dimensions $d \geq 3$, the assumption $P(\overline{N}_\infty = 0) = 1$ in conjunction with Proposition 1.4.3 gives us $\alpha \geq \alpha^* > 1/\pi_d > 0$, which implies

$$(\alpha - 1) \sum_{s=1}^{\infty} r_s = (\alpha - 1) \frac{\pi_d}{1 - \pi_d} > \frac{1}{1 - \pi_d} > 1, \tag{2.20}$$

so that (2.18) follows.

As for the second statement, we compute

$$
\sum_{t=t_0+1}^{T} \left[(\alpha-1)\sum_{j=1}^{t_0} r_j \mathcal{R}_{t-j} - \mathcal{R}_t \right] = (\alpha-1)\sum_{j=1}^{t_0} r_j \sum_{t=t_0+1}^{T} \mathcal{R}_{t-j} - \sum_{t=t_0+1}^{T} \mathcal{R}_t
$$

$$
= (\alpha-1)\sum_{j=1}^{t_0} r_j (V_{T-j} - V_{t_0-j}) - (V_T - V_{t_0}) \geq (\alpha-1)(V_T - t_0)) \sum_{j=1}^{t_0} r_j - V_T
$$

$$
\geq (1+\varepsilon)V_T - (\alpha-1)t_0 \sum_{j=1}^{t_0} r_j - V_T = \varepsilon V_T - c_4,
$$

where for the last but one inequality, we used that $V_{t_0-j} \leq t_0 - j$ and $V_{T-j} + j \geq V_T$ for all $1 \leq j \leq t_0$. $\qquad\square$

2.4 Proof of the main theorem

Proof of Theorem 1.6.2. The idea of the proof is to obtain some estimate of the form

$$
\liminf_{T\to\infty} \frac{\sum_{t=1}^{T} \mathcal{R}_t^{3/2}}{\sum_{t=1}^{T} \mathcal{R}_t} \geq C \text{ some constant, } P\text{-a.s..} \tag{2.21}
$$

This then implies

$$
\limsup_{t\to\infty} \mathcal{R}_t \geq C^2, \ P\text{-a.s.,} \tag{2.22}
$$

as can easily be verified by contradiction.

However, the only tool we have at hand to estimate $\mathcal{R}_t^{3/2}$ is Lemma 2.3.6, and we need to carry out several operations before arriving at (2.21).

First, we apply Lemma 2.3.6 to $j = 1, \dots t_0$, with t_0 from (2.18), and take the sum:

$$
\sum_{j=1}^{t_0} \left[(4\widetilde{c}_2 + 2\widetilde{c}_3)\mathcal{R}_{t-j}^{3/2} + \frac{3\widetilde{c}_2}{N_{t-j}} \right]
$$

$$
\geq \sum_{j=1}^{t_0} \sum_{x} \left[(\mathscr{P}_j \rho_{t-j}(x))^2 - P\left((\mathscr{P}_{j-1}\rho_{t-j+1}(x))^2 \big| \mathscr{F}_{t-j} \right) \right] + (\alpha-1)\sum_{j=1}^{t_0} r_j \mathcal{R}_{t-j}
$$

$$
= \sum_{j=1}^{t_0} \sum_{x} \left[(\mathscr{P}_{j-1}\rho_{t-j+1}(x))^2 - P\left((\mathscr{P}_{j-1}\rho_{t-j+1}(x))^2 \big| \mathscr{F}_{t-j} \right) \right]
$$

$$
+ \sum_{j=1}^{t_0} \sum_{x} \left[(\mathscr{P}_j \rho_{t-j}(x))^2 - (\mathscr{P}_{j-1}\rho_{t-j+1}(x))^2 \right] + (\alpha-1)\sum_{j=1}^{t_0} r_j \mathcal{R}_{t-j}
$$

$$
= \sum_{j=1}^{t_0} \left[Z_{j-1}(t-j+1) - Z_{j-1}(t-j) \right] + \sum_{x} (\mathscr{P}_{t_0}\rho_{t-t_0}(x))^2 - \mathcal{R}_t + (\alpha-1)\sum_{j=1}^{t_0} r_j \mathcal{R}_{t-j}
$$

$$
\geq \sum_{j=1}^{t_0} \left[Z_{j-1}(t-j+1) - Z_{j-1}(t-j) \right] + (\alpha-1)\sum_{j=1}^{t_0} r_j \mathcal{R}_{t-j} - \mathcal{R}_t.
$$

In the last equality, we made use of Remark 2.3.8.

Another summation, over $t = t_0 + 1, \ldots, T$, makes appear $V_T = \sum_{t=1}^{T} \mathscr{R}_t$ on the right hand side (we immediately replace a telescopic sum by the end terms and apply (2.19)):

$$\sum_{t_0+1}^{T} \sum_{j=1}^{t_0} \left[(4\widetilde{c}_2 + 2\widetilde{c}_3) \mathscr{R}_{t-j}^{3/2} + \frac{3\widetilde{c}_2}{N_{t-j}} \right] \geq \sum_{j=1}^{t_0} \left[Z_{j-1}(T - j + 1) - Z_{j-1}(t_0 - j + 1) \right] + \varepsilon V_T - c_4 \quad (2.23)$$

Now, if we divide by V_T and let T tend to infinity, the fact that by Theorem 1.5.3, $V_\infty = \infty$ makes disappear several terms: the sum over $3\widetilde{c}_2/N_{t-j}$ is finite by (1.13), and the one with the martingales $Z_\cdot(\cdot)$ vanishes by Lemma 2.3.7. This leads directly to (2.21) and concludes the proof. □

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