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# GAUSSIAN MEASURES OF DILATIONS OF CONVEX ROTATIONALLY SYMMETRIC SETS IN $\mathbb{C}^N$

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*Abstract*
We consider the complex case of the *S-inequality*. It concerns the behaviour of Gaussian measures of dilations of convex and rotationally symmetric sets in $\mathbb{C}^n$. We pose and discuss a conjecture that among all such sets measures of cylinders (i.e. the sets $\{z \in \mathbb{C}^n \mid |z_1| \leq p\}$) decrease the fastest under dilations.
Our main result in this paper is that this conjecture holds under the additional assumption that the Gaussian measure of the sets considered is not greater than some constant $c > 0.64$.

## Introduction

Let $\nu_n$ be the standard Gaussian measure on $\mathbb{C}^n$, i.e.

$$\nu_n(B) = \frac{1}{(2\pi)^n}\int_{j(B)} \exp\left(-\sum_{k=1}^{n}(x_k^2+y_k^2)\right) \mathrm{d}x_1\mathrm{d}y_1\ldots\mathrm{d}x_n\mathrm{d}y_n,$$

for any Borel set $B \subset \mathbb{C}^n$, where $j\colon \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$ is the standard isomorphism $j((x_1+iy_1,\ldots,x_n+iy_n)) = (x_1,y_1,\ldots,x_n,y_n)$. Denote for any $z=(z_1,\ldots,z_n), w=(w_1,\ldots,w_n) \in \mathbb{C}^n$ by $\langle w,z\rangle = \sum_{k=1}^n w_k\bar{z_k}$ a scalar product on $\mathbb{C}^n$ and the norm generated by it as $\|z\| = \sqrt{\langle z,z\rangle}$.
Let $A \subset \mathbb{C}^n$ be a set, which is

- convex,
- **rotationally symmetric**, i.e. for any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $a \in A$ implies that $\lambda a \in A$

and $P = \{z \in \mathbb{C}^n \mid |\langle z, v\rangle| \le p\}$ be a **cylinder** such that $\nu_n(A) = \nu_n(P)$, where $v \in \mathbb{C}^n$ has length 1 and $p \ge 0$ is a **radius** of $P$. We ask whether

$$\nu_n(tA) \ge \nu_n(tP), \qquad \text{for } t \ge 1,$$

i.e. whether the measure of dilations of cylinders grows the slowest among all convex rotationally symmetric sets.
The analogous question in $\mathbb{R}^n$ has an affirmative answer which was shown by R. Latała and K. Oleszkiewicz [5]. Following their method in the considered complex case we obtain a partial answer to the question. The main result is the following

**Theorem 1.** *There exists a constant $c > 0.64$ such that for any convex rotationally symmetric set $A \subset \mathbb{C}^n$, with measure $\nu_n(A) \le c$, and a cylinder $P = \{z \in \mathbb{C}^n \mid |z_1| \le p\}$ satisfying $\nu_n(A) = \nu_n(P)$, we have*

$$\nu_n(tA) \le \nu_n(tP), \qquad \text{for } 0 \le t \le 1. \tag{$*$}$$

The paper is organized as follows. In Section 1 we give the proof of the above theorem. In Section 2 we state some remarks concerning this theorem. Especially, we discuss the possibility of omitting the restriction on the measure assumed in Theorem 1, but weakening its assertion. Section 3 is devoted to proofs of some auxiliary lemmas which have slightly technical character.

# 1 Proof of the main result

Firstly, let us set up some notation. We put $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ for the standard norm of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. By $\gamma_n$ we denote the standard Gaussian measure in $\mathbb{R}^n$ and by $\gamma_n^+(A) := \underline{\lim}_{h\to 0+}(\gamma_n(A^h) - \gamma_n(A))/h$ the Gaussian perimeter of $A \subset \mathbb{R}^n$, where $A^h := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, A) \le h\}$ is a $h$-neighbourhood of $A$. Analogously, we define $\nu_n^+(A)$. Moreover, we will use the functions

$$\Phi(x) = \gamma_1((-\infty, x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t,$$
$$T(x) = 1 - \Phi(x).$$

Following the same procedure as in the real case, presented in detail in [5], we can reduce a proof of $(*)$ to some kind of an isoperimetric problem in $\mathbb{R}^3$. However, these estimations turn out to be insufficient and a constraint involving a boundedness of the measure from above by $c$ appears. For the sake of the reader's convenience, that reduction is briefly presented below.

(I) For any measurable set $A \subset \mathbb{C}^n$ let $\nu_A(t) := \nu_n(tA)$. Then Theorem 1 is equivalent to $\nu_A'(1) \ge \nu_P'(1)$, provided that $\nu_A(1) = \nu_P(1) \le c$. Since $P$ is a cylinder we have $\nu_P'(1) = p\nu_n^+(P)$.

(II) Convexity of $A$ gives $\nu_A'(1) \ge w\nu_n^+(A)$, where

$$w := \sup\{r \ge 0 \mid \{z \in \mathbb{C}^n \mid \|z\| < r\} \subset A\}.$$

The parameter $2w$ is in some sense the width of the set $A$.

(III) Rotational symmetry of $A$ gives that $A$ is included in some cylinder of the radius $w$. Indeed, by the definition of $w$ there is a point, say $a$, from the closure of $A$ such that $\|a\| = w$. Using the convexity of $A$ we infer the existence of the supporting hyperplane $\{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z, a\rangle = \|a\|^2\}$. Now the rotational symmetry of $A$ comes in and it yields that $A$ is included in a cylinder.

Namely, we have $A \subset \{z \in \mathbb{C}^n \mid |\langle z, a\rangle| \leq \|a\|^2\}$. Thanks to the invariance of the Gaussian measure with respect to unitary transformations we may assume without loss of generality that $a = \|a\|e_1$. Then

$$A \subset \{z \in \mathbb{C}^n \mid |z_1| \leq w\}.$$

Now we can apply Ehrhard's symmetrization [1], that is for a given point $z = x + iy$ with modulus less than $w$ we replace the whole section

$$A_z = \{(z_2, \dots, z_n) \in \mathbb{C}^{n-1} \mid (z, z_2, \dots, z_n) \in A\},$$

of our set with a real half-line $(-\infty, f(|z|))$ with the same Gaussian measure as $A_z$. In such a way we obtain a set in $\mathbb{R}^3$

$$\widetilde{A} = \left\{(x, y, t) \in \mathbb{R}^3 \mid t \leq f\left(\sqrt{x^2+y^2}\right), \sqrt{x^2+y^2} \leq w\right\}$$

where $f : [0, w] \longrightarrow \mathbb{R} \cup \{-\infty\}$,

$$f\left(\sqrt{x^2+y^2}\right) := \Phi^{-1}\left(\nu_{n-1}\left(A_z\right)\right).$$

The function $f$ is well defined (by the rotational symmetry of $A$), and, as $A$ is convex by Ehrhard's inequality [1], $f$ is concave and nonincreasing. Clearly, $\nu_n(A) = \gamma_3(\widetilde{A})$. The key property of this symmetrization is that $\nu_n^+(A) \geq \gamma_3^+(\widetilde{A})$. Obviously a symmetrized cylinder $P$ is a **cylinder** $\widetilde{P} = \{z \in \mathbb{R}^2 \mid |z| \leq p\} \times \mathbb{R}$ and $\nu_n^+(P) = pe^{-p^2/2} = \gamma_3^+(\widetilde{P})$.

Summing up, in order to prove Theorem 1 it is enough to show

**Theorem 2.** *There exists a constant $c > 0.64$ with the following property. Let $A \subset \mathbb{R}^3$ be a set of the form*

$$A = \left\{(x, y, t) \in \mathbb{R}^3 \mid t \leq f\left(\sqrt{x^2+y^2}\right), \sqrt{x^2+y^2} < w\right\},$$

*where $f : [0, w) \longrightarrow \mathbb{R}$ is a concave, nonincreasing, smooth function such that $f(x) \xrightarrow[x \to w-]{} -\infty$. Let $P = \{(x, y, t) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \leq p\} \subset \mathbb{R}^3$ be a cylinder with the same measure as $A$, that is, $\gamma_3(A) = \gamma_3(P) = 1 - e^{-p^2/2}$. Then*

$$w\gamma_3^+(A) \geq p\gamma_3^+(P), \tag{1}$$

*provided that $\gamma_3(A) \leq c$.*

*Proof.* Following [5], we define for fixed $x \in [0, w]$

$$A(x) = A \cup \{z \in \mathbb{R}^2 \mid |z| < x\} \times \mathbb{R},$$
$$P(x) = \{z \in \mathbb{R}^2 \mid |z| < a(x)\} \times \mathbb{R},$$

where the function $a(x)$ is defined by the equation

$$\gamma_3(A(x)) = \gamma_3(P(x)).$$

We have $\partial A(x) = B_1(x) \cup B_2(x)$, where $B_1(x) = \{(z, t) \in \mathbb{R}^2 \times \mathbb{R} \mid |z| = x, t \geq f(|z|)\}$, $B_2(x) = \{(z, t) \in \mathbb{R}^2 \times \mathbb{R} \mid |z| > x, t = f(|z|)\}$. Let

$$L(x) = w\gamma_3^+(B_2(x)) + x\gamma_3^+(B_1(x)) - a(x)\gamma_3^+(P(x)), \qquad x \in [0, w].$$

Since $A(w)$ is a cylinder with radius $w$, we have $L(w)=0$. Also note that $L(0)=w\gamma_3^+(A)-p\gamma_3^+(P)$. Therefore it suffices to prove that $L$ is nonincreasing.
We can easily calculate the terms which appear in the definition of $L$ in order to obtain $L'(x)$. Namely

$$\begin{aligned}\gamma_3^+(B_2(x)) &= \frac{1}{\sqrt{2\pi}}\int_x^w t\exp\left(-\frac{t^2+f(t)^2}{2}\right)\sqrt{1+f'(t)^2}\mathrm{d}t,\\ \gamma_3^+(B_1(x)) &= \frac{1}{\sqrt{2\pi}^3}\int_0^{2\pi}\int_{f(x)}^{\infty}\exp\left(-\frac{x^2+t^2}{2}\right)x\ \mathrm{d}t\mathrm{d}\phi\\ &= xe^{-x^2/2}(1-\Phi(f(x)))=xe^{-x^2/2}T(f(x)),\\ \gamma_3^+(A(x)) &= a(x)e^{-a(x)^2/2}.\end{aligned}$$

Putting these into the definition of $L$ we have

$$\begin{aligned}L(x)=&\frac{w}{\sqrt{2\pi}}\int_x^w t\exp\left(-\frac{t^2+f(t)^2}{2}\right)\sqrt{1+f'(t)^2}\mathrm{d}t+x^2e^{-x^2/2}T(f(x))\\ &-a(x)^2e^{-a(x)^2/2}.\end{aligned}$$

Moreover

$$\begin{aligned}\gamma_3(A(x))=&\gamma_3\left(\{z\in\mathbb{R}^2\mid |z|<x\}\times\mathbb{R}\right)\\ &+\gamma_3\left(\{(z,t)\in\mathbb{R}^2\times\mathbb{R}\mid |z|>x,t\le f(|z|)\}\right)\\ =&1-e^{-x^2/2}+\int_x^w te^{-t^2/2}\Phi(f(t))\mathrm{d}t.\end{aligned}$$

Thus

$$1-e^{-a(x)^2/2}=\gamma_3(P(x))=\gamma_3(A(x))=1-e^{-x^2/2}+\int_x^w te^{-t^2/2}\Phi(f(t))\mathrm{d}t,$$

and differentiating in $x$ we get

$$a'(x)a(x)e^{-a(x)^2/2}=xe^{-x^2/2}(1-\Phi(f(x)))=xe^{-x^2/2}T(f(x)).$$

This allows us to compute $L'$. We have

$$\begin{aligned}L'(x)=&-\frac{w}{\sqrt{2\pi}}x\exp\left(-\frac{x^2+f(x)^2}{2}\right)\sqrt{1+f'(x)^2}\\ &+e^{-x^2/2}\left(2xT(f(x))-x^2\frac{e^{-f(x)^2/2}}{\sqrt{2\pi}}f'(x)-x^3T(f(x))\right)\\ &-\left(2-a(x)^2\right)xe^{-x^2/2}T(f(x)).\end{aligned}$$

Hence $L'\le 0$ iff

$$w\sqrt{1+f'(x)^2}+xf'(x)\ge(a(x)^2-x^2)\sqrt{2\pi}e^{f(x)^2/2}T(f(x)),\qquad x\in[0,w].$$

Since $f' \le 0$ ($f$ is nonincreasing) and $\inf_{t\le 0}(w\sqrt{1+t^2}+xt)=\sqrt{w^2-x^2}$ we will have $L' \le 0$ if we show that

$$\sqrt{w^2-x^2} \ge (a(x)^2-x^2)\sqrt{2\pi}e^{f(x)^2/2}T(f(x)), \qquad x\in[0,w]. \tag{2}$$

Estimating $a(x)^2-x^2$ we can prove the above inequality in some special cases. Notice that monotonicity of $f$ implies $A(x)\subset\{z\in\mathbb{R}^2 \mid |z|<x\}\times\mathbb{R}\cup\{(z,t)\in\mathbb{R}^2\times\mathbb{R} \mid x\le|z|\le w, t\le f(x)\}$, hence

$$1-e^{-a(x)^2/2}=\gamma_3(A(x))\le(1-e^{-x^2/2})+(e^{-x^2/2}-e^{-w^2/2})\Phi(f(x)),$$

so

$$a(x)^2-x^2\le -2\ln\left(T(f(x))+\Phi(f(x))e^{-(w^2-x^2)/2}\right). \tag{3}$$

By this inequality the proof of (2) reduces to

$$\sqrt{w^2-x^2}\ge -2\sqrt{2\pi}e^{f(x)^2/2}T(f(x))\ln\left(T(f(x))+\Phi(f(x))e^{-(w^2-x^2)/2}\right). \tag{4}$$

In general the above inequality is not true. However, Lemma 1, which is proved in the last section, deals with some particular cases.
Let us introduce functions $F:\mathbb{R}\longrightarrow(0,\infty)$, $G:(0,\infty)\longrightarrow(0,\infty)$ given by the formulas

$$F(y)=-\sqrt{2\pi}e^{y^2/2}T(y)\ln T(y), \tag{5}$$

$$G(y)=\frac{y}{2(1-e^{-y^2/2})}. \tag{6}$$

Note that $F$ is increasing and onto (cf. Lemma 2). We will need the constant

$$H=F^{-1}\left(G\left(\sqrt{8/\pi}\right)\right).$$

**Lemma 1.** *Let either*

*(i)* $u\le\sqrt{8/\pi}$, $y\in\mathbb{R}$, *or*

*(ii)* $u>\sqrt{8/\pi}$, $y\le H$.

*Then*

$$-2\sqrt{2\pi}e^{y^2/2}T(y)\ln\left(T(y)+\Phi(y)e^{-u^2/2}\right)\le u.$$

Applying Lemma 1 for $u=\sqrt{w^2-x^2}$, $y=f(x)$, we get the desired inequality (4) for $x$ such that $\sqrt{w^2-x^2}\le\sqrt{8/\pi}$ or $\sqrt{w^2-x^2}>\sqrt{8/\pi}$ and $f(x)\le H$.
Therefore, it remains to prove (2) for $x$ satisfying $\sqrt{w^2-x^2}>\sqrt{8/\pi}$ and $f(x)>H$. Observe that

$$(a(x)^2-x^2)'=2(a(x)a'(x)-x)=2x\left(e^{(a(x)^2-x^2)/2}T(f(x))-1\right),$$

but thanks to (3) we get

$$e^{(a(x)^2-x^2)/2}<1/T(f(x)),$$

hence

$$(a(x)^2-x^2)'<0.$$

Thus the function $[0,w] \ni x \longmapsto a(x)^2 - x^2 \in [0,\infty)$ is decreasing. It yields

$$\sup_{x\in[0,w]} (a(x)^2 - x^2) = a(0)^2 = p^2.$$

Moreover, the function $x \longmapsto e^{f(x)^2/2}T(f(x))$ is nondecreasing on the interval $[0,w] \cap \{x | f(x) > 0\}$ as a composition of the nonincreasing function $f$ and the decreasing one $y \longmapsto e^{y^2/2}T(y)$ for $y > 0$ ([5, Lemma 1]). Consequently

$$\sup\left\{e^{f(x)^2/2}T(f(x)) \mid f(x) > H\right\} = e^{H^2/2}T(H).$$

Combining these two observations and using the assumption $c \geq \gamma_3(A) = \gamma_3(P) = 1 - e^{-p^2/2}$, that is $p^2 \leq -2\ln(1-c)$, we obtain that (2) holds for $x$ such that $\sqrt{w^2 - x^2} > \sqrt{8/\pi}$ and $f(x) > H$. Indeed

$$\begin{aligned}(a(x)^2 - x^2)\sqrt{2\pi}e^{f(x)^2/2}T(f(x)) &\leq \sqrt{2\pi}p^2 e^{H^2/2}T(H)\\ &\leq -2\sqrt{2\pi}\ln(1-c)e^{H^2/2}T(H)\\ &= \sqrt{\frac{8}{\pi}} < \sqrt{w^2 - x^2},\end{aligned}$$

where the last equality holds by the definition of the constant $c$. Namely, we set

$$c = 1 - \exp\left(-\frac{1}{\pi e^{H^2/2}T(H)}\right) > 0.64,$$

which completes the proof. □

**Remark 1.** I is very easy to verify that $c > 0.64$. Firstly, we check by direct computation that $G(\sqrt{8/\pi}) > F(0.7)$, whence $H > 0.7$ by virtue of the monotonicity of $F$. Secondly, we observe that the dependence $c$ on $H$ is increasing as it was mentioned that $y \longmapsto e^{y^2/2}T(y)$ for $y > 0$ decreases. Thus

$$c = 1 - \exp\left(-\frac{1}{\pi e^{0.7^2/2}T(0.7)}\right) > 0.64.$$

From the isoperimetric-like inequality (1) proved in Theorem 2 we have already inferred (cf. steps (I)-(III) presented at the very beginning of this section) that

$$\nu_n(A) = \nu_n(P) \leq c \quad \text{implies} \quad \nu_A'(1) \geq \nu_P'(1).$$

As it was said, this in turn gives the comparison of the measures of $A$ and of a cylinder $P$ when we shrink these sets by dilating them — Theorem 1. We can also use this implication in order to show what happens with measures when we expand our sets (the simple reasoning which ought to be repeated may be found in [4])

**Corollary 1.** *For any convex rotationally symmetric set $A \subset \mathbb{C}^n$, with measure $\nu_n(A) \leq c$, and a cylinder $P$ satisfying $\nu_n(A) = \nu_n(P)$, we have*

$$\nu_n(tA) \geq \nu_n(tP), \qquad \text{for } 1 \leq t \leq t_0, \tag{7}$$

*where $t_0 \geq 1$ satisfies $\nu_n(t_0 A) = c$.*

## 2 Some remarks

**Remark 2.** Generally, without the assumption on the measure of a set $A$ Theorem 2 fails. To see this let us consider a cylindrical frustum $A = \{(z,t) \in \mathbb{R}^2 \times \mathbb{R} \mid |z| \leq w, t \leq y\}$ with the radius $w$ and the height $y$. This is not exactly a set as in the assumptions of Theorem 2, that is, lying under a graph of a smooth concave function (there is a problem with smoothness), but an easy approximation argument will fill in the gap. Take a cylinder $P = \{z \in \mathbb{R}^2 \mid |z| \leq p\} \times \mathbb{R}$ with the same measure as $A$, which means

$$\Phi(y)(1-e^{-w^2/2}) = \gamma_3(A) = \gamma_3(P) = 1 - e^{-p^2/2}.$$

We show that for some large enough $w$ and $y$ there actually holds the reverse inequality to the one stated in Theorem 2

$$w\gamma_3^+(A) < p\gamma_3^+(P).$$

Indeed, let us fix the parameters of the cylindrical frustum such that

$$e^{-w^2/2} = T(y), \quad y > 0.$$

Thus $1 - e^{-w^2/2} = \Phi(y)$. To simplify some calculations, let us define a function

$$g(y) = \frac{1}{\sqrt{2\pi}e^{y^2/2}T(y)}.$$

Now, the relation between $w$ and $y$ may be written as $w^2 = -2\ln T(y) = y^2 + 2\ln\left(\sqrt{2\pi}g(y)\right)$. Furthermore, we have

$$y < g(y) < \sqrt{y^2+2}, \qquad y > 0,$$

where the left inequality is a standard estimation for $T(y)$ and the right one follows from [5, Lemma 2]. Therefore

$$\begin{aligned}
w\gamma_3^+(A) &= w\left(we^{-w^2/2}\Phi(y) + \frac{e^{-y^2/2}}{\sqrt{2\pi}}(1-e^{-w^2/2})\right)\\
&= T(y)\left(w^2\Phi(y) + w\Phi(y)\frac{e^{-y^2/2}}{\sqrt{2\pi}T(y)}\right) < T(y)\left(w^2\Phi(y) + wg(y)\right)\\
&< T(y)\left(w^2\Phi(y) + \sqrt{y^2 + 2\ln\left(\sqrt{2\pi}g(y)\right)}\sqrt{y^2+2}\right)\\
&\leq T(y)\left(w^2\Phi(y) + y^2 + \ln\left(\sqrt{2\pi}g(y)\right) + 1\right)\\
&= T(y)\left(w^2\left(1+\Phi(y)\right) + 1 - \ln\left(\sqrt{2\pi}g(y)\right)\right).
\end{aligned}$$

Let us choose $y$ such that

$$1 - \ln\left(\sqrt{2\pi}g(y)\right) < -2(1+\Phi(y))\ln\left(1+\Phi(y)\right).$$

Then

$$\begin{aligned}
w\gamma_3^+(A) &< T(y)\left(w^2\left(1+\Phi(y)\right) - 2(1+\Phi(y))\ln\left(1+\Phi(y)\right)\right)\\
&= -2T(y)\left(1+\Phi(y)\right)\ln\left(e^{-w^2/2}\left(1+\Phi(y)\right)\right)\\
&= p^2e^{-p^2/2} = p\gamma_3^+(P),
\end{aligned}$$

where the second equation holds since

$$e^{-p^2/2} = 1 - \Phi(y)(1 - e^{-w^2/2}) = T(y) + \Phi(y)e^{-w^2/2} = T(y)\left(1 + \Phi(y)\right).$$

In the previous remark we have seen that the assumption on the measure in Theorem 2 is essential. This assumption and the technique which have been used cause that the restriction on the measure also appears in Theorem 1. We may obtain a weaker version of the inequality $(*)$ dropping the inconvenient assumption $\gamma_3(A) \leq c$. This result reads as follows

**Theorem 3.** *There exists a constant $K = 3$ such that for any convex rotationally symmetric set $A \subset \mathbb{C}^n$ and a cylinder $P$ satisfying $\nu_n(A) = \nu_n(P)$, we have*

$$\nu_n((1 + K(t-1))A) \geq \nu_n(tP), \qquad \text{for } t \geq 1. \tag{8}$$

*Proof.* Let us denote $\ell(t) = 1 + K(t-1)$.
It suffices to prove (8) only for sets with big measure, i.e. $\nu_n(A) \geq c$, where $c$ is the constant from Theorem 1. Indeed, assume that (8) holds for all convex rotationally symmetric sets $A$ such that $\nu_n(A) \geq c$. We are going to show this inequality also for a set $A$ with the measure less than $c$. Let us fix such a set and take $t_0 > 1$ such that $\nu_n(t_0 A) = c$. From Corollary 1 we get

$$\nu_n(tA) \geq \nu_n(tP), \qquad t \leq t_0.$$

Now, we are to prove (8) for $t > t_0$. Let $Q$ be a cylinder with the same measure as $t_0 A$. Applying what we have assumed we obtain

$$\nu_n\left(\ell(t)(t_0 A)\right) \geq \nu_n(tQ), \qquad t \geq 1. \tag{9}$$

One can make two simple observations

$$\ell(t)t_0 < \ell(t_0 t),$$
$$\nu_n(Q) = \nu_n(t_0 A) \geq \nu_n(t_0 P) \Longrightarrow \nu_n(tQ) \geq \nu_n(t t_0 P).$$

Together with the inequality (9) this yields

$$\nu_n\left(\ell(t t_0)A\right) \geq \nu_n(t t_0 P), \qquad t \geq 1,$$

which is just the desired inequality.
Henceforth, we are going to deal with the proof of inequality (8) in the case of $\nu_n(A) \geq c$. The idea is to exploit the deep result of Latała and Oleszkiewicz concerning dilations in the real case. Namely, from Theorem 1 of [5] we have

$$\nu_n\left(\ell(t)A\right) \geq \nu_n\left(\ell(t)S\right), \qquad t \geq 1,$$

where

$$S = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |\mathrm{Re} z_1| \leq s\},$$

is a strip of the width $2s$ chosen so that $\nu_n(A) = \nu_n(S) = 1 - 2T(s)$. Therefore, we end the proof, providing that we show

$$\nu_n\left(\ell(t)S\right) \geq \nu_n(tP), \qquad t \geq 1.$$

This inequality in turn can be written more explicitly. We have

$$\nu_n\left(\ell(t)S\right) = 1 - 2T\left(\ell(t)s\right),$$

and using the relation $1-e^{-p^2/2}=\nu_n(P)=\nu_n(A)=\nu_n(S)=1-2T(s)$ we get $e^{-p^2/2}=2T(s)$. Hence

$$\nu_n(tP)=1-e^{-(tp)^2/2}=1-(2T(s))^{t^2}.$$

Thus it is enough to show that

$$(2T(s))^{t^2}\geq 2T(\ell(t)s), \qquad t\geq 1,\ s\geq s_0, \tag{10}$$

where $s_0$ is such that a strip with the width $2s_0$ has the measure $c$, i.e. $1-2T(s_0)=c$. Since $c>0.64$, it follows that $T(s_0)<0.18<T(0.9)$, so $s_0>0.9$.
Let us deal with the inequality (10). For $t$ close to 1 we will apply the Prékopa-Leindler inequality [2, Theorem 7.1]. To see this, let us fix $s\geq s_0$ and $t\geq 1$ and consider the functions

$$f(x)=\frac{2}{\sqrt{2\pi}}e^{-x^2/2}\mathbb{1}_{[\ell(t)s,\infty)}(x),$$
$$g(x)=\frac{2}{\sqrt{2\pi}}e^{-x^2/2}\mathbb{1}_{[0,\infty)}(x),$$
$$h(x)=\frac{2}{\sqrt{2\pi}}e^{-x^2/2}\mathbb{1}_{[s,\infty)}(x).$$

It is not hard to see that the inequality

$$f(x)^{1/t^2}g(y)^{1-1/t^2}\leq h\left(\frac{1}{t^2}x+\left(1-\frac{1}{t^2}\right)y\right),$$

holds for any $x,y\in\mathbb{R}$ if and only if $\ell(t)s\geq t^2s$, or equivalently $t\leq K-1=2$. Then, by virtue of Prékopa-Leindler inequality, we obtain

$$(2T(\ell(t)s))^{1/t^2}=\left(\int_{\mathbb{R}}f\right)^{1/t^2}\left(\int_{\mathbb{R}}g\right)^{1-1/t^2}\leq\int_{\mathbb{R}}h=2T(s).$$

Now we are left with the proof of (10) in the case of $t>2$ and $s\geq s_0$. To handle it, we use the asymptotic behaviour of the function $T$ and perform some calculations. In accordance with the standard estimate from above of the tail probability of the Gaussian distribution we get

$$T(\ell(t)s)<\frac{1}{\sqrt{2\pi}}\frac{1}{\ell(t)s}e^{-\ell(t)^2s^2/2},$$

whereas from Lemma 2 in [5]

$$T(s)>\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{s^2+2}}e^{-s^2/2}.$$

Therefore, in order to show (10) it is enough to prove

$$\left(\frac{2}{\sqrt{2\pi}}\right)^{t^2}\frac{1}{(s^2+2)^{t^2/2}}e^{-t^2s^2/2}\geq\frac{2}{\sqrt{2\pi}}\frac{1}{\ell(t)s}e^{-\ell(t)^2s^2/2},$$

which is equivalent to the inequality

$$\exp\left(\frac{s^2}{2}\left(\ell(t)^2-t^2\right)\right)\geq\left(\sqrt{\frac{\pi}{2}}\right)^{t^2-1}\frac{(s^2+2)^{t^2/2}}{\ell(t)s}, \qquad s\geq s_0,\ t\geq 2.$$

Taking the logarithm of both sides, putting the definition of $\ell(t) = 1 + K(t-1) = 3t - 2$ and simplifying we have to prove

$$\left(8s^2 - \ln\left(\frac{\pi}{2}\left(s^2+2\right)\right)\right)t^2 - 12s^2t + 4s^2 + \ln\left(\frac{\pi}{2}s^2\right) + 2\ln(3t-2) \geq 0.$$

Let us call the left hand side by $F(s,t)$. Notice that

$$\begin{aligned}\frac{\partial F}{\partial t}(s,t) &= 2\left(8s^2 - \ln\left(\frac{\pi}{2}\left(s^2+2\right)\right)\right)t - 12s^2 + \frac{2}{3t-2}\\ &> 2\left(5s^2 - \ln\left(\frac{\pi}{2}\left(s^2+2\right)\right)\right)t > 2\left(5s^2 - \frac{\pi}{2e}(s^2+2)\right)t\\ &= 2\left(\left(5-\frac{\pi}{2e}\right)s^2 - \frac{\pi}{e}\right)t \geq 2\left(\left(5-\frac{\pi}{2e}\right)s_0^2 - \frac{\pi}{e}\right)t\\ &> 2\left(\left(5-\frac{\pi}{2e}\right)\cdot 0.81 - \frac{\pi}{e}\right)t > 0,\end{aligned}$$

where in the first inequality we used only the assumption that $t > 2$ getting $-12s^2 > -6ts^2$ and neglected the term $\frac{2}{3t-2}$ as being positive, while in the second one we exploit the well-known inequality $\ln x \leq \frac{x}{e}$. Knowing that this derivative is positive, we will finish if we check that $F(s,2) > 0$. It can be done by direct computation

$$\begin{aligned}F(s,2) &= 4\left(8s^2 - \ln\left(\frac{\pi}{2}\left(s^2+2\right)\right)\right) - 24s^2 + 4s^2 + \ln s^2 + \ln\frac{\pi}{2} + 2\ln 4\\ &= 4\left(3s^2 - \ln\left(\frac{\pi}{2}\left(s^2+2\right)\right)\right) + \ln\left(8\pi s^2\right)\\ &> 4\left(\left(3-\frac{\pi}{2e}\right)s^2 - \frac{\pi}{e}\right) > 0.\end{aligned}$$

The proof is now complete. □

## 3 Technical lemmas

We are going to prove some rather technical lemmas which will help us with the proof of Lemma 1.

**Lemma 2.** *The function $F$, defined in* (5)*, is increasing and onto* $(0,\infty)$.

*Proof.* In order to prove that $F$ is increasing it suffices to show that $F$ is nondecreasing. Indeed, if $F$ was constant on some interval, it would be constant everywhere as $F$ is an analytic function. Clearly, $F$ is nondecreasing iff $1/F$ is nonincreasing. Notice that

$$\begin{aligned}\frac{1}{F(y)} &= \frac{-e^{-y^2/2}}{\sqrt{2\pi}}\frac{1}{T(y)\ln T(y)} = \frac{T'(y)}{T(y)\ln T(y)}\\ &= \frac{\left(-\ln T(y)\right)'}{-\ln T(y)} = \left(\ln\left(-\ln T(y)\right)\right)',\end{aligned}$$

thus $1/F$ is nonincreasing iff $y \longmapsto \ln\left(-\ln T(y)\right)$ is concave, that is for any $x, y \in \mathbb{R}$, $\lambda \in (0,1)$

$$-\ln T(\lambda x + (1-\lambda)y) \geq \left(-\ln T(x)\right)^{\lambda}\left(-\ln T(y)\right)^{1-\lambda}.$$

Since $\lim_{x\to-\infty}(-\ln T(x)) = 0$, we have

$$-\ln T(x) = \int_{-\infty}^{x} (-\ln T(t))'\,\mathrm{d}t = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}T(t)}\mathrm{d}t,$$

and the above inequality will hold by virtue of the Prékopa-Leindler inequality. We only need to check the assumptions, that is to verify whether the function $\ln \frac{e^{-t^2/2}}{\sqrt{2\pi}T(t)}$ is concave. Calculating the second derivative one can easily check that it is non-positive iff

$$\begin{aligned}0 &\geq T(t)^2 + \frac{e^{-t^2/2}}{\sqrt{2\pi}} t T(t) - \left(\frac{e^{-t^2/2}}{\sqrt{2\pi}}\right)^2 \\ &= \left(T(t) - \frac{e^{-t^2/2}}{\sqrt{2\pi}} \frac{\sqrt{t^2+4}-t}{2}\right)\left(T(t) + \frac{e^{-t^2/2}}{\sqrt{2\pi}} \frac{\sqrt{t^2+4}+t}{2}\right), \quad t \in \mathbb{R},\end{aligned}$$

which is equivalent to

$$T(t) \geq \frac{e^{-t^2/2}}{\sqrt{2\pi}} \frac{\sqrt{t^2+4}-t}{2}, \qquad t \in \mathbb{R}.$$

For $t \geq 0$ this follows from a well-known Komatsu's estimate (cf. [3], page 17). For $t < 0$ we have $T(t) > 1/2$, hence

$$2T(t)\sqrt{2\pi}e^{t^2/2} + t \geq \sqrt{2\pi}(1+t^2/2) + t > 0,$$

and

$$\begin{aligned}\left(2T(t)\sqrt{2\pi}e^{t^2/2} + t\right)^2 &> \left(\sqrt{2\pi}(1+t^2/2) + t\right)^2 \\ &= 2\pi\left(1+\frac{t^2}{2}\right)^2 + 2\sqrt{2\pi}\left(1+\frac{t^2}{2}\right)t + t^2 \\ &= 2\left(1+\frac{t^2}{2}\right)\left(\frac{\pi}{2}t^2 + \sqrt{2\pi}t + \pi\right) + t^2 \\ &> 2\left(\left(\sqrt{\frac{\pi}{2}}t+1\right)^2 + \pi - 1\right) + t^2 \\ &> 2(\pi-1) + t^2 > t^2 + 4.\end{aligned}$$

This completes the proof of the monotonicity of $F$.
$F$ is onto $(0,\infty)$ as

$$F(y) \xrightarrow[y\to-\infty]{} 0,$$
$$F(y) \xrightarrow[y\to+\infty]{} \infty.$$

□

**Lemma 3.** *The function $G$, defined in* (6)*, is increasing for* $u \geq \sqrt{8/\pi}$.

*Proof.* We have

$$G'(u) = \frac{1 - e^{-u^2/2} - u^2 e^{-u^2/2}}{2\left(1 - e^{-u^2/2}\right)^2},$$

so $G'(u) > 0$ iff $e^{u^2/2} > 1 + u^2$. This is true for $u^2 > 8/\pi$ since $e^{4/\pi} > 1 + 8/\pi$. □

*Proof of Lemma 1.* *(i)* Using the convexity of the function $-\ln$ we get

$$\begin{aligned} -2\sqrt{2\pi}e^{y^2/2}T(y)\ln\left(T(y) + \Phi(y)e^{-u^2/2}\right) \\ \leq 2\sqrt{2\pi}e^{y^2/2}T(y)\left(-T(y)\ln 1 - \Phi(y)\ln e^{-u^2/2}\right) \\ = \sqrt{2\pi}e^{y^2/2}T(y)\Phi(y)u^2 \leq \sqrt{\frac{\pi}{8}}u^2 \leq u, \end{aligned}$$

where we use $\sup_{y\in\mathbb{R}} \sqrt{2\pi}e^{y^2/2}T(y)\Phi(y) = \sqrt{\frac{\pi}{8}}$ (see Lemma 5 in [5]).

*(ii)* Since $T(y) + \Phi(y)e^{-u^2/2} = e^{-u^2/2} + (1 - e^{-u^2/2})T(y)$, we may also apply the convexity of $-\ln$ to points $1$, $T(y)$ with weights $e^{-u^2/2}$, $1 - e^{-u^2/2}$ and obtain

$$\begin{aligned} -2\sqrt{2\pi}e^{y^2/2}T(y)\ln\left(T(y) + \Phi(y)e^{-u^2/2}\right) \\ \leq -2\sqrt{2\pi}e^{y^2/2}T(y)\ln T(y)(1 - e^{-u^2/2}) \\ = \frac{F(y)}{G(u)}u \leq \frac{F(H)}{G\left(\sqrt{8/\pi}\right)}u = u. \end{aligned}$$

□

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