

RANDOM INTERLACEMENTS ON GALTON-WATSON TREESMARTIN TASSY¹*151 Thayer street, Department of mathematics Brown University Providence, RI, 02912*email: mtassy@math.brown.edu*Submitted June 5, 2010, accepted in final form November 2, 2010*

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Abstract

We study the critical parameter u^* of random interlacements on a Galton-Watson tree conditioned on the non-extinction event. We show that, for a given law of a Galton-Watson tree, the value of this parameter is a.s. constant and non-trivial. We also characterize this value as the solution of a certain equation.

1 Introduction

The aim of this note is to study random interlacements model on a Galton-Watson tree. We will mainly be interested in the critical parameter of the model. In particular we want to understand whether this parameter is non trivial (that is different from 0 and ∞), whether it is random and how it depends on the law of the Galton-Watson tree. Our main theorem answers all of these questions and even goes further by characterizing the critical parameter as the solution of a certain equation.

The random interlacements model was recently introduced on \mathbb{Z}^d , $d \geq 3$, by A.S. Sznitman in [Szn10] and generalised to arbitrary transient graphs by A. Teixeira in [Tei09]. It is a special dependent site-percolation model where the set \mathcal{S} of occupied vertices on a transient graph (G, \mathcal{E}) is constructed as the trace left on G by a Poisson point process on the space of doubly infinite trajectories modulo time shift. The density of the set \mathcal{S} is determined by a parameter $u > 0$ which comes as a multiplicative parameter of the intensity measure of the Poisson point process. In this paper, we will not need the complete construction of the random interlacements percolation. For our purposes it will be sufficient to know that the law Q_u^G of the vacant set $\mathcal{V} = G \setminus \mathcal{S}$ of the random interlacements at level u is characterized by

$$Q_u^G [K \subset \mathcal{V}] = e^{-u \text{cap}_G(K)}, K \subset G \text{ finite}, \quad (1.1)$$

where $\text{cap}_G(K)$ is the capacity of K in G (see Section 2 for definition). In addition to this formula we will need the description of the distribution of the vacant cluster containing a given vertex in the case when G is a tree given in [Tei09] which we state in Theorem 2 below.

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The critical parameter u_G^* of random interlacements on G is defined as:

$$u_G^* = \inf_{u \in \mathbb{R}^+} \left\{ u : Q_u^G\text{-a.s. all connected components of } \mathcal{V} \text{ are finite} \right\}. \tag{1.2}$$

In this article we take the graph G to be a Galton-Watson rooted tree T defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, conditioned on non-extinction. We denote by \emptyset the root of the tree, $(\rho_i)_{i \geq 0}$ the offspring distribution of the non-conditioned Galton-Watson process, f its generating function and q the probability of the extinction event. $\tilde{\mathbb{P}}$ stands for the conditional law of the Galton-Watson tree on the non-extinction event and $\tilde{\mathbb{E}}$ for the corresponding conditional expectation. We assume that T is supercritical that is

$$\sum_{i=0}^{\infty} i \rho_i > 1. \tag{A0}$$

If (A0) is satisfied, then $\tilde{\mathbb{P}}$ is well defined and T is $\tilde{\mathbb{P}}$ -a.s. transient, see for example Proposition 3.5 and Corollary 5.10 of [LP]. Thus the random interlacements are defined on T and in particular Q_u^T is $\tilde{\mathbb{P}}$ -a.s. well-defined.

To state our principal theorem we introduce \tilde{T} the sub-tree of T composed of the vertices which have infinite descendance. We recall the Harris decomposition (cf. Proposition 5.26 in [LP]): under $\tilde{\mathbb{P}}$, \tilde{T} is a Galton-Watson tree with generating function \tilde{f} given by

$$\tilde{f}(s) = \frac{f(q + (1-q)s) - q}{1-q}. \tag{1.3}$$

Theorem 1. *Let T be a Galton Watson tree with a law satisfying (A0). Then there exists a non-random constant $u^* \in (0, \infty)$ such that*

$$u_T^* = u^*, \tilde{\mathbb{P}}\text{-a.s.} \tag{1.4}$$

Moreover if we denote by $\mathcal{L}_\chi(u) = \tilde{\mathbb{E}} \left[e^{-u \text{cap}_T(\{\emptyset\})} \right]$ the annealed probability that the root is vacant at level u (cf 1.1), then u^* is the only solution on $(0, \infty)$ of the equation

$$(\tilde{f}^{-1})'(\mathcal{L}_\chi(u)) = 1. \tag{1.5}$$

The main difficulty in proving this theorem is the dependence present in the model. More precisely, for any $x \in T$, the probability that $x \in \mathcal{V}$ is given by

$$Q_u^T[x \in \mathcal{V}] = e^{-u \text{cap}_T(x)}.$$

The capacity $\text{cap}_T(x)$ depends on the whole tree T . It is thus not possible to construct the component of the vacant set containing the root of a given tree as a sequence of independent generations, which is a key property when proving an analogous statement for Bernoulli percolation on a Galton-Watson tree.

However, it turns out that despite this dependence, it is possible to construct a recurrence relation under the annealed measure $\tilde{\mathbb{P}} \otimes Q_u^T$ for a well chosen quantity related to the size of the cluster at a given point (see (4.11)). This is done in section 4. Using this recurrence relation, it is possible to find an annealed critical parameter u^* and show that it is non-trivial. In section 3 we prove that u_T^* is $\tilde{\mathbb{P}}$ -a.s. constant and thus that $u_T^* = u^*$ for $\tilde{\mathbb{P}}$ -a.e. tree. Section 2 introduces some preliminary definitions and recalls some useful results for random interlacements on trees.

2 Definitions and preliminary results

Let us introduce some notations first. For a given tree T with root \emptyset and a vertex $x \in T \setminus \{\emptyset\}$, we write \hat{x} for the closest ancestor of x in T , $|x|$ for its distance to the root, Z_x for the number of children of x and $\text{deg}_T(x)$ for its degree. We denote by T_x the sub-tree of T containing x and all its descendants. If T is a tree with root \emptyset , for every child x of \emptyset we will say that T_x is a descendant tree of T . For any infinite rooted tree T , we denote by \tilde{Z}_x the number of children of x in \tilde{T} . For a tree T and a vertex x of T , we denote by P_x^T the law of a simple discrete time random walk $(X_n)_{n \geq 0}$ started at x . For every set K in T , we use \tilde{H}_K to denote the hitting time of the set K defined by

$$\tilde{H}_K = \inf_{n \geq 1} \{n : X_n \in K\}. \tag{2.1}$$

We write e_K^T for the equilibrium measure of K in T and $\text{cap}_T(K)$ for its total mass, also called capacity of the set K :

$$e_K^T(x) = \text{deg}_T(x) P_x^T [\tilde{H}_K = \infty] 1_{x \in K}, \tag{2.2}$$

$$\text{cap}_T(K) = \sum_{x \in T} e_K^T(x) = \sum_{x \in K} \text{deg}_T(x) P_x^T [\tilde{H}_K = \infty]. \tag{2.3}$$

We also denote \mathcal{C}_x the connected component of \mathcal{V} containing x . According to Corollary 3.2 of [Tei09], the definition (1.2) of the critical parameter u_T^* is equivalent to

$$u_T^* = \inf_{u \in \mathbb{R}^+} \{u : Q_u^T [|\mathcal{C}_\emptyset| = \infty] = 0\}. \tag{2.4}$$

Finally we recall Theorem 5.1 of [Tei09] which identifies the law of the vacant cluster \mathcal{C}_x on a fixed tree T with the law of the vacant set left by inhomogeneous Bernoulli site percolation. This theorem also allows us to compare random interlacements on T and random interlacements on its descendant trees.

Theorem 2 (Theorem 5.1 of [Tei09]). *Let T be a transient rooted tree with locally bounded degree. For every vertex $x \in T$ we consider the function $h_T^x : T_x \rightarrow [0, 1]$ given by:*

$$h_T^x(z) = \text{deg}_T(z) P_z^{T_x} [\tilde{H}_{\{z, \hat{z}\}} = \infty] P_z^{T_x} [\tilde{H}_{\hat{z}} = \infty] 1_{z \neq x} \tag{2.5}$$

and $h_T^x(x) = 0$. Conditionally on $\{x \in \mathcal{V}\}$, $\mathcal{C}_x \cap T_x$ has the same law under Q_u^T as the open cluster of x in an inhomogeneous Bernoulli site-percolation on T_x where every site $z \in T_x$ is open independently with probability

$$p_u(z) = \exp(-uh_T^x(z)). \tag{2.6}$$

3 $\bar{\mathbb{P}}$ -a.s. constancy of u^*

In this section we prove that for a given Galton-Watson tree T satisfying (A0) the critical parameter u_T^* is $\bar{\mathbb{P}}$ -a.s constant. We will use Theorem 2 to prove a zero-one law for the event $\{Q_u^T [|\mathcal{C}_\emptyset| = \infty] = 0\}$. The proof of this zero-one law is based on the following definition and lemma which we learnt in [LP]. We present here its proof for sake of completeness.

Definition 3. We say that a property \mathscr{P} of a tree is inherited if the two following conditions are satisfied:

$$T \text{ has } \mathscr{P} \Rightarrow \text{all descendant trees of } T \text{ have } \mathscr{P}. \tag{3.1}$$

All finite trees have \mathcal{P} . (3.2)

The zero-one law for Galton-Watson trees associated to such properties is:

Lemma 4. *If \mathcal{P} is an inherited property then, for every Galton-Watson tree process T satisfying (A0), $\bar{\mathbb{P}}[T \text{ has } \mathcal{P}] \in \{0, 1\}$.*

Proof. Let E be the set of trees that have the property \mathcal{P} and Z_\emptyset the number of children of the root. Using condition (3.1) we can write

$$\begin{aligned} \mathbb{P}[T \in E] &= \mathbb{E}[\mathbb{P}[T \in E | Z_\emptyset]] \\ &\leq \mathbb{E}[\mathbb{P}[\{\forall x : |x| = 1, T_x \in E\} | Z_\emptyset]]. \end{aligned} \tag{3.3}$$

Conditionally on Z_\emptyset , the random trees $(T_x)_{x:|x|=1}$ are independent and have the same law as T . Hence the last inequality is equivalent to

$$\begin{aligned} \mathbb{P}[T \in E] &\leq \mathbb{E}[(\mathbb{P}[T \in E])^{Z_\emptyset}] \\ &\leq f(\mathbb{P}[T \in E]). \end{aligned} \tag{3.4}$$

By assumption (A0) we know that $\rho_i > 0$ for some $i \geq 2$. Therefore f is strictly convex on $(0, 1)$ with $f(q) = q$ and $f(1) = 1$. Hence from (3.4) we have $\mathbb{P}[T \in E] \in [0, q] \cup \{1\}$. Since all finite trees have \mathcal{P} , and $\mathbb{P}[T \text{ is finite}] = q$, we can deduce that $\mathbb{P}[T \in E] \in \{q, 1\}$ and consequently

$$\bar{\mathbb{P}}[T \in E] = \frac{\mathbb{P}[\{T \in E\} \cap \{|T| = \infty\}]}{\mathbb{P}[\{|T| = \infty\}]} \in \{0, 1\} \tag{3.5}$$

which finishes the proof of the lemma. □

Proof of (1.4). To prove that u_T^* is $\bar{\mathbb{P}}$ -a.s constant we show first that, for a tree T with root \emptyset , the property \mathcal{P}_u defined by

$$T \text{ has } \mathcal{P}_u \text{ iff } T \text{ is finite or } Q_u^T[|\mathcal{C}_\emptyset| = \infty] = 0, \tag{3.6}$$

is inherited. Since every finite tree has \mathcal{P}_u by definition, we just have to prove the statement

$$\{\exists x \in T, |x| = 1 : T_x \text{ has not } \mathcal{P}_u\} \Rightarrow \{T \text{ has not } \mathcal{P}_u\}. \tag{3.7}$$

Let x be a child of the root such that $|T_x| = \infty$ and T_x has not \mathcal{P}_u which can also be written $Q_u^{T_x}[|\mathcal{C}_x| = \infty] > 0$. Since $T_x \subset T$, T is not finite. We will show that $Q_u^T[|\mathcal{C}_\emptyset| = \infty] > 0$. From the formula (2.5) it follows that for every $z \in T_x \setminus \{x\}$, $h_T^x(z) = h_{T_x}^x(z)$. Using Theorem 2 this means that, conditionally on $\{x \in \mathcal{V}\}$, the law of the cluster $\mathcal{C}_x \cap T_x$ under Q_u^T and the law of the cluster \mathcal{C}_x under $Q_u^{T_x}$ are the same. In particular we have

$$Q_u^T[|\mathcal{C}_x \cap T_x| = \infty | x \in \mathcal{V}] = Q_u^{T_x}[|\mathcal{C}_x| = \infty | x \in \mathcal{V}]. \tag{3.8}$$

Moreover, we see from (2.5) that $h_T^x(z) = h_T^\emptyset(z)$ for every $z \in T_x \setminus \{x\}$. Applying Theorem 2 to the clusters \mathcal{C}_\emptyset and \mathcal{C}_x , we see that the law of $\mathcal{C}_\emptyset \cap T_x$ under $Q_u^T[|\cdot, x \in \mathcal{V}]$ is the same as the law of $\mathcal{C}_x \cap T_x$ under $Q_u^T[|\cdot, x \in \mathcal{V}]$. In particular we have

$$Q_u^T[|\mathcal{C}_\emptyset \cap T_x| = \infty | \emptyset, x \in \mathcal{V}] = Q_u^T[|\mathcal{C}_x \cap T_x| = \infty | x \in \mathcal{V}]. \tag{3.9}$$

Since on $\{\emptyset, x \in \mathcal{V}\}$, \emptyset and x are in the same open cluster, we can rewrite (3.8) and (3.9) as

$$\begin{aligned} Q_u^T [|\mathcal{C}_x \cap T_x| = \infty | \emptyset, x \in \mathcal{V}] &= Q_u^T [|\mathcal{C}_\emptyset \cap T_x| = \infty | \emptyset, x \in \mathcal{V}] \\ &= Q_u^T [|\mathcal{C}_x \cap T_x| = \infty | x \in \mathcal{V}] \\ &= Q_u^{T_x} [|\mathcal{C}_x| = \infty | x \in \mathcal{V}] \\ &> 0, \end{aligned} \quad (3.10)$$

by hypothesis. Finally, since

$$Q_u^T [\emptyset, x \in \mathcal{V}] = e^{-u \text{cap}_T(\{\emptyset, x\})} > 0, \quad (3.11)$$

this yields

$$0 < Q_u^T [|\mathcal{C}_\emptyset \cap T_x| = \infty | \emptyset, x \in \mathcal{V}] Q_u^T [\emptyset, x \in \mathcal{V}] \leq Q_u^T [|\mathcal{C}_\emptyset^u| = \infty] \quad (3.12)$$

which finishes the proof that \mathcal{P}_u is inherited.

We can now apply Theorem 3 and deduce that $\mathbb{P}^{\neq}(T \text{ has } \mathcal{P}_u) \in \{0, 1\}$. Thus for every $s \in \mathbb{Q}^+$, there exists a set $A_s \subset \Omega$ such that $\bar{\mathbb{P}}[A_s] = 1$ and $1_{Q_s^T[|\mathcal{C}_\emptyset| = \infty] = 0}$ is constant on A_s . This yields

$$T \rightarrow \inf \{s \in \mathbb{Q}^+ : Q_s^T [|\mathcal{C}_\emptyset| = \infty] = 0\} \quad (3.13)$$

is constant on $A = \bigcap_{s \in \mathbb{Q}^+} A_s$ with $\bar{\mathbb{P}}[A] = 1$. But, since $u \mapsto Q_u^T [|\mathcal{C}_\emptyset| = \infty]$ is decreasing, we also have:

$$\begin{aligned} u_*^T &= \inf \{u \in \mathbb{R}^+ : Q_u^T [|\mathcal{C}_\emptyset| = \infty] = 0\} \\ &= \inf \{u \in \mathbb{Q}^+ : Q_u^T [|\mathcal{C}_\emptyset| = \infty] = 0\}. \end{aligned} \quad (3.14)$$

It follows directly that u_* is \mathbb{P}^{\neq} -a.s constant. \square

4 Characterization of u^*

In this section we will show that u^* is non-trivial and can be obtained as the root of equation (1.5). In order to make our calculation more natural we will work with the modified tree T' obtained by attaching an additional vertex Δ to the root of T . This change is legitimate only if T and T' have the same critical parameter u^* , which is equivalent to

$$Q_u^T [|\mathcal{C}_\emptyset| = \infty] > 0 \text{ iff } Q_u^{T'} [|\mathcal{C}_\emptyset| = \infty] > 0. \quad (4.1)$$

To prove (4.1) we observe that by Theorem 2

$$Q_u^T [|\mathcal{C}_\emptyset| = \infty | \emptyset \in \mathcal{V}] = Q_u^{T'} [|\mathcal{C}_\emptyset| = \infty | \emptyset \in \mathcal{V}]. \quad (4.2)$$

Since $Q_u^T [\emptyset \in \mathcal{V}] > 0$ and $Q_u^{T'} [\emptyset \in \mathcal{V}] > 0$ this is equivalent to

$$\frac{Q_u^T [|\mathcal{C}_\emptyset| = \infty]}{Q_u^T [\emptyset \in \mathcal{V}]} = \frac{Q_u^{T'} [|\mathcal{C}_\emptyset| = \infty]}{Q_u^{T'} [\emptyset \in \mathcal{V}]} \quad (4.3)$$

so that (4.1) holds and $u_T^* = u_{T'}^*$.

If $|x| \geq 1$, $(T_x)'$ is isomorphic to the tree obtained by attaching \hat{x} to T_x . We will thus identify both trees and write T'_x for the tree $T_x \cup \{\hat{x}\}$.

For every tree T we define the random variable

$$\gamma(T) = P_{\emptyset}^{T'} [\tilde{H}_{\Delta} = \infty]. \tag{4.4}$$

The random variables $(\gamma(T_x))_{|x|=1}$ are related to the random variable $\chi(T) := \text{cap}_{T'}(\emptyset)$ by

$$\chi(T) = \text{cap}_T(\emptyset) \tag{4.5}$$

$$= \text{cap}_{T'}(\emptyset) = \sum_{x \in \tilde{T}: |x|=1} \gamma(T_x). \tag{4.6}$$

The second equality is an easy consequence of definition of the capacity and the third equality follows, using Markov property, from the following computation:

$$\begin{aligned} \chi(T) &= (Z_{\emptyset} + 1) P_{\emptyset}^{T'} [\tilde{H}_{\emptyset} = \infty] \\ &= (Z_{\emptyset} + 1) \sum_{x \in \tilde{T}: |x|=1} P_x^{T'} [\tilde{H}_{\emptyset} = \infty] P_{\emptyset}^{T'} [X_1 = x] \\ &= (Z_{\emptyset} + 1) \sum_{x \in \tilde{T}: |x|=1} \frac{1}{(Z_{\emptyset} + 1)} P_x^T [\tilde{H}_{\emptyset} = \infty] \\ &= \sum_{x \in \tilde{T}: |x|=1} P_x^{T'_x} [\tilde{H}_{\emptyset} = \infty] = \sum_{x \in \tilde{T}: |x|=1} \gamma(T_x). \end{aligned} \tag{4.7}$$

We also define \mathcal{L}_{γ} the Laplace transform of $T \rightarrow \gamma(T)$ under $\bar{\mathbb{E}}$.

The recursive structure of Galton-Watson tree implies that the random variables $(\gamma(T_x))_{|x|=1}$ are i.i.d. We can use this property and formula (1.3) to express relation (4.5) in terms of Laplace transforms. This yields

$$\begin{aligned} \mathcal{L}_{\chi}(u) &:= \bar{\mathbb{E}} [\exp(-u \text{cap}_{T'}(\emptyset))] \tag{4.8} \\ &= \bar{\mathbb{E}} \left[\bar{\mathbb{E}} \left[\exp(-u \sum_{x \in \tilde{T}: |x|=1} \gamma(T_x)) \mid \tilde{Z}_{\emptyset} \right] \right] \\ &= \bar{\mathbb{E}} \left[\bar{\mathbb{E}} \left[\prod_{x \in \tilde{T}: |x|=1} \exp(-u \gamma(T_x)) \mid \tilde{Z}_{\emptyset} \right] \right] \\ &= \bar{\mathbb{E}} \left[\bar{\mathbb{E}} [\exp(-u \gamma(T))]^{\tilde{Z}_{\emptyset}} \stackrel{(1.3)}{=} \tilde{f}(\mathcal{L}_{\gamma}(u)) \right]. \end{aligned}$$

Since $(\tilde{f}^{-1})' = \frac{1}{\tilde{f}' \circ \tilde{f}^{-1}}$, this allows us to write (1.5) as

$$\frac{1}{\tilde{f}' \circ \tilde{f}^{-1}(\tilde{f}(\mathcal{L}_{\gamma}(u)))} = 1 \tag{4.9}$$

and thus

$$\tilde{f}'(\mathcal{L}_{\gamma}(u)) = 1. \tag{1.5}'$$

Moreover, \tilde{f} and \tilde{f}' being bijective, the uniqueness of the solution is preserved. Thus from now on we will consider (1.5) and (1.5)' as equivalent.

We will now explicit a relation verified by the annealed probability that \mathcal{C}_\emptyset is infinite.

Proposition 5. *For a Galton-Watson process, $r^u := \mathbb{E} \left[Q_u^T \left(|\mathcal{C}_\emptyset| = \infty \right) \right]$ is the largest root in $[0, 1]$ of the equation*

$$\mathcal{L}_\chi(u) - r^u = \tilde{f}(\mathcal{L}_\gamma(u) - r^u). \quad (4.10)$$

Proof. For every vertex $x \in T$, we introduce the notation

$$\mathcal{D}_x = \left(\max_{y \in \mathcal{C}_x \cap \tilde{T}_x} |y| \right) - |x|, \quad (4.11)$$

the relative depth of the cluster containing x . Since $\{\mathcal{D}_\emptyset \geq n+1\} \subset \{\emptyset \in \mathcal{V}\}$, for every $n \in \mathbb{N}$, we have

$$Q_u^{T'} [\{\emptyset \in \mathcal{V}\} \cap \{\mathcal{D}_\emptyset < n+1\}] = Q_u^{T'} (\emptyset \in \mathcal{V}) - Q_u^{T'} (\mathcal{D}_\emptyset \geq n+1). \quad (4.12)$$

We can also write

$$\begin{aligned} Q_u^{T'} [\{\emptyset \in \mathcal{V}\} \cap \{\mathcal{D}_\emptyset < n+1\}] &= Q_u^{T'} [\mathcal{D}_\emptyset < n+1 | \emptyset \in \mathcal{V}] Q_u^{T'} [\emptyset \in \mathcal{V}] \\ &= Q_u^{T'} [\cap_{x \in \tilde{T}: \hat{x} = \emptyset} \{\mathcal{D}_x < n\} | \emptyset \in \mathcal{V}] Q_u^{T'} [\emptyset \in \mathcal{V}]. \end{aligned} \quad (4.13)$$

According to Theorem 2, we know that conditionally on $\{\emptyset \in \mathcal{V}\}$, under $Q_u^{T'}$, \mathcal{C}_\emptyset has the same law as a cluster obtained by Bernoulli site-percolation. Hence the random variables $(\mathcal{D}_x)_{x \in \tilde{T}: \hat{x} = \emptyset}$ are independent under $Q_u^{T'} [\cdot | \emptyset \in \mathcal{V}]$. Moreover for all vertices $x \in \tilde{T}$ such that $\hat{x} = \emptyset$ and every $z \in T'_x \setminus \{\emptyset\}$ we have the equality $h_T^x(z) = h_{T'_x}^x(z)$. This implies that

$$Q_u^{T'} [\mathcal{D}_x \geq n | \emptyset \in \mathcal{V}] = Q_u^{T'_x} [\mathcal{D}_x \geq n | \emptyset \in \mathcal{V}]. \quad (4.14)$$

So that we can rewrite (4.13) as

$$\begin{aligned} Q_u^{T'} [\{\emptyset \in \mathcal{V}\} \cap \{\mathcal{D}_\emptyset < n+1\}] &= \left(\prod_{x \in \tilde{T}: \hat{x} = \emptyset} Q_u^{T'_x} [\mathcal{D}_x < n | \emptyset \in \mathcal{V}] \right) Q_u^{T'} [\emptyset \in \mathcal{V}] \\ &= \left(\prod_{x \in \tilde{T}: \hat{x} = \emptyset} Q_u^{T'_x} [\mathcal{D}_x < n | \emptyset \in \mathcal{V}] \right) Q_u^{T'} [\emptyset \in \mathcal{V}] \quad (4.15) \\ &= \left(\prod_{x \in \tilde{T}: \hat{x} = \emptyset} \left(1 - \frac{Q_u^{T'_x} [\mathcal{D}_x \geq n]}{Q_u^{T'_x} [\emptyset \in \mathcal{V}]} \right) \right) Q_u^{T'} [\emptyset \in \mathcal{V}], \end{aligned}$$

where in the last equation we use the fact that $Q_u^{T'_x}$ -a.s we have the inclusion $\{\mathcal{D}_x \geq n\} \subseteq \{\emptyset \in \mathcal{V}\}$. According to (1.1) and (2.3), we have

$$\begin{aligned} Q_u^{T'_x} [\emptyset \in \mathcal{V}] &= \exp \left(-u \deg_{T'_x}(\emptyset) P_x^{T'_x} [\tilde{H}_\emptyset = \infty] \right) \quad (4.16) \\ &= \exp \left(-u P_x^{T'_x} [\tilde{H}_\emptyset = \infty] \right) \\ &= \exp(-u \gamma(T_x)). \end{aligned}$$

Using (4.5) and (4.16), (4.15) can be rewritten as

$$\begin{aligned}
 & Q_u^{T'} [\{\emptyset \in \mathcal{V}\} \cap \{\mathcal{D}_\emptyset < n + 1\}] \\
 &= \left(\prod_{x \in \tilde{T}: \hat{x} = \emptyset} \left(1 - Q_u^{T'_x} [\mathcal{D}_x \geq n] e^{u\gamma(T_x)} \right) \right) e^{-u \sum_{x \in \tilde{T}: \hat{x} = \emptyset} \gamma(T_x)} \\
 &= \prod_{x \in \tilde{T}: \hat{x} = \emptyset} \left(e^{-u\gamma(T_x)} - Q_u^{T'_x} [\mathcal{D}_x \geq n] \right).
 \end{aligned} \tag{4.17}$$

We can now finally prove (4.10). If we denote $r_n^u = \mathbb{E} [Q_u^{T'} (\mathcal{D}_\emptyset \geq n)]$, we have

$$\begin{aligned}
 \mathcal{L}_\chi(u) - r_{n+1}^u &:= \mathbb{E} [e^{-u \text{cap}_{r'}(\emptyset)} - Q_u^{T'} (\mathcal{D}_\emptyset \geq n + 1)] \\
 &= \mathbb{E} [Q_u^{T'} (\emptyset \in \mathcal{V}) - Q_u^{T'} (\mathcal{D}_\emptyset \geq n + 1)] \\
 &\stackrel{(4.12)}{=} \mathbb{E} [Q_u^{T'} [\{\emptyset \in \mathcal{V}\} \cap \{\mathcal{D}_\emptyset < n + 1\}]] \\
 &\stackrel{(4.17)}{=} \mathbb{E} \left[\prod_{x \in \tilde{T}: \hat{x} = \emptyset} \left(\exp(-u\gamma(T_x)) - Q_u^{T'_x} (\mathcal{D}_x \geq n) \right) \right] \\
 &= \tilde{f} (\mathcal{L}_\gamma(u) - r_n^u).
 \end{aligned} \tag{4.18}$$

The sequence $(r_n^u)_{n \in \mathbb{N}}$ is decreasing by definition. Therefore it converges and its limit $r^u = \mathbb{E} [Q_u^{T'} [\mathcal{D}_\emptyset = \infty]]$ verifies

$$\mathcal{L}_\chi(u) - r^u = \tilde{f} (\mathcal{L}_\gamma(u) - r^u). \tag{4.19}$$

The function \tilde{f} is strictly convex by (A0). Therefore the function

$$x \mapsto \mathcal{L}_\chi(u) - \tilde{f} (\mathcal{L}_\gamma(u) - x) \tag{4.20}$$

is strictly concave and the equation

$$\mathcal{L}_\chi(u) - x = \tilde{f} (\mathcal{L}_\gamma(u) - x). \tag{4.21}$$

has at most two roots in $[0, 1]$. According to (4.8), 0 is always a root. Assume that there exists another root x_0 in $(0, 1]$. Then using the concavity we have

$$x < \mathcal{L}_\chi(u) - \tilde{f} (\mathcal{L}_\gamma(u) - x) \tag{4.22}$$

on $(0, x_0)$. The sequence $(r_n^u)_{n \in \mathbb{N}}$ is positive and decreasing and verifies (4.18). Thus (4.22) and an easy recurrence shows that $\forall n \in \mathbb{N}, r_n^u \geq x_0$. Since r^u verifies (4.19), r^u is a root of (4.21) and r^u can only be x_0 in this case.

Finally, the tree T has locally finite degree and thus $\{\mathcal{D}_\emptyset = \infty\} = \{|\mathcal{C}_\emptyset| = \infty\}$. This yields

$$r^u = \mathbb{E} [Q_u^{T'} [|\mathcal{C}_\emptyset| = \infty]] \tag{4.23}$$

is the largest root of (4.10) in $[0, 1]$ which finishes the proof of Proposition 5. \square

We are now able to deduce non-triviality and (1.5)' from the deterministic study of the roots of the equality (4.10).

Proof of (1.5)'. According to (4.10), we have that

$$\mathcal{L}_\chi(u) - r^u = \tilde{f}(\mathcal{L}_\gamma(u) - r^u). \quad (4.24)$$

Using Taylor-Laplace formula, this can be rewritten:

$$\begin{aligned} 0 &= r^u - \mathcal{L}_\chi(u) + \tilde{f}(\mathcal{L}_\gamma(u) - r^u) \\ &\stackrel{(4.8)}{=} r^u - (\tilde{f}(\mathcal{L}_\gamma(u)) - \tilde{f}(\mathcal{L}_\gamma(u) - r^u)) \\ &= r^u \left(1 - \tilde{f}'(\mathcal{L}_\gamma(u)) + r^u \int_0^1 (1-t) \tilde{f}''(\mathcal{L}_\gamma(u) - tr^u) dt \right). \end{aligned} \quad (4.25)$$

From Proposition 5 and the definition of u^* , we can easily deduce that u^* is the supremum over u for which the function

$$g_u(x) = 1 - \tilde{f}'(\mathcal{L}_\gamma(u)) + x \int_0^1 (1-t) \tilde{f}''(\mathcal{L}_\gamma(u) - tx) dt \quad (4.26)$$

has a root in $(0, 1)$.

We first show that

$$g_u \text{ has a root in } (0, 1) \text{ iff } g_u(0) < 0. \quad (4.27)$$

According to (1.3), the function \tilde{f} is strictly convex. Therefore, for every $x \in (0, 1)$ we have

$$x \int_0^1 (1-t) \tilde{f}''(\mathcal{L}_\gamma(u) - tx) dt > 0. \quad (4.28)$$

In particular, we have $g_u(x) > g_u(0)$ for every $x \in (0, 1)$.

Moreover $\tilde{f}(0) - 0 = \tilde{f}(1) - 1 = 0$, thus the convexity yields

$$\forall x \in (0, 1), \tilde{f}(x) < x. \quad (4.29)$$

We can now deduce from (4.8) that for $\forall u \in (0, \infty)$, we have

$$\mathcal{L}_\chi(u) = \tilde{f}(\mathcal{L}_\gamma(u)) < \mathcal{L}_\gamma(u) < 1. \quad (4.30)$$

Replacing r^u by $\mathcal{L}_\gamma(u)$ in the first line and the third line of (4.25), we obtain

$$\mathcal{L}_\gamma(u) - \mathcal{L}_\chi(u) + \tilde{f}(0) = \mathcal{L}_\gamma(u) g_u(\mathcal{L}_\gamma(u)). \quad (4.31)$$

Since $\tilde{f}(0) = 0$, this yields

$$g_u(\mathcal{L}_\gamma(u)) = \frac{\mathcal{L}_\gamma(u) - \mathcal{L}_\chi(u)}{\mathcal{L}_\gamma(u)} > 0. \quad (4.32)$$

Since g_u is continuous, this finishes the proof of (4.27).

To finish the proof of (1.5)' and non-triviality of u^* we should show that (1.5)' admits exactly one solution $u_0 \in (0, \infty)$ and that

$$g_u(0) > 0 \text{ iff } u > u_0. \quad (4.33)$$

We have

$$g_u(0) = 1 - \tilde{f}'(\mathcal{L}_\gamma(u)) \quad (4.34)$$

with $\tilde{f}'(\cdot) = f'(q + (1 - q)\cdot)$. Since f' is continuous, we can write

$$\begin{aligned} \lim_{u \rightarrow \infty} \tilde{f}'(\mathcal{L}_\gamma(u)) &= \tilde{f}'\left(\lim_{u \rightarrow \infty} \mathcal{L}_\gamma(u)\right) = f'(q) \\ \lim_{u \rightarrow 0} \tilde{f}'(\mathcal{L}_\gamma(u)) &= \tilde{f}'\left(\lim_{u \rightarrow 0} \mathcal{L}_\gamma(u)\right) = f'(1). \end{aligned} \quad (4.35)$$

Moreover since $f(q) = q$ and $f(1) = 1$, the strict convexity of f given by (A0) implies that $f'(q) < 1$ and $f'(1) > 1$. According to (4.34), this means that

$$\lim_{u \rightarrow 0} g_u(0) < 0 < \lim_{u \rightarrow \infty} g_u(0). \quad (4.36)$$

Finally f' being increasing, \tilde{f}' is also increasing and $u \mapsto g_u(0)$ is increasing. Consequently, there exist a unique $u_0 \in (0, \infty)$ such that $g_{u_0}(0) = 0$ (which is equivalent to u_0 is a solution of (1.5)') and we have (4.33). This implies that $u^* = u_0$ and concludes our proof. \square

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