

COMPOSITIONS OF MAPPINGS OF INFINITELY DIVISIBLE DISTRIBUTIONS WITH APPLICATIONS TO FINDING THE LIMITS OF SOME NESTED SUBCLASSES

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Abstract

Let $\{X_t^{(\mu)}, t \geq 0\}$ be a Lévy process on \mathbb{R}^d whose distribution at time 1 is μ , and let f be a nonrandom measurable function on $(0, a), 0 < a \leq \infty$. Then we can define a mapping $\Phi_f(\mu)$ by the law of $\int_0^a f(t) dX_t^{(\mu)}$, from $\mathfrak{D}(\Phi_f)$ which is the totality of $\mu \in I(\mathbb{R}^d)$ such that the stochastic integral $\int_0^a f(t) dX_t^{(\mu)}$ is definable, into a class of infinitely divisible distributions. For $m \in \mathbb{N}$, let Φ_f^m be the m times composition of Φ_f itself. Maejima and Sato (2009) proved that the limits $\bigcap_{m=1}^{\infty} \Phi_f^m(\mathfrak{D}(\Phi_f^m))$ are the same for several known f 's. Maejima and Nakahara (2009) introduced more general f 's. In this paper, the limits $\bigcap_{m=1}^{\infty} \Phi_f^m(\mathfrak{D}(\Phi_f^m))$ for such general f 's are investigated by using the idea of compositions of suitable mappings of infinitely divisible distributions.

1 Introduction

Let $\mathcal{P}(\mathbb{R}^d)$ be the class of all probability distributions on \mathbb{R}^d . Throughout this paper, $\mathcal{L}(X)$ denotes the law of an \mathbb{R}^d -valued random variable X and $\hat{\mu}(z), z \in \mathbb{R}^d$, denotes the characteristic function of $\mu \in \mathcal{P}(\mathbb{R}^d)$. Also $I(\mathbb{R}^d)$ denotes the class of all infinitely divisible distributions on \mathbb{R}^d . $C_\mu(z), z \in \mathbb{R}^d$, denotes the cumulant function of $\mu \in I(\mathbb{R}^d)$, that is, $C_\mu(z)$ is the unique continuous function satisfying $\hat{\mu}(z) = e^{C_\mu(z)}$ and $C_\mu(0) = 0$. For $\mu \in I(\mathbb{R}^d)$ and $t > 0$, we call the distribution with characteristic function $\hat{\mu}(z)^t = e^{tC_\mu(z)}$ the t -th convolution of μ and write μ^t for it. We use the

Lévy-Khintchine triplet (A, ν, γ) of $\mu \in I(\mathbb{R}^d)$ in the sense that

$$C_\mu(z) = -2^{-1}\langle z, Az \rangle + i\langle \gamma, z \rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle(1 + |x|^2)^{-1} \right) \nu(dx), \quad z \in \mathbb{R}^d,$$

where $|\cdot|$ and $\langle \cdot, \cdot \rangle$ are the Euclidean norm and inner product on \mathbb{R}^d , respectively, A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a measure (called the Lévy measure) on \mathbb{R}^d satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty$. When we want to emphasize the Lévy-Khintchine triplet, we write $\mu = \mu_{(A, \nu, \gamma)}$.

We use stochastic integrals with respect to Lévy processes $\{X_t, t \geq 0\}$ of nonrandom measurable functions $f : [0, \infty) \rightarrow \mathbb{R}$, which are $\int_0^t f(s) dX_s, t \in [0, \infty)$. As the definition of stochastic integrals, we adopt the method in Sato [25, 26]. It is known that if f is locally square integrable on $[0, \infty)$, then $\int_0^t f(s) dX_s, t \in [0, \infty)$, is definable for any Lévy process $\{X_t\}$. The improper stochastic integral $\int_0^\infty f(s) dX_s$ is defined as the limit in probability of $\int_0^t f(s) dX_s$ as $t \rightarrow \infty$ whenever the limit exists. In our definition, $\{\int_0^t f(s) dX_s, t \in [0, \infty)\}$ is an additive process in law, which is not always càdlàg in t . If we take its càdlàg modification, the convergence of $\int_0^t f(s) dX_s$ above is equivalent to the almost sure convergence of the modification as $t \rightarrow \infty$.

Let $\{X_t^{(\mu)}, t \geq 0\}$ stand for a Lévy process on \mathbb{R}^d with $\mathcal{L}(X_1^{(\mu)}) = \mu$. Using this Lévy process, we can define a mapping

$$\Phi_f(\mu) = \mathcal{L} \left(\int_0^a f(t) dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\Phi_f) \subset I(\mathbb{R}^d), \tag{1.1}$$

for a nonrandom measurable function $f : [0, a) \rightarrow \mathbb{R}$, where $0 < a \leq \infty$ and $\mathfrak{D}(\Phi_f)$ is the domain of a mapping Φ_f that is the class of $\mu \in I(\mathbb{R}^d)$ for which $\int_0^a f(t) dX_t^{(\mu)}$ is definable in the sense above. Also, $\mathfrak{D}^0(\Phi_f)$ denotes the totality of $\mu \in I(\mathbb{R}^d)$ satisfying $\int_0^a |C_\mu(f(t)z)| dt < \infty$ for all $z \in \mathbb{R}^d$. For a mapping Φ_f , $\mathfrak{R}(\Phi_f)$ is its range that is $\{\Phi_f(\mu) : \mu \in \mathfrak{D}(\Phi_f)\}$. When we consider the composition of two mappings Φ_f and Φ_g , denoted by $\Phi_g \circ \Phi_f$, the domain of $\Phi_g \circ \Phi_f$ is $\mathfrak{D}(\Phi_g \circ \Phi_f) = \{\mu \in I(\mathbb{R}^d) : \mu \in \mathfrak{D}(\Phi_f) \text{ and } \Phi_f(\mu) \in \mathfrak{D}(\Phi_g)\}$. For $m \in \mathbb{N}$, Φ_f^m means the m times composition of Φ_f itself.

A set $H \subset \mathcal{P}(\mathbb{R}^d)$ is said to be closed under type equivalence if $\mathcal{L}(X) \in H$ implies $\mathcal{L}(aX + c) \in H$ for $a > 0$ and $c \in \mathbb{R}^d$. $H \subset I(\mathbb{R}^d)$ is called completely closed in the strong sense (abbreviated as c.c.s.s.) if H is closed under type equivalence, convolution, weak convergence and t -th convolution for any $t > 0$.

We list below several known mappings. In the following, $I_{\log}(\mathbb{R}^d)$ denotes the totality of $\mu \in I(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \log^+ |x| \mu(dx) < \infty$, where $\log^+ |x| = (\log |x|) \vee 0$.

(1) \mathcal{U} -mapping (Alf and O'Connor [1], Jurek [8]): Let

$$\mathcal{U}(\mu) = \mathcal{L} \left(\int_0^1 t dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\mathcal{U}) = I(\mathbb{R}^d),$$

and let $U(\mathbb{R}^d)$ be the Jurek class on \mathbb{R}^d . Then $U(\mathbb{R}^d) = \mathcal{U}(I(\mathbb{R}^d))$.

(2) Φ -mapping (Wolfe [30], Jurek and Vervaat [15], Sato and Yamazato [29]): Let

$$\Phi(\mu) = \mathcal{L} \left(\int_0^\infty e^{-t} dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\Phi) = I_{\log}(\mathbb{R}^d),$$

and let $L(\mathbb{R}^d)$ be the class of selfdecomposable distributions on \mathbb{R}^d . Then $L(\mathbb{R}^d) = \Phi(I_{\log}(\mathbb{R}^d))$.

(3) Υ -mapping (Barndorff-Nielsen and Thorbjørnsen [6], Barndorff-Nielsen et al. [4]): Let

$$\Upsilon(\mu) = \mathcal{L} \left(\int_0^1 \log \frac{1}{t} dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\Upsilon) = I(\mathbb{R}^d), \tag{1.2}$$

and let $B(\mathbb{R}^d)$ be the Goldie–Steutel–Bondesson class on \mathbb{R}^d . Then $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$.

(4) \mathcal{G} -mapping (Maejima and Sato [18]): Let $t = h(s) = \int_s^\infty e^{-u^2} du, s > 0$, and denote its inverse function by $s = h^*(t)$. Let

$$\mathcal{G}(\mu) = \mathcal{L} \left(\int_0^{\sqrt{\pi}/2} h^*(t) dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\mathcal{G}) = I(\mathbb{R}^d),$$

and let $G(\mathbb{R}^d)$ be the class of generalized type G distributions on \mathbb{R}^d . Then $G(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d))$.

(5) Ψ -mapping (Barndorff-Nielsen et al. [4]): Let $t = e(s) = \int_s^\infty u^{-1} e^{-u} du, s > 0$, and denote its inverse function by $s = e^*(t)$. Let

$$\Psi(\mu) = \mathcal{L} \left(\int_0^\infty e^*(t) dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\Psi) = I_{\log}(\mathbb{R}^d),$$

and let $T(\mathbb{R}^d)$ be the Thorin class on \mathbb{R}^d . Then $T(\mathbb{R}^d) = \Psi(I_{\log}(\mathbb{R}^d))$.

(6) \mathcal{M} -mapping (Aoyama et al. [3]): Let $t = m(s) = \int_s^\infty u^{-1} e^{-u^2} du, s > 0$, and denote its inverse function by $s = m^*(t)$. Let

$$\mathcal{M}(\mu) = \mathcal{L} \left(\int_0^\infty m^*(t) dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\mathcal{M}) = I_{\log}(\mathbb{R}^d).$$

We call $M(\mathbb{R}^d) := \mathcal{M}(I_{\log}(\mathbb{R}^d))$ the class M and it was actually introduced in Aoyama et al. [3] in the symmetric case.

Remark 1.1. Jurek [13] introduced the mapping

$$\mathcal{X}^{(e)}(\mu) = \mathcal{L} \left(\int_0^\infty t dX_{1-e^{-t}}^{(\mu)} \right),$$

which is the same as Υ in (1.2) by the time change of the driving Lévy process. In the same way, it holds that

$$\mathcal{G}(\mu) = \mathcal{L} \left(\int_0^\infty t dX_{\sqrt{\pi}/2-h(t)}^{(\mu)} \right).$$

Using this type of time change, we might avoid taking inverse functions as integrands of stochastic integral mappings. However, recently in Sato [27], Barndorff-Nielsen et al. [5] and other papers, they have used stochastic integral mappings whose integrands are some inverse functions and driving Lévy processes have original time parameter. In this paper, we also use this type of expressions.

Here we also introduce mappings $\Phi_\alpha, \alpha < 2$, (O'Connor [21, 22], Jurek [9, 10, 11], Jurek and Schreiber [14], Sato [27], Maejima et al. [16]). Let

$$t = \varphi_\alpha(s) = \int_s^1 u^{-\alpha-1} du, \quad s \geq 0,$$

and let $s = \varphi_\alpha^*(t)$ be its inverse function. Define

$$\Phi_\alpha(\mu) = \mathcal{L} \left(\int_0^{\varphi_\alpha(0)} \varphi_\alpha^*(t) dX_t^{(\mu)} \right).$$

Then,

$$\Phi_\alpha(\mu) = \begin{cases} \mathcal{L} \left(\int_0^{-1/\alpha} (1+at)^{-1/\alpha} dX_t^{(\mu)} \right), & \text{when } \alpha < 0, \\ \mathcal{L} \left(\int_0^\infty e^{-t} dX_t^{(\mu)} \right), & \text{when } \alpha = 0, \\ \mathcal{L} \left(\int_0^\infty (1+at)^{-1/\alpha} dX_t^{(\mu)} \right), & \text{when } 0 < \alpha < 2. \end{cases}$$

Furthermore, we introduce mappings $\Psi_{\alpha,\beta}, \alpha < 2, \beta > 0$. Let

$$t = G_{\alpha,\beta}(s) = \int_s^\infty u^{-\alpha-1} e^{-u^\beta} du, \quad s \geq 0,$$

and let $s = G_{\alpha,\beta}^*(t)$ be its inverse function. Define

$$\Psi_{\alpha,\beta}(\mu) = \mathcal{L} \left(\int_0^{G_{\alpha,\beta}(0)} G_{\alpha,\beta}^*(t) dX_t^{(\mu)} \right),$$

where

$$G_{\alpha,\beta}(0) = \begin{cases} \beta^{-1} \Gamma(-\alpha\beta^{-1}), & \text{when } \alpha < 0, \\ \infty, & \text{when } \alpha \geq 0. \end{cases}$$

These mappings are introduced first by Sato [27] for $\beta = 1$ and later by Maejima and Nakahara [17] for general $\beta > 0$. Due to Sato [27], Maejima and Nakahara [17], we have the domains $\mathfrak{D}(\Phi_\alpha)$ and $\mathfrak{D}(\Psi_{\alpha,\beta})$ as follows. Let $\beta > 0$.

$$\mathfrak{D}(\Phi_\alpha) = \mathfrak{D}(\Psi_{\alpha,\beta}) = \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha < 0, \\ I_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ I_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ I_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ I_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2, \end{cases}$$

where

$$\begin{aligned}
 I_\alpha(\mathbb{R}^d) &= \left\{ \mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \right\}, \quad \text{for } \alpha > 0, \\
 I_\alpha^0(\mathbb{R}^d) &= \left\{ \mu \in I_\alpha(\mathbb{R}^d) : \int_{\mathbb{R}^d} x \mu(dx) = 0 \right\}, \quad \text{for } \alpha \geq 1, \\
 I_1^*(\mathbb{R}^d) &= \left\{ \mu = \mu_{(A,v,\gamma)} \in I_1^0(\mathbb{R}^d) : \lim_{T \rightarrow \infty} \int_1^T t^{-1} dt \int_{|x|>t} x v(dx) \text{ exists in } \mathbb{R}^d \right\}.
 \end{aligned}$$

Since $\Phi_0 = \Phi$, $\Phi_{-1} = \mathcal{U}$, $\Psi_{-1,1} = \Upsilon$, $\Psi_{-1,2} = \mathcal{G}$, $\Psi_{0,1} = \Psi$ and $\Psi_{0,2} = \mathcal{M}$, the mappings Φ_α and $\Psi_{\alpha,\beta}$ are important. Also, Maejima and Nakahara [17] characterized the classes $\mathfrak{R}(\Psi_{\alpha,\beta})$, $\alpha < 1, \beta > 0$ by conditions of radial components in the polar decomposition of Lévy measures.

Define nested subclasses $L_m(\mathbb{R}^d), m \in \mathbb{Z}_+$ of $L(\mathbb{R}^d)$ in the following way: $\mu \in L_m(\mathbb{R}^d)$ if and only if for each $b > 1$, there exists $\rho_b \in L_{m-1}(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$, where $L_0(\mathbb{R}^d) := L(\mathbb{R}^d)$. Hereafter we denote the closure under weak convergence and convolution of a class $H \subset \mathcal{P}(\mathbb{R}^d)$ by \overline{H} . For $\alpha \in (0, 2]$, let $S_\alpha(\mathbb{R}^d)$ be the class of all α -stable distributions on \mathbb{R}^d and let $S(\mathbb{R}^d) = \bigcup_{\alpha \in (0,2)} S_\alpha(\mathbb{R}^d)$. Then, the limit $L_\infty(\mathbb{R}^d) := \bigcap_{m=0}^\infty L_m(\mathbb{R}^d)$ is known to be equal to $\overline{S(\mathbb{R}^d)}$. In Sato [24] or Rocha-Arteaga and Sato [23], this is proved via the following fact: $\mu = \mu_{(A,v,\gamma)} \in L_\infty(\mathbb{R}^d)$ if and only if

$$v(B) = \int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-\alpha-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where Γ is a measure on $(0, 2)$ satisfying

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma(d\alpha) < \infty,$$

and λ_α is a probability measure on $S := \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ for each $\alpha \in (0, 2)$, and $\lambda_\alpha(C)$ is measurable in $\alpha \in (0, 2)$ for every $C \in \mathcal{B}(S)$. This Γ is uniquely determined by μ and this λ_α is uniquely determined by μ up to α of Γ -measure 0. For the case in more general spaces, see Jurek [7]. For a set $A \in \mathcal{B}((0, 2))$, let $L_\infty^A(\mathbb{R}^d)$ denote the class of $\mu \in L_\infty(\mathbb{R}^d)$ with Γ satisfying $\Gamma((0, 2) \setminus A) = 0$. It is also known that for $m \in \mathbb{N}$, $\mathfrak{R}(\Phi^m) = L_{m-1}(\mathbb{R}^d)$. Hence $\bigcap_{m=1}^\infty \mathfrak{R}(\Phi^m) = L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$. In Maejima and Sato [18], nested subclasses $\mathfrak{R}(\mathcal{U}^m)$, $\mathfrak{R}(\Upsilon^m)$, $\mathfrak{R}(\Psi^m)$ and $\mathfrak{R}(\mathcal{G}^m)$, $m \in \mathbb{N}$, were studied and the limits of these nested subclasses were proved to be equal to $\overline{S(\mathbb{R}^d)}$, (see also Jurek [12]). Furthermore, Sato [28] proved that the mappings $\Psi_{\alpha,1}, \alpha \in (0, 2)$ produce smaller classes than $\overline{S(\mathbb{R}^d)}$ as the limit of iteration. Maejima and Ueda [19] showed that the mapping Φ_α has the same iterated limit as that of $\Psi_{\alpha,1}$ for $\alpha \in (0, 2)$. Maejima and Ueda [20] also constructed a mapping producing a larger class than $\overline{S(\mathbb{R}^d)}$, which is the closure of the class of semi-stable distributions with a fixed span.

The purpose of this paper is to find the limit of the nested subclasses $\mathfrak{R}(\Psi_{\alpha,\beta}^m), m \in \mathbb{N}$. For that, we start with the composition of $\Psi_{\alpha-\beta,\beta}$ and Φ_α , which will be used for characterizing the nested subclasses $\mathfrak{R}(\Psi_{\alpha,\beta}^m), m \in \mathbb{N}$.

2 Results

For $\beta > 0$, let

$$\mathcal{K}_\beta(\mu) = \mu^\beta = \mathcal{L} \left(\int_0^\beta dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\mathcal{K}_\beta) = I(\mathbb{R}^d).$$

The following lemma is trivial.

Lemma 2.1. For $\beta > 0$ and any mapping Φ_f defined by (1.1) with a locally square-integrable function f , we have

$$\mathcal{K}_\beta \circ \Phi_f = \Phi_f \circ \mathcal{K}_\beta.$$

Here

$$\mathfrak{D}(\mathcal{K}_\beta \circ \Phi_f) = \mathfrak{D}(\Phi_f \circ \mathcal{K}_\beta) = \mathfrak{D}(\Phi_f).$$

The following result on composition will be a key in the proof of the main theorem, Theorem 2.4.

Theorem 2.2. Let $\alpha \in (-\infty, 1) \cup (1, 2)$ and $\beta > 0$. Then

$$\Psi_{\alpha,\beta} = \mathcal{K}_\beta \circ \Phi_\alpha \circ \Psi_{\alpha-\beta,\beta} = \mathcal{K}_\beta \circ \Psi_{\alpha-\beta,\beta} \circ \Phi_\alpha,$$

including the equality of the domains.

Remark 2.3. Theorem 2.2 with $\beta = 1$ is included in Theorem 3.1 of Sato [27]. Also, the case $\alpha = 0$ was already proved by Aoyama et al. [2].

Our main result of this paper is the following theorem on the limits of the nested subclasses $\mathfrak{R}(\Psi_{\alpha,\beta}^m)$ which is $\bigcap_{m=1}^\infty \mathfrak{R}(\Psi_{\alpha,\beta}^m)$.

Theorem 2.4. Let $\beta > 0$. Then

$$\bigcap_{m=1}^\infty \mathfrak{R}(\Psi_{\alpha,\beta}^m) = \begin{cases} L_\infty(\mathbb{R}^d), & \text{for } \alpha \in (-\infty, 0], \\ L_\infty^{(\alpha,2)}(\mathbb{R}^d), & \text{for } \alpha \in (0, 1), \\ L_\infty^{(\alpha,2)}(\mathbb{R}^d) \cap I_1^0(\mathbb{R}^d), & \text{for } \alpha \in (1, 2) \setminus \{1 + n\beta : n \in \mathbb{N}\}. \end{cases}$$

Remark 2.5. Theorem 2.4 for the case $-1 \leq \alpha < 0, \beta > 0$ follows immediately from Theorem 3.4 of Maejima and Sato [18]. Maejima and Sato [18] also proved the case $\alpha = 0, \beta = 1$. Furthermore, the case $\beta = 1, \alpha \in (0, 2)$ was already proved by Sato [28]. The case $\alpha = 0$ is found in Aoyama et al. [2].

We also have the following.

Theorem 2.6. Let $\beta > 0$ and $\alpha \in (-\infty, 2) \setminus \{1 + n\beta : n \in \mathbb{Z}_+\}$. Then

$$\bigcap_{m=1}^\infty \mathfrak{R}(\Psi_{\alpha,\beta}^m) = \mathfrak{R}(\Psi_{\alpha,\beta}) \cap \overline{S(\mathbb{R}^d)}. \quad (2.1)$$

3 Proofs

We first prove Theorem 2.2.

Proof of Theorem 2.2. For $\mu \in I(\mathbb{R}^d)$, we have

$$\begin{aligned}
 & \beta \int_0^{\varphi_\alpha(0)} du \int_0^{G_{\alpha-\beta,\beta}(0)} \left| C_\mu \left(G_{\alpha-\beta,\beta}^*(v) \varphi_\alpha^*(u)z \right) \right| dv \\
 &= \beta \int_0^1 s^{-\alpha-1} ds \int_0^\infty |C_\mu(tsz)| t^{\beta-\alpha-1} e^{-t^\beta} dt \\
 &= \beta \int_0^1 s^{-\beta-1} ds \int_0^\infty |C_\mu(uz)| u^{\beta-\alpha-1} e^{-s^{-\beta}u^\beta} du \\
 &= \beta \int_0^\infty |C_\mu(uz)| u^{\beta-\alpha-1} du \int_0^1 s^{-\beta-1} e^{-s^{-\beta}u^\beta} ds \\
 &= \int_0^\infty |C_\mu(uz)| u^{-\alpha-1} du \int_{u^\beta}^\infty e^{-v} dv \\
 &= \int_0^\infty |C_\mu(uz)| u^{-\alpha-1} e^{-u^\beta} du = \int_0^{G_{\alpha,\beta}(0)} \left| C_\mu \left(G_{\alpha,\beta}^*(t)z \right) \right| dt.
 \end{aligned} \tag{3.1}$$

Let $\alpha < 0$. Then $\mathfrak{D}(\Psi_{\alpha,\beta}) = \mathfrak{D}(\mathcal{K}_\beta \circ \Psi_{\alpha-\beta,\beta} \circ \Phi_\alpha) = \mathfrak{D}(\mathcal{K}_\beta \circ \Phi_\alpha \circ \Psi_{\alpha-\beta,\beta}) = I(\mathbb{R}^d)$ and Propositions 3.4 and 2.17 of Sato [26] yields the finiteness of (3.1). Then we can use Fubini's theorem and have

$$\begin{aligned}
 \int_0^{G_{\alpha,\beta}(0)} C_\mu \left(G_{\alpha,\beta}^*(t)z \right) dt &= \beta \int_0^{\varphi_\alpha(0)} du \int_0^{G_{\alpha-\beta,\beta}(0)} C_\mu \left(G_{\alpha-\beta,\beta}^*(v) \varphi_\alpha^*(u)z \right) dv \\
 &= \beta \int_0^{G_{\alpha-\beta,\beta}(0)} dv \int_0^{\varphi_\alpha(0)} C_\mu \left(G_{\alpha-\beta,\beta}^*(v) \varphi_\alpha^*(u)z \right) du,
 \end{aligned} \tag{3.2}$$

by a similar calculation to (3.1). This yields that

$$\Psi_{\alpha,\beta}(\mu) = \mathcal{K}_\beta \circ \Phi_\alpha \circ \Psi_{\alpha-\beta,\beta}(\mu) = \mathcal{K}_\beta \circ \Psi_{\alpha-\beta,\beta} \circ \Phi_\alpha(\mu). \tag{3.3}$$

Let $\alpha \in [0, 1) \cup (1, 2)$. Let $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta})$. Note that the domains $\mathfrak{D}(\Psi_{\alpha,\beta})$ and $\mathfrak{D}(\Phi_\alpha)$ are the same and decreasing in $\alpha < 2$ with respect to set inclusion due to Remark to Theorem 2.8 of Sato [27]. Then $\mu \in \mathfrak{D}(\Psi_{\alpha-\beta,\beta}) \cap \mathfrak{D}(\Phi_\alpha)$. We have that

$$\beta \int_0^{\varphi_\alpha(0)} \left| C_{\Psi_{\alpha-\beta,\beta}(\mu)}(\varphi_\alpha^*(u)z) \right| du$$

and

$$\beta \int_0^{G_{\alpha-\beta,\beta}(0)} \left| C_{\Phi_\alpha(\mu)}(G_{\alpha-\beta,\beta}^*(v)z) \right| dv$$

are not greater than (3.1). Take into account that $\mathfrak{D}(\Psi_{\alpha,\beta}) = \mathfrak{D}^0(\Psi_{\alpha,\beta})$ for $\alpha \in (-\infty, 1) \cup (1, 2)$ and $\beta > 0$ due to Theorem 2.4 of Sato [27]. Then $\mu \in \mathfrak{D}^0(\Psi_{\alpha,\beta})$, which yields the finiteness of

(3.1). Therefore $\Psi_{\alpha-\beta,\beta}(\mu) \in \mathfrak{D}(\mathcal{K}_\beta \circ \Phi_\alpha)$ and $\Phi_\alpha(\mu) \in \mathfrak{D}(\mathcal{K}_\beta \circ \Psi_{\alpha-\beta,\beta})$. Hence $\mu \in \mathfrak{D}(\mathcal{K}_\beta \circ \Phi_\alpha \circ \Psi_{\alpha-\beta,\beta}) \cap \mathfrak{D}(\mathcal{K}_\beta \circ \Psi_{\alpha-\beta,\beta} \circ \Phi_\alpha)$.

If $\mu \in \mathfrak{D}(\mathcal{K}_\beta \circ \Psi_{\alpha-\beta,\beta} \circ \Phi_\alpha)$, then $\mu \in \mathfrak{D}(\Phi_\alpha) = \mathfrak{D}(\Psi_{\alpha,\beta})$.

Let $\mu \in \mathfrak{D}(\mathcal{K}_\beta \circ \Phi_\alpha \circ \Psi_{\alpha-\beta,\beta})$. Then $\Psi_{\alpha-\beta,\beta}(\mu) \in \mathfrak{D}(\Phi_\alpha)$. Let $\mu = \mu_{(A,v,\gamma)}$ and $\Psi_{\alpha-\beta,\beta}(\mu) = \tilde{\mu} = \tilde{\mu}_{(\tilde{A},\tilde{v},\tilde{\gamma})}$. If $\alpha = 0$, then

$$\infty > \int_{|x|>1} \log|x| \tilde{v}(dx) = \int_0^\infty t^{\beta-1} e^{-t^\beta} dt \int_{|tx|>1} \log|tx| v(dx),$$

which yields that $\int_{|tx|>1} \log|tx| v(dx) < \infty$ a.e. $t > 0$. Hence $\mu \in I_{\log}(\mathbb{R}^d) = \mathfrak{D}(\Psi_{0,\beta})$. If $\alpha \in (0, 1) \cup (1, 2)$, then

$$\infty > \int_{|x|>1} |x|^\alpha \tilde{v}(dx) = \int_0^\infty t^{\beta-\alpha-1} e^{-t^\beta} dt \int_{|tx|>1} |tx|^\alpha v(dx),$$

which yields that $\int_{|tx|>1} |x|^\alpha v(dx) < \infty$ a.e. $t > 0$. Hence $\mu \in I_\alpha(\mathbb{R}^d) = \mathfrak{D}(\Psi_{\alpha,\beta})$ for $\alpha \in (0, 1)$. Let $\alpha \in (1, 2)$. Then $\tilde{\mu} \in \mathfrak{D}(\Phi_\alpha) = I_\alpha^0(\mathbb{R}^d)$. It follows that

$$\tilde{\gamma} = - \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \tilde{v}(dx) = - \int_0^\infty t^{\beta-\alpha-1} e^{-t^\beta} dt \int_{\mathbb{R}^d} \frac{tx|tx|^2}{1+|tx|^2} v(dx)$$

and

$$\tilde{\gamma} = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty t^{\beta-\alpha} e^{-t^\beta} dt \left\{ \gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|tx|^2} - \frac{1}{1+|x|^2} \right) v(dx) \right\}.$$

Hence

$$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty t^{\beta-\alpha} e^{-t^\beta} \left(\gamma + \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} v(dx) \right) dt = 0,$$

which yields that $\gamma + \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} v(dx) = 0$. Therefore $\mu \in I_\alpha^0(\mathbb{R}^d) = \mathfrak{D}(\Psi_{\alpha,\beta})$.

Thus we conclude that $\mathfrak{D}(\Psi_{\alpha,\beta}) = \mathfrak{D}(\mathcal{K}_\beta \circ \Psi_{\alpha-\beta,\beta} \circ \Phi_\alpha) = \mathfrak{D}(\mathcal{K}_\beta \circ \Phi_\alpha \circ \Psi_{\alpha-\beta,\beta})$ for $\alpha \in [0, 1) \cup (1, 2)$. If $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta}) = \mathfrak{D}^0(\Psi_{\alpha,\beta})$, then (3.1) is finite and we have (3.2) and (3.3). \square

Let

$$\mathcal{C}_\alpha(\mathbb{R}^d) := \left\{ \mu = \mu_{(A,v,\gamma)} \in I(\mathbb{R}^d) : \lim_{r \rightarrow \infty} r^\alpha \int_{|x|>r} v(dx) = 0 \right\},$$

$$\mathcal{C}_\alpha^0(\mathbb{R}^d) := \mathcal{C}_\alpha(\mathbb{R}^d) \cap I_1^0(\mathbb{R}^d).$$

We need the following lemma in the proof below.

Lemma 3.1 (Corollary 4.2 of Maejima and Ueda [19]). *Let $H \subset I(\mathbb{R}^d)$ be c.c.s.s.*

(i) *If $\alpha \leq 0$ and $H \supset S(\mathbb{R}^d)$, then*

$$\bigcap_{m=1}^{\infty} \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) = \overline{S(\mathbb{R}^d)}.$$

(ii) If $0 < \alpha < 1$ and $H \supset \bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)$, then

$$\bigcap_{m=1}^{\infty} \Phi_\alpha^m (H \cap \mathfrak{D}(\Phi_\alpha^m)) = \overline{\bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d)}.$$

(iii) If $1 < \alpha < 2$ and $H \supset \bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)$, then

$$\bigcap_{m=1}^{\infty} \Phi_\alpha^m (H \cap \mathfrak{D}(\Phi_\alpha^m)) = \overline{\bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)}.$$

Here $\overline{S(\mathbb{R}^d)}$ and $\overline{\bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)}$ are c.c.s.s., (see e.g. Proposition 3.12 (i) and Theorem 3.20 of Maejima and Ueda [19]).

To prove Theorem 2.4, it is sufficient to show the following, due to Theorem 4.6 of Maejima and Ueda [19] that is Theorem 2.4 with the replacement of $\Psi_{\alpha, \beta}$ by Φ_α .

Theorem 3.2. Let $\beta > 0$ and $\alpha \in (-\infty, 2) \setminus \{1 + n\beta : n \in \mathbb{Z}_+\}$. Then we have

$$\bigcap_{m=1}^{\infty} \mathfrak{R}(\Psi_{\alpha, \beta}^m) = \bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_\alpha^m). \quad (3.4)$$

Proof. Lemma 2.1 and Theorem 2.2 yield that for $\alpha \in (-\infty, 1) \cup (1, 2)$, $\beta > 0$ and $m \in \mathbb{N}$,

$$\begin{aligned} \mathfrak{R}(\Psi_{\alpha, \beta}^m) &= \mathfrak{R}(\Phi_\alpha^m \circ \Psi_{\alpha-\beta, \beta}^m \circ \mathcal{K}_\beta^m) \\ &= \Phi_\alpha^m \left(\mathfrak{R}(\Psi_{\alpha-\beta, \beta}^m \circ \mathcal{K}_\beta^m) \cap \mathfrak{D}(\Phi_\alpha^m) \right) \\ &= \Phi_\alpha^m \left(\Psi_{\alpha-\beta, \beta}^m \left(\mathfrak{R}(\mathcal{K}_\beta^m) \cap \mathfrak{D}(\Psi_{\alpha-\beta, \beta}^m) \right) \cap \mathfrak{D}(\Phi_\alpha^m) \right) \\ &= \Phi_\alpha^m \left(\Psi_{\alpha-\beta, \beta}^m \left(\mathfrak{R}(\mathcal{K}_\beta^m) \cap \mathfrak{D}(\Psi_{\alpha-\beta, \beta}^m) \right) \cap \mathfrak{D}(\Phi_\alpha^m) \right) \\ &= \Phi_\alpha^m \left(\Psi_{\alpha-\beta, \beta}^m \left(I(\mathbb{R}^d) \cap \mathfrak{D}(\Psi_{\alpha-\beta, \beta}^m) \right) \cap \mathfrak{D}(\Phi_\alpha^m) \right) \\ &= \Phi_\alpha^m \left(\mathfrak{R}(\Psi_{\alpha-\beta, \beta}^m) \cap \mathfrak{D}(\Phi_\alpha^m) \right). \end{aligned} \quad (3.5)$$

Fix any $\beta > 0$.

We first show

$$\bigcap_{m=1}^{\infty} \mathfrak{R}(\Psi_{\alpha, \beta}^m) = \bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_\alpha^m) \quad \text{for all } \alpha < (n\beta) \wedge 1, \quad (3.6)$$

for each $n \in \mathbb{N}$ by induction. Let $n = 1$. For $\alpha < \beta \wedge 1$, Proposition 3.2 of Maejima and Sato [18] entails that $H := \bigcap_{k=1}^{\infty} \mathfrak{R}(\Psi_{\alpha-\beta, \beta}^k)$ is c.c.s.s. Also, Lemma 3.7 of Maejima and Sato [18] yields that $H \supset S(\mathbb{R}^d)$. It follows from (3.5) that for $m \in \mathbb{N}$,

$$\Phi_\alpha^m (H \cap \mathfrak{D}(\Phi_\alpha^m)) \subset \mathfrak{R}(\Psi_{\alpha, \beta}^m) \subset \mathfrak{R}(\Phi_\alpha^m).$$

Thus Lemma 3.1 yields (3.6) with $n = 1$. Assume that (3.6) is true for $n - 1$ in place of n with

$n \geq 2$. Then for $\alpha < (n\beta) \wedge 1$, it follows that $\alpha - \beta < ((n-1)\beta) \wedge 1$. Therefore

$$\begin{aligned} \bigcap_{k=1}^{\infty} \mathfrak{R}(\Psi_{\alpha-\beta, \beta}^k) &= \bigcap_{k=1}^{\infty} \mathfrak{R}(\Phi_{\alpha-\beta}^k) \\ &= \begin{cases} \overline{S(\mathbb{R}^d)}, & \text{when } \alpha - \beta \leq 0 \\ \overline{\bigcup_{\gamma \in [\alpha-\beta, 2]} S_{\gamma}(\mathbb{R}^d) \cap \mathcal{C}_{\alpha-\beta}(\mathbb{R}^d)}, & \text{when } 0 < \alpha - \beta < 1, \end{cases} \end{aligned}$$

by the assumption of induction and Lemma 3.1. When $\alpha - \beta \leq 0$, it follows from (3.5) that for $m \in \mathbb{N}$,

$$\Phi_{\alpha}^m \left(\overline{S(\mathbb{R}^d)} \cap \mathfrak{D}(\Phi_{\alpha}^m) \right) \subset \mathfrak{R}(\Psi_{\alpha, \beta}^m) \subset \mathfrak{R}(\Phi_{\alpha}^m),$$

which yields (3.6) by Lemma 3.1. When $0 < \alpha - \beta < 1$, it follows from (3.5) that for $m \in \mathbb{N}$,

$$\Phi_{\alpha}^m \left(\overline{\bigcup_{\gamma \in [\alpha-\beta, 2]} S_{\gamma}(\mathbb{R}^d)} \cap \mathfrak{D}(\Phi_{\alpha}^m) \right) \subset \mathfrak{R}(\Psi_{\alpha, \beta}^m) \subset \mathfrak{R}(\Phi_{\alpha}^m), \quad (3.7)$$

since $\mathfrak{D}(\Phi_{\alpha}^m) \subset \mathfrak{D}(\Phi_{\alpha}) \subset I_{\alpha}(\mathbb{R}^d) \subset I_{\alpha-\beta}(\mathbb{R}^d) \subset \mathcal{C}_{\alpha-\beta}(\mathbb{R}^d)$. Thus Lemma 3.1 yields (3.6). Then (3.6) is true for all $n \in \mathbb{N}$, namely, (3.4) holds for all $\alpha < 1$.

We next show

$$\bigcap_{m=1}^{\infty} \mathfrak{R}(\Psi_{\alpha, \beta}^m) = \bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_{\alpha}^m) \quad \text{for all } \alpha \in (1 + (n-1)\beta, 1 + n\beta) \cap (-\infty, 2), \quad (3.8)$$

for each $n \in \mathbb{Z}_+$ by induction. If $n = 0$, then $\alpha < 1$ and we have just shown the case. Assume that (3.8) holds for $n-1$ in place of n with $n \geq 1$. Then for $\alpha \in (1 + (n-1)\beta, 1 + n\beta) \cap (-\infty, 2)$, it follows that $\alpha - \beta \in (1 + (n-2)\beta, 1 + (n-1)\beta) \cap (-\infty, 2)$. Then the assumption of induction and Lemma 3.1 yields that

$$\begin{aligned} \bigcap_{k=1}^{\infty} \mathfrak{R}(\Psi_{\alpha-\beta, \beta}^k) &= \bigcap_{k=1}^{\infty} \mathfrak{R}(\Phi_{\alpha-\beta}^k) \\ &= \begin{cases} \overline{S(\mathbb{R}^d)}, & \text{when } \alpha - \beta \leq 0 \\ \overline{\bigcup_{\gamma \in [\alpha-\beta, 2]} S_{\gamma}(\mathbb{R}^d) \cap \mathcal{C}_{\alpha-\beta}(\mathbb{R}^d)}, & \text{when } 0 < \alpha - \beta < 1, \\ \overline{\bigcup_{\gamma \in [\alpha-\beta, 2]} S_{\gamma}(\mathbb{R}^d) \cap \mathcal{C}_{\alpha-\beta}^0(\mathbb{R}^d)}, & \text{when } 1 < \alpha - \beta < 2. \end{cases} \end{aligned}$$

When $\alpha - \beta < 1$, (3.8) holds by the same argument as above. When $1 < \alpha - \beta < 2$, the same inclusion as (3.7) follows from (3.5), since $\mathfrak{D}(\Phi_{\alpha}^m) \subset \mathfrak{D}(\Phi_{\alpha}) \subset I_{\alpha}^0(\mathbb{R}^d) \subset I_{\alpha-\beta}^0(\mathbb{R}^d) \subset \mathcal{C}_{\alpha-\beta}^0(\mathbb{R}^d)$. Therefore Lemma 3.1 yields (3.8). Then (3.8) is true for all $n \in \mathbb{Z}_+$. Thus (3.4) holds for all $\alpha \in (1, 2) \setminus \{1 + n\beta : n \in \mathbb{N}\}$. \square

We finally prove Theorem 2.6.

Proof of Theorem 2.6. (3.5) with $m = 1$ yields that $\mathfrak{R}(\Psi_{\alpha, \beta}) \subset \mathfrak{R}(\Phi_{\alpha})$. Also we have

$$\bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_{\alpha}^m) = \mathfrak{R}(\Phi_{\alpha}) \cap \overline{S(\mathbb{R}^d)},$$

which is Theorem 5.2 of Maejima and Ueda [19]. Then we have

$$\mathfrak{R}(\Psi_{\alpha,\beta}) \cap \overline{S(\mathbb{R}^d)} \subset \mathfrak{R}(\Phi_\alpha) \cap \overline{S(\mathbb{R}^d)} = \bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_\alpha^m) = \bigcap_{m=1}^{\infty} \mathfrak{R}(\Psi_{\alpha,\beta}^m),$$

where the last equality follows from Theorem 3.2. Furthermore, we have that

$$\bigcap_{m=1}^{\infty} \mathfrak{R}(\Psi_{\alpha,\beta}^m) \subset \mathfrak{R}(\Psi_{\alpha,\beta}),$$

and that

$$\bigcap_{m=1}^{\infty} \mathfrak{R}(\Psi_{\alpha,\beta}^m) = \bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_\alpha^m) = \mathfrak{R}(\Phi_\alpha) \cap \overline{S(\mathbb{R}^d)} \subset \overline{S(\mathbb{R}^d)}.$$

Thus $\bigcap_{m=1}^{\infty} \mathfrak{R}(\Psi_{\alpha,\beta}^m) \subset \mathfrak{R}(\Psi_{\alpha,\beta}) \cap \overline{S(\mathbb{R}^d)}$. Therefore (2.1) holds. \square

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