DOI: 10.1214/ECP.v14-1461

Elect. Comm. in Probab. **14** (2009), 224–231

ELECTRONIC COMMUNICATIONS in PROBABILITY

ALMOST SURE LIMIT THEOREM FOR THE MAXIMA OF STRONGLY DEPENDENT GAUSSIAN SEQUENCES

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Submitted December 2, 2008, *accepted in final form* April 27, 2009

AMS 2000 Subject classification: 60F05; 62E20; 62F12; 62M10
Keywords: Almost sure central limit theorem, Strongly dependent sequence, Logarithmic average

Abstract

In this paper, we prove an almost sure limit theorem for the maxima of strongly dependent Gaussian sequences under some mild conditions. The result is an expansion of the weakly dependent result of E. Csáki and K. Gonchigdanzan.

1 Introduction and main result

In past decades, the almost sure central limit theorem (ASCLT) has been studied for independent and dependent random variables more and more profoundly. Cheng et al.[CPQ98], Fahrner and Stadtmüller[FS98] and Berkes and Csáki[BC01] considered the ASCLT for the maximum of i.i.d. random variables. An influential work is Csáki and Gonchigdanzan[CG02], which proved an almost sure limit theorem for the maximum of stationary weakly dependent sequence.

Theorem A. Let $X_1, X_2, \cdots$ be a standardized stationary Gaussian sequence with $r_n = Cov(X_1, X_{n+1})$ satisfying $r_n \log n(\log\log n)^{1+\varepsilon} = O(1)$ as $n \to \infty$. Let $M_k = \max_{i\le k} X_i$. If $a_n = (2\log n)^{1/2}$, $b_n = (2\log n)^{1/2} - \frac{1}{2}(2\log n)^{-1/2}(\log\log n + \log(4\pi))$, then

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k} I(a_k(M_k - b_k) \le x) = \exp(-e^{-x}) \ \ a.s., \tag{1}$$

where I is indicator function.

Shouquan Chen and Zhengyan Lin[CL06] extended the results in [CG02] to the non-stationary case.

Leadbetter et al [LLR83] showed the following theorem.

Theorem B. Let $X_1, X_2, \cdots$ be a standardized stationary Gaussian sequence with $r_n = Cov(X_1, X_{n+1})$ and $M_n = \max_{1\le i\le n} X_i$. Let $a_n = (2\log n)^{1/2}$ and $b_n = (2\log n)^{1/2} - \frac{1}{2}(2\log n)^{-1/2}(\log\log n +$

[1] RESEARCH SUPPORTED BY THE SCIENTIFIC RESEARCH FUND OF SICHUAN UNIVERSITY OF SCIENCE & ENGINEERING UNDER GRANT 2007RZ014.

$\log(4\pi)$). If $r_n \log n \to r > 0$, then

$$\lim_{n\to\infty} P\Big(a_n(M_n - b_n) \le x\Big) = \int_{-\infty}^{\infty} \exp\Big(-e^{-x-r+\sqrt{2r}z}\Big)\phi(z)dz, \tag{2}$$

where and in the sequel ϕ is standard normal density.
In the paper, we consider the ASCLT version of (2). The theorem below is useful in our proof.
Theorem C. [Leadbetter et al., 1983, Theorem 4.2.1, Normal Comparison Lemma] Suppose $X_1, X_2, \cdots, X_n$ are standard normal variables with covariance matrix $\Lambda^1 = (\Lambda^1_{ij})$, and $Y_1, Y_2, \cdots, Y_n$ similarly with covariance $\Lambda^0 = (\Lambda^0_{ij})$, and $\rho_{ij} := \max(|\Lambda^1_{ij}|, |\Lambda^0_{ij}|)$, assuming that $\max_{i\ne j}\rho_{ij} =: \delta < 1$. Further, let $u_1, \cdots, u_n$ be real numbers. Then

$$\begin{aligned}&|P(X_j \le u_j, j = 1, \cdots, n) - P(Y_j \le u_j, j = 1, \cdots, n)| \\ &\le K_1 \sum_{1\le i<j\le n} |\Lambda^1_{ij} - \Lambda^0_{ij}| \exp\left(-\frac{u_i^2 + u_j^2}{2(1+\rho_{ij})}\right)\end{aligned} \tag{3}$$

with some positive constant K_1 depending only on δ.
Throughout this paper, $\xi_1, \xi_2, \cdots$ is stationary dependent Gaussian sequence and $M_n = \max_{1\le i\le n}\xi_i$, $M_{k,n} = \max_{k+1\le i\le n}\xi_i$. Let $r_n = Cov(\xi_1, \xi_{n+1})$. If

$$r_n \log n \to r \ge 0, \text{ as } n \to \infty. \tag{4}$$

$\xi_1, \xi_2, \cdots$ was called as dependent: weakly dependent for $r = 0$ and strongly dependent for $r > 0$. Let

$$\rho_n = \frac{r}{\log n}, r \text{ defined in } (4). \tag{5}$$

In the paper, a very natural and mild assumption is

$$|r_n - \rho_n| \log n (\log\log n)^{1+\varepsilon} = O(1). \tag{6}$$

We mainly consider the ASCLT of the maximum of stationary Gaussian sequence satisfying (4), under the mild condition (6), which is crucial to consider other versions of the ASCLT such as that of the maximum of non-stationary strongly dependent sequence and the function of the maximum. In the sequel, $a = O(b)$ is denoted by $a \ll b$, C is a constant which may change from line to line. The main result is as follows.
Theorem. Let $\{\xi_n\}$ be a sequence of stationary standard Gaussian random variables with covariances $r_{ij} = r_{|j-i|}$ satisfying (4). $M_k = \max_{i\le k}\xi_i$. The definitions of a_n, b_n is the same as in Theorem A. Assume $r_{ij} = r_{|j-i|}$ satisfies (6). Then

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k} I\Big(a_k(M_k - b_k) \le x\Big) = \int_{-\infty}^{+\infty} \exp\Big(-e^{-x-r+\sqrt{2r}z}\Big)\phi(z)dz \quad a.s.. \tag{7}$$

Remark 1. When $r = 0$, clearly, Theorem induces Theorem A. When $r > 0$, $\xi_1, \xi_2, \cdots$ is strongly dependent. We mainly focus on the proof of Theorem 1 for this case.
Remark 2. In the above definition of ρ_n, when $n = 1$, the definition is incompatible. In the paper, we mainly consider the case of $n \to \infty$. So here, n may be assumed in a neighborhood of $+\infty$ and the incompatibility doesn't result in the invalidation of our argument.

2 Auxiliary lemmas

In this section, we present and prove some lemmas which are useful in our proof of the main result.

Lemma 2.1. Assume $|r_n-\rho_n|\log n(\log\log n)^{1+\varepsilon}=O(1)$. Let the constants u_n be such that $n(1-\Phi(u_n))$ is bounded where Φ is standard normal distribution function. Then

$$\sup_{1\le k\le n} k\sum_{j=1}^{n}|r_j-\rho_n|\exp\left(-\frac{u_k^2+u_n^2}{2(1+|\omega_j|)}\right)\ll(\log\log n)^{-(1+\varepsilon)}, \tag{8}$$

and

$$\sup_{1\le k\le n} k\sum_{j=1}^{n}|r_j-\rho_n|\exp\left(-\frac{u_n^2}{1+|\omega_j|}\right)\ll(\log\log n)^{-(1+\varepsilon)}, \tag{9}$$

where $\omega_j=\max\{|r_j|,\rho_n\}$.

Proof. The proof of (8). According to Leadbetter et al.[LLR83], we have $\sigma_1(k):=\sup_{k<m\le n}|r_m|<1$ when $r_n\to 0$. By assumption, we have $\sigma_2(k):=\sup_{k<m\le n}|\rho_m|<1$. Therefore, we have $\sigma(k):=\sup_{k<m\le n}|\omega_m|<1$. By assumption again, $n(1-\Phi(u_n))\le K$ for some constant $K>0$. Define $\{v_n\}$ by $v_n=u_n$, if $n\le K$ and $n(1-\Phi(v_n))=K$, if $n>K$. Then clearly $u_n\ge v_n$ and hence

$$k\sum_{j=1}^{n}|r_j-\rho_n|\exp\left(-\frac{u_k^2+u_n^2}{2(1+|\omega_j|)}\right)\le k\sum_{j=1}^{n}|r_j-\rho_n|\exp\left(-\frac{v_k^2+v_n^2}{2(1+|\omega_j|)}\right). \tag{10}$$

Then it is sufficient to prove (8) for the sequence $\{v_n\}$. Using a usual fact

$$1-\Phi(x)\sim\frac{\phi(x)}{x}, x\to\infty, \tag{11}$$

we can write that

$$\exp\left(-\frac{v_n^2}{2}\right)\sim\frac{K\sqrt{2\pi}v_n}{n}, v_n\sim(2\log n)^{1/2} \tag{12}$$

Define α to be $0<\alpha<(1-\sigma(0))/(1+\sigma(0))$. Write

$$\begin{aligned}
&k\sum_{j=1}^{n}|r_j-\rho_n|\exp\left(-\frac{v_k^2+v_n^2}{2(1+|\omega_j|)}\right)\\
&=k\sum_{1\le j\le[n^\alpha]}|r_j-\rho_n|\exp\left(-\frac{v_k^2+v_n^2}{2(1+|\omega_j|)}\right)+k\sum_{[n^\alpha]<j\le n}|r_j-\rho_n|\exp\left(-\frac{v_k^2+v_n^2}{2(1+|\omega_j|)}\right)\\
&=T_1+T_2.
\end{aligned} \tag{13}$$

Using (12), we have

$$\begin{aligned}
T_1&\le kn^\alpha\exp\left(-\frac{v_k^2+v_n^2}{2(1+\sigma(0))}\right)=kn^\alpha\left(\exp\left(-\frac{v_k^2+v_n^2}{2}\right)\right)^{1/(1+\sigma(0))}\\
&\ll kn^\alpha\left(\frac{v_kv_n}{kn}\right)^{1/(1+\sigma(0))}\ll k^{1-1/(1+\sigma(0))}n^{\alpha-1/(1+\sigma(0))}(\log k\log n)^{1/2(1+\sigma(0))}\\
&\ll n^{1+\alpha-2/(1+\sigma(0))}(\log n)^{1/(1+\sigma(0))}.
\end{aligned} \tag{14}$$

Since $1+\alpha-2/(1+\sigma(0))<0$, we know $T_1 \ll n^{-\delta}$, for some $\delta>0$, uniformly for $1\le k\le n$. For the estimation of the bound of T_2, we can let $p=[n^\alpha]$. We have

$$\begin{aligned}T_2 &\le k\exp\left(-\frac{v_k^2+v_n^2}{2(1+\sigma(p))}\right)\sum_{p+1\le j\le n}|r_j-\rho_n| \\ &= \frac{nk}{\log n}\exp\left(-\frac{v_k^2+v_n^2}{2(1+\sigma(p))}\right)\frac{\log n}{n}\sum_{p+1\le j\le n}|r_j-\rho_n|. \end{aligned} \tag{15}$$

As $r_n\log n\to r$, there must be a constant C such that $r_n\log n\le C$, for $n\ge 1$. Using (12), similarly to the proof of Lemma 6.4.1 in Leadbetter et al.[LLR83], it can be shown

$$\begin{aligned}&\frac{nk}{\log n}\exp\left(-\frac{v_k^2+v_n^2}{2(1+\sigma(p))}\right)\\ &\le \frac{nk}{\log n}\exp\left(-\frac{v_k^2+v_n^2}{2(1+C/\log n^\alpha)}\right)\\ &= \frac{nk}{\log n}\left(\exp\left(-\frac{v_k^2}{2}\right)\right)^{1/(1+C/\log n^\alpha)}\left(\exp\left(-\frac{v_n^2}{2}\right)\right)^{1/(1+C/\log n^\alpha)}\\ &\ll C\left(\frac{n^2}{\log n}\left(\frac{v_n}{n}\right)^{2/(1+C/\log n^\alpha)}\right)^{1/2}\left(\frac{k^2}{\log k}\left(\frac{v_k}{k}\right)^{2/(1+C/\log n^\alpha)}\right)^{1/2}\\ &\ll Cn^{(C/\log n^\alpha)/(1+C/\log n^\alpha)} = O(1),\end{aligned} \tag{16}$$

and

$$\frac{\log n}{n}\sum_{p+1\le j\le n}|r_j-\rho_n| \le \frac{1}{\alpha n}\sum_{p+1\le j\le n}|r_j\log j - r| + r\frac{1}{n}\sum_{p+1\le j\le n}\left|1-\frac{\log n}{\log j}\right|. \tag{17}$$

Consider the first term on the right-hand side, using (6), we have

$$\frac{1}{\alpha n}\sum_{p+1\le j\le n}|r_j\log j-r| \ll \frac{1}{\alpha n}\sum_{p+1\le j\le n}(\log\log j)^{-(1+\varepsilon)} \ll (\log\log n)^{-(1+\varepsilon)}. \tag{18}$$

According to Leadbetter et al.[LLR83](page 135), we can write

$$\frac{1}{n}\sum_{p+1\le j\le n}\left|1-\frac{\log n}{\log j}\right| = O\left(\frac{1}{\alpha\log n}\int_0^1|\log x|dx\right) \ll (\log\log n)^{-(1+\varepsilon)}. \tag{19}$$

Combining (15), (16), (17), (18)and (19), we have $T_2 \ll (\log\log n)^{-(1+\varepsilon)}$. Clearly (8) follows, (9) does similarly. The proof is completed.

Lemma 2.2. Let $\tilde{\xi}_1,\tilde{\xi}_2,\cdots,\tilde{\xi}_n$ be standard stationary Gaussian variables with constant covariance $\rho_n = r/\log n$ and $\xi_1,\xi_2,\cdots,\xi_n$ satisfy the conditions of the Theorem. Denote $\tilde{M}_n=\max_{i\le n}\tilde{\xi}_i$ and $M_n=\max_{i\le n}\xi_i$. Assume $n(1-\Phi(u_n))$ is bounded and (6) is satisfied. Then

$$|E(I(M_n\le u_n)-I(\tilde{M}_n\le u_n))| \ll (\log\log n)^{-(1+\varepsilon)} \tag{20}$$

Proof. Using Theorem C and Lemma 2.1, the proof can be gained simply.

Lemma 2.3. Let $\eta_1, \eta_2, \cdots$ be a sequence of bounded random variables. If

$$Var\left(\sum_{k=1}^{n}\frac{1}{k}\eta_k\right) \ll \log^2 n(\log\log n)^{-(1+\varepsilon)} \text{ for some } \varepsilon > 0, \tag{21}$$

then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}(\eta_k - E\eta_k) = 0 \quad a.s. \tag{22}$$

Proof. The proof can be found in Csáki and Gonchigdanzan[CG02].

3 Proof of main result

The proof of Theorem. When $a_n = (2\log n)^{1/2}$, $b_n = (2\log n)^{1/2} - \frac{1}{2}(2\log n)^{-1/2}(\log\log n + \log(4\pi))$, we have $u_n = x/a_n + b_n$ satisfying $n(1-\Phi(u_n)) < C$. Under the assumptions, we firstly show

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}(I(M_k \le u_k) - P(M_k \le u_k)) = 0 \quad a.s. \tag{23}$$

Using Lemma 2.3, it is sufficient to prove

$$Var\left(\sum_{k=1}^{n}\frac{1}{k}I(M_k \le u_k)\right) \ll \log^2 n(\log\log n)^{-(1+\varepsilon)} \text{ for some } \varepsilon > 0. \tag{24}$$

Let $\zeta, \zeta_1, \zeta_2, \cdots$ be independent standard normal variables. Obviously $(1-\rho_k)^{1/2}\zeta_1 + \rho_k^{1/2}\zeta$, $(1-\rho_k)^{1/2}\zeta_2 + \rho_k^{1/2}\zeta, \cdots$ have constant covariance $\rho_k = r/\log k$. Define

$$\begin{aligned} M_k(\rho_k) &= \max_{1\le i\le k}((1-\rho_k)^{1/2}\zeta_i + \rho_k^{1/2}\zeta) = (1-\rho_k)^{1/2}\max(\zeta_1, \zeta_2, \cdots, \zeta_k) + \rho_k^{1/2}\zeta \\ &=: (1-\rho_k)^{1/2}M_k(0) + \rho_k^{1/2}\zeta. \end{aligned}$$

Using the well-known c_2- inequality, the left-hand side of (24) can be written as

$$\begin{aligned} &Var\left(\sum_{k=1}^{n}\frac{1}{k}I(M_k \le u_k) - \sum_{k=1}^{n}\frac{1}{k}I(M_k(\rho_k) \le u_k) + \sum_{k=1}^{n}\frac{1}{k}I(M_k(\rho_k) \le u_k)\right) \\ &\le 2\Bigg(Var\left(\sum_{k=1}^{n}\frac{1}{k}I(M_k(\rho_k) \le u_k)\right) \\ &\quad + Var\left(\sum_{k=1}^{n}\frac{1}{k}I(M_k \le u_k) - \sum_{k=1}^{n}\frac{1}{k}I(M_k(\rho_k) \le u_k)\right)\Bigg) \\ &=: L_1 + L_2. \end{aligned}$$

We will show $L_i \ll \log^2 n(\log\log n)^{-(1+\varepsilon)}$, i=1,2. For $i=1$, Write L_1 as

$$\begin{aligned}
&E\left(\sum_{k=1}^{n}\frac{1}{k}(I(M_k(\rho_k)\leq u_k)-P(M_k(\rho_k)\leq u_k))\right)^2\\
&=E\left(\sum_{k=1}^{n}\frac{1}{k}(I(M_k(0)\leq(1-\rho_k)^{-1/2}(u_k-\rho_k^{1/2}\zeta))\right.\\
&\quad\left.-P(M_k(0)\leq(1-\rho_k)^{-1/2}(u_k-\rho_k^{1/2}\zeta))\right)^2\\
&=\int_{-\infty}^{+\infty}E\left(\sum_{k=1}^{n}\frac{1}{k}(I(M_k(0)\leq(1-\rho_k)^{-1/2}(u_k-\rho_k^{1/2}z))\right.\\
&\quad\left.-P(M_k(0)\leq(1-\rho_k)^{-1/2}(u_k-\rho_k^{1/2}z))\right)^2 d\Phi(z)\\
&=:\int_{-\infty}^{+\infty}E\left(\sum_{k=1}^{n}\frac{1}{k}\eta_k\right)^2 d\Phi(z). \qquad (25)
\end{aligned}$$

Write

$$\begin{aligned}
&E\left(\sum_{k=1}^{n}\frac{1}{k}\eta_k\right)^2=\sum_{k=1}^{n}\frac{1}{k^2}E|\eta_k|^2+2\sum_{1\leq k<l\leq n}\frac{|E(\eta_k\eta_l)|}{kl}\\
&=:H_1+H_2. \qquad (26)
\end{aligned}$$

$H_1 \ll \sum_{k=1}^{n}\frac{1}{k^2}<\infty$. For H_2, note

$$\begin{aligned}
|E(\eta_k\eta_l)| &\leq |Cov(I(M_k(0)\leq(1-\rho_k)^{-1/2}(u_k-\rho_n^{1/2}z)),\\
&\qquad I(M_l(0)\leq(1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z))-I(M_{k,l}(0)\leq(1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z))|\\
&\ll E|I(M_l(0)\leq(1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z)-I(M_{k,l}(0)\leq(1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z)|\\
&=P(M_{k,l}(0)\leq(1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z))-P(M_l(0)\leq(1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z))\\
&=\Phi^{l-k}((1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z))-\Phi^{l}((1-\rho_l)^{-1/2}(u_l-\rho_l^{1/2}z))\\
&\leq\frac{k}{l}.
\end{aligned}$$

So, we have

$$H_2\ll\sum_{1\leq k<l\leq n}\frac{1}{kl}\left(\frac{k}{l}\right)\ll\log n\ll(\log n)^2(\log\log n)^{-(1+\varepsilon)}. \qquad (27)$$

Combining (25), (26) and (27), we can get $L_1 \ll \log^2 n(\log\log n)^{-(1+\varepsilon)}$. For $i=2$, write L_2 as

$$\begin{aligned}
& Var\left(\sum_{k=1}^{n}\frac{1}{k}(I(M_k(\rho_k)\le u_k)-I(M_k\le u_k))\right)\\
&\le E\left(\sum_{k=1}^{n}\frac{1}{k}(I(M_k(\rho_k)\le u_k)-I(M_k\le u_k))\right)^2\\
&= E\left(\sum_{k=1}^{n}\frac{1}{k^2}(I(M_k(\rho_k)\le u_k)-I(M_k\le u_k))^2\right)\\
&\quad +2\sum_{1\le i<j\le n}\frac{|E((I(M_i(\rho_i)\le u_i)-I(M_i\le u_i))(I(M_j(\rho_j)\le u_j)-I(M_j\le u_j)))|}{ij}\\
&=: J_1+J_2. \qquad (28)
\end{aligned}$$

Obviously $J_1<\infty$. To estimate J_2, using Lemma 2.2, we have

$$\begin{aligned}
&|E((I(M_i(\rho_i)\le u_i)-I(M_i\le u_i))(I(M_j(\rho_j)\le u_j)-I(M_j\le u_j)))|\\
&\le |E(I(M_j(\rho_j)\le u_j)-I(M_j\le u_j))| \ll (\log\log j)^{-(1+\varepsilon)}.
\end{aligned}$$

So

$$\begin{aligned}
J_2 &\ll \sum_{j=3}^{n}\frac{1}{j(\log\log j)^{1+\varepsilon}}\sum_{i=1}^{j-1}\frac{1}{i} \ll \sum_{j=3}^{n}\frac{\log j}{j(\log\log j)^{1+\varepsilon}}\\
&\ll \log n\sum_{j=3}^{n}\frac{1}{j(\log\log j)^{1+\varepsilon}} \ll (\log n)^2(\log\log n)^{-(1+\varepsilon)}. \qquad (29)
\end{aligned}$$

Combining (28) and (29) induces $L_2 \ll \log^2 n(\log\log n)^{-(1+\varepsilon)}$.
Secondly, according to Leadbetter et al.[LLR83](page 136), we have $P\{a_n(M_n-b_n)\le x\}\to \int_{-\infty}^{\infty}\exp(-e^{-x-r+\sqrt{2r}z})\phi(z)dz$, as $n\to\infty$. Clearly this induces

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}P(M_k\le u_k)\to\int_{-\infty}^{\infty}\exp(-e^{-x-r+\sqrt{2r}z})\phi(z)dz\ a.s.,$$

as $n\to\infty$. The conclusion follows.

Acknowledgements The author would like to thank the referee for his careful reading of the manuscript and many constructive comments which improving the paper greatly and the editor for his careful working.

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