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SOME LIL TYPE RESULTS ON THE PARTIAL SUMS AND TRIMMED SUMS WITH MULTIDIMENSIONAL INDICES

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Abstract
Let $\{X, X_{\mathrm{n}}; n \in \mathrm{N}^d\}$ be a field of i.i.d. random variables indexed by d-tuples of positive integers and let $S_{\mathrm{n}} = \sum_{\mathrm{k} \le \mathrm{n}} X_{\mathrm{k}}$. We prove some strong limit theorems for S_{n}. Also, when $d \ge 2$ and $h(\mathrm{n})$ satisfies some conditions, we show that there are no LIL type results for $S_{\mathrm{n}}/\sqrt{|\mathrm{n}|h(\mathrm{n})}$.

1 Introduction and main results

Let N^d be the set of d-dimensional vectors $\mathrm{n} = (n_1, \dots, n_d)$ whose coordinates $n_1, \dots, n_d$ are natural numbers. The symbol $\le$ means coordinate-wise ordering in N^d. For $\mathrm{n} \in \mathrm{N}^d$, we define $|\mathrm{n}| = \prod_{i=1}^d n_i$. Let X be a random variable, $c(x)$ be a non-decreasing function and $\mathcal{F}(x) = \mathsf{P}(|X| \ge x)$, $B(x) = \mathrm{inv}c(x) := \sup\{t > 0 : c(t) < x\}$, $\psi(x) = (B(x)/\mathcal{F}(x))^{1/2}$, $\phi(x) = \mathrm{inv}\psi(x)$. For $\mathrm{n} \in \mathrm{N}^d$, we define $c_{\mathrm{n}} = c(|\mathrm{n}|)$, $h(\mathrm{n}) = h(|\mathrm{n}|)$, etc.
The present paper proves some strong limit theorems for the partial sums with multidimensional indices. Before we state our main results, some previous work should be introduced. Let $\{X, X_n; n \ge 1\}$ be a sequence of real-valued independent and identically distributed (*i.i.d.*) random variables, and let $S_n = \sum_{i=1}^n X_i$, $n \ge 1$. Define $Lx = \log_e \max\{e, x\}$ and $LLx = L(Lx)$ for $x \in R$. The classical Hartman-Wintner law of the iterated logarithm states that

$$\limsup_{n\to\infty} \frac{\pm S_n}{\sqrt{2nLLn}} = \sigma \quad a.s.$$

if and only if $\mathsf{E}X = 0$ and $\sigma^2 = \mathsf{E}X^2 < \infty$. Starting with the work of Feller (1968) there has been quite some interest in finding extensions of the Hartman-Wintner LIL to the infinite variance case. To cite the relevant work on the two sided LIL behavior for real-valued random variables, let us first recall some definitions introduced by Klass (1976). As above let $X : \Omega \to \mathrm{R}$ be a random variable and assume that $0 < \mathsf{E}|X| < \infty$. Set

$$H(t) := \mathsf{E}X^2 I\{|X| \le t\} \quad \text{and} \quad M(t) := \mathsf{E}|X| I\{|X| > t\}, t \ge 0.$$

Then it is easy to see that the function

$$G(t) := t^2/(H(t) + tM(t)), t > 0$$

is continuous and increasing and the function K is defined as its inverse function. Moreover, one has for this function K that as $x \nearrow \infty$

$$K(x)/\sqrt{x} \nearrow (\mathsf{E}X^2)^{1/2} \in]0, \infty] \tag{1.1}$$

and

$$K(x)/x \searrow 0. \tag{1.2}$$

Set $\gamma_n = \sqrt{2}K(n/LLn)LLn$. Klass (1976, 1977) established a one-sided LIL result with respect to this sequence which also implies the two-sided LIL result if $\mathsf{E}X = 0$,

$$\limsup_{n\to\infty} |S_n|/\gamma_n = 1 \quad \text{a.s.} \tag{1.3}$$

if and only if

$$\sum_{n=1}^{\infty} \mathsf{P}(|X| \geq \gamma_n) < \infty. \tag{1.4}$$

But since it can be quite difficult to determine $\{\gamma_n\}$ and (1.4) may be not satisfied, Einmahl and Li (2005) addressed the following modified forms of the LIL behavior problem.
PROBLEM 1 Give a sequence, $a_n = \sqrt{nh(n)}$, where h is a slowly varying non-decreasing function, we ask: When do we have with probability 1, $0 < \limsup_{n\to\infty} |S_n|/a_n < \infty$?
PROBLEM 2 Consider a non-decreasing sequence c_n satisfying $0 < \liminf_{n\to\infty} c_n/\gamma_n < \infty$. When do we have with probability 1, $0 < \limsup_{n\to\infty} |S_n|/c_n < \infty$? If this is the case, what is the cluster set $C(\{S_n/c_n; n \geq 1\})$?
Theorem 1 and Theorem 3 in Einmahl and Li (2005) solved the problems above. The reader is also referred to their paper for some other references on LIL.
Now, let $\{X, X_{\mathrm{n}}, \mathrm{n} \in \mathrm{N}^d\}$ be *i.i.d.* random variables and $d \geq 2$. It is interesting to ask whether there are some two-sided LIL behavior for $S_{\mathrm{n}} = \sum_{\mathrm{k}\leq\mathrm{n}} X_{\mathrm{k}}$ $(d \geq 2)$ with finite expectation and infinite variance. For example, does the two-sided Klass LIL still hold for S_{n} when $d \geq 2$? The following one of main results of the present paper answers this question.

Theorem 1.1. *Let $d \geq 2$. We have*

$$\limsup_{\mathrm{n}\to\infty} \frac{|S_{\mathrm{n}}|}{\gamma_{\mathrm{n}}} = \begin{cases} \infty \text{ a.s.} & \text{if } \mathsf{E}X^2(\log|X|)^{d-1}/\log_2|X| = \infty \\ \sqrt{d} \text{ a.s.} & \text{if } \mathsf{E}X^2(\log|X|)^{d-1}/\log_2|X| < \infty \end{cases}.$$

Remark 1.1. Here and below, γ_{n} denotes $\gamma_{|\mathrm{n}|}$. Also, from Theorem 1.1, we see that for $d \geq 2$,

$$\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} = \sqrt{d} \text{ a.s.}$$

if and only if

$$\mathsf{E}X = 0 \quad \text{and} \quad \mathsf{E}X^2(\log|X|)^{d-1}/\log_2|X| < \infty.$$

This says that the two-sided Klass LIL is reduced to Wichura's LIL (Wichura(1973)).
The proof of Theorem 1.1 is based on the following Theorem 1.2, which says that in general there is no two-sided LIL behavior for $S_{\mathrm{n}} = \sum_{\mathrm{k}\leq\mathrm{n}} X_{\mathrm{k}}$ $(d \geq 2)$ with a wide class of normalizing sequences if the variance is infinite.
Let the function $c(x)$, $c_n = c(n)$ satisfy the following conditions.

$$c_n/\sqrt{n} \nearrow \infty, \tag{1.5}$$

$$\forall\varepsilon > 0, \exists\, m_\varepsilon > 0: \quad c_n/c_m \leq (1+\varepsilon)(n/m), \quad n \geq m \geq m_\varepsilon. \tag{1.6}$$

Theorem 1.2. *Let $d \geq 2$ and $c_n = \sqrt{nh(n)}$ satisfy (1.5) and (1.6). Moreover, suppose that $h(n)$ satisfies*

$$\frac{LLn}{h(n)} \max_{1\leq i\leq n} \frac{h(i)}{(Li)^{d-1}} = o(1) \quad \text{as } n \to \infty. \tag{1.7}$$

Then, the following statements are equivalent:
(1). we have

$$\mathsf{E}X = 0, \quad \sum_{n=1}^{\infty} (Ln)^{d-1} \mathsf{P}\Big(|X| \geq \sqrt{nh(n)}\Big) < \infty; \tag{1.8}$$

(2). we have

$$\limsup_{\mathrm{n}\to\infty} \frac{|S_{\mathrm{n}}|}{\sqrt{|\mathrm{n}|h(\mathrm{n})}} < \infty \text{ a.s.}; \tag{1.9}$$

(3). we have

$$\limsup_{\mathrm{n}\to\infty} \frac{|S_{\mathrm{n}}|}{\sqrt{|\mathrm{n}|h(\mathrm{n})}} = 0 \text{ a.s.} \tag{1.10}$$

Remark 1.2: Now, we take a look at the condition (1.7). We claim that $h(n)$ satisfies (1.7) when $LLn/h(n) \searrow 0$ as $n \to \infty$. To see this, we let $N(\varepsilon)$ denote an integer such that $LLn/(Ln)^{d-1} \leq \varepsilon$ when $n \geq N(\varepsilon)$. Then, we have

$$\begin{aligned}
&\frac{LLn}{h(n)} \max_{1\leq i\leq n} \frac{h(i)}{(Ln)^{d-1}} \leq \frac{LLn}{h(n)} \max_{1\leq i\leq N(\varepsilon)} \frac{h(i)}{(Li)^{d-1}} + \frac{LLn}{h(n)} \max_{N(\varepsilon)\leq i\leq n} \frac{h(i)}{(Li)^{d-1}} \\
&\leq \frac{LLn}{h(n)} \max_{1\leq i\leq N(\varepsilon)} \frac{h(i)}{(Li)^{d-1}} + \frac{LLn}{h(n)} \max_{N(\varepsilon)\leq i\leq n} \frac{h(i)}{LLi} \frac{LLi}{(Li)^{d-1}} \\
&\leq \frac{LLn}{h(n)} \max_{1\leq i\leq N(\varepsilon)} \frac{h(i)}{(Li)^{d-1}} + \varepsilon \to 0,
\end{aligned}$$

as $n \to \infty$, $\varepsilon \to 0$. Theorem 1.2 can also be seen as a supplement to the Marcinkiewicz strong law of large numbers for multidimensional indices ($d \geq 2$). For example, we can take $h(x) = (LLx)^r$, $r > 1$, $h(x) = (Lx)^r$, $r > 0$ and $h(x) = \exp((Lx)^\tau)$, $0 < \tau < 1$ etc. Some other known results, such as some results of Smythe (1973), Gut (1978, 1980) and Li (1990), are reobtained by Theorem 1.2. Here we only introduce the results by Li (1990). Let $\mathcal{Q}$ be the class of positive non-decreasing and continuous functions g defined on $[0, \infty)$ such that for some constant $K(g) > 0$, $g(xy) \leq K(g)(g(x) + g(y))$ for all $x, y > 0$ and $x/g(x)$ is non-decreasing whenever x is sufficiently large. If $g \in \mathcal{Q}$ and $d \geq 2$, Li (1990) showed that if $g(x) \nearrow \infty$, then

$$\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}| / \sqrt{|\mathrm{n}| g(|\mathrm{n}|) L_2 |\mathrm{n}|} < \infty \quad \text{a.s.}$$

if and only if

$$\mathsf{E}X = 0, \mathsf{E}X^2 (L|X|)^{d-1} / (g(|X|) L_2 |X|) < \infty.$$

Remark 1.3: We see from Theorem 1.1 that the Klass LIL does not hold when the variance is infinite and $d \geq 2$. So it is interesting to find other normalizing sequences instead of γ_{n}. But this seems too difficult to find them. Also, from Theorem 1.2, we see that many two sided LIL results for the sum of a sequence of random variables do not hold for the sum of a field of random variables ($d \geq$2). This is because that condition (1.8) usually implies $\alpha_0 = 0$, where α_0 is defined in Theorem 2.1 below. Of course, there maybe exist a random variable X with infinite variance and a normalizing sequence $\sqrt{nh(n)}$ such that condition (1.8) holds and $0 < \alpha_0 < \infty$ when $d \geq 2$. However, it seems too difficult to find them. Instead, we give the following theorem, which is an answer to PROBLEM 1 when S_n is replaced by S_{n}, $d \geq 2$.

Theorem 1.3. *Let $d \geq 2$. Suppose that $h(x)$ is a slowly varying non-decreasing function. Then we have*

$$0 < \limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}| / \sqrt{|\mathrm{n}| h(\mathrm{n})} < \infty \quad \text{a.s.} \tag{1.11}$$

if and only if (1.8) holds and

$$0<\lambda:=\limsup_{x\to\infty}\frac{\Psi^{-1}(xLLx)}{x^2LLx}H(x)<\infty, \tag{1.12}$$

where $H(x)=\mathsf{E}X^2I\{|X|\le x\}$ *and* $\Psi(x)=\sqrt{xh(x)}$.

Remark 1.4. We refer the reader to Einmahl and Li (2005) for some similar conditions as (1.12). We can see from (3.1) that λ is usually equal to 0 under (1.8).

The remaining part of the paper is organized as follows. In Section 2, we state and prove a general result on the LIL for the trimmed sums, from which our main results in Section 1 can be obtained. In Section 3, Theorems 1.1-1.3 are proved. Throughout, C denotes a positive constant and may be different in every place.

2 Some LIL results for trimmed sums

In this section, we prove a slightly more general theorem. Moreover, we will see that if some "maximal" random variables are removed from S_{n}, the two sided LIL for $d\ge 2$ may hold again. Now we introduce some notations. For an integer $r\ge 1$ and $|\mathrm{n}|\ge r$, let $X_{\mathrm{n}}^{(r)}=X_{\mathrm{m}}$ if $|X_{\mathrm{m}}|$ is the r-th maximum of $\{|X_{\mathrm{k}}|;\mathrm{k}\le\mathrm{n}\}$ (0 if $r>|\mathrm{n}|$). Let $S_{\mathrm{n}}=\sum_{\mathrm{k}\le\mathrm{n}}X_{\mathrm{k}}$ and ${}^{(r)}S_{\mathrm{n}}=S_{\mathrm{n}}-(X_{\mathrm{n}}^{(1)}+\cdots+X_{\mathrm{n}}^{(r)})$ (0 if $r>|\mathrm{n}|$) be the trimmed sums. ${}^{(0)}S_{\mathrm{n}}$ is just S_{n}. Let $L_q^{(d)}$ denote the space of all real random variables X such that

$$J_q^{(d)}:=\int_0^\infty (Lt)^{d-1}\big(t\mathsf{P}\big(|X|>t\big)\big)^q\frac{dt}{t}<\infty.$$

And let $B(x):=c^{-1}(x)$ denote the inverse function of $c(x)$. Throughout the whole section we assume that $c(x)$ is an non-decreasing function and $\{c_n\}$ is a sequence of positive real numbers satisfying conditions (1.5) and (1.6). Finally, let $C_n:=n\mathsf{E}XI\{|X|\le c_n\}$.

Theorem 2.1 *Let* $d\ge 2$, $r\ge 0$. *Suppose that* $B(|X|)\in L_{r+1}^{(d)}$. *Set*

$$\alpha_0=\sup\Big\{\alpha\ge 0:\sum_{n=1}^\infty n^{-1}(Ln)^{d-1}\exp\Big(-\frac{\alpha^2c_n^2}{2n\sigma_n^2}\Big)=\infty\Big\},$$

where $\sigma_n^2=H(\delta c_n)=\mathsf{E}X^2I\{|X|\le\delta c_n\}$ *and* $\delta>0$. *Then we have with probability 1,*

$$\limsup_{\mathrm{n}\to\infty}|{}^{(r)}S_{\mathrm{n}}-C_{\mathrm{n}}|/c_{\mathrm{n}}=\alpha_0. \tag{2.1}$$

Remark 2.1: (The Feller and Pruitt example). Let $\{X,X_{\mathrm{n}},\mathrm{n}\in\mathrm{N}^d\}$ $(d\ge 2)$ be *i.i.d.* random variables with the common symmetric probability density function

$$f(x)=\frac{1}{|x|^3}I\{|x|\ge 1\}.$$

We have $H(x)=\log x$, $x\ge 1$ and chose $c_n=\sqrt{nLnLLn}$. One can easily check that $B(|X|)\in L_{r+1}^{(d)}$ when $r\ge(d-1)$, and $\sigma_n^2\sim 2^{-1}Ln$ as $n\to\infty$. Moreover, by Lemma 2.2 below, we have $C_n=o(c_n)$. So, if $r\ge(d-1)$, with probability 1,

$$\limsup_{\mathrm{n}\to\infty}|{}^{(r)}S_{\mathrm{n}}|/\sqrt{|\mathrm{n}|(L\mathrm{n})LL\mathrm{n}}=\sqrt{d}.$$

Remark 2.2. We continue to consider the Feller and Pruitt example. Let $\{X,X_{\mathrm{n}},\mathrm{n}\in\mathrm{N}^d\}$ $(d\ge 2)$ be defined in Remark 2.1. Is there any sequence $c_n=\sqrt{nh(n)}$ satisfying (1.5) and (1.6) such that $0<\limsup_{\mathrm{n}\to\infty}|S_{\mathrm{n}}|/c_{\mathrm{n}}<\infty$ a.s. ? The answer is negative. We will prove that for any sequence $c_n=\sqrt{nh(n)}$ satisfying (1.5) and (1.6), $\limsup_{\mathrm{n}\to\infty}|S_{\mathrm{n}}|/c_{\mathrm{n}}<\infty$ a.s. implies $\limsup_{\mathrm{n}\to\infty}|S_{\mathrm{n}}|/c_{\mathrm{n}}=0$ a.s. To prove this, we should first note that $\limsup_{\mathrm{n}\to\infty}|S_{\mathrm{n}}|/c_{\mathrm{n}}<\infty$ a.s. implies $\sum_{\mathrm{n}\in\mathrm{N}^d}\mathsf{P}(|X|\ge c_{\mathrm{n}})<\infty$ by the Borel-Cantelli lemma. So $\sum_{n=1}^\infty(Ln)^{d-1}\mathsf{P}(|X|\ge c_n)<\infty$. And since $\mathsf{P}(|X|\ge x)=x^{-2}$

for $|x| > 1$, we have $\sum_{n=1}^{\infty}(Ln)^{d-1}/(nh(n)) < \infty$. This implies $\sum_{i=1}^{\infty} i^{d-1}/h(2^i) < \infty$. Hence $\sum_{i=n}^{2n} i^{d-1}/h(2^i) = o(1)$. It follows that $n^d = o(h(2^{2n}))$ which in turn implies $h(n) \geq (Ln)^d$ for n large. Note that $\sigma_n^2 \sim 2^{-1}Ln$. So $\alpha_0 = 0$. We end the proof by Theorem 2.1 and the fact $C_n = o(c_n)$, implied by Lemma 2.2 below.

To prove Theorem 2.1, we need the following lemmas. Recall the functions $\mathcal{F}(x)$ and $\phi(x)$ defined in Section 1.

Lemma 2.1. *$B(|X|) \in L^d_{r+1}$ if and only if*

$$\int_0^{\infty} (Lt)^{d-1}\Big(tP\Big(|X| > \varepsilon c_t\Big)\Big)^{r+1}\frac{dt}{t} < \infty \quad (\forall \varepsilon > 0).$$

And if $B(|X|) \in L^d_{r+1}$, then for $k > 2 + 2r$ and any $\delta > 0$

$$\int_0^{\infty} (Lt)^{d-1}t^{k-1}\mathcal{F}^k(\phi(\delta t))dt < \infty,$$

and for Q large enough (say $Q > 4 + 4r$),

$$\int_0^{\infty} x^{-1}(Lt)^{d-1}\Big(\frac{\phi(x)}{c(x)}\Big)^Q dx < \infty.$$

Proof. See Zhang (2002).

Lemma 2.2. *If $B(|X|) \in L^d_{r+1}$, then for any $\tau > 0$ and $\beta > 2$*

$$\mathsf{E}|X|I\{\phi(n) \leq |X| \leq c_n\} = o(c_n/n) \quad \text{as } n \to \infty, \tag{2.2}$$

$$\mathsf{E}|X|^{\beta}I\{|X| \leq \tau c_n\} = o(c_n^{\beta}/n) \quad \text{as } n \to \infty. \tag{2.3}$$

If $B(|X|) \in L^d_{r+1}$ and $c_n/c_m \leq C(n/m)^{\mu}$, $n \geq m$, where $\mu = (1+r)^{-1} \vee \nu$, for some $0 < \nu < 1$, then

$$\mathsf{E}|X|I\{|X| \geq \phi(n)\} = o(c_n/n) \quad \textit{as } n \to \infty, \tag{2.4}$$

Proof. We prove (2.4) first. If $r = 0$, then $\mu = 1$. So

$$\begin{aligned}
&c_n^{-1}n\mathsf{E}|X|I\{|X| \geq \phi(n)\} \\
&\leq n\mathcal{F}(\phi(n)) + c_n^{-1}n\mathsf{E}|X|I\{|X| \geq c_n\} \\
&\leq n\mathcal{F}(\phi(n)) + c_n^{-1}n\sum_{j=n}^{\infty} c_j\mathsf{P}(c_{j-1} < |X| \leq c_j) \\
&\leq n\mathcal{F}(\phi(n)) + \sum_{j=n}^{\infty} j\mathsf{P}(c_{j-1} < |X| \leq c_j) \\
&= o(1).
\end{aligned}$$

If $r > 0$, then $\mu < 1$, and

$$\begin{aligned}
&c_n^{-1}n\mathsf{E}|X|I\{|X| \geq \phi(n)\} \\
&\leq c_n^{-1}n\sum_{j=n}^{\infty} c_j\mathsf{P}\Big(\phi(j-1) < |X| \leq \phi(j)\Big) \\
&\leq Cn\sum_{j=n}^{\infty} \frac{j^{\mu}}{n^{\mu}}\mathsf{P}\Big(\phi(j-1) < |X| \leq \phi(j)\Big) \\
&\leq Cn\mathcal{F}(\phi(n-1)) + Cn^{1-\mu}\sum_{j=n}^{\infty} j^{\mu-1}\mathcal{F}(\phi(j)) \\
&=: J_1 + J_2.
\end{aligned}$$

We can infer $J_1 \to 0$ from Lemma 2.1. And

$$J_2 \le Cn^{1-\mu}\sum_{j=n}^{\infty} j^{\mu-2} j\mathcal{F}(\phi(j)) = o(1)n^{1-\mu}\sum_{j=n}^{\infty} j^{\mu-2} = o(1).$$

Therefore, (2.4) is true.
The proof of (2.2) is easy. So we omit it. Now we prove (2.3). By (1.5),

$$\begin{aligned}
&c_n^{-\beta} n\mathsf{E}|X|^\beta I\{|X| \le \tau c_n\}\\
&\le c_n^{-\beta} n \sum_{j=1}^{n} c_n^\beta \mathsf{P}(\tau c_{j-1} < |X| \le \tau c_j)\\
&\le Cn\sum_{j=1}^{n} n^{-\beta/2} j^{\beta/2}\mathsf{P}(\tau c_{j-1} < |X| \le \tau c_j)\\
&\le Cn\sum_{j=1}^{n} n^{-\beta/2} j^{\beta/2-2} j\mathsf{P}(|X| \ge \tau c_j)\\
&= o(1)n\sum_{j=1}^{n} n^{-\beta/2} j^{\beta/2-2}\\
&= o(1).
\end{aligned}$$

Lemma 2.3. *Define*

$$\alpha_0^{'} = \sup\Big\{\alpha \ge 0 : \sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\exp\Big(-\frac{\alpha^2 c_n^2}{2n\widetilde{\sigma}_n^2}\Big) = \infty\Big\},$$

where $\widetilde{\sigma}_n^2 = EX^2 I\{|X| \le \phi(\delta n)\}$, $\delta > 0$. *Let* α_0 *be defined in Theorem 2.1 and* $B(|X|) \in L_{r+1}^d$. *Then* $\alpha_0 = \alpha_0^{'}$.
Proof. It can be proved by Lemma 2.1 that $\phi(n)/c_n \to 0$. Let $\Delta_n = \mathsf{E}X^2 I\{\phi(\delta n) < |X| \le \delta c_n\}$. For any $\omega > 0$, we have

$$\exp\Big(-\frac{\alpha^2 c_n^2}{2n\sigma_n^2}\Big) \le \exp\Big(-\frac{\alpha^2 c_n^2}{2n(1+\omega)\widetilde{\sigma}_n^2}\Big) + \exp\Big(-\frac{\alpha^2 c_n^2}{2n(1+\omega^{-1})\Delta_n}\Big). \tag{2.5}$$

To see (2.5), we can assume that $\Delta_n \le \sigma_n^2(1+\omega^{-1})^{-1}$, otherwise (2.5) holds spontaneously. But $\Delta_n \le \sigma_n^2(1+\omega^{-1})^{-1}$ implies $\widetilde{\sigma}_n^2(1+\omega) \ge \sigma_n^2$, we see that (2.5) is always right. By Lemma 2.1 and the trivial inequality $\exp(-x) \le Cx^{-Q}$ for any $Q > 0$ when x large enough, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\exp\Big(-\frac{\alpha^2 c_n^2}{2n(1+\omega^{-1})\Delta_n}\Big) \le C\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\Big(\frac{n\Delta_n}{c_n^2}\Big)^Q\\
&\le C\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\Big(n\mathcal{F}(\phi(\delta n))\Big)^Q < \infty.
\end{aligned}$$

Therefore, $\alpha_0 \le \sqrt{1+\omega}\alpha_0^{'}$. It is obvious that $\alpha_0^{'} \le \alpha_0$. Since we can choose ω arbitrarily small, we see that $\alpha_0^{'} = \alpha_0$. □
Lemma 2.4. *Let* $n_j = [(1+\varepsilon)^j]$, $j \ge 1$, $\varepsilon > 0$. *Suppose that* $B(|X|) \in L_{r+1}^d$. *Then we have:*

$$\sum_{j=1}^{\infty} j^{d-1}\exp\Big(-\frac{\alpha^2 c_{n_j}^2}{2n_j\widetilde{\sigma}_{n_j}^2}\Big)\begin{cases} = \infty & \text{if } \alpha < \alpha_0\\ < \infty & \text{if } \alpha > \alpha_0\end{cases}.$$

Proof. Let $\alpha < \alpha_0$. We have

$$\infty \;=\; \sum_{j=j_0}^{\infty}\sum_{n=n_j+1}^{n_{j+1}} n^{-1}(Ln)^{d-1}\exp\Big(-\frac{\alpha^2 c_n^2}{2n\widetilde{\sigma}_n^2}\Big) \le C\sum_{j=j_0}^{\infty} j^{d-1}\exp\Big(-\frac{\alpha^2 c_{n_j}^2}{2n_j\widetilde{\sigma}_{n_{j+1}}^2}\Big).$$

Since $n_{j+1}/n_j = O(1)$, and δ is a arbitrary number, we see that

$$\sum_{j=j_0}^{\infty} j^{d-1} \exp\Big(-\frac{\alpha^2 c_{n_j}^2}{2n_j\widetilde{\sigma}_{n_j}^2}\Big) = \infty.$$

Another part of the lemma follows similarly. $\square$

The last lemma comes from Einmahl and Mason [2], p 293.

Lemma 2.5 *Let $X_1, \cdots, X_m$ be independent mean zero random variables satisfying for some $M > 0$, $|X_i| \le M$, $1 \le i \le m$. If the underlying probability space $(\Omega, \Re, P)$ is rich enough, one can define independent normally distributed mean zero random variables $V_1, \ldots, V_m$ with $\mathsf{Var}(V_i) = \mathsf{Var}(X_i)$, $1 \le i \le m$, such that*

$$\mathsf{P}\Big(|\sum_{i=1}^{m}(X_i - V_i)| \ge \delta\Big)) \le c_1 \exp(-c_2\delta/M),$$

here c_1 and c_2 are positive universal constants.

We are ready to prove Theorem 2.1 now.

Proof of Theorem 2.1. First we prove

$$\limsup_{\mathrm{n}\to\infty} |^{(r)}S_{\mathrm{n}} - C_{\mathrm{n}}|/c_{\mathrm{n}} \le \alpha_0 \text{ a.s.} \tag{2.6}$$

Obviously it can be assumed that $\alpha_0 < \infty$. Let $\theta > 1$ and θ^j denote $[\theta^j]$. By the definition of α_0, we can easily show that

$$\sum_{j=1}^{\infty} j^{d-1} \exp\Big(-\frac{2\alpha_0^2 c_{\theta^{j+1}}^2}{2\theta^j\sigma_{\theta^j}^2}\Big) < \infty.$$

So $\theta^j\sigma_{\theta^j}^2/c_{\theta^{j+1}}^2 \to 0$ as $j \to \infty$. This implies $n\sigma_n^2 = o(c_n^2)$. Recall the definition of $\phi(x)$ in Section 1. Throughout the proofs, we let $\theta^{\mathrm{i}} = (\theta^{i_1}, \cdots, \theta^{i_d})$, $\phi(\theta^{\mathrm{i}}) = \phi(|\theta^{\mathrm{i}}|)$ etc. Let

$$S_{1,\mathrm{n}}(\mathrm{i}) = \sum_{\mathrm{k}\le\mathrm{n}} X_{\mathrm{k}} I\{|X_{\mathrm{k}}| \le \phi(\theta^{\mathrm{i}})\}, \quad S_{2,\mathrm{n}}(\mathrm{i}) = \sum_{\mathrm{k}\le\mathrm{n}} X_{\mathrm{k}} I\{|X_{\mathrm{k}}| \le \varepsilon c_{\theta^{\mathrm{i}}}\}, \quad \varepsilon > 0.$$

We have

$$\begin{aligned}
&\sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(\max_{\mathrm{m}\le\theta^{\mathrm{i}}} |^{(r)}S_{\mathrm{m}} - C_{\mathrm{m}}| \ge (\alpha_0 + 6\varepsilon + 3\varepsilon r)c_{\theta^{\mathrm{i}}}\Big)\\
&\le \sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(\max_{\mathrm{m}\le\theta^{\mathrm{i}}} |^{(r)}S_{\mathrm{m}} - S_{2,\mathrm{m}}(\mathrm{i})| \ge \varepsilon r c_{\theta^{\mathrm{i}}}\Big)\\
&\quad + \sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(\max_{\mathrm{m}\le\theta^{\mathrm{i}}} |S_{2,\mathrm{m}}(\mathrm{i}) - S_{1,\mathrm{m}}(\mathrm{i})| \ge \varepsilon(2r+3)c_{\theta^{\mathrm{i}}}\Big)\\
&\quad + \sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(\max_{\mathrm{m}\le\theta^{\mathrm{i}}} |S_{1,\mathrm{m}}(\mathrm{i}) - C_{\mathrm{m}}| \ge (\alpha_0 + 3\varepsilon)c_{\theta^{\mathrm{i}}}\Big)\\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

And

$$\begin{aligned}
I_1 &\le \sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(|X_{\theta^{\mathrm{i}}}^{(r+1)}| \ge \varepsilon c_{\theta^{\mathrm{i}}}\Big) \le \sum_{\mathrm{i}\in\mathrm{N}^d} \Big(|\theta^{\mathrm{i}}|\mathcal{F}(\varepsilon c_{\theta^{\mathrm{i}}})\Big)^{r+1}\\
&\le C\sum_{j=1}^{\infty} j^{d-1}\Big(\theta^j\mathcal{F}(\varepsilon c_{\theta^j})\Big)^{r+1} \le C\sum_{j=1}^{\infty} j^{-1}(Lj)^{d-1}\Big(j\mathcal{F}(\varepsilon c_j)\Big)^{r+1} < \infty,
\end{aligned}$$

$$\begin{aligned}
I_2 &\le \sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(\sharp\{|X_{\mathrm{k}}| \ge \phi(\theta^{\mathrm{i}}); \mathrm{k} \le \theta^{\mathrm{i}}\} \ge 2r+3\Big) \le C\sum_{\mathrm{i}\in\mathrm{N}^d} \Big(|\theta^{\mathrm{i}}|\mathcal{F}(\phi(\theta^{\mathrm{i}}))\Big)^{2r+3}\\
&\le C\sum_{j=1}^{\infty} j^{d-1}\Big(\theta^j\mathcal{F}(\phi(\theta^j))\Big)^{2r+3} \le C\sum_{j=1}^{\infty} j^{-1}(Lj)^{d-1}\Big(j\mathcal{F}(\phi(j))\Big)^{2r+3} < \infty.
\end{aligned}$$

By Lemma 2.2, we have $(S_{1,\theta^{\mathrm{i}}}(\mathrm{i}) - C_{\theta^{\mathrm{i}}}/c_{\theta^{\mathrm{i}}} \to 0$ in probability. Therefore, by a version of the Lévy inequalities (cf. Lemma 2 and Remark 6 in Li and Tomkins (1998)) and (2.2),

$$\begin{aligned} I_3 &\le C\sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(|S_{1,\theta^{\mathrm{i}}}(\mathrm{i}) - \mathsf{E}S_{1,\theta^{\mathrm{i}}}(\mathrm{i})| \ge (\alpha_0+2\varepsilon)c_{\theta^{\mathrm{i}}}\Big) \\ &\le C\sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(|T(\mathrm{i})| \ge (\alpha_0+\varepsilon)c_{\theta^{\mathrm{i}}}\Big) + C\sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(|S_{1,\theta^{\mathrm{i}}}(\mathrm{i}) - \mathsf{E}S_{1,\theta^{\mathrm{i}}}(\mathrm{i}) - T(\mathrm{i})| \ge \varepsilon c_{\theta^{\mathrm{i}}}\Big) \\ &=: I_{31} + I_{32}, \end{aligned}$$

where $T(\mathrm{i}) = \sum_{\mathrm{k}\le\theta^{\mathrm{i}}} Y_{\mathrm{k}}$, and $\{Y_{\mathrm{k}}, \mathrm{k} \le \theta^{\mathrm{i}}\}$ are i.i.d. normal random variables with mean zero and variance $\mathsf{Var}\Big(XI\{|X| \le \phi(\theta^{\mathrm{i}})\}\Big)$, $\mathrm{i} \in \mathrm{N}^d$. Now, by Lemma 2.2 and Lemma 2.5, for q large enough,

$$I_{32} \le C\sum_{j=1}^{\infty} j^{d-1}\Big(\frac{\phi(\theta^j)}{c_{\theta^j}}\Big)^q \le C\sum_{j=1}^{\infty} j^{-1}(Lj)^{d-1}\Big(\frac{\phi(j)}{c_j}\Big)^q < \infty.$$

From the tail probability estimator of the standard normal distribution and Lemma 2.4, we have

$$I_{31} \le C\sum_{\mathrm{i}\in\mathrm{N}^d} \exp\Big(-\frac{(\alpha_0+\varepsilon)^2 c_{\theta^{\mathrm{i}}}^2}{2|\theta^{\mathrm{i}}|H(\phi(\theta^{\mathrm{i}}))}\Big) \le C\sum_{j=1}^{\infty} j^{d-1}\exp\Big(-\frac{(\alpha_0+\varepsilon)^2 c_{\theta^j}^2}{2\theta^j H(\phi(\theta^j))}\Big) < \infty.$$

Then, by the Borel-Cantelli lemma,

$$\limsup_{\mathrm{i}\to\infty} \frac{\max_{\mathrm{m}\le\theta^{\mathrm{i}}} |{}^{(r)}S_{\mathrm{m}} - C_{\mathrm{m}}|}{c_{\theta^{\mathrm{i}}}} \le \alpha_0 \quad \text{a.s.}$$

A standard argument and (1.6) yield

$$\limsup_{\mathrm{n}\to\infty} \frac{|{}^{(r)}S_{\mathrm{n}} - C_{\mathrm{n}}|}{c_{\mathrm{n}}} \le \alpha_0 \quad \text{a.s.}$$

So, we only need to prove

$$\limsup_{\mathrm{n}\to\infty} |{}^{(r)}S_{\mathrm{n}} - C_{\mathrm{n}}|/c_{\mathrm{n}} \ge \alpha_0 \text{ a.s.} \tag{2.7}$$

Case 1: $\alpha_0 < \infty$. To prove (2.7), it is sufficient to show that for every $\varepsilon > 0$, there is a $\theta_0 > 0$ such that when $\theta > \theta_0$,

$$\limsup_{\mathrm{i}\to\infty} \frac{{}^{(r)}S_{\theta^{\mathrm{i}}} - C_{\theta^{\mathrm{i}}}}{c_{\theta^{\mathrm{i}}}} \ge \alpha_0 - \varepsilon \quad \text{a.s.} \tag{2.8}$$

But if we prove that for every $\varepsilon > 0$ and θ large enough,

$$\limsup_{\mathrm{i}\to\infty} \frac{S_{1,\theta^{\mathrm{i}}}(\mathrm{i}) - C_{\theta^{\mathrm{i}}}}{c_{\theta^{\mathrm{i}}}} \ge \alpha_0 - \varepsilon \quad \text{a.s.} \tag{2.9}$$

then, by $I_1 < \infty$ and $I_2 < \infty$, we can see that (2.8) holds. Now we come to prove (2.9). Obviously, it can be assumed that $\alpha_0 > 0$. Let $\mathrm{N_i} = \{\mathrm{n} : \theta^{\mathrm{i}\text{-}1} < \mathrm{n} \le \theta^{\mathrm{i}}\}$, $\mathrm{N}_{\mathrm{i}}^c = \{\mathrm{n} : \mathrm{n} \le \theta^{\mathrm{i}}\} - \mathrm{N_i}$ and

$$S_3(\mathrm{i}) = \sum_{\mathrm{k}\in\mathrm{N_i}} X_{\mathrm{k}} I\{|X_{\mathrm{k}}| \le \phi(\theta^{\mathrm{i}})\}, \quad S_4(\mathrm{i}) = \sum_{\mathrm{k}\in\mathrm{N}_{\mathrm{i}}^c} X_{\mathrm{k}} I\{|X_{\mathrm{k}}| \le \phi(\theta^{\mathrm{i}})\}.$$

Note that $\alpha_0 < \infty$. Just as the proof of $I_3 < \infty$ and by the Borel-Cantelli lemma, we have

$$\limsup_{\mathrm{i}\to\infty} \frac{|S_4(\mathrm{i}) - \mathsf{E}S_4(\mathrm{i})|}{c_{\theta^{\mathrm{i}}}} \le \alpha_0\theta^{-1} \quad \text{a.s.}$$

So, in order to prove (2.9), by the Borel-Cantelli lemma, we only need to show that for every $\varepsilon > 0$ and θ large enough,

$$\sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(\Big|\frac{S_3(\mathrm{i}) - \mathsf{E}S_3(\mathrm{i})}{c_{\theta^{\mathrm{i}}}}\Big| \ge \alpha_0 - \varepsilon\Big) = \infty.$$

By Lemma 2.5 and note that $I_{32} < \infty$, it suffices to prove

$$\sum_{\mathrm{i}\in \mathrm{N}^d} \mathsf{P}\Big(\Big|\frac{T_3(\mathrm{i})}{c_{\theta^{\mathrm{i}}}}\Big| \geq \alpha_0 - \varepsilon\Big) = \infty \tag{2.10}$$

for every $\varepsilon > 0$ and θ large enough, where $T_3(\mathrm{i}) = \sum_{\mathrm{k}\in \mathrm{N}_\mathrm{i}} Y_\mathrm{k}$ and $\{Y_\mathrm{k}, \mathrm{k} \in \mathrm{N}_\mathrm{i}\}$ are i.i.d. normal random variables with mean zero and variance $\mathsf{Var}\Big(XI\{|X| \leq \phi(\theta^{\mathrm{i}})\}\Big)$, $\mathrm{i} \in \mathrm{N}^d$. That is, we shall prove that

$$\sum_{\mathrm{i}\in \mathrm{N}^d} \mathsf{P}\Big(\Big|\frac{H^{'}(\phi(\theta^{\mathrm{i}}))N}{c_{\theta^{\mathrm{i}}}}\Big| \geq \alpha_0 - \varepsilon\Big) = \infty,$$

where $H^{'}(\phi(\theta^{\mathrm{i}})) \sim \Big(|\theta^{\mathrm{i}}|(1-\theta^{-1})^d \mathsf{Var}\Big(XI\{|X| \leq \phi(\theta^{\mathrm{i}})\}\Big)\Big)^{1/2}$ denotes the square root of the variance of $T_3(\mathrm{i})$ and N denotes a standard normal random variable. Note that

$$\frac{n\mathsf{E}X^2 I\{|X| \leq c_n\}}{c_n^2} = o(1) \quad \text{as } n \to \infty,$$

we can get for $|\mathrm{i}|$ large enough,

$$\mathsf{P}\Big(\Big|\frac{H^{'}(\phi(\theta^{\mathrm{i}}))N}{c_{\theta^{\mathrm{i}}}}\Big| \geq \alpha_0 - \varepsilon\Big) \geq C \exp\Big(-\frac{(\alpha_0 - \varepsilon/2)^2 c_{\theta^{\mathrm{i}}}^2}{2|\theta^{\mathrm{i}}| H(\phi(\theta^{\mathrm{i}}))}\Big).$$

By Lemma 2.4,

$$\begin{aligned}\sum_{\mathrm{i}\in \mathrm{N}^d} \mathsf{P}\Big(\Big|\frac{H^{'}(\phi(\theta^{\mathrm{i}}))N}{c_{\theta^{\mathrm{i}}}}\Big| \geq \alpha_0 - \varepsilon\Big) &\geq C \sum_{\mathrm{i}\in \mathrm{N}^d} \exp\Big(-\frac{(\alpha_0 - \varepsilon/2)^2 c_{\theta^{\mathrm{i}}}^2}{2|\theta^{\mathrm{i}}| H(\phi(\theta^{\mathrm{i}}))}\Big) \\ &\geq C \sum_{j=1}^{\infty} j^{d-1} \exp\Big(-\frac{(\alpha_0 - \varepsilon/2)^2 c_{\theta^j}^2}{2\theta^j H(\phi(\theta^j))}\Big) = \infty,\end{aligned}$$

which implies (2.10). So (2.7) holds.

Case 2: $\alpha_0 = \infty$. Obviously, it is enough to verify

$$\limsup_{\mathrm{i}\to\infty} \frac{S_{1,\theta^{\mathrm{i}}}(\mathrm{i}) - C_{\theta^{\mathrm{i}}}}{c_{\theta^{\mathrm{i}}}} = \infty \text{ a.s.} \tag{2.11}$$

We first assume

$$\limsup_{\mathrm{i}\to\infty} \frac{S_4(\mathrm{i}) - \mathsf{E}S_4(\mathrm{i})}{c_{\theta^{\mathrm{i}}}} < \infty \text{ a.s.}$$

Then, by (2.2) and the Borel-Cantelli lemma, in order to prove (2.11), it suffices to show

$$\sum_{\mathrm{i}\in \mathrm{N}^d} \mathsf{P}\Big(S_3(\mathrm{i}) - \mathsf{E}S_3(\mathrm{i}) \geq \varepsilon c_{\theta^{\mathrm{i}}}\Big) = \infty \quad \text{for every } \varepsilon > 10. \tag{2.12}$$

The same as above, we only need to prove

$$\sum_{\mathrm{i}\in \mathrm{N}^d} \mathsf{P}\Big(T_3(\mathrm{i}) \geq \varepsilon c_{\theta^{\mathrm{i}}}\Big) = \infty \quad \text{for every } \varepsilon > 10.$$

Set

$$N_0 = \{\mathrm{i} : \frac{c_{\theta^{\mathrm{i}}}}{H^{'}(\phi(\theta^{\mathrm{i}}))} \geq 1\} \quad \text{and} \quad N_0^c = \{\mathrm{i} : \frac{c_{\theta^{\mathrm{i}}}}{H^{'}(\phi(\theta^{\mathrm{i}}))} < 1\}.$$

If Card $N_0^c = \infty$, we have

$$\sum_{\mathrm{i}\in N_0^c} \mathsf{P}\Big(T_3(\mathrm{i}) \geq \varepsilon c_{\theta^{\mathrm{i}}}\Big) = \sum_{\mathrm{i}\in N_0^c} \mathsf{P}\Big(N \geq \frac{\varepsilon c_{\theta^{\mathrm{i}}}}{H^{'}(\phi(\theta^{\mathrm{i}}))}\Big) \geq \sum_{\mathrm{i}\in N_0^c} \mathsf{P}\Big(N \geq \varepsilon\Big) = \infty.$$

So (2.12) holds. Therefore we can assume that Card $N_0^c < \infty$. By the tail probability estimator of the normal distribution, we have

$$\mathsf{P}(N \ge x) \ge Cx^{-1} \exp\Big(-\frac{x^2}{2}\Big) \ge C\exp(-x^2), \quad x \ge 10.$$

And by Lemma 2.3, Lemma 2.4 and Card $N_0^c < \infty$, $\alpha_0 = \infty$,

$$\sum_{\mathrm{i}\in\mathrm{N}^d} \mathsf{P}\Big(N \ge \frac{\varepsilon c_{\theta^{\mathrm{i}}}}{H'(\phi(\theta^{\mathrm{i}}))}\Big) \ge \sum_{\mathrm{i}\in N_0} \mathsf{P}\Big(N \ge \frac{\varepsilon c_{\theta^{\mathrm{i}}}}{H'(\phi(\theta^{\mathrm{i}}))}\Big) \ge \sum_{\mathrm{i}\in N_0} \exp\Big(-\Big(\frac{\varepsilon c_{\theta^{\mathrm{i}}}}{H'(\phi(\theta^{\mathrm{i}}))}\Big)^2\Big)$$
$$\ge C\sum_{j=1}^{\infty} j^{d-1}\exp\Big(-\Big(\frac{\varepsilon c_{\theta^j}}{H'(\phi(\theta^j))}\Big)^2\Big) = \infty,$$

which implies (2.12). Therefore we have (2.11).
It remains for us to prove (2.11) when

$$\limsup_{\mathrm{i}\to\infty} \frac{S_4(\mathrm{i}) - \mathsf{E}S_4(\mathrm{i})}{c_{\theta^{\mathrm{i}}}} = \infty \text{ a.s.} \tag{2.13}$$

By using (2.2), we have

$$\limsup_{\mathrm{i}\to\infty} \frac{S_4(\mathrm{i}) - C_{\theta^{\mathrm{i}\text{-}1}}}{c_{\theta^{\mathrm{i}}}} = \infty \text{ a.s.} \tag{2.14}$$

Hence if we show that

$$\lim_{\mathrm{i}\to\infty} \frac{\sum_{\mathrm{k}\in\mathrm{N}_{\mathrm{i}}^c} X_{\mathrm{k}} I\{\phi(\theta^{\mathrm{i}\text{-}1}) \le |X_{\mathrm{k}}| \le \phi(\theta^{\mathrm{i}})\}}{c_{\theta^{\mathrm{i}}}} = 0 \quad a.s. \tag{2.15}$$

then, together with (2.14), (2.11) is proved.
Now we prove (2.15). The same as above (using (2.2) and Lemma 2.5), it suffices to show

$$I := \sum_{\mathrm{i}\in\mathrm{N}^d} \exp\Big(-\frac{\varepsilon c_{\theta^{\mathrm{i}}}^2}{|\theta^{\mathrm{i}}|\mathsf{E}X^2 I\{\phi(\theta^{\mathrm{i}\text{-}1}) \le |X| \le \phi(\theta^{\mathrm{i}})\}}\Big) < \infty \quad \text{for every } \varepsilon > 0.$$

But this follows from Lemma 2.1 and

$$\begin{aligned} I &\le C\sum_{\mathrm{i}\in\mathrm{N}^d}\Big(\frac{|\theta^{\mathrm{i}}|\mathsf{E}X^2 I\{\phi(\theta^{\mathrm{i}\text{-}1}) \le |X| \le \phi(\theta^{\mathrm{i}})\}}{c_{\theta^{\mathrm{i}}}^2}\Big)^Q \\ &\le C\sum_{\mathrm{i}\in\mathrm{N}^d}\Big(|\theta^{\mathrm{i}}|\mathcal{F}(\phi(\theta^{\mathrm{i}\text{-}1}))\Big)^Q < \infty \end{aligned}$$

for some large Q. The proof of Theorem 2.1 is terminated now. □

3 Proofs of main results in Section 1

Since the proof of Theorem 1.1 is based on Theorem 1.2, we shall prove Theorem 1.2 first.

Proof of Theorem 1.2: The proofs of (3)⇒(2) is obviously. From (1.6), we see that $c_n \le Cn$. So by the law of larger numbers and the Borel-Cantelli lemma, it is easy to see that (2)⇒(1). Now, we show that (1)⇒ (3). Recall $C_n = n\mathsf{E}XI\{|X| \le c_n\}$. From Lemma 2.2, it holds that $C_n = o(c_n)$. By Theorem 2.1, it suffices to show $\alpha_0 = 0$, which will be implied by

$$\frac{LLj}{h(j)}\mathsf{E}X^2 I\{|X| \le \sqrt{jh(j)}\} = o(1) \tag{3.1}$$

as $j \to \infty$. Now we come to prove it. By (1.8),

$$\sum_{j=1}^{\infty} j(Lj)^{d-1}\mathsf{P}\Big(c_{j-1} < |X| \le c_j\Big) < \infty.$$

Then

$$\sum_{k=1}^{n} \min_{i\le k} \frac{i(Li)^{d-1}}{c_i^2} c_k^2 \mathsf{P}\Big(c_{k-1} < |X| \le c_k\Big) \le C \quad \text{for some } C>0 \text{ and } n\ge 1.$$

That is

$$\mathsf{E}X^2 I\{|X|\le c_n\} \le C \max_{j\le n} \frac{h(j)}{(Lj)^{d-1}},$$

which together with (1.7), implies (3.1). The proof is completed. □

Proof of Theorem 1.1: If $\mathsf{E}X^2(\log|X|)^{d-1}/\log_2|X| < \infty$, then $\sigma^2 = \mathsf{E}X^2 < \infty$ since $d \ge 2$. We have that $K(n/LLn)LLn \sim \sigma\sqrt{nLLn}$. So from the classical LIL (c.f. Wichura (1973)) we can get $\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} = \sqrt{d}$ a.s.

Now, we assume that $\mathsf{E}X^2(\log|X|)^{d-1}/\log_2|X| = \infty$. If $\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} < \infty$ a.s. and $\mathsf{E}X^2 < \infty$, then $\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\sqrt{|\mathrm{n}|LLn} < \infty$ a.s., which implies $\mathsf{E}X^2(\log|X|)^{d-1}/\log_2|X| < \infty$ by Kolmogorov's 0-1 law and the Borel-Cantelli lemma. By the contradiction, we must have either $\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} = \infty$ a.s. or $\mathsf{E}X^2 = \infty$. We claim that $\mathsf{E}X^2 = \infty$ implies

$$\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} = \infty \text{ a.s.} \tag{3.2}$$

If (3.2) is not true, then by Kolmogorov's 0-1 law, $\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} =: C < \infty$ a.s. So we have

$$\sum_{n=1}^{\infty} (Ln)^{d-1} \mathsf{P}(|X| \ge \gamma_n) < \infty. \tag{3.3}$$

By Lemma 2.2, we obtain

$$n\mathsf{E}|X|I\{|X|\ge\gamma_n\} = o(\gamma_n) \quad \text{and} \quad n\mathsf{E}X^2 I\{|X|\le\gamma_n\} = o(\gamma_n^2). \tag{3.4}$$

Obviously γ_n satisfies conditions (1.5) and (1.6). Moreover, when $\mathsf{E}X^2=\infty$, we have $LLn/h(n) \searrow 0$, where $h(n) := 2K^2(n/LLn)(LLn)^2/n$. So, by Theorem 1.2 and Remark 1.2, we have

$$\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} = 0 \text{ a.s.} \tag{3.5}$$

Next, we prove that under (3.3), we can get $\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} \ge \sqrt{d}$ a.s. By Theorem 2.1, it suffices to prove that

$$\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1} \exp\Big(-\frac{\alpha^2\gamma_n^2}{2nH(\gamma_n)}\Big) = \infty \quad \text{for every } \alpha < \sqrt{d}. \tag{3.6}$$

Obviously, if we have

$$H(\gamma_n) \ge (\frac{1}{2}-\varepsilon)\frac{\gamma_n^2}{nLLn} \quad \text{for every } \varepsilon > 0 \tag{3.7}$$

when n large enough, then (3.6) holds. Now we prove (3.7). By (3.4) and the definition of the K-function,

$$\begin{aligned} H(\gamma_n) &\ge H(K(n/LLn)) + K(n/LLn)\mathsf{E}|X|I\{K(n/LLn) < |X| \le \gamma_n\} \\ &= \frac{K^2(n/LLn)LLn}{n} - K(n/LLn)\mathsf{E}|X|I\{|X|>\gamma_n\} \\ &\ge (\frac{1}{2}-\varepsilon)\frac{\gamma_n^2}{nLLn}. \end{aligned}$$

Therefore (3.7) holds and $\limsup_{\mathrm{n}\to\infty} |S_{\mathrm{n}}|/\gamma_{\mathrm{n}} \ge \sqrt{d}$ a.s. But this contradicts (3.5). So we have (3.2). We complete the proof of Theorem 1.1. □

Proof of Theorem 1.3. Note that (1.11) implies (1.8) by the law of larger numbers and the Borel-Cantelli lemma. Hence in order to prove the theorem, it is sufficient to prove that under (1.8),

$$C\lambda^{1/2} \leq \limsup_{\mathrm{n}\to\infty} \frac{|S_{\mathrm{n}}|}{\sqrt{|\mathrm{n}|h(\mathrm{n})}} \leq (2d\lambda)^{1/2} \quad a.s. \tag{3.8}$$

for some $C > 0$.
Now, we come to prove the upper bound. Obviously we can assume that $\lambda < \infty$. It will be shown that under (1.8) and $\lambda < \infty$,

$$A := \sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1} \exp\Big(-\frac{\varepsilon c_n^2}{n\Delta_n}\Big) < \infty, \quad \forall\varepsilon > 0, \tag{3.9}$$

where $\Delta_n = \mathsf{E}X^2 I\{c_n/LLn \leq |X| \leq c_n\}$, $c_n = \sqrt{nh(n)}$. Clearly, we have $H(c_n/LLn) \leq Ch(n)/LLn$ when $\lambda < \infty$. Therefore $\Delta_n \leq H(c_n) \leq Ch(n(LLn)^2)/LLn$. Also by a property of the slowly varying function, we have $h(n)/h(n(LLn)^2) \geq C(LLn)^{-1/2}$. So, by the inequality $\exp(-x) \leq Cx^{-1}\exp(-x/2)$ for $x > 0$,

$$\begin{aligned}
A &\leq C\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\frac{n\Delta_n}{c_n^2}\exp\Big(-\frac{\varepsilon c_n^2}{2n\Delta_n}\Big)\\
&\leq C\sum_{n=1}^{\infty}(Ln)^{d-1}\frac{\mathsf{E}|X|^3 I\{|X|\leq c_n\}}{c_n^3}LLn\exp\Big(-\frac{\varepsilon c_n^2}{2n\Delta_n}\Big)\\
&\leq C\sum_{n=1}^{\infty}(Ln)^{d-1}\frac{\mathsf{E}|X|^3 I\{|X|\leq c_n\}}{c_n^3}LLn\exp\Big(-\frac{Ch(n)LLn}{h(n(LLn)^2)}\Big)\\
&\leq C\sum_{n=1}^{\infty}(Ln)^{d-1}\frac{\mathsf{E}|X|^3 I\{|X|\leq c_n\}}{c_n^3}\\
&\leq C\sum_{n=1}^{\infty}(Ln)^{d-1}\sum_{k=1}^{n}\frac{c_k^3}{c_n^3}\mathsf{P}(c_{k-1}\leq |X|\leq c_k)\\
&\leq C\sum_{k=1}^{\infty}\mathsf{P}(c_{k-1}\leq |X|\leq c_k)\sum_{n=k}^{\infty}\frac{k^{3/2}}{n^{3/2}}(Ln)^{d-1}\\
&\leq C\sum_{k=1}^{\infty}k(Lk)^{d-1}\mathsf{P}(c_{k-1}\leq |X|\leq c_k)\\
&< \infty.
\end{aligned}$$

In the above inequalities, (1.5) is used.
Since $H(c_n/LLn) \leq (\lambda+\varepsilon)h(n)/LLn$ for $\forall\varepsilon > 0$ and n large enough, we can easily obtain that

$$\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\exp\Big(-\frac{\alpha^2 c_n^2}{2nH(c_n/LLn)}\Big) < \infty$$

for $\alpha > (2d\lambda+\varepsilon)^{1/2}$ and $\forall\varepsilon > 0$. Then using the following inequality

$$\exp\Big(-\frac{a}{x+y}\Big) \leq \exp\Big(-\frac{a}{(1+\delta)x}\Big) + \exp\Big(-\frac{a}{(1+\delta^{-1})y}\Big)$$

for any $a, x, y, \delta > 0$, and together with (3.9), we have

$$\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\exp\Big(-\frac{\alpha^2 c_n^2}{2nH(c_n)}\Big) < \infty$$

for $\alpha > (2d\lambda+\varepsilon)^{1/2}$ and $\forall\varepsilon > 0$. The upper bound is proved now by Theorem 2.1.

Next, we shall prove the lower bound in (3.8). Clearly, it can be assumed that $\lambda > 0$. By Theorem 2.1, it is enough to check that there exists a positive constant C_1 such that

$$\sum_{n=1}^{\infty} n^{-1}(Ln)^{d-1}\exp\Big(-\frac{\alpha^2 h(n)}{H(c_n)}\Big) = \infty \quad \text{for any } \alpha < (C_1\lambda)^{1/2}. \tag{3.10}$$

The arguments in Einmal and Li (2005) will be used. We can find a subsequence $m_k \nearrow \infty$ so that

$$H(c_{m_k}) \geq \lambda(1-\frac{1}{k})\frac{h(m_k)}{LLm_k} \quad \text{and} \quad h(m_k) \geq (1-\frac{1}{k})h(2m_k), \quad k \geq 1.$$

Thus, we have

$$H(c_n) \geq \lambda(1-\frac{1}{k})^2\frac{h(n)}{LLn}, \quad m_k \leq n \leq n_k := 2m_k,$$

which in turn implies that

$$\sum_{n=m_k}^{n_k}\frac{(Ln)^{d-1}}{n}\exp\Big(-\frac{\alpha^2 h(n)}{H(c_n)}\Big) \geq d^{-1}\Big[(Ln_k)^d-(Lm_k)^d\Big](Ln_k)^{-\alpha^2/\{\lambda(1-1/k)^2\}}$$
$$\geq C(Lm_k)^{d-1-2\varepsilon} \to \infty$$

for $\alpha < (\varepsilon\lambda)^{1/2}$ and $0 < \varepsilon < 1/2$. Hence (3.10) holds with any $0 < C_1 < 1/2$. The proof of Theorem 1.3 is completed. $\square$

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