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# Erratum to "A discontinuity adjustment for subdistribution function confidence bands applied to right-censored competing risks data"*

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**Abstract:** In this erratum, we correct Theorem 5.1 in [1], by mending the limit distribution of the Aalen-Johansen estimator under discontinuous survival distributions.

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We consider the same competing risks setup as in [1], i.e., we assume that there are $k \in \mathbb{N}$ competing risks and $n \in \mathbb{N}$ i.i.d. random event times $T_1, ..., T_n$, which are independently right-censored and distributed as a random variable $T \sim S$. Here, $S$ denotes the survival function, i.e., $S(t) = P(T > t)$ for all $t \geq 0$; $S$ need not be continuous. Then, we denote the probability that an individual is under observation at time $t-$, that is, *just before* time $t$, by $\bar{H}(t) = P(\min(T, C) \geq t) = S(t-)G(t-)$ for all $t \geq 0$. Here, $C \sim G$ with survival function $G(t) = P(C > t)$ denotes a generic censoring time which is assumed to be independent of $T$. Also, $t \mapsto f(t-)$ denotes the left-continuous version of a right-continuous function $f : [0, \infty) \to \mathbb{R}$. Furthermore, let $\widehat{A}_j$ denote the cause-specific Nelson-Aalen estimator for the cumulative hazard function $A_j$ of type $j$ events, $\widehat{S}$ the Kaplan-Meier estimator for the survival function $S$, and $\widehat{F}_j = \int_0^\cdot \widehat{S}(u-)\mathrm{d}\widehat{A}_j(u)$ the Aalen-Johansen estimator for the cumulative incidence function $F_j = \int_0^\cdot S(u-)\mathrm{d}A_j(u)$ for all $j \in \{1, ..., k\}$, see [1] for details. In addition to the assumptions in [1], it is actually required that $\bar{H}(K) > 0$ for $K \geq 0$ to ensure finite variances $\sigma_j^2(K), j \in \{1, \ldots, k\}$, in Theorem 4.1 therein.

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Theorem 5.1 in [1] states for $k = 2$ competing risks that

$$\sqrt{n}(\widehat{F}_1 - F_1) \xrightarrow{d} U_{F_1}$$

as $n \to \infty$ on the càdlàg space $D[0, K]$ equipped with the sup-norm, where $U_{F_1}$ is a zero-mean Gaussian process with covariance function

$$\begin{aligned}\sigma^2_{F_1} : (s,t) \mapsto &\int_0^{s\wedge t} \frac{(1 - F_2(u-) - F_1(s))(1 - F_2(u-) - F_1(t))}{\bar{H}(u)} \frac{\mathrm{d}A_1(u)}{1 - \Delta A(u)} \\ &+ \int_0^{s\wedge t} \frac{(F_1(u-) - F_1(s))(F_1(u-) - F_1(t))}{\bar{H}(u)} \frac{\mathrm{d}A_2(u)}{1 - \Delta A(u)} \\ &+ \sum_{u \in D, u \le s,t} \frac{S^2(u-)}{\bar{H}(u)} \frac{\Delta A_1(u) \Delta A_2(u)}{(1 - \Delta A(u))^2},\end{aligned}$$

where $A = \sum_{j=1}^k A_j$ and $D = \{t \in [0, K] : \Delta A(t) > 0\}$ is the set of discontinuity time points. However, we found that the right-continuous versions $F_1, F_2, S$ must appear in the covariance function above in all occurrences of $F_1(u-), F_2(u-), S(u-)$.

In order to prove this, we go one step back: By Theorem 4.1 in [1],

$$\sqrt{n}\left(\widehat{A}_1 - A_1, ..., \widehat{A}_k - A_k\right) \xrightarrow{d} (U_1, ..., U_k)$$

holds as $n \to \infty$ on the product space $D^k[0, K]$ equipped with the max-sup norm, where $U_1, ..., U_k$ are zero-mean Gaussian-martingales with

$$\begin{aligned}cov(U_j(t), U_j(s)) &= \int_0^{t\wedge s} \frac{1 - \Delta A_j(u)}{\bar{H}(u)} \,\mathrm{d}A_j(u) =: \sigma^2_j(t \wedge s), \\ cov(U_j(t), U_\ell(s)) &= -\int_0^{t\wedge s} \frac{\Delta A_\ell(u)}{\bar{H}(u)} \,\mathrm{d}A_j(u) =: \sigma_{j\ell}(t \wedge s)\end{aligned}$$

for all $t, s \in [0, K], j, \ell \in \{1, ..., k\}, j \neq \ell$. We further note that the limit $(U_1, ..., U_k)$ is separable since $G^{uc}_1, ..., G^{uc}_k$ and $\overline{G}$ in Appendix A of [1] are tight, which follows by the main empirical central limit theorems in [2], as in Example 3.10.20.

Now it holds that

$$\begin{aligned}&\sqrt{n}(\widehat{F}_1(t) - F_1(t)) \\ &= \sqrt{n}\left(\int_0^t \widehat{S}(u-)\mathrm{d}\widehat{A}_1(u) - \int_0^t S(u-)\mathrm{d}A_1(u)\right) \\ &= \int_0^t \widehat{S}(u-)\mathrm{d}\sqrt{n}(\widehat{A}_1 - A_1)(u) + \int_0^t \sqrt{n}(\widehat{S} - S)(u-)\mathrm{d}A_1(u) \\ &= \sqrt{n}(\widehat{A}_1 - A_1)(t)\widehat{S}(t) - \int_0^t \sqrt{n}(\widehat{A}_1 - A_1)(u)\mathrm{d}\widehat{S}(u) + \int_0^t \sqrt{n}(\widehat{S} - S)(u-)\mathrm{d}A_1(u)\end{aligned}$$

for all $t \in [0, K]$ by integration by parts, that is

$$\int_0^t f(v-)\,\mathrm{d}g(v) = (gf)(t) - (gf)(0-) - \int_0^t g(v)\,\mathrm{d}f(v)$$

for $f \in BV_1[0, K], g \in D[0, K]$, where $BV_1[0, K]$ denote the set of all càdlàg functions $D[0, K]$ of total variation bounded by 1. As in Example 3.10.33 in [2], the functional delta method yields

$$\left(\sqrt{n}(\widehat{A}_1 - A_1), \sqrt{n}(\widehat{S} - S)\right) \xrightarrow{d} \left(U_1, -S(.)\int_0^{\cdot} \frac{S(v-)}{S(v)}\mathrm{d}(U_1 + U_2)(v)\right)$$

as $n \to \infty$ on $D^2[0, K]$, where the integral is defined by integration by parts since $U_1 + U_2$ is not of bounded variation. Hence, we get

$$\left(\sqrt{n}(\widehat{A}_1 - A_1), \sqrt{n}(\widehat{S} - S), \widehat{S}\right) \xrightarrow{d} \left(U_1, -S(.)\int_0^{\cdot} \frac{S(v-)}{S(v)}\mathrm{d}(U_1 + U_2)(v), S\right) \quad (1)$$

as $n \to \infty$ on $D^2[0, K] \times BV_1[0, K]$ by Slutsky's lemma. Note that the map

$$\psi : D^2[0, K] \times BV_1[0, K] \to D[0, K],$$
$$(\tilde{A}, \tilde{B}, \tilde{C}) \mapsto \tilde{A}(.)\tilde{C}(.) - \int_0^{\cdot} \tilde{A}\mathrm{d}\tilde{C} - \int_0^{\cdot} \tilde{B}(u-)\mathrm{d}A_1(u)$$

is continuous on $D^2[0, K] \times \{S\}$. Thus, an application of the continuous mapping theorem and changing the order of integration result in

$$\begin{aligned}
&\sqrt{n}(\widehat{F}_1 - F_1) \\
&\xrightarrow{d} U_1(.)S(.) - \int_0^{\cdot} U_1\mathrm{d}S - \int_0^{\cdot} S(u-)\int_0^{u-} \frac{S(v-)}{S(v)}\mathrm{d}(U_1 + U_2)(v)\mathrm{d}A_1(u) \\
&= \int_0^{\cdot} S(u-)\mathrm{d}U_1(u) - \int_{[0,.)} \frac{S(v-)}{S(v)} \int_{(v,.]} S(u-)\mathrm{d}A_1(u)\mathrm{d}(U_1 + U_2)(v) \\
&= \int_0^{\cdot} S(u-)\mathrm{d}U_1(u) - \int_{[0,.)} \frac{S(v-)}{S(v)}(F_1(.) - F_1(v))\mathrm{d}(U_1 + U_2)(v) \\
&= \int_0^{\cdot} \frac{S(u-)}{S(u)}\left(S(u) - F_1(.) + F_1(u)\right)\mathrm{d}U_1(u) + \int_0^{\cdot} \frac{F_1(u) - F_1(.)}{1 - \Delta A(u)}\mathrm{d}U_2(u) \\
&= \int_0^{\cdot} \frac{1 - F_2(u) - F_1(.)}{1 - \Delta A(u)}\mathrm{d}U_1(u) + \int_0^{\cdot} \frac{F_1(u) - F_1(.)}{1 - \Delta A(u)}\mathrm{d}U_2(u)
\end{aligned}$$

as $n \to \infty$ on $D[0, K]$.

**Theorem 1** (Corrected Theorem 5.1 in [1])**.** *As $n \to \infty$, we have on the càdlàg space $D[0, K]$*

$$\sqrt{n}(\widehat{F}_1 - F_1) \xrightarrow{d} U_{F_1} = \int_0^{\cdot} \frac{1 - F_2(u) - F_1(.)}{1 - \Delta A(u)}\mathrm{d}U_1(u) + \int_0^{\cdot} \frac{F_1(u) - F_1(.)}{1 - \Delta A(u)}\mathrm{d}U_2(u),$$

*where $U_{F_1}$ is a zero-mean Gaussian process with covariance function*

$$\sigma^2_{F_1} : (s,t) \mapsto \int_0^{s\wedge t} \frac{(1-F_2(u)-F_1(s))(1-F_2(u)-F_1(t))}{\bar{H}(u)} \frac{\mathrm{d}A_1(u)}{1-\Delta A(u)} + \int_0^{s\wedge t} \frac{(F_1(u)-F_1(s))(F_1(u)-F_1(t))}{\bar{H}(u)} \frac{\mathrm{d}A_2(u)}{1-\Delta A(u)} + \sum_{u\in D, u\leq s,t} \frac{S^2(u)}{\bar{H}(u)} \frac{\Delta A_1(u)\Delta A_2(u)}{(1-\Delta A(u))^2}.$$

The covariance function can be calculated analogously to Appendix E of [1]. Here, the last sum may be simplified to $\sum_{u\in D, u\leq s,t} \frac{S(u-)}{G(u-)} \Delta A_1(u)\Delta A_2(u)$.

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## References

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