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Homogenization of the variational principle for discrete random maps*

Andrew Krieger† Georg Menz† Martin Tassy‡

Dedicated to the memory of Thomas Ligget

Abstract

We consider homogenization of random surfaces and study the variational principle for graph homomorphisms from subsets of $\mathbb{Z}^m$ into $\mathbb{Z}$, where the underlying uniform measure is perturbed by a random potential. Motivated by the theories of random walks in random potentials, we assume that random potential is stationary, ergodic, and bounded in L^1. We show that the variational principle holds in probability and that the entropy functional homogenizes, i.e. is independent of the values taken by the random potential. The main ingredients in the argument are the existence of the quenched surface tension, the equivalence of the quenched and the annealed surface tension, and robustness of the surface tension under change in boundary data. These ingredients are deduced by a combination of a superadditive ergodic theorem and combinatorial results, especially the Kirszbraun theorem.

Keywords: variational principles; limit shapes; random surfaces; entropy; local surface tension; homogenization; subadditive ergodic theorem.
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†Department of Mathematics, University of California, Los Angeles.
E-mail: krieger.1618@gmail.com,georg.menz@gmail.com
‡Dartmouth College, Hanover. E-mail: mtassy@math.dartmouth.edu

1 Introduction

The broader scope of this article is the study of limit shapes as a limiting behavior of discrete systems. Limit shapes are a well-known and studied phenomenon in statistical physics and combinatorics (e.g. [24]). Among others, models that exhibit limit shapes include domino tilings and dimer models (e.g. [25, 12, 13]), polymer models (e.g. [6, 4]), lozenge tilings (e.g. [16, 35, 48]), Ginzburg-Landau models (e.g. [17, 22]), Gibbs models (e.g. [44]), the Ising model (e.g. [18, 10]), asymmetric exclusion processes (e.g. [20]), sandpile models (e.g.[32]), the six vertex model (e.g. [7, 14, 42]), and the Young tableaux (e.g. [34, 47, 41]).

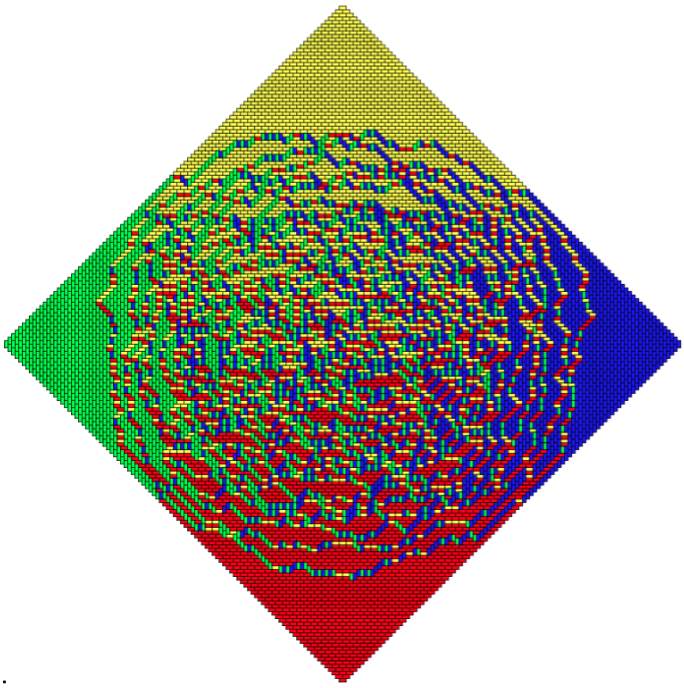


Figure 1: An Aztec diamond for domino tilings. The combinatorics of the model is similar to Lipschitz functions from $\mathbb{Z}^2$ to $\mathbb{Z}$ (see [13]).

Limit shapes appear in stiff models whenever fixed boundary conditions force a certain response of the system. The numerous examples in the literature and many simulations show that the existence of limit shapes is a universal phenomenon. Among many possible references, let us just mention [27, 8, 36, 37, 26]. Several new approaches were developed recently to make methods more robust; see for example [11, 9, 14, 1]. In [36], variational principles were studied in target spaces where the usual cluster swapping methods do not work.

In this article we explore a new direction and show the robustness of the variational principle in a random potential. The basic objects for our model are graph homomorphisms from finite subsets of the m-dimensional lattice $\mathbb{Z}^m$ into $\mathbb{Z}$, also called height functions. The physical motivation behind our model is the study of interfaces that separates two phases. An example of an interface would be the surface that separates a fluid droplet from its surrounding air, or domain walls in a ferromagnet at low temperatures. We consider an effective model which means that the interface has no overlap and can be represented as the graph of a height function $f : R \to \mathbb{R}$, for some domain R. Additionally, we assume that the interface is forced to have a certain height at the boundary, which is for example the case when describing the membrane of a drum. As an interface strives to minimize the local surface tension, the shape f of the interface can be macroscopically described as the minimizer of the variational problem

$$\min_{f, f_{|\partial R}=g} \int_R \mathrm{ent}(\nabla f(x), f(x), x) dx, \tag{1.1}$$

where ent denotes the local surface tension. For details we refer to the standard literature on interface models, e.g. [21].

In this article, the interface is modeled microscopically via a height function $h : \mathbb{Z}^m \supset R_n \to \mathbb{Z}$, allowing only height differences of ± 1 between neighboring sites. Without the random perturbation the model could be interpreted as degenerate $\nabla\phi$ model, where the potential ϕ is a degenerate double well potential given by

$$\phi(z) = \begin{cases} 0, & \text{if } |z| = 1, \\ \infty, & \text{else.} \end{cases}.$$

As the microscopic ensemble μ_ω of height functions, we consider the uniform measure perturbed by a random potential ω acting on the height space $\mathbb{Z}$. The random potential ω makes certain heights more attractive for the height function h than other heights. The role of the random potential ω is to model the pinning of the interface at certain height levels, caused for example by random heterogeneities in the medium like defects, dislocations, dopants, and vacancies. The pinning of interfaces plays an important role in a variety of phenomena, including grain growth, martensitic phase transitions, ferroelectricity, dislocations and fracture (see e.g. [46] and the references therein).

In two dimensions and without random potential, $\mathbb{Z}$-homomorphisms are equivalent to a special case of the six-vertex model, where all vertex weights are identical, i.e. the square-ice model. The limiting behavior of the $\mathbb{Z}$-homomorphism model without random potential is well-studied; see for example [3, 40, 39]. As we add the random potential on height levels, this perturbation is highly nontrivial for the square-ice model. The likelihood of placing a specific ice-molecule at a certain position would depend in a subtle way on its environment via the associated height function. A direct way to approach stochastic homogenization on the square-ice model would be to add the random potential onto the underlying lattice $\mathbb{Z}^m$, modulating the likelihood of placing different molecules depending on the lattice location $x \in \mathbb{Z}^m$. In the meantime of the review process of this article, a study on deterministic periodic homogenization of the six-vertex model was published (see [19]).

Another reason for introducing a random potential is to test the robustness of the methods used to prove variational principles and similar results. Several nice properties do not carry over from the unperturbed model: exact computations like those in [13] are prohibitively difficult, *a priori* proofs of concentration (e.g. the martingale method of [12]) do not seem to apply, and there are no obvious global symmetries. To overcome those obstacles we make use of ergodicity and homogenization.

The random potential is inspired by homogenization of random walks in random environment (see e.g. the survey [5]). Indeed, the bridge model of [23], i.e. transient random walks in random environment conditioned to start and end at prescribed boundary values, is a special case of the $\mathbb{Z}$-homomorphism in random potential model with dimension $m = 1$. The bridge model exhibits asymptotically different maximal order statistics than does the bridge model originating from the simple random walk. The model considered in this article is a natural extension of bridges to “random sheets.” It would be interesting to extend the results of [23] to higher dimensions and compare against the Gaussian free field.

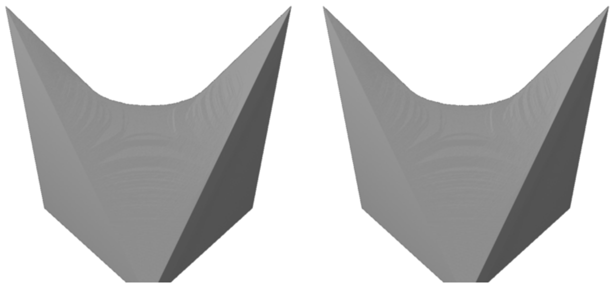
Before summarizing the mathematical results of the article, let us motivate the model further by discussing empirical results from computer simulation; cf. Figure 2 and 3. We generated random environments $(\omega_e)_{e\in E(S_n)}$ according to various distributions (e.g. i.i.d. Gaussian) and for various box sizes (up to a 1000×1000-vertex box). We chose boundary data $h_{\partial S_n}$, then we sampled a height function $h \in M(S_n, h_{\partial S_n})$ according to the random measure μ_ω using the Markov chain Monte Carlo method.

We call attention to a few details from the simulations. The two height functions in Figure 2b are drawn from two different measures μ_ω, where the random potentials ω are sampled such that $\{\omega_e \,|\, e \in E(\mathbb{Z})\}$ are i.i.d. with $\mathbb{P}(\omega_e = 1) = \mathbb{P}(\omega_e = -1) = \frac{1}{2}$. Although the exact value of ω varies in the two samples, the randomly chosen height functions in the pictures appear to be macroscopically identical. This is a good indicator that this model homogenizes. By this we mean that the macroscopic features measure μ_ω do not depend (in the limit, except with negligible probability) on the exact choice of ω. Rather those macroscopic features of μ_ω only depend on the distribution of ω and the boundary data. Indeed, the main results of this article apply to the random potential from Figure 2b, so we know that this model homogenizes.

The three height functions in Figure 2c are sampled from three different measures μ_ω, where $\omega_e \sim \mathcal{N}(0, 1)$ are i.i.d. standard normal variables. Notice that these three height functions differ macroscopically, depending on the realization of ω. This does not contradict the results of this article because the random potential is unbounded. We expect (but have not proven) that this model fails to homogenize when the random potential is unbounded, with energetic pinning effects from ω overwhelming the entropic effects from the underlying combinatorial $\mathbb{Z}$-homomorphism model. If the random measure μ_ω does not homogenize, then the limit shape under μ_ω may depend on the actual values of ω, and thus be may be a non-trivial random variable (with respect to the randomness that determines ω). However, we conjecture that the arctic circle, i.e. the boundary between the frozen and liquid region, still homogenizes and is independent of the law of the perturbation even in the case of an unbounded random potential (see Figure 3).

In order to understand the behavior underlying the simulations in Figure 2, we prove two main results: a profile theorem (see Theorem 2.22) and a variational principle (see Theorem 2.23). A third related result, namely a large deviations principle (see Theorem 2.24), is not difficult to prove by the same methods, but we omit it for brevity. These results hold with high probability for fixed environments ω. They establish that, for typical samples of ω, there holds a conclusion similar to the profile theorem or variational principle for the non-random model studied in our companion article [31]. Indeed the purpose of the companion article was to distill, simplify, and explain the steps involved in proving these results. Understanding the methods in the companion article will help to understand the general outline of the proofs in this article. From a high-level perspective, the main results are similar to the simpler case studied in [31].

(a) A height function sampled without random potential.



(b) Two height functions sampled from μ_ω, under two different samples of ω with a.s. bounded distribution. Specifically, the random variables $\{\omega_e \,|\, e \in E(\mathbb{Z})\}$ are i.i.d. with $\mathbb{P}(\omega_e = 1) = \mathbb{P}(\omega_e = -1) = \frac{1}{2}$.

(c) Three height functions sampled from μ_ω, under three different samples of ω with a.s. unbounded distribution. Specifically, the random variables $\{\omega_e \,|\, e \in E(\mathbb{Z})\}$ are i.i.d. standard normal variables. One can observe the pinning of the surface to certain height levels.

Figure 2: Height functions h, sampled from random measures μ_ω, which in turn are derived from randomly sampled fields ω. The height functions are rendered as 3D solids with the random surface $\{(x, y, h(x, y)) \,|\, (x, y) \in S_n\}$ as their "top" face. Boundary values of h are fixed, and the behavior of h on the interior of the domain S_n depends on ω. If the law of ω is bounded (as is the case in Figure 2b), then the results in this article imply that with high probability, the macroscopic behavior of h does not depend on the specific realization of ω, but only on its distribution.

Let us now briefly discuss the main results of this article. We start with the profile theorem. It asymptotically characterizes the cardinality of the set of height functions h_{R_n} that are uniformly close to a particular macroscopic state h_R (also called asymptotic height profile later on). Without random potential, the profile theorem states (cf. [31, Theorem 15]) that

$$\begin{aligned}\operatorname{Ent}_{R_n}\Big(\Big\{&\text{height functions } h_{R_n} : R_n \to \mathbb{Z} \\ &\qquad\qquad \text{with } \|h_{R_n} - \tilde{h}_R\|_\infty < \varepsilon\Big\}\Big) \\ \approx \operatorname{Ent}_R(h_R)\,,&\end{aligned} \tag{1.2}$$

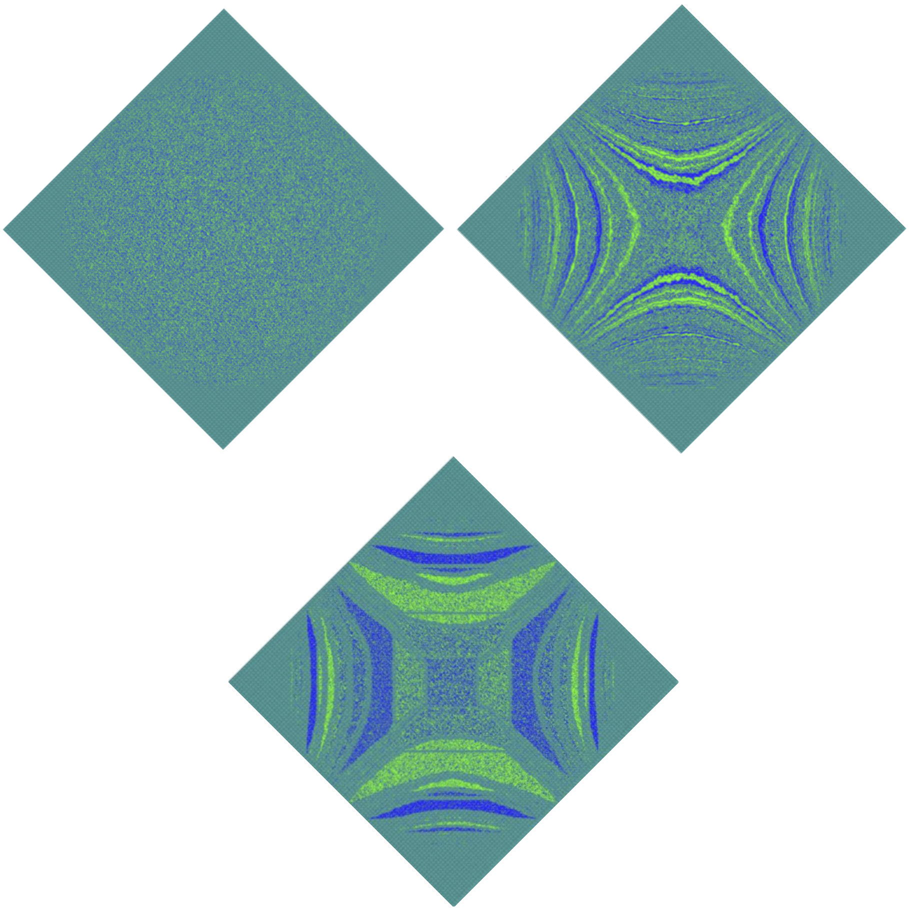


Figure 3: The height functions h are rendered from the birds-eye perspective and satisfy the same boundary conditions as in Figure 2. The first picture has no random field, the second has an iid. random field uniformly distributed on the set $\{1,\ldots,10\}$. In the third picture, the random field is idd. standard normal. By zooming into the picture, it seems that the *arctic circle*, i.e. the boundary between the frozen and the liquid region, seems not to be effected by the law of the random field.

where (for details see Section 2):

$$R_n \subset \mathbb{Z}^m \text{ is such that } \tfrac{1}{n}R_n \text{ converges to } R\,,$$

$$\tilde{h}_R(z) = nh_R(\tfrac{z}{n}) \text{ is a rescaled version of } h_R\,,$$

$$\mathrm{Ent}_{R_n}(M) = -\frac{1}{|R_n|}\log|M|\,, \tag{1.3}$$

$$\mathrm{Ent}_R(h_R) = \int_R \mathrm{ent}(\nabla h_R(x))\,, \text{ and} \tag{1.4}$$

$$\mathrm{ent} : [-1,1]^m \to \mathbb{R} \text{ is determined by the combinatorics of the } \mathbb{Z}\text{-homomorphism model.}$$

In the setting of homogenization we substitute the uniform measure on the set of microscopic height functions with a random measure μ_ω that is characterized by the random potential ω. The quantity $\mathrm{Ent}_{R_n}(M)$ from (1.3) is dependent on ω, and is therefore a random variable. Specifically, the cardinality $|M|$ is replaced by a partition function of μ_ω on the set of height functions. The quantity $\mathrm{Ent}_R(h_R)$ from the right-hand side of (1.2) is replaced by $\mathrm{Ent}_{R,\mathsf{an}}(h_R)$, the annealed macroscopic entropy (see Definition 2.21). Likewise the local surface tension $\mathrm{ent}(\cdot)$ in the definition (1.4) is replaced by the annealed local surface tension $\mathrm{ent}_{\mathsf{an}}(\cdot)$. In both cases, "annealed" means that the influence of the random field ω is averaged out i.e. $\mathrm{ent}_{\mathsf{an}}(s) := \mathbb{E}[\mathrm{ent}(s,\omega)]$. Therefore $\mathrm{ent}_{\mathsf{an}}(\cdot)$ and $\mathrm{Ent}_{R,\mathsf{an}}(\cdot)$ are non-random. Turning back to the conclusion (1.2) of the profile theorem, the left-hand side is a non-trivial random variable, but the limiting quantity on the right-hand side is not random.

Let's turn to the second main result, namely the variational principle. Recall that the profile theorem measures the set of height functions that stay close to a target asymptotic height profile over the entire domain. The variational principle instead measures the whole set of height functions with certain boundary values. Without random potential (cf. [31, Theorem 16]), the result is

$$\begin{aligned}&\mathrm{Ent}_{R_n}\big(\big\{h_{R_n}:R_n\to\mathbb{Z}\,\big|\,h_{R_n}|_{\partial R_n}\text{ is close to }h_{\partial R}\big\}\big)\\&\qquad\approx\inf_{h_R}\mathrm{Ent}_R(h_R)\,,\end{aligned}\tag{1.5}$$

where "close" means close in the supremum norm after rescaling, and where the infimum runs over all asymptotic height profiles consistent with the given boundary data $h_{\partial R}:\partial R\to\mathbb{R}$. In the setting of homogenization, i.e. adding a random potential to the uniform measure, the necessary modifications to this approximate identity are analogous to those for the profile theorem above: $\mathrm{Ent}_{R_n}(\cdot)$ becomes a random variable dependent on ω just as above, and $\mathrm{Ent}_R(h_R)$ is again replaced by the non-random quantity $\mathrm{Ent}_{R,\mathsf{an}}(h_R)$. Hence, it follows from our main result that the variational principle homogenizes. Coming back to our initial motivation of studying interfaces, the variational principle vindicates the Ansatz (1.1), i.e. that an interface can be described macroscopically by minimizing the overall surface tension.

Let's now discuss the large deviations principle. Let $h_{\partial R}$ be an asymptotic boundary height function and let A be a Borel set in the space of asymptotic height functions with boundary values given by $h_{\partial R}$, equipped with the supremum norm. Without random potential the large deviations principle states (cf. [31, Theorem 17]):

$$\begin{aligned}&-\frac{1}{|R_n|}\log\mu_n\big(\big\{h_{R_n}:R_n\to\mathbb{Z}\,\big|\,\text{after rescaling, }h_{R_n}\in A\big\}\big)\\&\qquad\approx\inf_{h_R\in A}\mathrm{Ent}_R(h_R)-E_0\,,\end{aligned}\tag{1.6}$$

where μ_n is the uniform measure on the set of (microscopic) height functions with appropriate boundary values, where $E_0 := \inf_{h_R}\mathrm{Ent}_R(h_R)$ is the infimum of the entropy over all asymptotic height functions with boundary values $h_{\partial R}$.

In the setting of homogenization, the large deviation principle needs to be adapted in an analogous way as for the profile theorem and the variational principle: The uniform measure μ_n is replaced by the random measure $\mu_{n,\omega}$, and $\mathrm{Ent}_R(\cdot)$ is replaced by the annealed macroscopic entropy. Then the large deviation principle holds again with respect to sample ω of the random field with high probability. Because the rate functional homogenizes, i.e. it is independent of the realization ω of the random field, the large deviation principle would also homogenize.

It is natural to ask whether the infima in (1.5) and (1.6) admit a minimizer, and if so whether the minimizer is unique. Existence follows from convexity of the integrand function $\mathrm{ent}_{\mathsf{an}}(\cdot)$ (Lemma 3.16 establishes convexity of $\mathrm{ent}_{\mathsf{an}}(\cdot)$). Uniqueness of $h_{R,\min}$ follows from strict convexity of $\mathrm{ent}_{\mathsf{an}}(\cdot)$; see for example [13]. Strict convexity remains an open question for this model.

The proofs of the main results are based on two main ingredients: existence and characterization of the quenched local surface tension and robustness of the entropy.

The first main ingredient is the existence of the quenched local surface tension $\mathrm{ent}_{\mathsf{an}}(s,\omega)$. Without random potential existence follows from superadditivity by application of Fekete's lemma. With random potential we turn to a superadditive ergodic theorem instead. Superadditivity and translation invariance are enough to establish existence of the quenched local surface tension. Ergodicity is used to characterize the quenched local surface tension. At slopes $s \neq 0$, translating a domain R_n by $z \in \mathbb{Z}^m$ implies shifting the boundary heights by $s \cdot z$, and the random potential is ergodic with respect to this kind of height shift. Since the quenched local surface tension is translation invariant, it follows that it is almost surely equal to its expectation, the annealed local surface tension. The same conclusion holds in the case $s = 0$, which we show using an argument with credit to Marek Biskup.

The second main ingredient in proving the main results of this article is robustness. In Section 3 we prove several results, which serve to control the change in the microscopic entropy $\mathrm{Ent}_{R_n}(A,\omega)$ as the set of height functions A changes. For example, when

$$A = \left\{ h_{R_n} \,\middle|\, h_{R_n}|_{\partial R_n} = h_{\partial R_n} \right\}$$

is defined by boundary data $h_{\partial R_n}$, we consider the effect of changing the boundary data. In order to control the change of microscopic entropy, the main idea is to use the Kirszbraun theorem (see Theorem 3.1). It allows to extend height functions on a domain to height functions on the larger domain. This provides an injection between the two sets of height functions, and it remains to control the energetic effect contributed by the newly added edges in the larger domain. When the Kirszbraun theorem is not useful, we fall back to combinatorial results.

The proof of the robustness results illustrate a primary source of difficulty: passing from combinatorial estimates on the number of height functions to control over energetic effects arising from the random potential. In the example discussed above, every height function in the smaller (in the sense of cardinality) set admits an extension in the larger set. It is not difficult to compare the total energy of an extension to that of the original height function, using the assumption that the random potential is bounded.

After applying the two ingredients listed above, it remains to apply approximations of Lipschitz functions and compactness of the space of asymptotic height functions (with fixed boundary values). For these last steps of the argument we follow Sections 5 through 8 of [31], with some modifications needed to account for the random potential. Because the proof is largely the same as in our companion article we do not go into great detail for these steps.

The rest of this article is organized as follows.

- In Section 2 we define the precise setting and state the main results.
- In Section 3 we state and prove key results about the local surface tension.
- In Section 4 we prove the first main result, namely the profile theorem.
- In Section 5 we prove the second main result, namely the variational principle.
- In Section 6 we state a few open problems and directions for further research.

Notation and conventions

For the convenience of the reader, we summarize the basic notation that we use throughout this article.

- $|A|$ denotes either the cardinality or the Lebesgue measure of the set A, depending on context.
- $S_n := \{-n, -(n-1), \dots, n-1, n\}^m \subset \mathbb{Z}^m$ denotes a hypercube in the lattice, centered at the origin.
- For $z, z' \in \mathbb{Z}^m$, $z \sim z'$ means that z and z' are nearest-neighbors (i.e. the ℓ^1 distance $\sum_{i=1}^m |z_i - z_i'|_1$ is exactly 1).
- For $S \subset \mathbb{Z}^m$, $\partial S := \{z \in S \,|\, \exists \tilde{z} \in \mathbb{Z}^m \setminus S,\, \tilde{z} \sim z\}$ is the (interior) boundary of S.
- $e_{zz'}$ is the unoriented edge between neighbors $z \sim z'$ in $\mathbb{Z}^m$.
- For $h : \mathbb{Z}^m \to \mathbb{Z}$ and $e = e_{zz'} \in E(\mathbb{Z}^m)$, we abuse notation and write $h(e)$ to denote the edge $e_{h(z),h(z')} \in E(\mathbb{Z})$.
- τ_w denotes the shift by $w \in \mathbb{Z}^m$ on edges of the graph $\mathbb{Z}^m$. That is, $\tau_w e_{zz'} = e_{z+w,z'+w}$.
- $s \in \mathbb{R}^m$ denotes a vector satisfying $|s|_\infty \leq 1$.
- For a vector $\mathbf{x}$ of parameters, we denote with $\theta_{\mathbf{x}}(\varepsilon)$ a smooth, non-negative function that may depend on the parameters $\mathbf{x}$ and satisfies $\lim_{\varepsilon \to 0} \theta_{\mathbf{x}}(\varepsilon) = 0$. If the function does not depend on any parameters, we also write θ (see also the notation explained in [31, Section 2.4]).

2 Setting and main results

In this section we describe the model under study, introduce related notation, and state the main results of this article. The setting, notation, and main results are similar to those of the companion article [31].

2.1 Basic definitions

Throughout the sequel, we fix a dimension $m \in \mathbb{N}$, a macroscopic domain $R \subset \mathbb{R}^m$, and a sequence of microscopic domains $R_n \subset \mathbb{Z}^m$ satisfying these assumptions:

Assumption 2.1 (Assumptions on domain R and R_n)**.** *We assume that $R \subset \mathbb{R}^m$ is compact and connected, that R is the closure of its interior, and that the boundary of R has zero Lebesgue measure. We assume that $R_n \subset \mathbb{Z}^m$ is contained in R after rescaling, i.e. that $\frac{1}{n} R_n \subset R$, although this is just a simplifying assumption. Moreover, we assume that $\frac{1}{n} R_n \to R$ in the Hausdorff metric, i.e. the metric on $\{A \subset \mathbb{R}^m\}$ defined by*

$$d_H(A,B) := \left(\sup_{x \in A} \inf_{y \in B} |x-y|_1 \right) \vee \left(\sup_{y \in B} \inf_{x \in A} |x-y|_1 \right).$$

Now, we define precisely the height functions in our model.

Definition 2.2 (Height function and lifted height function)**.** *A height function on R_n is a graph homomorphism $h_{R_n} : R_n \to \mathbb{Z}$. In other words, if $z, w \in R_n$ and $z \sim w$, then $|h_{R_n}(z) - h_{R_n}(w)| = 1$, and for any $z = (z_1, \dots, z_m) \in R_n$,*

$$h_{R_n}(z) \equiv z \pmod 2, \quad \textit{i.e. } h_{R_n}(z) \equiv \sum_{i=1}^m z_i \pmod 2. \tag{2.1}$$

Let $E(G)$ denote the set of undirected nearest-neighbor edges on a graph G. To a height function $h_{R_n} : R_n \to \mathbb{Z}$ we associate the lifted height function $\hat{h}_{R_n} : E(R_n) \to E(\mathbb{Z})$ via the formula $\hat{h}_{R_n}(e_{z,z'}) := e_{h_{R_n}(z), h_{R_n}(z')}$. As graph-homophormisms map nearest-neighbors to nearest-neighbors, the lifted height function $\hat{h}$ is well defined. With a slight abuse of notation we write $h_{R_n}(e)$ instead of $\hat{h}_{R_n}(e)$. It becomes clear from the argument, which is either a vertex x or an edge e, if the height function $h_{R_n}(x)$ or lifted height function $\hat{h}_{R_n}(e)$ is considered.

The condition (2.1) states that a height function preserves the parity of the lattice $\mathbb{Z}^m$. Indeed, every graph homomorphism either preserves parity at all points or inverts parity at all points, since the source space $\mathbb{Z}^m$ and the target space $\mathbb{Z}$ are both bipartite. Our main results are also valid without the parity-preserving condition, but for the same reasons as outlined in [31, Section 2.1] we include it for simplicity.

We introduce the following symbols to refer to sets of height functions:

Definition 2.3 (Sets of height functions)**.** *Let R_n be a microscopic domain as above, let $h_{R_n} : R_n \to \mathbb{Z}$ be a boundary height function, and let $\delta > 0$. We define:*

$$
\begin{aligned}
M(R_n) &:= \left\{ h_{R_n} : R_n \to \mathbb{Z} \,\middle|\, h_{R_n} \text{ is a height function} \right\}, \\
M(R_n, h_{\partial R_n}) &:= \left\{ h_{R_n} \in M(R_n) \,\middle|\, h_{R_n}|_{\partial R_n} = h_{\partial R_n} \right\}, \\
M(R_n, h_{\partial R_n}, \delta) &:= \left\{ h_{R_n} \in M(R_n) \,\middle|\, \sup_{z \in \partial R_n} |h_{R_n}(z) - h_{\partial R_n}(z)| < \delta n \right\}, \text{ and} \\
B(R_n, h_R, \delta) &:= \left\{ h_{R_n} \in M(R_n) \,\middle|\, \sup_{z \in R_n} |h_R(\tfrac{1}{n} z) - \tfrac{1}{n} h_{R_n}(z)| < \delta \right\}.
\end{aligned}
$$

In the last definition, the expression "$h_R(\frac{1}{n}z)$" makes sense because of the assumption that $\frac{1}{n}R_n \subset R$ in Assumption 2.1.

The limiting object for convergent sequences of height functions is:

Definition 2.4 (Asymptotic height function)**.** *We call a function $h_R : R \to \mathbb{R}$ an asymptotic height function if h_R is Lipschitz with Lipschitz constant at most 1, with respect to the ℓ^1-norm on $\mathbb{R}^m$; that is, if*

$$
\operatorname{Lip}(h_R) := \sup_{x \neq y \in R} \frac{|h_R(x) - h_R(y)|}{|x - y|_1} \leq 1\,.
$$

Likewise, if $h_{\partial R} : \partial R \to \mathbb{R}$ is 1-Lipschitz (with respect to the ℓ^1-norm), we call $h_{\partial R}$ an asymptotic boundary height function.

The limit of height functions is defined as follows.

Definition 2.5 (Convergence of height functions)**.** *Given a sequence of height functions $h_{R_n} : R_n \to \mathbb{Z}$ and an asymptotic height function $h_R : R \to \mathbb{R}$, we say that h_{R_n} converges in the scaling limit to h_R if*

$$
\lim_{n \to \infty} \sup_{z \in R_n} \sup_{\substack{x \in R \\ |x - \frac{1}{n}z|_1 \leq d_n}} \left| \frac{1}{n} h_{R_n}(z) - h_R(x) \right| = 0\,,
$$

where $d_n := d_H(\frac{1}{n}R_n, R)$.

Finally, we define the following sets of asymptotic height functions:

Definition 2.6 (Sets of asymptotic height functions)**.** *Let $R \subset \mathbb{R}^m$ be a domain satisfying Assumption 2.1, let $h_{\partial R} : \partial R \to \mathbb{R}$ be an asymptotic boundary height function, and let $\delta > 0$. We define:*

$$
\begin{aligned}
M(R) &:= \left\{ h_R : R \to \mathbb{R} \,\middle|\, h_R \text{ is an asymptotic height function} \right\}, \\
M(R, h_{\partial R}) &:= \left\{ h_R : R \to \mathbb{R} \,\middle|\, h_R|_{\partial R} = h_{\partial R} \right\}, \\
M(R, h_{\partial R}, \delta) &:= \left\{ h_R : R \to \mathbb{R} \,\middle|\, \forall x \in \partial R\,,\ |h_R(x) - h_{\partial R}(x)| \leq \delta \right\}, \text{ and} \\
B(R, \tilde{h}_R, \delta) &:= \left\{ h_R : R \to \mathbb{R} \,\middle|\, \forall x \in R\,,\ |h_R(x) - \tilde{h}_R(x)| < \delta \right\}.
\end{aligned}
$$

2.2 Defining the entropy

In order to define the local surface tension, both quenched and annealed, we fix a family of canonical height functions with fixed slope. These are the linear and affine height functions, so called because they approximate linear and affine functions of real variables.

Definition 2.7 (Affine and linear height functions)**.** *For $s \in [-1,1]^m$, $b \in \mathbb{R}$, and $n \in \mathbb{N}$, we define the affine height function $h^{s\cdot x+b} \in M(\mathbb{Z}^m)$ as*

$$h^{s\cdot x+b}(z) := [s \cdot z + b]_{z \bmod 2} \quad \text{for all } z \in \mathbb{Z}^m\,,$$

where for $t \in \mathbb{R}$ and $z \in \mathbb{Z}^m$, $[t]_{z \bmod 2}$ is the integer with the same parity as z that is closest to t. (In the ambiguous case, namely when t is an integer having opposite parity as z, we choose arbitrarily but consistently to "round up" and set $[t]_{z \bmod 2} = (t+1)$.) For $s \in [-1,1]^m$, the linear height function $h^s \in M(\mathbb{Z}^m)$ is given by $h^s = h^{s\cdot x+0}$, i.e.

$$h^s(z) := [s \cdot z]_{z \bmod 2} \quad \text{for all } z \in \mathbb{Z}^m\,,$$

Remark 2.8. The symbol "x" in the superscript "$s \cdot x + b$" is a formal variable, used so that the superscript resembles a meaningful expression instead of, say, the less intuitive pair (s, b). It is not difficult to verify that the functions defined above are graph homomorphisms. We refer the reader to [31, Lemma 7] for the details.

Until now, the setup has been the same as in the companion article [31]. Let us now turn to homogenization and to the new contributions of this article. The main change in the model is that instead of the uniform measure on $M(R_n, h_{\partial R_n})$ we consider a noisy perturbation μ_ω of the uniform measure, where $\omega = (\omega_e)_{e\in E(\mathbb{Z})}$ denotes a random field. In contrast to some other models in homogenization, the noise ω acts on the height space $\mathbb{Z}$ and will make certain height levels more and other levels less attractive, modeling a pinning effect caused for example by random inhomogenities in the material or medium.

Assumption 2.9 (Random field ω)**.** *We consider a real-valued random potential*

$$\omega = (\omega_e)_{e\in E(\mathbb{Z})} \in \mathbb{R}^{E(\mathbb{Z})}$$

defined on the set of edges $E(\mathbb{Z})$ of $\mathbb{Z}$. We assume that ω satisfies the following assumptions:

- *The random field ω is almost surely finite, and moreover the random variable C_ω defined by*

  $$C_\omega := 1 \vee \sup_{e\in E(\mathbb{Z})} |\omega_e|$$

  is in L^1, i.e. $\mathbb{E}[C_\omega] < \infty$.
- *The random field ω is shift invariant. This means that for any finite number of edges $e_1, \dots e_k \in E(\mathbb{Z})$, any integer $z \in \mathbb{Z}$, and any bounded and measurable function $\xi : \mathbb{R}^k \to \mathbb{R}$,*

  $$\mathbb{E}\big[\xi(\omega_{e_1}, \dots, \omega_{e_k})\big] = \mathbb{E}\big[\xi(\omega_{\tau_z(e_1)}, \dots, \omega_{\tau_z(e_k)})\big]\,,$$

  where $\tau_z : E(\mathbb{Z}) \to E(\mathbb{Z})$ is the shift by z (as per the Notation and Conventions above).
- *Moreover, the random field ω is ergodic with respect to the set of shifts $\{\tau_z \,|\, z \in \mathbb{Z},\, z \equiv 0 \pmod 2\}$. This means that if $E \subset \Omega$ is a shift invariant event, i.e. if $E = \tau_2^{-1}(E)$, then $\mathbb{P}(E) \in \{0, 1\}$.*
- *We assume w.l.o.g. (as a matter of normalization) that*

  $$\mathbb{E}[\omega_{e_{0,1}}] = 0\,,$$

  where $e_{0,1}$ is the edge from 0 to 1 in $\mathbb{Z}$.

Example 2.10. The simplest non-trivial example of a random field ω that satisfies Assumption 2.9 is the i.i.d. field. Let X denote a bounded (real) random variable with mean 0, and let $(\omega_e)_{e\in E(\mathbb{Z})}$ denote a family of i.i.d. copies of X.

Remark 2.11. The assumptions of shift invariance and ergodicity are standard in homogenization literature; see for example the "usual conditions" for the random conductance model from [5, Definition 3.1]. However we point out one difference: the random field ω is ergodic with respect to the even shifts $\{\tau_z \,|\, z \equiv 0 \pmod 2\}$. This is a stronger condition than being ergodic with respect to the full set of shifts $\{\tau_z \,|\, z \in \mathbb{Z}\}$. This requirement is due to the earlier assumption made in Definition 2.2 that height functions preserve parity. As such, we cannot simply shift a height function up or down by 1 in the height space; if $h_{S_n}(z) = k \in \mathbb{Z}$, then there is no (parity-preserving) height function "$\tau_1 h_{S_n}$" such that $\tau_1 h_{S_n}(z) = k+1$. More concretely, the family of measure-preserving translations used in the proof of Lemma 3.14 below includes all of the shifts $\{\tau_z \,|\, z \equiv 0 \pmod 2\}$ and none of the shifts $\{\tau_z \,|\, z \equiv 1 \pmod 2\}$, hence the stronger ergodicity assumption is technically required.

In this article we study the random surfaces in the random potential defined by ω. In homogenization one considers two different situations: In the quenched case, one considers the measure μ_ω for fixed ω. In the annealed case, one takes the expectation with respect to ω. Our goal is to show that the variational principle holds with high probability. With that context in mind, we define the quenched Hamiltonian $H_{R_n}(\cdot) = H_{R_n}(\cdot,\omega)$ and the quenched measure μ_ω as follows:

Definition 2.12 (The quenched Hamiltonian)**.** *For finite subsets $R_n \subset \mathbb{Z}^m$, We define the Hamiltonian H_{R_n} as follows: for a fixed boundary height function $h_{\partial R_n} : \partial R_n \to \mathbb{Z}$, and for any height function $h_{R_n} \in M(R_n, h_{\partial R_n})$ and any realization ω of the random field,*

$$H_{R_n}(h_{R_n},\omega) = \sum_{e\in E(R_n)} \omega_{h_{R_n}(e)}, \tag{2.2}$$

where $E(R_n) = \{e_{x,y} \,|\, x, y \in R_n\}$ is the edge set of the subgraph of $\mathbb{Z}^m$ induced by R_n. The edge $h_{R_n}(e) \in E(\mathbb{Z})$ is given by the lifted height function, i.e. $h_{R_n}(e_{x,y}) := e_{h_{R_n}(x),h_{R_n}(y)}$ (see also Definition 2.2).

Definition 2.13 (Quenched Gibbs measure)**.** *Given a realization ω of the random field and a set $A \subset M(R_n)$ of height functions, the partition function $Z_\omega(A)$ is given by*

$$Z_\omega(A) = \sum_{h_{R_n}\in A} \exp\bigl(H_{R_n}(h_{R_n},\omega)\bigr).$$

For a fixed boundary data function $h_{\partial R_n} \in M(\partial R_n)$, the quenched Gibbs measure μ_ω on $M(R_n, h_{\partial R_n})$ is defined by

$$\mu_\omega(h_{R_n}) = \frac{1}{Z_\omega\bigl(M(R_n,h_{\partial R_n})\bigr)} \exp\bigl(H_{R_n}(h_{R_n},\omega)\bigr).$$

Remark 2.14. If one chooses the constant field $\omega = \mathbf{0} = (0)_{e\in E(\mathbb{Z})}$, then the associated quenched Gibbs measure $\mu_{\mathbf{0}}$ is the uniform measure on $M(R_n, h_{\partial R_n})$. In this case one recovers the variational principle of [31].

Remark (The role of the random field ω)**.** The values ω_e of the random field ω modulate the likelihood of seeing certain height levels. More precisely, let us consider two edges $e, \tilde{e} \in E(\mathbb{Z})$ that denote two different height levels. Let us assume that $\omega_e \gg \omega_{\tilde{e}}$. We consider two height function h and $\tilde{h}$, the first one oscillating around the height level e and the second one around the height level $\tilde{e}$. From (2.2) we get that the Hamiltonian $H_{R_n}(h) \gg H_{R_n}(\tilde{h})$. Because in the quenched Gibbs measure μ_ω height functions with larger Hamiltonian are exponentially more likely, we will observe the height function h more often compared to $\tilde{h}$ when sampling from μ_ω.

Now let us introduce the microscopic entropy of our model. Again there are two situations: first, the quenched case, defined for a fixed realization ω and the annealed case.

Definition 2.15 (Quenched and annealed microscopic entropy)**.** *Given a domain $R_n \subset \mathbb{Z}^m$ and a finite non-empty subset $A \subset M(R_n)$, the quenched microscopic entropy $\mathrm{Ent}_{R_n}(A,\omega)$ is given by*

$$\mathrm{Ent}_{R_n}(A,\omega) := -\frac{1}{|R_n|}\log Z_\omega(A) \\ \left(= -\frac{1}{|R_n|}\log \sum_{h_{R_n}\in A} \exp\big(H_{R_n}(h_{R_n},\omega)\big)\right).$$

The annealed microscopic entropy $\mathrm{Ent}(R_n, h_{\partial R_n})$ is given by

$$\mathrm{Ent}_{R_n,\mathsf{an}}(A) := \mathbb{E}\big[\mathrm{Ent}_{R_n}(A,\omega)\big].$$

Remark 2.16. As in Remark 2.14, if one chooses the constant field $\omega = \mathbf{0}$, then the quenched microscopic entropy $\mathrm{Ent}_{R_n}(M(R_n, h_{\partial R_n}), \mathbf{0})$ is the same as the microscopic entropy of [31].

Next, we define the local surface tension. As with the microscopic entropy, the local surface tension admits both a quenched and an annealed version.

Definition 2.17 (Quenched microscopic and local surface tension)**.** *The quenched local surface tension is the a.s.-limit*

$$\mathrm{ent}(s,\omega) := \lim_{n\to\infty} \mathrm{ent}_n(s,\omega)\,, \tag{2.3}$$

where $\mathrm{ent}_n(s,\omega)$ is the quenched microscopic surface tension, defined by

$$\mathrm{ent}_n(s,\omega) := \mathrm{Ent}_{S_n}\left(M(S_n, h^s_{\partial S_n}),\omega\right).$$

Recall from Notation and Conventions above that $S_n = \{-n,\dots,n\}^m$, and note that the existence of the limit in (2.3) *is the content of Lemma 3.14.*

Definition 2.18 (Annealed microscopic and local surface tension)**.** *The annealed microscopic surface tension $\mathrm{ent}_{n,\mathsf{an}}(s)$ is given by*

$$\mathrm{ent}_{n,\mathsf{an}}(s) := \mathbb{E}\left[\mathrm{ent}_n(s,\omega)\right],$$

and the annealed local surface tension $\mathrm{ent}_{\mathsf{an}}(s)$ is given by

$$\mathrm{ent}_{\mathsf{an}}(s) := \mathbb{E}\left[\mathrm{ent}(s,\omega)\right].$$

Remark 2.19. Similarly to Remark 2.14 and Remark 2.16, we obtain back the local surface tension for the uniform measure if we consider a constant random field $\omega = \mathbf{0}$. In the case of random potential, it follows from Assumption 2.9 and Lemma 3.2 that $\mathrm{ent}_n(s,\omega)$ is uniformly integrable and therefore that $\mathrm{ent}_{n,\mathsf{an}}$ and $\mathrm{ent}_{\mathsf{an}}$ are well-defined.

Remark 2.20. It is not hard to see that the annealed local surface tension is also the limit of the annealed microscopic surface tension. Indeed, from Assumption 2.9 the quenched microscopic surface tension $\mathrm{ent}_n(s,\omega)$ is dominated by an L^1 function (see Lemma 3.2). Therefore, the dominated convergence theorem implies that

$$\lim_{n\to\infty}\mathrm{ent}_{n,\mathsf{an}}(s) = \lim_{n\to\infty}\mathbb{E}\left[\mathrm{ent}_n(s,\omega)\right] = \mathbb{E}\left[\lim_{n\to\infty}\mathrm{ent}_n(s,\omega)\right] = \mathrm{ent}_{\mathsf{an}}(s)\,.$$

The annealed macroscopic entropy is defined by:

Definition 2.21 (Annealed macroscopic entropy)**.** *Given an asymptotic height function $h_R \in M(R, h_{\partial R})$, the annealed macroscopic entropy $\mathrm{Ent}_{R,\mathsf{an}}(h_R)$ is defined by*

$$\mathrm{Ent}_{R,\mathsf{an}}(h_R) := \int_R \mathrm{ent}_{\mathsf{an}}(\nabla h_R(x))\, dx\,.$$

The first main result of this article is the profile theorem:

Theorem 2.22 (Profile theorem)**.** *Recall that $C_\omega := 1 \vee \sup_{e \in E(\mathbb{Z})} |\omega_e|$ is by Assumption 2.9 an L^1 random variable. Then for any $h_R \in M(R, h_{\partial R})$ and any $\eta > 0$, there exist functions $\theta_{h_R}(\delta)$ and $\theta_{h_R,\delta}(\frac{1}{n})$ with $\theta_{h_R}(\delta) \to 0$ as $\delta \to 0$ and $\theta_{h_R,\delta}(\frac{1}{n}) \to 0$ as $n \to \infty$ such that*

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\Bigg(\Big|\mathrm{Ent}_{R_n}\big(B(R_n, h_R, \delta), \omega\big) - \mathrm{Ent}_{\mathsf{an}}(R, h_R)\Big| \\ \geq \eta + C_\omega \theta_{h_R}(\delta) + C_\omega \theta_{h_R,\delta}\big(\tfrac{1}{n}\big)\Bigg) = 0\,.\end{aligned} \tag{2.4}$$

The second main result is the variational principle:

Theorem 2.23 (Variational principle)**.** *The random variables*

$$\mathrm{Ent}_{R_n}(M(R_n, h_{\partial R_n}, \delta), \omega)$$

converge in probability to the infimum of $\mathrm{Ent}_{\mathsf{an}}(R, h_R)$ over asymptotic height functions $h_R \in M(R, h_{\partial R})$, i.e. for every $\eta > 0$,

$$\begin{aligned}\limsup_{\delta\to 0}\, \limsup_{n\to\infty}\, \mathbb{P}\Bigg(\Big|\,\mathrm{Ent}_{R_n}\big(M(R_n, h_{\partial R_n}, \delta), \omega\big) \\ - \inf_{h_R \in M(R, h_{\partial R})} \mathrm{Ent}_{R,\mathsf{an}}(h_R)\Big| \geq \eta\Bigg) = 0\,.\end{aligned}$$

The third main result, which we state but do not prove, is the large deviations principle. The notation introduced below is standard for large deviations theory.

Theorem 2.24 (Large deviations principle)**.** *Consider the space $M(R)$ of asymptotic height functions on R, endowed with the topology of uniform convergence. For $\delta > 0$ and $n \in \mathbb{N}$, define a random probability measure $\mu_{\delta,n}(\cdot, \omega)$ on $M(R)$ by*

$$\mu_{\delta,n}(A, \omega) := \frac{Z_\omega\big(\{h_{R_n} \in M(R_n, h_{\partial R_n}, \delta) \,|\, \tilde{h}_{R_n} \in A\}\big)}{Z_\omega\big(M(R_n, h_{\partial R_n}, \delta)\big)}\,,$$

where $\tilde{h}_{R_n} \in M(R)$ denotes the asymptotic height function given by rescaling and interpolating $h_{R_n} \in M(R_n)$, i.e. $\tilde{h}_{R_n}(\frac{1}{n}z) = \frac{1}{n} h_{R_n}(z)$ for $z \in R_n$.

Then the measures $\mu_{\delta,n}$ satisfy a large deviations principle in probability with rate functional I given by

$$I(h_R) := \begin{cases} \mathrm{Ent}_{R,\mathsf{an}}(h_R) - E & \text{if } h_R \in M(R, h_{\partial R})\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

where $E := \inf_{h_R \in M(R, h_{\partial R})} \mathrm{Ent}_{R,\mathsf{an}}(h_R)$. Specifically, this means that for any Borel set $A \subset M(R)$,

$$\limsup_{\delta\to 0}\, \limsup_{n\to\infty}\, \mathbb{P}\bigg(\frac{1}{|R_n|} \log \mu_{\delta,n}(A) \;\geq\; -\inf_{h_R \in A^\circ} I(h_R)\bigg) = 0$$

and

$$\limsup_{\delta\to 0}\, \limsup_{n\to\infty}\, \mathbb{P}\bigg(\frac{1}{|R_n|} \log \mu_{\delta,n}(A) \;\leq\; -\inf_{h_R \in \overline{A}} I(h_R)\bigg) = 0\,,$$

where A° denotes the interior of A and $\overline{A}$ denotes the closure.

3 The quenched and annealed local surface tension

The purpose of this section is to establish several fundamental properties of the quenched entropy and local surface tension of our model. We proceed as follows:

- In Section 3.1 we state the Kirszbraun theorem, used heavily in the rest of this section and beyond.
- In Section 3.2 we derive robustness of the entropy and local surface tension under boundary value changes.
- In Section 3.3 we prove the existence of the quenched local surface tension and the equivalence between the quenched and annealed local surface tension.
- In Section 3.4 we study the local surface tension as a function $s \mapsto \mathrm{ent}_{\mathsf{an}}(s)$, and we show that this function is convex and continuous.

3.1 Kirszbraun theorem

We consider a discrete analogue of the classical Kirszbraun theorem of [30]. The classical theorem gives a condition under which a Lipschitz continuous function can be extended from a subset of a domain to the entirety of that domain. Likewise, the Kirszbraun theorem for graph homomorphisms gives a condition under which a $\mathbb{Z}$-valued graph homomorphism may be extended from a subset of a domain to the entire domain. Note that the property of being a $\mathbb{Z}$-valued graph homomorphism is stronger than the Lipschitz property with constant 1, since if $z \sim \tilde{z}$ are two adjacent points in the domain of a graph homomorphism $h : S \to \mathbb{Z}$, then $h(z) \neq h(\tilde{z})$.

Theorem 3.1 (Kirszbraun)**.** *Let Λ be a connected region of $\mathbb{Z}^m$, let S be a subset of Λ, and let $\bar{h} : S \to \mathbb{Z}$ be a graph homomorphism that preserves parity. There exists a graph homomorphism $h : \Lambda \to \mathbb{Z}$ such that $h = \bar{h}$ on S if and only if for all $x, y \in S$,*

$$d_{\mathbb{Z}}(\bar{h}(x), \bar{h}(y)) \leq d_\Lambda(x, y), \tag{3.1}$$

where $d_{\mathbb{Z}}$ and d_Λ denote respectively the graph distance on $\mathbb{Z}$ and on $\Lambda \subset \mathbb{Z}^m$.

This is a well-known result (see e.g. [44, Lemma 4.3.1]), and we omit the proof from this article. As an illustration of the usefulness of the Kirszbraun theorem, we prove the following lemma, which justifies the choice of the normalizing factor $\frac{1}{|R_n|}$ in Definition 2.15:

Lemma 3.2. *Almost surely (in terms of the distribution $\mathbb{P}$ of the random field ω),*

$$-\log(2) - 2mC_\omega \leq \mathrm{Ent}_{R_n}\big(M(R_n, h_{\partial R_n}), \omega\big) \leq 2mC_\omega \,.$$

Proof. As a corollary of the Kirszbraun theorem (Theorem 3.1), there is always at least one height function $h_0 \in M(R_n, h_{\partial R_n})$. So,

$$\begin{aligned} \mathrm{Ent}_{R_n}\big(M(R_n, h_{\partial R_n}), \omega\big) &\leq -\frac{1}{|R_n|} \log \sum_{h \in \{h_0\}} \exp\left(\sum_{e \in E(R_n)} \omega_{e_{h(x),h(y)}} \right) \\ &\leq \frac{|E(R_n)|}{|R_n|} C_\omega \\ &\leq 2mC_\omega \,. \end{aligned}$$

On the other hand, we overestimate the cardinality of $M(R_n, h_{\partial R_n})$ as follows: enumerate the points of the interior of R_n, in such a way that each point x_i is adjacent to the previous point x_{i-1} (and the first point x_1 is adjacent to $x_0 \in \partial R_n$). For each point x_i

in the enumeration, we require that $h(x_i) = h(x_{i-1}) \pm 1$, so there are at most 2 choices for $h(x_i)$. All together, $|M(R_n, h_{\partial R_n})| \leq 2^{|R_n|}$. It follows that

$$\begin{aligned}
\mathrm{Ent}(R_n, h_{\partial R_n}, \omega) &\geq -\frac{1}{|R_n|} \log\Big(|M(R_n, h_{\partial R_n})| \exp\big(C_\omega |E(R_n)|\big)\Big) \\
&\geq -\frac{1}{|R_n|} \log 2^{|R_n|} - \frac{|E(R_n)|}{|R_n|} C_\omega \\
&\geq -\log(2) - mC_\omega. \qquad \square
\end{aligned}$$

In the sequel, we will usually use the Kirszbraun theorem in the following setting. Given two domains $R_{n_1} \subset R_{n_2} \subset \mathbb{Z}^m$, a height function $h_{R_{n_1}} \in M(R_{n_1})$, and a boundary height function $h_{\partial R_{n_2}} \in M(\partial R_{n_2})$, there exists an extension $\tilde{h}_{R_{n_2}} \in M(R_{n_2})$ with $\tilde{h}_{R_{n_2}}|_{R_{n_1}} = h_{R_{n_1}}$ and $\tilde{h}_{R_{n_2}}|_{\partial R_{n_2}} = h_{\partial R_{n_2}}$ if and only if

$$|h_{R_{n_1}}(z_1) - h_{\partial R_{n_2}}(z_2)| \leq |z_1 - z_2|_1 \quad \text{for all } z_1 \in \partial R_{n_1}, z_2 \in \partial R_{n_2}\,.$$

3.2 Robustness of the quenched entropy

In this section we show that the quenched microscopic entropy and local surface tensions are robust, in the sense that small changes in boundary values cause small changes in the numeric value of the entropy. There are two steps in proving these robustness results: First, just as for the unperturbed model of [31], compare the two sets of height functions associated with the two boundary value functions, perhaps by exhibiting an injection from one set into the second or by estimating cardinalities directly. Second, show that individual height functions from each of the two sets contribute comparable amounts to the entropy after applying the random potential, e.g. by showing that every height function in one set admits a “similar” height function in the second set, whose Hamiltonian value is not much different; this step is sometimes straightforward and other times quite subtle.

Before stating the results of this section, let us recall the definition of the quenched microscopic surface tension $\mathrm{ent}_n(s, \omega)$ and how it is related to the quenched microscopic entropy $\mathrm{Ent}_{R_n}(A, \omega)$ of some set $A \subset M(R_n)$ of height functions. By Definition 2.15 it holds that

$$\mathrm{Ent}_{R_n}(A, \omega) = -\frac{1}{|R_n|} \log \sum_{h \in A} \exp\left(\sum_{e_{x,y} \in E(R_n)} \omega_{h(e_{x,y})}\right),$$

where $h(e_{x,y}) = e_{h(x),h(y)}$ denotes the lifted height function. We recall that $M(S_n, h^s_{\partial S_n})$ denotes the set of height functions with linear boundary condition (see also Definition 2.7). By Definition 2.17 it holds

$$\mathrm{ent}_n(s, \omega) := \mathrm{Ent}_{R_n}(M(S_n, h^s_{\partial S_n}), \omega).$$

The next statement shows the robustness of the microscopic local surface tension under small changes in the boundary condition, if the l_∞ norm $|s|\infty$ of the slope s is bounded away from 1.

Lemma 3.3. *Let $\alpha > 0$, let $s \in \mathbb{R}^m$ with $|s|_\infty \leq 1 - \alpha$, let $\varepsilon \in (0, \frac{\alpha}{2})$, let $n \in \mathbb{N}$ with $n \geq (1 - \frac{2\varepsilon}{\alpha})^{-1}$, and let $h_{\partial S_n} \in M(\partial S_n, s, \varepsilon)$. Write*

$$n^+ := \left\lceil (1 + \tfrac{2\varepsilon}{\alpha}) n \right\rceil \qquad \text{and} \qquad n^- := \left\lfloor (1 - \tfrac{2\varepsilon}{\alpha}) n \right\rfloor.$$

(We remark that $1 \leq n^- < n < n^+$.) Then,

$$\begin{aligned}
\mathrm{ent}_{n^+}(s, \omega) - C_\omega\, \theta_m\big(\tfrac{\varepsilon}{\alpha}\big) &\leq \mathrm{Ent}_{S_n}\big(M(S_n, h_{\partial S_n}), \omega\big) \\
&\leq \mathrm{ent}_{n^-}(s, \omega) + C_\omega\, \theta_m\big(\tfrac{\varepsilon}{\alpha}\big)\,.
\end{aligned} \tag{3.2}$$

Here, $\theta_m : \mathbb{R} \to \mathbb{R}$ denotes a universal function, that may depend on the dimension m of the lattice, satisfying $\lim_{\epsilon\to 0} \theta_m(\varepsilon) = 0$. The constant $C_\omega := 1 \vee \sup_{e\in E(\mathbb{Z})} |\omega_e|$, defined in the Assumption 2.9, is an upper bound on the magnitude of the random field ω that may depend on the realization ω of the random field.

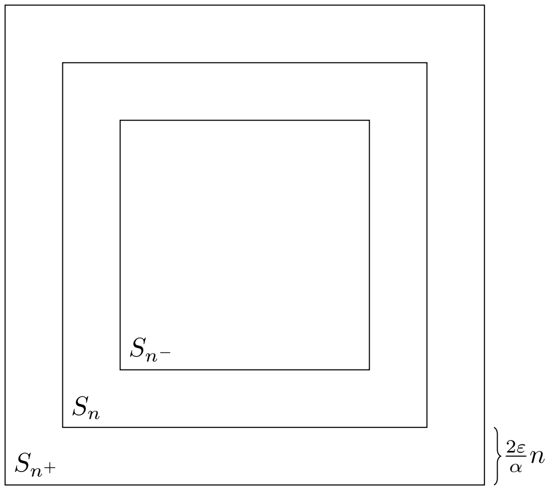


Figure 4: Nested domains from Lemma 3.3.

Proof of Lemma 3.3. We prove the inequality

$$\mathrm{ent}_{n^+}(s,\omega) - C_\omega\theta_m\big(\tfrac{\varepsilon}{\alpha}\big) \le \mathrm{Ent}_{S_n}\big(M(S_n, h_{\partial S_n}),\omega\big).$$

The proof of the reverse inequality is similar.

Note that the smaller square $S_n = \{-n, -(n-1), \dots, n-1, n\}^m$ is contained inside the larger square S_{n^+}, and that

$$|x-y|_1 \ge \tfrac{2\varepsilon}{\alpha} n \qquad \text{whenever } x \in \partial S_n \text{ and } y \in \partial S_{n^+}\,. \tag{3.3}$$

We construct an injection from $M(S_n, h_{\partial S_n})$ into $M(S_{n^+}, h^s_{\partial S_{n^+}})$ using the Kirszbraun theorem, Theorem 3.1. Let $h_{S_n} \in M(S_n, h_{\partial S_n})$, let $x \in \partial S_n$, and let $y \in \partial S_{n^+}$. By the definitions of $M(S_n, h_{\partial S_n})$ and of $h^s_{\partial S_n}$,

$$\begin{aligned}
&\big|h_{S_n}(x) - h^s_{S_{n^+}}(y)\big| \\
&\qquad \le \big|h_{S_n}(x) - s\cdot x\big| + \big|s\cdot(x-y)\big| + \big|h^s_{S_{n^+}}(y) - s\cdot y\big| \\
&\qquad \le \varepsilon n + |s|_\infty |x-y|_1 + 1.
\end{aligned}$$

By hypothesis $|s|_\infty \le 1-\alpha$ and by (3.3), $\varepsilon n \le \frac{\alpha}{2}|x-y|_1$. Therefore for $n \ge \frac{2}{\alpha}$,

$$|h_{S_n}(x) - h^s_{S_{n^+}}(y)| \le |x-y|_1\,,$$

so h_{S_n} admits an extension $h_{S_{n^+}} \in M(S_{n^+}, h^s_{\partial S_{n^+}})$. The map $h_{S_n} \mapsto h_{S_{n^+}}$ is an injection from $M(S_n, h_{\partial S_n})$ into $M(S_{n^+}, h^s_{\partial S_{n^+}})$. The existence of such an injection implies immediately that

$$\begin{aligned}
\mathrm{Ent}_{S_n}\big(M(S_n, h_{\partial S_n}),\omega\big) &\ge \frac{|S_n|}{|S_{n^+}|}\,\mathrm{Ent}_{S_{n^+}}\big(M(S_{n^+}, h^s_{\partial S_{n^+}}),\omega\big) \\
&\qquad - \frac{2mC_\omega(|S_{n^+}| - |S_n|)}{|S_n|} \\
&= \mathrm{Ent}_{S_{n^+}}\big(M(S_{n^+}, h^s_{\partial S_{n^+}}),\omega\big) - C_\omega\theta_m\big(\tfrac{\varepsilon}{\alpha}\big)\,.
\end{aligned}$$

This proves the first inequality of (3.2). As mentioned at the beginning of the proof, the other inequality is similar. Since $n^- < n$, one extends height functions from $M(S_{n^-}, h_{\partial S^s_{n^-}})$ to $M(S_n, h_{\partial S_n})$. We omit the details. $\square$

Lemma 3.3 does not extend to the case where $|s|_\infty = 1$. As $|s|_\infty \to 1$ the ratio of the box sizes $\frac{|S_{n^+}|}{|S_n|} \approx 1 + \frac{\varepsilon}{\alpha}$ and the error bound $\theta(\frac{\varepsilon}{\alpha})$ both diverge. Fundamentally these difficulties come from the Kirszbraun theorem. When $|s|_\infty$ is close to 1, the "margin" $S_{n^+} \setminus S_n$ must be large in order to connect $h_{\partial S_n}$ to $h_{\partial S_{n^+}}$ and when $|s|_\infty = 1$, such an extension is not generally possible. Therefore we take a different approach for $|s|_\infty \approx 1$, using elementary combinatorics to count the number of height functions. The two following calculations are intermediate results used to prove the robustness lemma, Lemma 3.6.

Lemma 3.4 (Counting height functions near $|s|_\infty = 1$)**.** *Let $\varepsilon > 0$. Let $s \in \mathbb{R}^m$ with $1 - \varepsilon < |s|_\infty \leq 1$, and let $h_{\partial S_n} \in M(\partial S_n, h^s_{\partial S_n}, \varepsilon)$. Then,*

$$\frac{1}{|S_n|} \log\big|M(S_n, h_{\partial S_n})\big| = \theta(\varepsilon)\,. \tag{3.4}$$

Proof of Lemma 3.4. Fix a coordinate index $1 \leq i \leq m$ such that $|s_i| > 1 - \varepsilon$, and assume without loss of generality that $s_i > 1 - \varepsilon$. Decompose S_n into $(2n+1)^{m-1}$ lines in the i^{th} coordinate direction. Along each such line h_{S_n} must increase by at least $2(1-2\varepsilon)n$. Therefore, the $2n$ edges in the line split into two subsets: at least $2(1-2\varepsilon)n$ "increasing" edges, and at most $4\varepsilon n$ "decreasing" edges. Counting each line independently, we conclude that

$$\big|M(S_n, h_{\partial S_n})\big| \leq \binom{2n}{\lceil 4\varepsilon n \rceil}^{(2n+1)^{m-1}}.$$

The conclusion (3.4) follows immediately. For a more verbose version of this proof, see [31, Lemma 21]. $\square$

Lemma 3.5 (Height functions at slope $|s|_\infty = 1$)**.** *Let $s' \in \mathbb{R}^m$ with $|s'|_\infty = 1$. Then $|M(S_n, h^{s'}_{\partial S_n})| = 1$, and the sole element of $M(S_n, h^{s'}_{\partial S_n})$ is the canonical height function $h^{s'}_{S_n}$.*

Proof of Lemma 3.5. As in the proof of Lemma 3.4, fix a coordinate index $1 \leq i \leq m$ such that $|s_i| = 1$. Decompose S_n into lines in the i^{th} coordinate direction. Along each line, any height function $h_{S_n} \in M(S_n, h^{s'}_{\partial S_n})$ must increase by exactly $2n$. Since h_{S_n} is a graph homomorphism, that is only possible if h_{S_n} increases along every edge, i.e. $h_{S_n}(x+1, y) - h_{S_n}(x, y) = 1$ for $x = -n, \ldots, n-1$. It follows that $|M(S_n h^{s'}_{\partial S_n})| \leq 1$. To complete the proof, observe that $h^{s'}_{S_n} \in M(S_n, h^{s'}_{\partial S_n})$. $\square$

Having recorded Lemma 3.4 and 3.5, we return to establishing robustness results. As in Lemma 3.3, our goal is to compare the microscopic surface tension $\mathrm{ent}_n(s, \omega) := \mathrm{Ent}_{S_n}(M(S_n, h^s_{\partial S_n}), \omega)$ and the entropy $\mathrm{Ent}_{S_n}(M(S_n, h_{\partial S_n}), \omega)$ associated to an "approximately affine" boundary height function $h_{\partial S_n} \in M(\partial S_n, h^s_{\partial S_n}, \varepsilon)$. The difference is that Lemma 3.3 took $|s|_\infty \leq 1 - \alpha$ and the lemma below takes $|s|_\infty > 1 - \alpha$.

Lemma 3.6. *Let $\varepsilon > 0$. Let $s, s' \in \mathbb{R}^m$ with $|s|_\infty \leq 1$, $|s'|_\infty = 1$, and $|s - s'|_\infty < \varepsilon$. Let $n \in \mathbb{N}$ be sufficiently large (specifically, $n \geq \frac{1}{\varepsilon}$) and let $h_{\partial S_n} \in M(\partial S_n, h^s_{\partial S_n}, \varepsilon)$. Then:*

$$\big|\mathrm{Ent}(S_n, h_{\partial S_n}, \omega) - \mathrm{ent}_n(s', \omega)\big| \;\leq\; C_\omega\, \theta(\varepsilon)\,.$$

Because of the $\theta(\varepsilon)$ error term, Lemma 3.6 will not be useful for slopes s with $|s|_\infty$ far from 1.

Remark 3.7 (Comment about the proof)**.** There are two ingredients to the proof. The first is counting results of Lemma 3.4 and Lemma 3.5, and the second is a comparison between the Hamiltonian $H_{S_n}(h_{S_n},\omega)$ of a generic height function $h_{S_n} \in M(S_n, h_{\partial S_n})$ and the Hamiltonian $H_{S_n}(h^{s'}_{S_n},\omega)$ of the unique element $h^{s'}_{S_n} \in M(S_n, h^{s'}_{\partial S_n})$. Since proofs were already given for the two lemmas, most of the argument below is spent on the comparison of Hamiltonians.

The comparison of Hamiltonians is also fundamentally a combinatorial argument that relies on the rigidity caused by the slopes s and s' being close to (or on) the boundary of the slope space $[-1,1]^m$. It is surprising that such a subtle argument is (apparently) needed in the case of homogenization, since the two counting lemmas are sufficient in the uniform case, and these lemmas are not very complicated to prove.

The subtlety is similar to that of the proof of Lemma 3.9 below. In both cases, the difficulty arises when comparing Hamiltonians for two height functions defined on the same domain S_n. In comparison, the proof of Lemma 3.3 (which has a similar statement to the current Lemma 3.6) is based on extending height functions from one domain to another larger domain via the Kirszbraun theorem. Comparing the Hamiltonian of a height function on a large domain to the Hamiltonian of the same function on a restricted domain is simple, since the difference is exactly relatable to the difference in domains.

Proof of Lemma 3.6. As mentioned above, we will compare the Hamiltonians $H_{S_n}(h_{S_n},\omega)$ and $H_{S_n}(h^{s'}_{S_n},\omega)$, where $h_{S_n} \in M(S_n, h_{\partial S_n})$ and $h^{s'}_{S_n} \in M(S_n, h^{s'}_{\partial S_n})$. More precisely, we will later deduce the inequality

$$\left|H_{S_n}(h_{S_n},\omega) - H_{S_n}(h^{s'}_{S_n},\omega)\right| \le 210m^2(2n+1)^m C_\omega \varepsilon\,. \tag{3.5}$$

Given that (3.5) holds, the proof is straight-forward: For one inequality, we calculate

$$\begin{aligned}
&\mathrm{Ent}_{S_n}\big(M(S_n,h_{\partial S_n}),\omega\big)\\
&\quad= -\frac{1}{|S_n|}\log \sum_{h_{S_n}\in M(S_n,h_{\partial S_n})} \exp\big(H_{S_n}(h_{S_n},\omega)\big)\\
&\quad\overset{(3.5)}{\le} -\frac{1}{|S_n|}\log \sum_{h_{S_n}\in M(S_n,h_{\partial S_n})} \exp\bigg(H_{S_n}(h^{s'}_{S_n},\omega)\\
&\qquad\qquad\qquad\qquad\qquad\qquad - 210m^2(2n+1)^m C_\omega\varepsilon\bigg)\\
&\overset{Lemma\ 3.4}{\le} \quad -\frac{1}{|S_n|}H_{S_n}\big(h^{s'}_{S_n},\omega\big) + \theta(\varepsilon) + 210m^2 C_\omega\varepsilon\\
&\quad= \mathrm{Ent}_{S_n}\big(M(S_n,h^{s'}_{S_n}),\omega\big) + C_\omega\theta(\varepsilon)\,.
\end{aligned}$$

The opposite inequality is derived in the same way, which concludes the proof of Lemma 3.6 up to the verification of (3.5).

For convenience, let us use for the remaining argument the following convention: When denoting the Hamiltonian of $H(h_{S_n},\omega)$ we just write $H(h_{S_n})$, omitting the dependency on the random field ω.

Verification of (3.5): Heuristically, the estimate (3.5) makes sense. Because the slopes s and s' are ε-close to each other, and s' has slope 1, every height function $h_{S_n} \in M(S_n, h_{\partial S_n})$ has to behave similar to the canonical height function $h^{s'}_{S_n}$ of slope s'. Therefore, the difference in the associated energies, as measured by the Hamiltonian $H_{S_n}(h_{S_n})$ and $H_{S_n}(h^{s'}_{S_n})$, should vanish as $\varepsilon \to 0$.

To make this argument rigorous one needs to precisely estimate the number of heights that each height function h_{S_n} visits, i.e. the set $\{h_{S_n}(e) \,|\, e \in E(S_n)\}$ with multiplicities, and compare to the corresponding set for $h^{s'}_{S_n}$. This is relatively straight-forward on a one-dimensional lattice but unfortunately becomes much more subtle on a higher-dimensional lattice. To see why, consider the decomposition of the box S_n into lines. This leads a decomposition of the edges in $E(S_n)$ into *parallel edges* within a line, and *cross edges* connecting two lines. Without cross edges the one-dimensional argument would easily carry over, but controlling the cross edges is necessary as well. This control is accomplished by the sets G_y below.

To begin the rigorous verification of (3.5), pick an arbitrary height function $h_{S_n} \in M(S_n, h_{\partial S_n})$. As mentioned above, we decompose S_n into lines parallel to one of the coordinate axes. Assume by symmetry that $s = (s_1, s_2, \ldots, s_m)$ and $s' = (s'_1, \ldots, s'_m)$ satisfy $s'_1 = 1$ and (therefore) $s_1 > 1 - \varepsilon$. For $y \in \{-n, \ldots, n\}^{m-1}$ let ℓ_y denote the line in the first coordinate direction through $(0, y)$ in S_n, i.e.

$$\ell_y := \big\{(-n, y),\, (-n+1, y),\, \ldots,\, (n-1, y),\, (n, y)\big\}\,.$$

Observe that S_n is the disjoint union of the $(2n+1)^{m-1}$ lines ℓ_y. In particular, the Hamiltonian $H_{S_n}(h_{S_n})$ decomposes with respect to the lines ℓ_y as

$$\begin{aligned} H_{S_n}(h_{S_n}) &:= \sum_{e \in E(S_n)} \omega_{h_{S_n}(e)} \\ &= \sum_y \left(\sum_{e \in E(\ell_y)} \omega_{h_{S_n}(e)} + \frac{1}{2} \sum_{y' \sim y} \sum_{e \in \tilde{E}_{y,y'}} \omega_{h_{S_n}(e)} \right) \\ &= \sum_y \tilde{H}_{\ell_y}(h_{S_n})\,, \end{aligned} \tag{3.6}$$

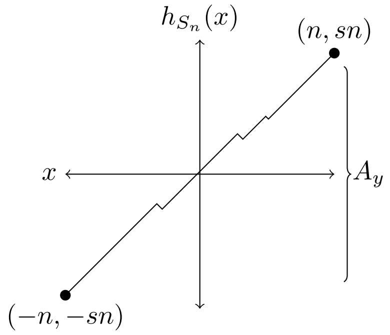
where $\tilde{E}_{y,y'}$ is the set of edges in $E(S_n)$ with one endpoint in ℓ_y and the other in $\ell_{y'}$ (we call these *cross edges*), and where $\tilde{H}_{\ell_y}$ is defined to be the parenthesized quantity from the line above. Note that the factor $\frac{1}{2}$ is necessary because each cross edge in $\tilde{E}_{y,y'}$ also contributes to $\tilde{H}_{\ell'_y}(h_{S_n})$, so without the factor $\frac{1}{2}$ the contributions from the cross edges would be double-counted.

We define two families of sets $A_y \subset E(\mathbb{Z})$ and $G_y \subset A_y$, indexed by points $y \in \{-n, \ldots, n\}^{m-1}$. In terms of the heuristic argument above, these sets roughly correspond to the heights visited by h_{S_n} and $h^{s'}_{S_n}$, although in fact both A_y and G_y are subsets of $\{h_{S_n}(e) \,|\, e \in E(S_n)\}$.

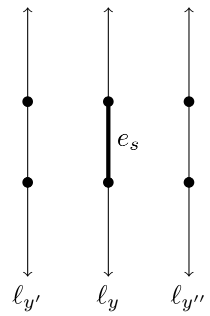
Let A_y denote the edges $e \in E(\mathbb{Z})$ that lie inside the interval from $(s \cdot (-n, y) + 2\varepsilon n)$ to $(s \cdot (+n, y) - 2\varepsilon n)$. Based on the boundary conditions and homomorphism property of h_{S_n} and $h^{s'}_{S_n}$, every edge $e \in A_y$ occurs both in the image $\{h_{S_n}(\tilde{e}) \,|\, \tilde{e} \in E(\ell_y)\}$ and in the image $\{h^{s'}_{S_n}(\tilde{e}) \,|\, \tilde{e} \in E(\ell_y)\}$ of the corresponding lifted height functions (see also Definition 2.2). (The factors of 2 in the definition of A_y are necessary since the boundary height function $h_{\partial S_n}$ may differ from $h^s_{\partial S_n}$ by up to εn, in addition to s_1 differing from 1 by up to ε.) The situation in dimension $m = 1$ is illustrated in Figure 5a.

We define $G_y \subset A_y$ in the following way: These are the edges $e \in A_y \subset E(\mathbb{Z})$ satisfying these three constraints with respect to h_{S_n} (illustrated in Figure 5b):

- e occurs with multiplicity 1 in the multi-set $\{h_{S_n}(\tilde{e}) \,|\, \tilde{e} \in E(\ell_y)\}$. (By choice of A_y, e occurs with multiplicity ≥ 1.) Write e_s for the unique edge $e_s \in E(\ell_y)$ such that $h_{S_n}(e_s) = e$.



(a) Here h_{S_n} is a one-dimensional height function with slope $s \geq 1 - \varepsilon$. The set A_y comprises the $1-4\varepsilon$ fraction of the $2n$ edges in ℓ_y, centered around 0. (The central height in higher dimensions is instead $s \cdot (0, y)$.) Both h_{S_n} and $h^{s'}_{S_n}$ must contain all of these edges in their image. They might contain additional edges.



(b) The three lines are ℓ_y in the center and two of its neighbors, $\ell_{y'}$ and $\ell_{y''}$. The highlighted edge is the edge $e_s \in E(\ell_y)$ for $e \in G_y$, i.e. the unique edge in ℓ_y with $h_{S_n}(e_s) = e$. There is also an edge $e_{s'}$ (not shown), satisfying the corresponding uniqueness property for $h^{s'}_{S_n}$. Finally, all six highlighted vertices are good, i.e. each vertex has a unique height within its line.

Figure 5: Figures relating to the proof of Lemma 3.6.

- Both endpoints of e occur with multiplicity 1 in the multi-set $\{h_{S_n}(z) \mid z \in \ell_y\}$.
- For each endpoint z of e_s and each neighboring vertex $z' \sim z$ that lies in $S_n \setminus \ell_y$, $h_{S_n}(z')$ occurs with multiplicity 1 in the multi-set $\{h_{S_n}(\tilde{z}) \mid \tilde{z} \in \ell_{y'}\}$ for the line $\ell_{y'}$ that contains it.

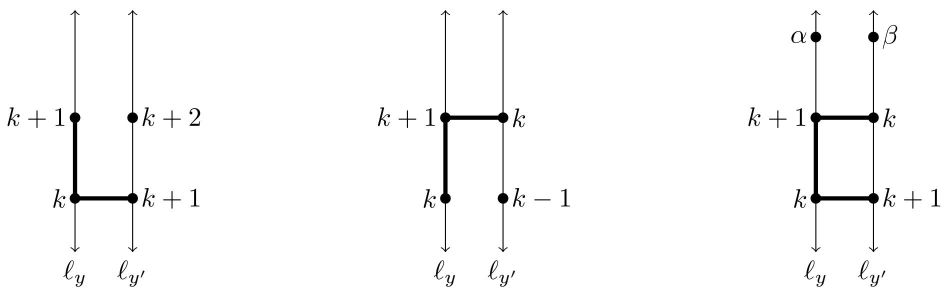
Further on in the argument, we will call elements of G_y "good" edges. We will call a vertex $z \in \ell_y$ "good" if its height $h_{S_n}(z)$ occurs in with multiplicity 1 in $\{h_{S_n}(\tilde{z}) \mid \tilde{z} \in \ell_y\}$, and likewise for $z' \in \ell_{y'}$.

Later on, we will need that for an arbitrary "good" edge $e \in G_y$ it holds:

$$\sum_{y' \sim y} \sum_{\substack{\tilde{e} \in \tilde{E}_{y,y'} \\ h_{S_n}(\tilde{e}) = e}} \omega_{h_{S_n}(\tilde{e})} = \big|\{y' \sim y\}\big| \, \omega_e. \tag{3.7}$$

Note that $|\{y' \sim y\}| \leq 2m$ for all y, with equality unless y is a boundary point (implicitly we assume that $y' \in \{-n, \ldots, n\}^{m-1}$). Argument for (3.7): We observe that for each $y' \sim y$, by using the second and third constraints and considering cases, there is a unique cross edge $e_{s,y'}$ between ℓ_y and $\ell_{y'}$ such that $h_{S_n}(e_{s,y'}) = e$. For a proof of this simple fact, we refer to Figure 6. The identity (3.7) follows then immediately.

We will also need to count $|G_y|$. Heuristically, since the slope s is close to 1, G_y must be a large subset of $E(\ell_y)$. To be precise, recall that $|A_y| \geq 2n - 4\lceil \varepsilon n \rceil$ by construction, and that G_y is the subset of edges $e \in A_y$ that satisfy the three constraints above. The second constraint actually implies the first, so to count G_y we simply count how many edges in A_y satisfy the last two constraints. Actually we count the complement, i.e. how many edges do not satisfy these two constraints. Indeed, each "bad" vertex in ℓ_y (in the sense described after the constraints) causes at most two edges in $E(\ell_y)$ to violate the second constraint. Likewise, each "bad" vertex in an adjacent line ℓ'_y causes at most two edges in $E(\ell_y)$ to violate the second constraint. All other edges in A_y are "good," i.e. are included in G_y.



(a) Case 1 (both adjacent height values larger): Clearly there is one edge between ℓ_y and $\ell_{y'}$ that is mapped to $e = e_{k,k+1}$. Suppose that another cross edge has heights k and $k+1$. Then its left endpoint would have either height k or height $k+1$, which contradicts the fact that the two labelled vertices in ℓ_y are "good," i.e. that their heights occur only once in ℓ_y.

(b) Case 2 (both adjacent height values smaller): Again there is one edge between ℓ_y and $\ell_{y'}$ that is mapped to $e = e_{k,k+1}$, and again no other vertices in ℓ_y can have either height k or height $k+1$.

(c) Case 3 (cannot occur because $e \in G_y$): Here there would be two edges between the lines that both map to $e_{k,k+1}$. But since the vertex at height k in ℓ_y is "good", the vertex labelled α must have height $k+2$. Likewise since the vertex at height $k+1$ in $\ell_{y'}$ is "good", vertex β must have height $k-1$. Since $\alpha \sim \beta$, this violates the graph homomorphism property.

Figure 6: Consideration of cases for part of the proof of Lemma 3.6. The claim to be shown is: given $e \in G_y$ (say $e = e_{k,k+1}$), there is a unique cross edge $e_{s,y'} \in \tilde{E}_{y,y'}$ which is mapped to e by the height function h_{S_n}. In the figure, the vertices are labelled by their heights, i.e. by the values of h_{S_n}. The bolded edge in ℓ_y is $e_s \in E(\ell_y)$, i.e. the unique edge in ℓ_y with $h_{S_n}(e_s) = e$. In Figure 6a and Figure 6b, the bolded edge between the lines is the unique edge between the lines with height $e_{k,k+1}$. Figure 6c shows two such edges, but in fact this case cannot occur. By the homomorphism property, these three cases exhaust the possibilities for heights on the two vertices in $\ell_{y'}$ that are adjacent to the endpoints of e_s.

It remains to count the "bad" vertices in any line ℓ_y. Since $s_1 > 1-\varepsilon$ and since h_{S_n} approximates the slope-s height function $h^s_{S_n}$ on ∂S_n, the height values $h_{S_n}(-n,y)$ and $h_{S_n}(+n,y)$ on the endpoints of ℓ_y differ by at least $2n - 4\varepsilon n$. Since h_{S_n} is a graph homomorphism, it maps the $2n+1$ vertices in ℓ_y surjectively onto the set of $\geq 2n - 4\lceil \varepsilon n \rceil + 1$ integers between the heights of the endpoints. By the pigeonhole principle, at most $8\lceil \varepsilon n \rceil$ of these integers occur with multiplicity ≥ 2, i.e. at most $8\lceil \varepsilon n \rceil$ vertices are "bad." Thus

$$\begin{aligned} |G_y| &\geq |A_y| - 2\left|\left\{\text{“bad” vertices in } \ell_y \text{ or } \ell_{y'} \text{ (for } y' \sim y)\right\}\right| \\ &\geq \underbrace{2n - 4\lceil \varepsilon n \rceil}_{|A_y|} - 2 \cdot \underbrace{(2m+1)}_{\text{\# lines}} \cdot \underbrace{8\lceil \varepsilon n \rceil}_{\text{“bad” vertices per line}} \\ &= 2n - (32m+20)\lceil \varepsilon n \rceil \\ &\geq 2n - 52m\lceil \varepsilon n \rceil\,. \end{aligned} \tag{3.8}$$

Now we work towards the Hamiltonian estimate (3.5). Let $e \in G_y$, and recall that e_s is the unique edge in $E(\ell_y)$ such that $h_{S_n}(e_s) = e$, and that $e_{s,y'}$ is the unique cross edge between ℓ_y and $\ell_{y'}$ such that $h_{S_n}(e_{s,y'}) = e$. As a result (recall the definitions of $\tilde{H}_{\ell_y}$ and

$\tilde{E}_{y,y'}$ from (3.6) above):

$$\begin{aligned}\tilde{H}_{\ell_y}(h_{S_n}) &= \Big(\sum_{\tilde{e}\in E(\ell_y)} \omega_{h_{S_n}(\tilde{e})}\Big) + \frac{1}{2}\Big(\sum_{y'\sim y}\sum_{\tilde{e}\in\tilde{E}_y} \omega_{h_{S_n}(\tilde{e})}\Big) \\ &\overset{(3.7)}{=} \Big(\sum_{e\in G_y} \omega_e + \sum_{\substack{\tilde{e}\in E(\ell_y)\\ h_{S_n}(\tilde{e})\notin G_y}} \omega_{h_{S_n}(\tilde{e})}\Big) \\ &\qquad + \frac{1}{2}\Big(\big|\{y'\sim y\}\big|\sum_{e\in G_y}\omega_e + \sum_{y'\sim y}\sum_{\substack{\tilde{e}\in\tilde{E}_{y,y'}\\ h_{S_n}(\tilde{e})\notin G_y}} \omega_{h_{S_n}(\tilde{e})}\Big)\,,\end{aligned}$$

so

$$\begin{aligned}&\Big|\tilde{H}_{\ell_y}(h_{S_n}) - \big(\tfrac{1}{2}\big|\{y'\sim y\}\big|+1\big)\sum_{e\in G_y}\omega_e\Big| \\ &\quad\le C_\omega\big(|E(\ell_y)| - |G_y|\big) + \frac{1}{2}\sum_{y'\sim y} C_\omega\big(|\tilde{E}_{y,y'}| - |G_y|\big) \\ &\quad\overset{(3.8)}{\le} 52mC_\omega\lceil\varepsilon n\rceil + \frac{1}{2}\sum_{y'\sim y} C_\omega\big(52m\lceil\varepsilon n\rceil+1\big) \\ &\quad\le 52mC_\omega\lceil\varepsilon n\rceil(1+m) + mC_\omega \\ &\quad\le 104m^2C_\omega\lceil\varepsilon n\rceil + mC_\omega \\ &\quad\le 105m^2(2n+1)C_\omega\varepsilon\,.\end{aligned}$$

In the last line, we assume that $n\ge\frac{1}{\varepsilon}$, so that $(2n+1)\varepsilon\ge\lceil\varepsilon n\rceil\ge 1$.

Because $s'_1=1$, $h^{s'}_{S_n}|_{\ell_y}$ is an injection, the three bullet points above are also satisfied with $h^{s'}_{S_n}$ in place of h_{S_n}. Therefore the calculation above also applies with $h^{s'}_{S_n}$ in place of h_{S_n}, so

$$\Big|\tilde{H}_{\ell_y}(h^{s'}_{S_n}) - \big(\tfrac{1}{2}\big|\{y'\sim y\}\big|+1\big)\sum_{e\in G_y}\omega_e\Big| \le 105m^2(2n+1)C_\omega\varepsilon\,.$$

By the triangle inequality,

$$\big|\tilde{H}_{\ell_y}(h_{S_n}) - \tilde{H}_{\ell_y}(h^{s'}_{S_n})\big| \le 210m^2(2n+1)C_\omega\varepsilon\,.$$

By summing over $y\in\{-n,\cdots,n\}^{m-1}$, we get the desired inequality (3.5), i.e.

$$\big|H_{S_n}(h_{S_n}) - H_{S_n}(h^{s'}_{S_n})\big| \le 210m^2(2n+1)^mC_\omega\varepsilon\,.$$

Both Lemma 3.3 and Lemma 3.6 imply that the microscopic entropy is robust to changes in boundary data, but they apply in different regimes. The former result applies when the boundary data has slope s with norm $|s|_\infty$ bounded away from 1, and the latter when the slope s has norm close to 1. For convenience later on, we combine the two results into a single theorem.

Theorem 3.8. *For any $\varepsilon\in(0,\frac{1}{9})$ and any slope $s\in[-1,1]^m$, there exist $A=A(s,\varepsilon)>0$, $B=B(s,\varepsilon)>0$, and $n_0=\lceil\frac{1}{\varepsilon}\rceil\in\mathbb{N}$ such that, for any $n\ge n_0$ and any boundary height function $h_{\partial S_n}\in M(\partial S_n, h^s_{\partial S_n},\varepsilon)$,*

$$\begin{aligned}\mathrm{ent}_{An}(s,\omega) - C_\omega\,\theta(\varepsilon) &\le \mathrm{Ent}\big(M(S_n,h_{\partial S_n}),\omega\big) \\ &\le \mathrm{ent}_{Bn}(s,\omega) + C_\omega\,\theta(\varepsilon).\end{aligned}$$

Moreover, the functions $A(s,\varepsilon)$ *and* $B(s,\varepsilon)$ *are bounded away from* 0 *and* ∞ *uniformly in* s *and* ε*. More precisely,*

$$1 \leq A(s,\varepsilon) \leq \left(1 + 2\varepsilon^{1/2} + \tfrac{1}{n}\right) < \infty$$

and

$$0 < \left(1 - 2\varepsilon^{1/2} - \tfrac{1}{n}\right) < B(s,\varepsilon) \leq 1.$$

Proof of Theorem 3.8. Take $\alpha = \varepsilon^{1/2}$, and proceed according to two cases. For slopes s with $|s|_\infty \leq 1-\alpha$, use Lemma 3.3 to choose $A = n^+/n \approx (1+2\varepsilon^{1/2})$ and $B = n^-/n \approx (1-2\varepsilon^{1/2})$. Note that $\varepsilon < \frac{1}{9}$ implies that $\varepsilon < \frac{\alpha}{2}$ and $n \geq \frac{1}{\varepsilon} \geq (1-2\varepsilon^{1/2})^{-1}$, as required by the lemma. Moreover $1 - 2\varepsilon^{1/2} - \frac{1}{n} > \frac{2}{9}$, so B is indeed bounded away from 0. The error terms $\theta(\frac{\varepsilon}{\alpha})$ from the lemma are equivalent to $\theta(\varepsilon^{1/2}) = \theta(\varepsilon)$.

For slopes with $|s|_\infty > 1-\alpha$, take $A = B = 1$ and apply Lemma 3.6 twice, using $\alpha = \varepsilon^{1/2}$ in place of ε: once for the boundary height function $h_{\partial S_n}$ given in the statement of the theorem, and once for the canonical boundary height function $h^s_{\partial S_n}$. The estimate on

$$\left|\mathrm{Ent}(M(S_n, h_{\partial S_n}), \omega) - \mathrm{ent}_n(s,\omega)\right|$$

follows from the triangle inequality. □

The robustness results above focused on boundary height functions that differed at macroscopic scale, i.e. $|h_{\partial S_n} - \tilde{h}_{\partial S_n}|_u \leq \varepsilon n$. For boundary height functions with sub-linear differences, we will derive stronger robustness results. Lemma 3.9 addresses the case where the two boundary height functions differ at only a single point on ∂S_n, and Corollary 3.11 extends to the sub-linear case (actually, only to $|h_{\partial S_n} - \tilde{h}_{\partial S_n}|_u = o(\frac{n}{\log n})$, but that is sufficient for our purposes.)

Lemma 3.9 (Robustness for minimally different boundary height functions)**.** *Fix* $n \in \mathbb{N}$*, and let* $h^+_{\partial S_n}$ *and* $\tilde{h}^-_{\partial S_n}$ *be two boundary height functions on the hypercube* S_n *which differ at exactly one point* $z_0 \in \partial S_n$*, i.e.* $h^+_{\partial S_n}|_{S_n \setminus \{z_0\}} = h^-_{\partial S_n}|_{S_n \setminus \{z_0\}}$ *and* $h^+_{\partial S_n}(z_0) = h^-_{\partial S_n}(z_0) + 2$*.*

Then,

$$\left|\mathrm{Ent}_{S_n}\left(M(S_n, h^+_{\partial S_n}), \omega\right) - \mathrm{Ent}_{S_n}\left(M(S_n, h^-_{\partial S_n}), \omega\right)\right| \leq \frac{4mC_\omega + \log(2n)}{|S_n|}\,.$$

Remark 3.10. The $\log(2n)$ term is necessary at least in some extreme cases. For example, suppose that $\omega \equiv 0$, $m = 1$, $z_0 = -n$, $h^+_{\partial S_n}(-n) = 2$, $h^-_{\partial S_n}(-n) = 0$, and $h^\pm_{\partial S_n}(n) = 2n$. Then $\mathrm{Ent}_{S_n}(M(S_n, h^+_{\partial S_n}), \mathbf{0}) = -\frac{1}{n}\log(2n)$ and $\mathrm{Ent}_{S_n}(M(S_n, h^-_{\partial S_n}), \mathbf{0}) = 0$; cf. Lemma 3.4 and Lemma 3.5 for calculations.

Proof of Lemma 3.9. For concreteness and w.l.o.g., we assume that the boundary values at z_0 are $h^-_{\partial S_n}(z_0) = 0$ and $h^+_{\partial S_n}(z_0) = 2$. (Technically this assumption is only valid if z_0 has even parity because we require that height functions preserve parity, and one should instead assume e.g. that $h^\pm_{\partial S_n}(z_0) \in \{1,3\}$ in the other case. For simplicity we ignore this detail in the rest of the proof.)

Consider the line $z_0, z_1, \ldots, z_{2n}$ of points in S_n starting from z_0 and going into S_n, perpendicular to the boundary. Classify each height function $h^+_{S_n} \in M(S_n, h^+_{\partial S_n})$ based on the number of initial "up" steps, i.e.

$$k_{\mathsf{up}}(h^+_{S_n}) := \max\left\{\tilde{k} \geq 0 \,|\, h^+_{S_n}(z_k) = h^+_{S_n}(z_{k-1}) + 1 \text{ for } 1 \leq k \leq \tilde{k}\right\}.$$

Note that from our initial assumption, $h^+_{S_n}(z_k) = k + 2$ for $0 \leq k \leq k_{\mathsf{up}}$. Necessarily $k_{\mathsf{up}}(h^+_{S_n}) < 2n$, since if $h^+_{S_n}$ went up along all $2n$ edges, then the values $h^-_{S_n}(z_{2n}) = h^+_{S_n}(z_{2n}) = 2n + 2$ and $h^-_{S_n}(z_0) = 0$ would violate the Kirszbraun theorem.

On the line segment $\{z_0, \ldots, z_{n_{\mathsf{up}}}\} \subset S_n$, $h^+_{S_n}$ is "too high," in the sense that no height function in $M(S_n, h^-_{\partial S_n})$ can match it. But by the Kirszbraun theorem, there exists $h^-_{S_n} \in M(S_n, h^-_{\partial S_n})$ such that $h^-_{S_n}(z_{n_{\mathsf{up}}+1}) = h^+_{S_n}(z_{n_{\mathsf{up}}+1})$. In fact, we may define $h^-_{S_n}$ by

$$h^-_{S_n}(z) = \begin{cases} k = h^+_{S_n}(z) - 2, & \text{if } z = z_k \text{ for } 0 \leq k \leq k_{\mathsf{up}}, \text{ and} \\ h^+_{S_n}(z), & \text{otherwise}. \end{cases}$$

It follows that $h^+_{S_n}$ and $h^-_{S_n}$ have the same Hamiltonian, except for the contribution from the edges incident to a vertex z_k ($0 \leq k \leq k_{\mathsf{up}}$). There are $(2m-1)(k_{\mathsf{up}}+1)$ such edges, which leads to the naive estimate $|H_{S_n}(h^+_{S_n}, \omega) - H_{S_n}(h^-_{S_n}, \omega)| \leq (2m-1)(k_{\mathsf{up}}+1)C_\omega$. This estimate is not useful because k_{up} on the right-hand side leads to an error of order n in the worst case.

However, a more careful estimate is possible. Indeed, both $h^+_{S_n}$ and $h^-_{S_n}$ map the edges e in question to the same collection of edges $\{e_{k,k+1} \,|\, 0 \leq k \leq k_{\mathsf{up}}\} \subset E(\mathbb{Z})$, with each $e_{k,k+1}$ repeated about $2m-1$ times. For a heuristic argument we refer to Figure 7. We use the ad-hoc notation h^+, h^- and k to denote $h^+_{S_n}$, $h^-_{S_n}$, and $k_{\mathsf{up}}(h^+)$, respectively. We get that

$$\begin{aligned} H_{S_n}(h^+, \omega) - H_{S_n}(h^-, \omega) = \sum_{i=0}^{k} \sum_{\substack{x \sim z_i \\ x \neq z_{i+1}, z_{i-1}}} & \omega_{h^+(e_{x,z_i})} - \omega_{h^-(e_{x,z_i})} \\ & + \sum_{i=0}^{k} \omega_{h^+(e_{z_i,z_{i+1}})} - \omega_{h^-(e_{z_i,z_{i+1}})}, \end{aligned} \tag{3.9}$$

where $h^+(e_{x,y}) = \omega_{h^+(x),h^+(y)}$ and $h^-(e_{x,y}) = \omega_{h^-(x),h^-(y)}$ (see Definition 2.2). From the construction it follows that (cf. Figure 7)

$$h^+(z_i) = \begin{cases} i+2, & \text{for all } 0 \leq i \leq k \\ k+1, & \text{for } i = k+1, \end{cases}$$

and

$$h^-(z_i) = \begin{cases} i, & \text{for all } 0 \leq i \leq k \\ k+1, & \text{for } i = k+1. \end{cases}$$

Additionally, we observe that for any nearest neighbor $x \sim z_i$ such that $x \neq z_l$ with $0 \leq i, l \leq k$, it holds (cf. Figure 7)

$$h^+(x) = h^-(x) = i + 1.$$

Therefore, we can rewrite (3.9) as

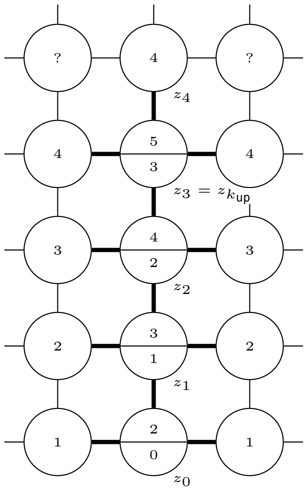
$$\begin{aligned} H_{S_n}(h^+, \omega) - H_{S_n}(h^-, \omega) = (2m-2) & \sum_{i=0}^{k} \omega_{e_{i+1,i+2}} - \omega_{e_{i+1,i}} \\ & + \sum_{i=0}^{k-1} \omega_{e_{i+2,i+3}} - \omega_{e_{i,i+1}} \\ & + \omega_{e_{k+1,k+2}} - \omega_{e_{k,k+1}}. \end{aligned}$$

Taking advantage that the edges are undirected, i.e. $e_{i,j} = e_{j,i}$, we observe that both sums telescope. Hence, we obtain

$$\begin{aligned} H_{S_n}(h^+, \omega) - H_{S_n}(h^-, \omega) = (2m-2)\omega_{e_{k+1,k+2}} & - (2m-2)\omega_{e_{1,0}} \\ & + 2\omega_{e_{k+1,k+2}} - \omega_{e_{k,k+1}} - \omega_{e_{0,1}}. \end{aligned}$$

This yields the improved inequality

$$\left|H_{S_n}(h^+_{S_n},\omega) - H_{S_n}(h^-_{S_n},\omega)\right| \le 4mC_\omega. \tag{3.10}$$



(a) The values of the height functions $h^+_{S_n}$ and $h^-_{S_n}$ from the proof of Lemma 3.9. On the vertices $z_0,\dots,z_{k_{\mathsf{up}}}$ where the two height functions differ, the larger value is the height that $h^+_{S_n}$ takes and the smaller value is $h^-_{S_n}$. Here $k_{\mathsf{up}} = 3$, since $h^+_{S_n}$ increases across the first three edges in the center line. The $(2m-1)(k_{\mathsf{up}}+1)$ shaded edges are exactly the set up edges incident to any of $z_0,\dots,z_{k_{\mathsf{up}}}$, and these are the only edges on which $h^\pm_{S_n}$ differ.

$e \in E(\mathbb{Z})$	$h^+_{S_n}$	$h^-_{S_n}$
$e_{0,1}$	0	$2m-1$
$e_{1,2}$	$2m-2$	$2m-1$
$e_{2,3}$	$2m-1$	$2m-1$
$e_{3,4}$	$2m-1$	$2m-1$
$\vdots$	$\vdots$	$\vdots$
$e_{k_{\mathsf{up}},k_{\mathsf{up}}+1}$	$2m-1$	$2m-1$
$e_{k_{\mathsf{up}}+1,k_{\mathsf{up}}+2}$	$2m$	0

(b) Number of shaded edges on which $h^+_{S_n}$, $h^-_{S_n}$ attain certain heights. For example, from the last row of the table: $h^+_{S_n}(e) = e_{k_{\mathsf{up}}+1,k_{\mathsf{up}}+2}$ for all $2m$ edges incident on $z_{k_{\mathsf{up}}}$. In the difference $H_{S_n}(h^+_{S_n}) - H_{S_n}(h^-_{S_n})$, the bulk of the height values in the table cancel, leaving only boundary terms. That is why the bound in (3.10) does not depend on k_{up}.

Figure 7: Explanation of inequality (3.10) from the proof of Lemma 3.9.

Now, we turn to the entropy inequality. For $0 \le k < 2n$, let

$$M_k := \left\{h^+_{S_n} \in M(S_n, h^+_{\partial S_n}) \,\middle|\, k_{\mathsf{up}}(h^+_{S_n}) = k\right\}.$$

Then the sets M_k ($0 \le k < 2n$) partition $M(S_n, h^+_{\partial S_n})$, so

$$\begin{aligned}
&\mathrm{Ent}_{S_n}\big(M(S_n, h^+_{\partial S_n}),\omega\big)\\
&\qquad = -\frac{1}{|S_n|}\log\sum_{k=0}^{2n-1}\sum_{h^+_{S_n}\in M_k}\exp\big(H_{S_n}(h^+_{S_n},\omega)\big)\\
&\qquad \overset{(3.10)}{\ge} -\frac{1}{|S_n|}\log\sum_{k=0}^{2n-1}\sum_{h^+_{S_n}\in M_k}\exp\big(H_{S_n}(h^-_{S_n},\omega)+4mC_\omega\big)\\
&\qquad \ge -\frac{1}{|S_n|}\log\sum_{k=0}^{2n-1}\sum_{h^-_{S_n}\in M(S_n,h^-_{\partial S_n})}\exp\big(H_{S_n}(h^-_{S_n},\omega)+4mC_\omega\big)\\
&\qquad = \mathrm{Ent}_{S_n}\big(M(S_n, h^-_{\partial S_n}),\omega\big) - \frac{4mC_\omega + \log(2n)}{|S_n|}.
\end{aligned}$$

The reverse inequality is derived by exchanging the roles of $h^{\pm}_{\partial S_n}$, considering the number k_{down} of initial downward steps of $h^{-}_{\partial S_n}$ on the line $\{z_0, \dots, z_{2n}\}$, and proceeding as before with the necessary changes. □

Lemma 3.9 applies only when the two boundary height function $h^{+}_{S_n}$ and $h^{-}_{S_n}$ differ minimally. However by applying Lemma 3.9 repeatedly, we can compare two more different height functions. That idea is captured in the following corollary.

Corollary 3.11 (Robustness with respect to sub-linear height differences)**.** *Let* $h_{\partial S_n}$ *and* $\tilde{h}_{\partial S_n}$ *be boundary height functions on* S_n*, and let* $M = \|h_{\partial S_n} - \tilde{h}_{\partial S_n}\|_\infty$*. Then*

$$\begin{aligned} &\Big|\mathrm{Ent}_{S_n}\big(M(S_n, h_{\partial S_n}), \omega\big) - \mathrm{Ent}_{S_n}\big(M(S_n, \tilde{h}_{\partial S_n}), \omega\big)\Big| \\ &\qquad \le \frac{M}{2}\big(4mC_\omega + \log(2n)\big)\frac{|\partial S_n|}{|S_n|}\,. \end{aligned}$$

Remark 3.12. The main idea of the proof is to interpolate the boundary height function from $h_{\partial S_n}$ to $\tilde{h}_{\partial S_n}$, where each step in the interpolation changes the value of the boundary height function at exactly one boundary point. Note that each interpolation step changes the height by 2 at that distinguished boundary point, which is the reason for the factor $\frac{M}{2}$ rather than simply M. Given such an interpolation, all that remains is to apply Lemma 3.9 and the triangle inequality.

Proof of Corollary 3.11. We claim that there exists a finite sequence $h^{(1)}_{\partial S_n}, \dots, h^{(k)}_{\partial S_n}$ such that each pair $h^{(j)}_{\partial S_n}$ and $h^{(j+1)}_{\partial S_n}$ differ at exactly one point, such that $h^{(1)}_{\partial S_n} = h_{\partial S_n}$ and $h^{(k)}_{\partial S_n} = \tilde{h}_{\partial S_n}$, and such that $k \le \frac{M}{2}|\partial S_n|$. Each element of the sequence is constructed from the previous element by a “flip” operation: Given a (boundary) height function $h^{(j)}_{\partial S_n}$ and a vertex $z_j \in \partial S_n$ where all the neighboring vertices $z' \in \partial S_n$, $z' \sim z_j$ have the same height $h_{\partial S_n}(z') = a \in \mathbb{Z}$, the height function $h^{(j+1)}_{\partial S_n}$ is identical to $h^{(j)}_{\partial S_n}$ on $\partial S_n \setminus \{z_j\}$ and takes the other valid value on z_j. Specifically, if $h^{(j)}_{\partial S_n}(z_j) = a+1$, then $h^{(j+1)}_{\partial S_n}(z_j) = a-1$; otherwise $h^{(j+1)}_{\partial S_n}(z_j) = a+1$.

It remains to show that the vertices $z_1, \dots, z_{k-1}$ can be chosen so that $h^{(k)}_{\partial S_n} = \tilde{h}_{\partial S_n}$ and so that $k \le \frac{M}{2}|\partial S_n|$. To prove both these points, consider the metric $d : M(\partial S_n) \times M(\partial S_n) \to \mathbb{Z}$ defined by

$$d(h'_{\partial S_n}, h''_{\partial S_n}) := \sum_{z \in \partial S_n} \big|h'_{\partial S_n}(z) - h''_{\partial S_n}(z)\big|\,.$$

As long as $d(h^{(j)}_{\partial S_n}, \tilde{h}_{\partial S_n}) > 0$, we will find a vertex z_j for which the flip operation both is valid and decreases the distance d. Towards this end, let $E_j := \{z \in \partial S_n \,|\, h^{(j)}_{\partial S_n}(z) > \tilde{h}_{\partial S_n}(z)\}$. If $E_j \ne \varnothing$, choose $z_j := \mathrm{argmax}_{z \in E_j}\, h^{(j)}_{\partial S_n}$.

We claim that flipping at z_j is valid, and more specifically that for all neighbors $z' \sim z_j$ in ∂S_n, $h^{(j)}_{\partial S_n}(z') = h^{(j)}_{\partial S_n}(z_j) - 1$. Indeed, there are two cases. If $h^{(j)}_{\partial S_n}(z') = \tilde{h}_{\partial S_n}(z')$ for any $z' \sim z_j$, then necessarily $\tilde{h}_{\partial S_n}(z_j) = h^{(j)}_{\partial S_n}(z_j) - 2$ and $\tilde{h}_{\partial S_n}(z') = h^{(j)}_{\partial S_n}(z') = h^{(j)}_{\partial S_n}(z_j) - 1$ for all $z' \sim z$. Otherwise all $z' \sim z$ are also in E_j, so the claim follows since z_j maximizes $h^{(j)}_{\partial S_n}$ over E_j. So as claimed, it is valid to flip the height function $h^{(j)}_{\partial S_n}$ at z_j, and this flip decreases the difference $|h^{(j+1)}_{\partial S_n}(z_j) - \tilde{h}_{\partial S_n}(z_j)|$ by two, and therefore decreases the distance $d(h^{(j+1)}_{\partial S_n}, \tilde{h}_{\partial S_n})$ by two. If E_j is empty, use instead the set $F_j := \{z \in \partial S_n \,|\, h^{(j)}_{\partial S_n}(z) < \tilde{h}_{\partial S_n}(z)\}$, pick $z_j := \mathrm{argmin}_{z \in F_j}\, h^{(j)}_{\partial S_n}$, and repeat the argument, changing inequalities and signs accordingly. If F_j is also empty, then $h^{(j)}_{\partial S_n} = \tilde{h}_{\partial S_n}$ and the process is complete. At most $\frac{1}{2}d(h_{\partial S_n}, \tilde{h}_{\partial S_n}) \le \frac{M}{2}|\partial S_n|$ steps are needed in total, since each step decreases the distance by 2. To complete the proof of the corollary, apply Lemma 3.9 to each pair $\{h^{(j)}_{\partial S_n}, h^{(j+1)}_{\partial S_n}\}$ and use the triangle inequality. □

3.3 Existence and equivalence of quenched and annealed local surface tension

Recall from Definition 2.17 that the quenched local surface tension is defined as the limit of the quenched microscopic surface tension. Because of the random potential ω, the existence of this limit is not obvious. We prove the existence of the limit using an ergodic theorem for almost superadditive random families.

First, we introduce the notation needed for stating the ergodic theorem. Let $\mathcal{B}$ denote the set of all (non-empty) boxes in $\mathbb{Z}^m$, i.e.

$$\mathcal{B} = \left\{ \big([a_1, b_1) \times \cdots \times [a_m, b_m) \big) \cap \mathbb{Z}^m \;\middle|\; a_1 < b_1, \ldots, a_m < b_m \in \mathbb{Z}^m \right\}.$$

Note that the sets $S_n := [-n, n]^m \cap \mathbb{Z}^m$ are included in $\mathcal{B}$. We say that a family of L^1 random variables $F = (F_B)_{B \in \mathcal{B}}$ is *almost superadditive* if, for any finitely many disjoint boxes $B_1, \ldots, B_n \in \mathcal{B}$ whose union $B = B_1 \cup \cdots \cup B_n$ also lies in $\mathcal{B}$,

$$F_B \ge \sum_{i=1}^{n} F_{B_i} - A \sum_{i=1}^{n} |\partial B_i| \quad \text{(a.s.)}, \tag{3.11}$$

where $A = A(\omega) : \Omega \to [0, \infty)$ is an L^1 random variable, and where $\partial B_i = \{x \in B_i \mid \exists y \in \mathbb{Z}^m \setminus B_i,\, x \sim y\}$ is the inner boundary of B_i.

Theorem 3.13 (Ergodic theorem for almost superadditive random families)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\tau = (\tau_u)_{u \in \mathbb{Z}^m}$ be a family of measure-preserving transformations on Ω, and let $F = (F_B)_{B \in \mathcal{B}}$ be a family of L^1 random variables satisfying the following three conditions:*

- *F is almost superadditive, i.e. F satisfies* (3.11)*,*
- *For all $u \in \mathbb{Z}^m$,*
$$\lim_{n \to \infty} \sup_{u \in \mathbb{Z}^m} \frac{1}{|S_n|} \Big| F_{u + S_n} - F_{S_n} \circ \tau_u \Big| = 0, \tag{3.12}$$
where $u + B = \{u + x \mid x \in B\}$ is the translation of B by u.
- *The quantity $\tilde{\gamma}(F) = \limsup_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}]$ is finite.*

Then the limit $\lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n}$ exists almost surely and in L^1. If moreover $\{\tau_u\}_{u \in \mathbb{Z}^m}$ is ergodic, then the limit is

$$\lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n} = \tilde{\gamma}(F).$$

This theorem is based on [2, Theorem 2.4], which is a multidimensional extension of the subadditive ergodic theorem proven in [29, 33] among many other sources. The version stated here is adapted to notion of almost superadditivity that the quenched microscopic entropy satisfies. For completeness, we give a proof of this version of the ergodic theorem in Appendix A. Now let us turn to the application of this ergodic theorem:

Lemma 3.14 (Existence of the quenched local surface tension)**.** *For almost every realization ω of the random field, the limit* (2.3) *exists.*

The proof is a straightforward application of the ergodic theorem.

Proof of Lemma 3.14. Fix $s \in [-1,1]^m$. Let the family of measure-preserving transformations $\tau = (\tau_u)_{u\in\mathbb{Z}^m}$ be given by

$$\big(\tau_u \omega\big)_e := \omega_{e-[s\cdot u]_{u \bmod 2}} \quad \text{for } e \in E(\mathbb{Z}) \text{ and } u \in \mathbb{Z}^m. \tag{3.13}$$

Define the random process $F = (F_B)_{B\in\mathcal{B}}$ by

$$F_B := -|B| \operatorname{Ent}\big(M(B, h^s_{\partial B}), \omega\big) = \log Z_\omega\big(M(B, h^s_{\partial B})\big)\,.$$

Now we verify the hypotheses of the ergodic theorem (Theorem 3.13). First, the fact that $|\omega_e| \le C_\omega$ for all edges $e \in E(\mathbb{Z})$ implies that each variable F_B ($B \in \mathcal{B}$) is in L^1.

Next, the almost superadditivity property (3.11) follows from distributivity:

$$\begin{aligned}\sum_{i=1}^n F_{B_i} &= \log \prod_{i=1}^n \sum_{h_{B_i}\in M(B_i, h^s_{\partial B_i})} \exp\big(H_{B_i}(h_{B_i},\omega)\big)\\ &= \log \sum_{\substack{h_{B_1}\in M(B_1,h^s_{\partial B_1})\\ \cdots \\ h_{B_n}\in M(B_n,h^s_{\partial B_n})}} \exp\left(\sum_{i=1}^n H_{B_i}(h_{B_i},\omega)\right).\end{aligned}$$

The final sum is indexed by n-tuples of height functions, i.e. it is the sum over the Cartesian product of the sets $M(B_i, h^s_{\partial B_i})$. This Cartesian product is a subset of $M(B, h_B)$, so

$$\sum_{i=1}^n F_{B_i} \le \log \sum_{h_B\in M(B,h^s_{\partial B})} \exp\left(\sum_{i=1}^n H_{B_i}(h_B|_{B_i},\omega)\right). \tag{3.14}$$

The quantity on the right-hand side of (3.14) differs from F_B by at most $mC_\omega \sum_{i=1}^n |\partial B_i|$, since the Hamiltonian terms in (3.14) do not include edges that cross from one box B_i to another box B_j. This error term satisfies (3.11).

Now let us show that F satisfies the translation invariance estimate (3.12). For $h_{\partial(u+B)} \in M(\partial(u+B))$, consider the shifted boundary height function $\Psi_u h_{\partial(u+B)} \in M(\partial B)$ defined by

$$(\Psi_u h_{\partial(u+B)})(z) := h_{\partial(u+B)}(u+z) - \lfloor s\cdot u\rfloor \qquad \text{for } z \in \partial B\,.$$

Since both $h^s_{\partial B}$ and $h^s_{\partial(u+B)}$ are rounded to the nearest integer (of appropriate parity), the shifted boundary height function $\Psi_u h^s_{\partial(u+B)}$ may not agree exactly with $h^s_{\partial B}$. However, it holds that

$$\big|\Psi_u h^s_{\partial(u+B)}(z) - h^s_{\partial B}(z)\big| \le 4 \qquad \text{for all } z \in \partial B\,.$$

Therefore by Corollary 3.11,

$$\big|F_{u+B} - F_B\circ \tau_u\big| \le |B|\,\theta\big(\tfrac{1}{n}\big)\,.$$

The last condition to check is $\tilde\gamma(F) = \limsup_{n\to\infty} \frac{1}{|S_n|}\mathbb{E}[F_{S_n}] < \infty$, which follows from boundedness of the quenched entropy. Indeed by Lemma 3.2, the inequality $F_B \le m|B|C_\omega$ holds almost surely, so $\tilde\gamma(F) \le \mathbb{E}(C_\omega) < \infty$.

At this point we have checked all the hypotheses of the ergodic theorem (Theorem 3.13). From the ergodic theorem we conclude that the pointwise limit

$$\operatorname{ent}(s,\omega) = \lim_{n\to\infty} \operatorname{ent}_n(s,\omega) = \lim_{n\to\infty} \frac{1}{|S_n|} F_{S_n}(\omega)$$

exists almost surely. In addition, when $s \ne 0$, the family of measure-preserving transformations $(\tau_u)_{u\in\mathbb{Z}^m}$ is ergodic with respect to $\mathbb{P}$, since the family includes every shift $\omega \mapsto (\omega_{k+e})_{e\in E(\mathbb{Z})}$ for $k \in \mathbb{Z}$. Therefore whenever $s \ne 0$, the limit $\operatorname{ent}(s,\omega)$ is almost surely equal to its expectation, $\mathbb{E}[\operatorname{ent}(s,\omega)] = \operatorname{ent}_{\mathsf{an}}(s)$. ☐

The failure of ergodicity in the case $s = 0$ is evident from the definition of $(\tau_u)_{u\in\mathbb{Z}^m}$ in (3.13): there we have $(\tau_u\omega)_e := \omega_{e-[s\cdot u]_{u \bmod 2}}$ for each $e \in E(\mathbb{Z})$. When $s = 0$ the quantity $s\cdot u$ is zero even as $u \to \infty$, so the entire family of transformations $(\tau_u)_{u\in\mathbb{Z}^m}$ is actually finite rather than ergodic. As such, a different argument is needed for $s = 0$. The authors would like to thank Marek Biskup for suggesting the following argument.

Lemma 3.15 (Equivalence of quenched and annealed local surface tension)**.** *For almost every ω, it holds that*

$$\mathrm{ent}(s,\omega) = \mathrm{ent}_{\mathsf{an}}(s). \tag{3.15}$$

Moreover, the quenched microscopic surface tension $\mathrm{ent}_n(s,\omega)$ converges in L^1 to $\mathrm{ent}_{\mathsf{an}}(s)$.

Proof of Lemma 3.15. For $s \neq 0$, the desired identity (3.15) follows from the ergodic theorem, as mentioned at the end of the proof of Lemma 3.14.

For $s = 0$, we will establish translation invariance of $\mathrm{ent}(s,\omega)$ directly. First we replace the environmental shift τ_2 by a shift in heights, i.e.

$$\mathrm{Ent}_{S_n}\big(M(S_n, h^0_{\partial S_n})\big)\circ\tau_2 = \mathrm{Ent}_{S_n}\big(M(S_n, h^{0\cdot x+2}_{\partial S_n})\big)\,. \tag{3.16}$$

This identity is justified simply by expanding definitions; both sides are equal to $-\frac{1}{|S_n|}\log\sum_{h_{S_n}}\exp(\sum_e \omega_{h_{S_n}(e)+2})$, where the first sum runs over $h_{S_n}\in M(S_n, h^0_{\partial S_n})$ and the second runs over $e \in E(S_n)$.

Now, the square S_n sits inside of S_{n+2}. The boundary values $h^{0\cdot x+2}_{\partial S_n}$ and $h^0_{\partial S_{n+2}}$ satisfy the Kirszbraun criterion (3.1); in fact, each $h \in M(S_n, h^{0\cdot x+2}_{\partial S_n})$ admits a unique extension $\tilde h$ in $M(S_{n+2}, h^0_{\partial S_{n+2}})$. Since $\tilde h$ is an extension of h to a domain with $O(n^{m-1})$ more points and $O(n^{m-1})$ more edges, the Hamiltonians satisfy

$$\big|H_{S_n}(h,\omega) - H_{S_{n+2}}(\tilde h,\omega)\big| \le cn^{m-1}C_\omega$$

for some $c > 0$. Therefore

$$\begin{aligned}
&\mathrm{Ent}_{S_n}\big(M(S_n, h^{0\cdot x+2}_{\partial S_n}),\omega\big)\\
&\qquad\ge -\frac{1}{|S_n|}\log\sum_{h\in M(S_n, h^{0\cdot x+2}_{\partial S_n})}\exp\big(H_{S_{n+2}}(\tilde h,\omega)\big) - \frac{cC_\omega}{n}\\
&\qquad\ge \mathrm{Ent}_{S_{n+2}}\big(M(S_{n+2}, h^0_{\partial S_{n+2}}),\omega\big) - \frac{cC_\omega}{n}\,.
\end{aligned} \tag{3.17}$$

Now, we combine (3.16) and (3.17) and send $n\to\infty$, which yields

$$\mathrm{ent}(0,\omega)\circ\tau_2 \ge \mathrm{ent}(0,\omega)\,.$$

By a similar argument with τ_2 replaced by τ_{-2}, we conclude that $\mathrm{ent}(0,\tau_2\omega) = \mathrm{ent}(0,\omega)$, i.e. $\mathrm{ent}(0,\omega)$ is invariant under τ_2. Since the distribution $\mathbb{P}$ of ω is ergodic with respect to τ_2 (cf. Assumption 2.9), this implies that $\mathrm{ent}(0,\omega) = \mathbb{E}[\mathrm{ent}(0,\omega)] = \mathrm{ent}_{\mathsf{an}}(0)$ almost surely. $\square$

3.4 Convexity and continuity

The last results that we need about the annealed local surface tension $\mathrm{ent}_{\mathsf{an}}(s)$ are that is is convex and continuous as a function of the slope s. Convexity allows us to apply standard analytic techniques to conclude that the macroscopic entropy functional $\mathrm{Ent}_{R,\mathsf{an}}(\cdot)$ is lower semi-continuous (see, for example, [13, Section 2]). By semi-continuity, there exists a (perhaps non-unique) minimizer of the entropy functional, so the minimum in the variational principle (Theorem 2.23) is achieved.

Lemma 3.16. *The function $s \mapsto \mathrm{ent}_{\mathsf{an}}(s)$ is convex for $s \in (-1,1)^m$.*

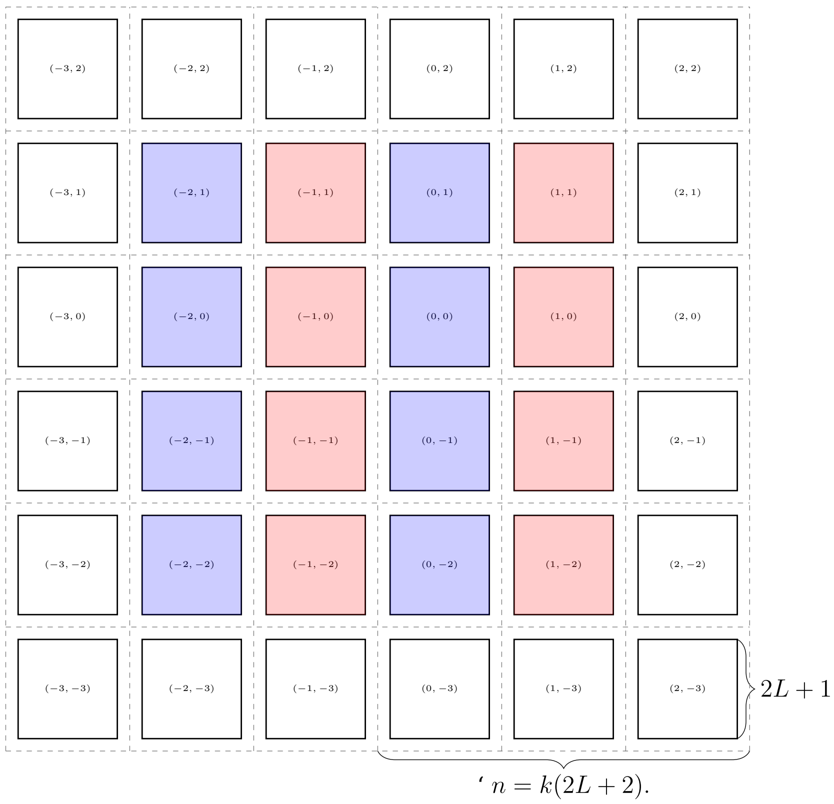


Figure 8: The large box S_{2n+1} is decomposed into $(2k)^2$-many small boxes $S_{2L+1}(i,j)$ separated by the grid S' (see the dashed lines). The large box S_{2n+1} is endowed with linear boundary condition of slope s_1. The blue boxes are endowed with an affine boundary condition of slope s_0, and the red boxes with slope s_2. In the figure $k=3$.

Remark 3.17. The proof follows a standard argument for the uniform case and is based on buckled height functions (see e.g. [44]).

Proof of Lemma 3.16. We shall prove that for any choice of fixed coordinates

$$s_1, \dots, s_{i-1}, s_{i+1}, \dots s_m \in [-1,1]^{m-1},$$

the single-variate functions $s_i \mapsto \mathrm{ent}_{\mathsf{an}}((s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_m)$ are convex. It follows from elementary analysis that $s \mapsto \mathrm{ent}_{\mathsf{an}}(s)$ is a convex function on the m-dimensional domain $[-1,1]^m$. To simplify notation, we state the proof in the case $m=2$, The proof generalizes to higher dimensions.

We choose $u_0, u_1, u_2, v \in [-1,1]$ such that such that

$$u_1 = \frac{1}{2}u_0 + \frac{1}{2}u_2.$$

Our goal is to prove that

$$\mathrm{ent}_{\mathsf{an}}((u_1,v)) \le \frac{1}{2}\,\mathrm{ent}_{\mathsf{an}}((u_0,v)) + \frac{1}{2}\,\mathrm{ent}_{\mathsf{an}}((u_2,v)), \tag{3.18}$$

from which the desired convexity follows.

We proceed as follows. First, let us consider the box

$$S_{2n+1} := \{-n, \dots, n\}^2,$$

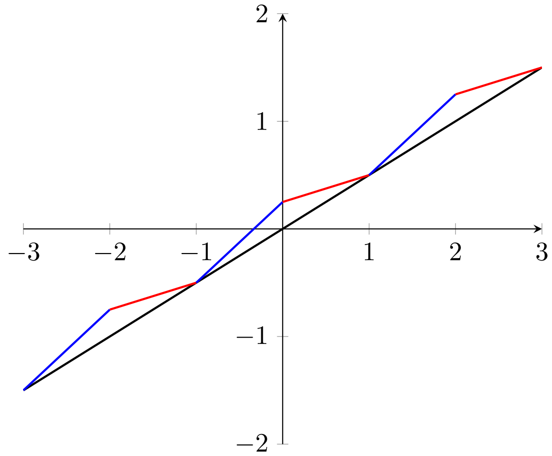


Figure 9: Illustration of *buckling* in one dimension. In the figure, the blue parts have slope $\frac{3}{4}$ and the red parts have slope $\frac{1}{4}$ resulting in an average slope of $\frac{1}{2}$.

which is centered at 0 and has side length $2n+1$. On S_{2n+1} we consider the negative normalized log-partition function of height functions with linear boundary values of slope $s_1 = (u_1, \nu)$. By definition this equals to the quenched microscopic surface tension $\operatorname{ent}_n(s_1, \omega)$. By Lemma 3.15 it holds $\lim_{n\to\infty} \operatorname{ent}_n(s_1, \omega) = \operatorname{ent}_{\mathsf{an}}(s_1)$, which recovers the left hand side of the desired inequality (3.18).

Now, we subdivide S_{2n+1} into $(2k)^2$ many smaller boxes $S_{2L+1}(i,j)$, $-k \le i, j < k$ of side length $2L+1$. The cubes are separated by the grid $S' := (2k+2)\mathbb{Z}$. For details we refer to the Figure 8. This decomposition implies the relation

$$n = k(2L+2). \tag{3.19}$$

The grid S' is introduced for convenience as it allows to work with centered sub-boxes $S_{2L+1}(i,j)$ with odd side length. Centered boxes are a requirement of the ergodic theorem that we used to prove Lemma 3.14 (the existence of the quenched local surface tension) and Lemma 3.15 (the equivalence of the quenched and annealed local surface tension). One could generalize those lemmas but the formulation and proof would become more subtle. Instead, we choose to add the grid S' to the decomposition of the hypercube S_{2n+1}. Because $|S'| \le 2(2k+1)n$, the effect of S' will be asymptotically negligible.

On the colored boxes $S_{2L+1}(i,j)$, $-k+1 \le i, j < k-1$, we define a *buckled* boundary condition via a piece-wise affine boundary height function g that alternates between the slope $s_0 = (u_0, v)$ and $s_2 = (u_2, v)$ from left to right. We will give the precise definition of the boundary height function g in the next paragraph. The boundary height function g *buckles* around the slope $s_1 = (u_1, v) = (\frac{1}{2}u_0 + \frac{1}{2}u_2, v)$ on a macroscopic scale. We refer to Figure 9 for an illustration of buckling. In the block decomposition of Figure 8, boxes with boundary slope s_0 are colored in blue whereas boxes with boundary slope s_2 are colored in red. The quenched entropy of every inner box $S_{2L+1}(i,j)$ is therefore either given by $\operatorname{ent}_{2L+1}(s_0, \tau_{i,j}\omega)$ or $\operatorname{ent}_{2L+1}(s_2, \tau_{i,j}\omega)$ for an appropriate shift operator $\tau_{i,j}$, depending on the position of the box.

In this paragraph, we explicitly construct the buckled boundary height function g. In the first read-through, this construction might be skipped. We define the set M_n as

$$M_n = S_{2n+1-2(2L+2)} \cap \left(S' \cup \bigcup_{-k+1 \le i,j < k-1} \partial S_{2l+1}(i,j) \right).$$

The set M_n consists out of the boundary of the colored boxes and the grid S' that is contained in the centered box of side length $2n+1-2(2L+2)$. We now construct a function $\hat{g} : M_n \to \mathbb{Z}$ as a piece-wise affine height function. We assume w.l.o.g. that $k-1$ is even. We start with the left-most vertical line of M_n, and define for $-(k-1)(2L+2) \le y \le (k-1)(2L+2)$ (cf. Definition 2.7)

$$\hat{g}(-(k-1)(2L+2), y) = [vy - a]_{(-(k-1)(2L+2),y) \bmod 2},$$

where $a = \frac{1}{2}(u_0 + u_2)(k-1)(2L+2) = u_1(k-1)(2L+2)$. For $-k+1 \le l < k-1$ we define the number

$$b(l) := \begin{cases} u_1(k-1+l)(2L+2), & \text{if } k-1+l \text{ is even,} \\ u_1(k-1+l-1)(2L+2) + u_0(2L+2), & \text{if } k-1+l \text{ is odd.} \end{cases}$$

Now, for $(x,y) \in M_n$ such that $l(2L+2) < x \le (l+1)(2L+2)$ we set

$$\hat{g}(x,y) = \begin{cases} [u_0(x - l(2L+2) + b(l) + vy - a]_{(x,y) \bmod 2}, & \text{if } l \text{ is even,} \\ [u_2(x - l(2L+2) + b(l) + vy - a]_{(x,y) \bmod 2}, & \text{if } l \text{ is odd.} \end{cases}$$

By definition, the map $\hat{g}$ is a height function on the set M_n. The buckled boundary height function g is defined as the restriction of $\hat{g}$ onto the boundaries of the colored boxes.

Let us turn to the question of how to relate $\operatorname{ent}_n(s_1, \omega)$ to the right hand side of (3.18). We claim that there exists a height function $\bar{g} : S_{2n+1} \to \mathbb{N}$ on the large box S_{2n+1} with the following properties: On the boundary of S_{2n+1} the function $\bar{g}$ is the linear boundary height function with slope s_1; and on the boundary of the colored boxes, i.e. on $\bigcup_{-k+1 \le i,j < k-1} \partial S_{2L+1}(i,j))$, the function $\bar{g}$ coincides with the buckled boundary height function g. Indeed, the buckled boundary height function g extends by construction to the height-function $\hat{g}$ on the set M_n. Then, $\hat{g}$ can be extended to a height function on S_{2n+1} with the desired properties using Kirszbraun theorem. Consider an arbitrary height function $h : \bigcup_{-k+1 \le i,j < k-1} S_{2L+1}(i,j) \to \mathbb{Z}$ that is defined on the colored boxes with buckled boundary condition g. This height function can be extended to a height function $\bar{h}$ on the large box S_{2n+1} such by setting $\bar{h}(x,y) = \bar{g}(x,y)$ for $(x,y) \in S_{2n+1} \backslash \bigcup_{-k+1 \le i,j < k-1} S_{2L+1}(i,j)$. By construction, the extended height function $\bar{h}$ will satisfy a linear boundary condition on S_{2n+1} with slope s_1. This allows to under-count the number of height functions on S_{2n+1} and deduce, similar to the proof of Lemma 3.3, the following estimate:

$$\begin{aligned} \operatorname{ent}_n(s_1, \omega) \le \frac{(2L+1)^2}{n^2} & \sum_{\substack{-k+1 \le i,j < k-1 \\ i \text{ is even}}} \operatorname{ent}_{2L+1}(s_0, \tau_{i,j}\omega) \\ & + \frac{(2L+1)^2}{n^2} \sum_{\substack{-k+1 \le i,j < k-1 \\ i \text{ is odd}}} \operatorname{ent}_{2L+1}(s_2, \tau_{i,j}\omega) \\ & \quad + C_\omega \, C \, \frac{2(2k+1)n}{n^2} + C_\omega \, C \, \frac{(2L+1)n}{n^2}. \end{aligned}$$

The first and second term counts the contribution of the blue and red boxes, respectively. The third term estimates the energetic effect of under-counting on the grid S'. The last term estimates the energetic effect of under-counting on the outer ring of uncolored boxes $S_{2L+1}(i,j)$ with either $i \in \{-k,k\}$ or $j \in \{-k,k\}$. By Lemma 3.15 we observe that

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{ent}_n(s_1,\omega) = \mathrm{ent}_{an}(s_1),\\
&\lim_{L\to\infty} \mathrm{ent}_{2l+1}(s_0,\tau_{i,j}\omega) = \mathrm{ent}_{an}(s_0), \qquad \text{and}\\
&\lim_{L\to\infty} \mathrm{ent}_{2l+1}(s_2,\tau_{i,j}\omega) = \mathrm{ent}_{an}(s_2).
\end{aligned}$$

Therefore, a combination of (3.19) and taking the limit $k, L \to \infty$ yields the desired inequality (3.18). □

4 Profile theorem

Before proving the profile theorem, Theorem 2.22, in its full generality, it is useful to prove a special case of the theorem with the extra assumptions that the asymptotic height function is piecewise affine, on a domain which is of a collection of simplices. In this special case it is not difficult to relate the microscopic entropy $\mathrm{Ent}_{R_n}(B(R_n,h_R,\delta),\omega)$ to the quenched microscopic surface tension $\mathrm{ent}_n(s,\omega)$, and then to derive the desired conclusion (2.4). The special case is stated in Lemma 4.3 below, after some necessary notation is introduced in Definitions 4.1 and 4.2.

Definition 4.1 (Simplices of scale ℓ; cf. [31, Definition 27] and [44, Section 5.2.1]). *Let $\mathrm{Sym}(m)$ denote the group of permutations on $\{1,\dots,m\}$, and for $w = (w_1,\dots,w_m) \in \mathbb{R}^m$, let $\lfloor w \rfloor$ denote the integer point $\lfloor w \rfloor := (\lfloor w_1 \rfloor,\dots,\lfloor w_m \rfloor)$. Let $v \in \mathbb{Z}^m$, let $\sigma \in \mathrm{Sym}(m)$, and let $\ell > 0$. Define $C(v,\sigma)$ to be the closure of the set*

$$\left\{ w \in \mathbb{R}^m \,\middle|\, \lfloor w \rfloor = v \text{ and } w_{\sigma(1)} - \lfloor w_{\sigma(1)} \rfloor > \cdots > w_{\sigma(m)} - \lfloor w_{\sigma(m)} \rfloor \right\},$$

and define the simplex of scale ℓ to the scaled set

$$\ell C(v,\sigma) := \{\ell w \,|\, w \in C(v,\sigma)\}.$$

Definition 4.2 (Piecewise affine asymptotic height functions). *Let $\Delta_1,\dots,\Delta_k$ be simplices of scale ℓ and let $K = \Delta_1 \cup \cdots \cup \Delta_k$ be their union. We say that an asymptotic height function $h_K \in M(K)$ is piecewise affine if each restriction $h_K|_{\Delta_i}$ is an affine function, i.e. if there exist $s_i \in [-1,1]^m$ and $b_i \in \mathbb{R}$ such that $h_K|_{\Delta_i}(x) = s_i \cdot x + b_i$ for all $x \in \Delta_i$. We write*

$$\begin{aligned}
M_{\mathsf{aff}}(K) &= \left\{ h_K \in M(K) \;\middle|\; h_K \text{ is piecewise affine} \right\}\\
M_{\mathsf{aff}}(K,h_{\partial K}) &= M_{\mathsf{aff}}(K) \cap M(K,h_{\partial K}).
\end{aligned}$$

Lemma 4.3 (Profile theorem, simplicial case). *Let $\Delta_1,\dots,\Delta_k$ be simplices of scale ℓ and let $K = \Delta_1 \cup \cdots \cup \Delta_k$ be their union.*

For any $h_K \in M_{\mathsf{aff}}(K,h_{\partial K})$ and any $\eta > 0$, there exists $\varepsilon = \varepsilon_0(h_K,\eta)$ such that, for any $\varepsilon \in (0,\varepsilon_0]$ and any $p_{\mathsf{max}} \in (0,1)$, there exists $n_0 = n_0(h_K,\eta,\varepsilon,p_{\mathsf{max}})$ such that for all $n \geq n_0$,

$$\begin{aligned}
\mathbb{P}\Big(\Big| &\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\varepsilon\ell),\omega\big) - \mathrm{Ent}_{K,\mathsf{an}}(h_K) \Big| \\
&> \eta + C_\omega \theta_{h_K}(\varepsilon) + C_\omega \theta_{h_K,\varepsilon}\big(\tfrac{1}{n}\big) \Big) < p_{\mathsf{max}}.
\end{aligned} \tag{4.1}$$

Proof. We will prove two bounds on the quenched microscopic entropy $\mathrm{Ent}_{K_n}(B(K_n,h_{K_n},\varepsilon\ell),\omega)$: an upper bound

$$\begin{aligned}
\mathbb{P}\Big(&\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\varepsilon\ell),\omega\big) > \mathrm{Ent}_{K,\mathsf{an}}(h_K) \\
&+ \eta + C_\omega \theta_{h_K}(\varepsilon) + C_\omega \theta_{h_K,\varepsilon}\big(\tfrac{1}{n}\big) \Big) \leq \theta_{h_K,\eta,\varepsilon}\big(\tfrac{1}{n}\big)
\end{aligned} \tag{4.2}$$

and a lower bound

$$\mathbb{P}\Big(\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\varepsilon\ell),\omega\big) < \mathrm{Ent}_{K,\mathsf{an}}(h_K) \\ - \eta - C_\omega\theta_{h_K}(\varepsilon) - C_\omega\theta_{h_K,\varepsilon}\big(\tfrac{1}{n}\big)\Big) \le \theta_{h_K,\eta,\varepsilon}\big(\tfrac{1}{n}\big)\,. \tag{4.3}$$

Assuming that both (4.2) and (4.3) hold, the conclusion (4.1) follows immediately by taking n_0 large enough based on the two $\theta_{h_k,\eta,\varepsilon}(\frac{1}{n})$ terms and applying the union bound on probabilities. So first let us verify the upper bound (4.2), and later we will verify the lower bound (4.3). For (4.2) we undercount the set of height functions $B(K_n,h_{K_n},\varepsilon\ell)$. We choose a fine mesh of hypercubes $Q_{i,n}$ that approximate K_n and consider only those height functions that agree with the canonical boundary height functions $h^{s_i\cdot x+b_i}_{\partial Q_{i,n}}$ on $\partial Q_{i,n}$, where $s_i\in[-1,1]^m$ and $b_i\in\mathbb{R}$ are chosen such that $s_i\cdot x+b_i = h_K|_{Q_i}$. The mesh size is small enough that every such height function is in $B(K_n,h_{K_n},\varepsilon\ell)$.

To be precise, let $q=\frac{1}{4}\varepsilon\ell$ be the mesh size. Let $Q_1,\dots,Q_k\subset\mathbb{R}^m$ enumerate the set of hypercubes in $\mathbb{R}^m$ that have side length q, have vertices in $q\mathbb{Z}^m$, and lie entirely in one of the simplices Δ_j. That last property ensures that there exist $s_i\in[-1,1]^m$ and $b_i\in\mathbb{R}$ such that

$$h_K(x) = s_i\cdot x + b_i \qquad \text{for all } x\in Q_i\,.$$

For $n\in\mathbb{N}$, let $Q_{i,n} := \{z\in\mathbb{Z}^m \mid \frac{1}{n}z\in Q_i\}$. Then as desired, for any choice of height functions

$$\big(h_{Q_{i,n}}\big)_{i=1}^k \in \prod_{i=1}^k M\big(Q_{i,n}, h^{s_i\cdot x+b_i}_{\partial Q_{i,n}}\big)\,,$$

there exists at least one extension $h_{K_n}\in M(K_n)$ to the whole of K_n (i.e. $h_{K_n}|_{Q_{i,n}} = h_{Q_{i,n}}$ for each $i=1,\dots,k$), and any such extension lies in $B(K_n,h_K,\varepsilon\ell)$ by choice of q. Therefore,

$$\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\varepsilon\ell),\omega\big) \le \frac{1}{k}\sum_{i=1}^k \mathrm{Ent}_{Q_{i,n}}\big(M(Q_{i,n},h^{s_i\cdot x+b_i}_{\partial Q_{i,n}}),\omega\big) \\ + C_\omega\theta_m(\varepsilon) + C_\omega\theta_{m,\varepsilon,\ell}\big(\tfrac{1}{n}\big)\,, \tag{4.4}$$

where the θ error terms come from the contribution of the set $K_n\setminus\bigcup_{i=1}^k Q_{i,n}$. For each $i=1,\dots,k$, let us abuse notation and write "qn" to denote the side length of the hypercube $Q_{i,n}$. (In fact, the actual product $q\cdot n$ is generally not an integer, but the quantity we call qn satisfies $|qn-q\cdot n|<1$.) Consider $Q_{i,n}$ as a translate $Q_{i,n}=v_i+S_{qn}$ for $v_i\in\mathbb{Z}^m$. Then the boundary values $h^{s_i\cdot x+b_i}_{\partial Q_{i,n}}$ are close to the translated values of $h^{s_i}_{\partial S_{qn}}$; in particular, for $z\in\partial S_{qn}$,

$$\Big|h^{s_i\cdot x+b_i}_{\partial Q_{i,n}}(v_i+z) - \Big(h^{s_i}_{\partial S_{qn}}(z) + \lfloor s_i\cdot v_i + nb_i\rfloor\Big)\Big| \le 4\,. \tag{4.5}$$

(A non-zero error occurs when s_i is irrational, or more generally when qns_i is not integral or has the wrong parity.) By Corollary 3.11 it follows that

$$\mathrm{Ent}_{Q_{i,n}}\big(M(Q_{i,n},h^{s_i\cdot x+b_i}_{\partial Q_{i,n}}),\omega\big) = \mathrm{ent}_{qn}(s_i,\tau_{\lfloor s_i\cdot v_i+nb_i\rfloor}\omega) + C_\omega\theta_m\big(\tfrac{1}{n}\big)\,. \tag{4.6}$$

Combining (4.4) and (4.6) and abbreviating $\tau_{i,n} := \tau_{\lfloor s_i\cdot v_i+nb_i\rfloor}$ yields

$$\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\varepsilon\ell),\omega\big) \\ \le \frac{1}{k}\sum_{i=1}^k \mathrm{ent}_{qn}(s_i,\tau_{i,n}\omega) + C_\omega\theta_m(\varepsilon) + C_\omega\theta_{m,\varepsilon,\ell}\big(\tfrac{1}{n}\big)\,. \tag{4.7}$$

We note that the sequences $\{\mathrm{ent}_{qn}(s_i,\tau_{i,n}\omega)\}_{n\in\mathbb{N}}$ may not necessarily converge to $\mathrm{ent}_{\mathsf{an}}(s)$ as $n\to\infty$, despite the almost-sure convergence result of Lemma 3.15, due to the potential shifts $\tau_{i,n}$. However, since each $\mathrm{ent}_{qn}(s_i,\cdot)\to\mathrm{ent}_{\mathsf{an}}(s_i)$ in L^1, we can apply the Markov bound:

$$\begin{aligned}
&\mathbb{P}\bigg(\bigg|\frac{1}{k}\sum_{i=1}^{k}\mathrm{ent}_{qn}(s_i,\tau_{i,n}\omega)-\frac{1}{k}\sum_{i=1}^{k}\mathrm{ent}_{\mathsf{an}}(s_i)\bigg|>\eta\bigg)\\
&\qquad\le\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\eta}\left\|\mathrm{ent}_{qn}(s_i,\cdot)-\mathrm{ent}_{\mathsf{an}}(s_i)\right\|_{L^1}\\
&\qquad=\theta_{h_K,\eta,\varepsilon,\ell}\left(\tfrac{1}{n}\right).
\end{aligned}\tag{4.8}$$

The last step in verifying (4.2) is to compare $\mathrm{Ent}_{K,\mathsf{an}}(h_K)$ to a sum involving $\mathrm{ent}_{\mathsf{an}}(s_i)$. This is straightforward: because h_K is affine on each hypercube Q_i, the integrand $x\mapsto\mathrm{ent}_{\mathsf{an}}(\nabla h_K(x))$ in the macroscopic entropy is constant on each Q_i, so

$$\begin{aligned}
\mathrm{Ent}_{K,\mathsf{an}}(h_K)&\overset{\text{def.}}{=}\frac{1}{|K|}\int_K\mathrm{ent}_{\mathsf{an}}(\nabla h_K(x))\,dx\\
&=\frac{1}{k}\sum_{i=1}^{k}\frac{1}{|Q_i|}\int_{Q_i}\mathrm{ent}_{\mathsf{an}}\big(\nabla h_K|_{Q_i}\big)+\theta_K(\varepsilon)\\
&=\frac{1}{k}\sum_{i=1}^{k}\mathrm{ent}_{\mathsf{an}}(s_i)+\theta_K(\varepsilon)\,.
\end{aligned}\tag{4.9}$$

The only error is from the contribution of the region $K\setminus\bigcup_{i=1}^{k}Q_i$. Combining inequalities (4.7), (4.8), and (4.9) proves the desired upper bound (4.2), i.e.

$$\begin{aligned}
\mathbb{P}\Big(&\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\varepsilon\ell),\omega\big)>\mathrm{Ent}_{K,\mathsf{an}}(h_K)\\
&+\eta+\theta_{h_K}(\varepsilon)+\theta_{h_K,\varepsilon}\left(\tfrac{1}{n}\right)\Big)\le\theta_{h_K,\eta,\varepsilon}\left(\tfrac{1}{n}\right).
\end{aligned}$$

Now we turn to the lower bound (4.3). Similar to before, let $q=\varepsilon^{1/2}\ell$ and let $Q_1,\dots,Q_k$ enumerate the hypercubes that have side length q, have vertices in $q\mathbb{Z}^m$, and lie entirely inside of one of the simplices Δ_j. Note that the side length q is different now compared to above when we were justifying the upper bound (4.2), and hence $Q_1,\dots,Q_k$ denotes a different set of hypercubes.

To prove (4.3) we overcount height functions, using the same idea as in the companion article [31]. In summary, define a subset of "exceptional" points $E_n\subset K_n$ as follows: let

$$G_n=\bigcup_{i=1}^{k}\partial Q_{i,n}\,,\quad U_n=K_n\setminus\bigcup_{i=1}^{k}Q_{i,n}\,,\quad\text{and}\quad E_n=G_n\cup U_n\,.$$

Informally, G_n is the "grid" formed by the boundaries of the hypercubes and U_n is the "uncovered" region, i.e. the part of K_n that is not covered by the hypercubes. We group height functions $h_{K_n}\in B(K_n,h_K,\varepsilon\ell)$ based on their values on the set E_n. For each fixed assignment of heights $h_{K_n}|_{E_n}\in M(E_n)$, the entropy of the set of extensions to the hypercubes $\bigcup_1^k Q_n\approx K_n\setminus E_n$ is asymptotically equal to the macroscopic entropy $\mathrm{Ent}_{K,\mathsf{an}}(h_K)$. The set E_n is not too large, so even after counting all admissible assignments $h_{K_n}|_{E_n}$, the resulting asymptotics match (4.3).

To make the above argument rigorous, let $\mathrm{Adm}(E_n)$ denote the set of admissible height functions on E_n, i.e. those height functions $h_{E_n}\in M(E_n)$ that admit an extension to a height function in $B(K_n, h_K, \varepsilon\ell)$. There is an obvious injection from $B(K_n, h_K, \varepsilon\ell)$ into

$$\biguplus_{h_{E_n} \in \mathrm{Adm}(E_n)} \prod_{i=1}^{k} M\big(Q_{i,n}, h_{E_n}|_{\partial Q_{i,n}}\big)\,, \tag{4.10}$$

where "$\biguplus$" denotes the disjoint union (so for distinct height functions h_{E_n} and h_{E_n} in $\mathrm{Adm}(E_n)$, the product sets $\prod_1^k M\big(Q_{i,n}, h_{E_n}|_{\partial Q_{i,n}}\big)$ and $\prod_1^k M\big(Q_{i,n}, h_{E_n}|_{\partial Q_{i,n}}\big)$ are considered disjoint inside the set from (4.10)). It follows that

$$\begin{aligned} &Z_\omega\big(B(K_n, h_K, \varepsilon\ell), \omega\big) \\ &\qquad \le \sum_{h_{E_n} \in \mathrm{Adm}(E_n)} Z_\omega\left(\prod_{i=1}^{k} M\big(Q_{i,n}, h_{E_n}|_{Q_{i,n}}\big)\right) \\ &\qquad \le \big|\mathrm{Adm}(E_n)\big| \max_{h_{E_n} \in \mathrm{Adm}(E_n)} Z_\omega\left(\prod_{i=1}^{k} M\big(Q_{i,n}, h_{E_n}|_{Q_{i,n}}\big)\right). \end{aligned}$$

Therefore

$$\begin{aligned} &\mathrm{Ent}_{K_n}\big(B(K_n, h_K, \varepsilon\ell), \omega\big) \\ &\qquad \ge \min_{h_{E_n} \in \mathrm{Adm}(E_n)} \sum_{i=1}^{k} \frac{|Q_{i,n}|}{|K_n|} \mathrm{Ent}_{Q_{i,n}}\big(M(Q_{i,n}, h_{E_n}|_{Q_{i,n}}), \omega\big) \\ &\qquad\quad - \frac{\log|\mathrm{Adm}(E_n)|}{|K_n|}\,. \end{aligned} \tag{4.11}$$

Clearly $\frac{|Q_{i,n}|}{|K_n|} = \frac{1}{k} + \theta_m(\varepsilon) + \theta_{m,\varepsilon,\ell}\big(\frac{1}{n}\big)$.

To control $|\mathrm{Adm}(E_n)|$, we argue as follows. First, $\frac{|G_n|}{|K_n|} = \theta_m(q) = \theta_m(\varepsilon)$ and $\frac{|U_n|}{|K_n|} = \theta_m(\varepsilon)$. Second, for an arbitrary base point $z_0 \in E_n$, there are at most $2\varepsilon\ell n + 1$ admissible values for $h_{E_n}(z_0)$ if $h_{E_n} \in \mathrm{Adm}(E_n)$, since h_{E_n} must extend to a height function in the ball $B(K_n, h_K, \varepsilon\ell)$. Third, the set E_n is connected, so for each of the admissible values of $h_{E_n}(z_0)$, there are at most $2^{|E_n|}$ height functions in $\mathrm{Adm}(E_n)$ taking that value at z_0. Putting these observations together, we conclude that $\frac{1}{|K_n|}\log|\mathrm{Adm}(E_n)| = \theta_m(\varepsilon) + \theta_{m,\varepsilon,\ell}\big(\frac{1}{n}\big)$.

Applying these asymptotic results in (4.11) yields

$$\begin{aligned} &\mathrm{Ent}_{K_n}\big(B(K_n, h_K, \varepsilon\ell), \omega\big) \\ &\qquad \ge \min_{h_{E_n} \in \mathrm{Adm}(E_n)} \frac{1}{k} \sum_{i=1}^{k} \mathrm{Ent}_{Q_{i,n}}\big(M(Q_{i,n}, h_{E_n}|_{Q_{i,n}}), \omega\big) \\ &\qquad\quad - \theta_m(\varepsilon) - \theta_{m,\varepsilon,\ell}\big(\tfrac{1}{n}\big)\,. \end{aligned} \tag{4.12}$$

Whenever $h_{E_n} \in \mathrm{Adm}(E_n)$,

$$\max_{z \in E_n} \big|h_K(\tfrac{1}{n}z) - \tfrac{1}{n}h_{E_n}(z)\big| < \varepsilon\ell\,,$$

so for each $i = 1, \ldots, k$, by analogy to (4.5),

$$\max_{z \in \partial S_{qn}} \Big|\big(h_{E_n}(v_i + z) - \lfloor s_i \cdot v_i + qnb_i \rfloor\big) - h^{s_i}_{\partial S_{qn}}(z)\Big| \le \varepsilon\ell n\,.$$

We apply Theorem 3.8 to the height function

$$\big(z \mapsto h_{E_n}(v_i + z) - \lfloor s_i \cdot v_i + qnb_i \rfloor\big) \in M(S_{qn})$$

to conclude that

$$\begin{aligned}&\mathrm{Ent}_{Q_{i,n}}\big(M(Q_{i,n},h_{E_n}|_{\partial Q_{i,n}}),\omega\big)\\ &\qquad\geq \mathrm{ent}_{Aqn}\big(s_i,\tau_{\lfloor s_i\cdot v_i+qnb_i\rfloor}\omega\big)-C_\omega\theta(\varepsilon)\,.\end{aligned}\tag{4.13}$$

The two almost-sure inequalities (4.12) and (4.13), the probability estimate (4.8), and the macroscopic bound (4.9) together imply the desired lower bound (4.3), which completes the proof of Lemma 4.3. □

The remainder of the proof of the profile theorem (Theorem 2.22) for general asymptotic height functions follows closely the proof in Section 6 of the companion article [31]. Below we state an approximation result (Theorem 4.4), which concludes that any asymptotic height function h_R admits a "good" approximation h_K satisfying the hypotheses of Lemma 4.3 above. Following that result are three robustness lemmas (Lemma 4.5, Lemma 4.6, and Lemma 4.7). With these tools it is straightforward to reduce the general case of Theorem 2.22 to the special case of Lemma 4.3. The approximation result (Theorem 4.4) is unchanged from the companion article, which should be expected because the random potential in the current model does not affect the class of limit objects that our model admits, i.e. domains satisfying Assumption 2.1 and asymptotic height functions. It is similar to [13, Lemma 2.2] or [43, Theorem 1]. A proof of it is given in the companion article [31].

Theorem 4.4 (Simplicial Rademacher theorem)**.** *Let $R\subseteq\mathbb{R}^m$ be a region satisfying Assumption 2.1, and let $h_R\in M(R,h_{\partial R})$ be an asymptotic height function on R. For any $\varepsilon>0$ and any $\ell>0$ sufficiently small (depending on ε), we may choose a simplex domain $K=\Delta_1\cup\cdots\cup\Delta_k\subseteq R$ of scale ℓ (see Definition 4.1) and a piecewise affine asymptotic height function $h_K:K\to\mathbb{R}$ (that is, an asymptotic height function such that each restriction $h_K|_{\Delta_i}:\Delta_i\to\mathbb{R}$ is affine) that satisfy the following properties:*

1. *$|R\setminus K|<\varepsilon$ and $d_H(K,R)<\varepsilon$, where we recall that for subsets of $\mathbb{R}^m$, $|\cdot|$ denotes the Lebesgue measure and $d_H(\cdot,\cdot)$ denotes Hausdorff metric;*
2. *$\max_{x\in K}|h_K(x)-h_R(x)|<\frac{1}{2}\varepsilon\ell$; and*
3. *on at least a $(1-\varepsilon)$ fraction of the points in K (by Lebesgue measure), the gradients $\nabla h_K(x)$ and $\nabla h_R(x)$ agree to within ε, i.e. $\frac{1}{|K|}\big|\big\{x\in K\;\big|\;|\nabla h_K(x)-\nabla h_R(x)|_2\geq\varepsilon\big\}\big|<\varepsilon$.*

Now we turn to the robustness lemmas, which will be used when applying Theorem 4.4 to approximate h_R by another asymptotic height function. The three lemmas below are almost direct analogues of Lemmas 35, 36, and 37 from [31] respectively.

Lemma 4.5 (Robustness of macroscopic entropy under approximations)**.** *Let $\varepsilon>0$, and let $\tilde R\subseteq R\subset\mathbb{R}^m$ be sets meeting the assumptions from Assumption 2.1 with $|R\setminus\tilde R|<\varepsilon$. Let $h_{\tilde R}\in M(\tilde R)$ and $h_R\in M(R)$ be such that*

$$\left|\left\{x\in\tilde R\;\middle|\;\big|\nabla h_{\tilde R}(x)-\nabla h_R(x)\big|_2\geq\varepsilon\right\}\right|<\varepsilon\,.\tag{4.14}$$

Then,

$$\mathrm{Ent}_{R,\mathsf{an}}(h_R)=\mathrm{Ent}_{\tilde R,\mathsf{an}}(h_{\tilde R})+\theta_m(\varepsilon)\,.$$

Proof. Recall from Definition 2.21 that

$$\mathrm{Ent}_{R,\mathsf{an}}(h_R):=\frac{1}{|R|}\int_R\mathrm{ent}_{\mathsf{an}}\big(\nabla h_R(x)\big)\,dx\,,$$

and likewise for $\mathrm{Ent}_{\tilde R,\mathsf{an}}(h_{\tilde R})$. The conclusion follows from three observations: first that the domains of integration are bounded sets with small symmetric difference, second that the function $s\mapsto\mathrm{ent}_{\mathsf{an}}(s)$ is continuous, and third that the functions ∇h_R and $\nabla h_{\tilde R}$ almost agree (as per (4.14)) on most of the intersection of their domains (by measure). □

Lemma 4.6 (Robustness of microscopic entropy under change in profile)**.** *Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $R \subset \mathbb{R}^m$ satisfy Assumption 2.1, and let $R_n \subset \mathbb{Z}^m$ satisfy $\frac{1}{n}R_n \subset R$. Let $h_R, \tilde{h}_R \in M(R)$ be two asymptotic height functions such that $\sup_{x\in R}|h_R(x) - \tilde{h}_R(x)| \le \varepsilon$. Then,*

$$\mathrm{Ent}_{R_n}\big(B(R_n, h_R, 2\varepsilon), \omega\big) \le \mathrm{Ent}_{R_n}\big(B(R_n, \tilde{h}_R, \varepsilon), \omega\big)\,.$$

Proof. For any fixed ω, the functional $\mathrm{Ent}_{R_n}(\cdot, \omega) : M(R_n) \to \mathbb{R}$ is monotonic, and it follows from Definition 2.3 that

$$B(R_n, \tilde{h}_R, \varepsilon) \subseteq B(R_n, h_R, 2\varepsilon)\,. \qquad \square$$

Lemma 4.7 (Robustness of microscopic entropy under domain approximations)**.** *Let $c \in (0,1]$, $\varepsilon \in (0,1]$, and $n \in \mathbb{N}$. Let $\tilde{R} \subset R \subset \mathbb{R}^m$ and $\tilde{R}_n \subset R_n \subset \mathbb{Z}^m$ satisfy these assumptions:*

$$\begin{aligned}
\tfrac{1}{n}R_n &\subset R\,, & \tfrac{1}{n}\tilde{R}_n &\subset \tilde{R}\,,\\
d_H(\tfrac{1}{n}R_n, R) &= \theta_R(\varepsilon)\,, & d_H(\tfrac{1}{n}\tilde{R}_n, \tilde{R}) &= \theta_R(\varepsilon)\,,\\
\frac{|R_n|}{n^m|R|} &= 1 + \theta_R(\varepsilon) + \theta_{R,\varepsilon}\big(\tfrac{1}{n}\big)\,, & \frac{|\tilde{R}_n|}{n^m|\tilde{R}|} &= 1 + \theta_R(\varepsilon) + \theta_{R,\varepsilon}\big(\tfrac{1}{n}\big)\,,\\
\frac{|R|}{|\tilde{R}|} &= 1 + \theta_R(\varepsilon)\,.
\end{aligned}$$

Let $h_R \in M(R)$ be an asymptotic height function with $\mathrm{Lip}(h_R) \le 1 - c\varepsilon$. Then,

$$\begin{aligned}
&\mathrm{Ent}_{\tilde{R}_n}\big(B(\tilde{R}_n, h_R, \varepsilon), \omega\big) - C_\omega\theta_R(\varepsilon) - C_\omega\theta_{R,\varepsilon}\big(\tfrac{1}{n}\big)\\
&\quad\le \mathrm{Ent}_{R_n}\big(B(R_n, h_R, \varepsilon), \omega\big)\\
&\quad\le \mathrm{Ent}_{\tilde{R}_n}\big(B(\tilde{R}_n, h_R, \tfrac{c}{3}\varepsilon^2), \omega\big) + C_\omega\theta_R(\varepsilon) + C_\omega\theta_{R,\varepsilon}\big(\tfrac{1}{n}\big)\,.
\end{aligned} \tag{4.15}$$

Proof. We prove the two inequalities in (4.15) separately. For the first inequality, observe that the map

$$\begin{aligned}
B(R_n, h_R, \varepsilon) &\to B(\tilde{R}_n, h_R, \varepsilon)\\
h_R &\mapsto h_R|_{\tilde{R}}
\end{aligned}$$

is not generally an injection, but it is at most $(2^{|R_n\setminus\tilde{R}_n|})$-to-1 (by the graph homomorphism property and connectedness of R_n). For any $h_{R_n} \in B(R_n, h_R, \varepsilon)$,

$$H_{R_n,\omega}(h_{R_n}) \le H_{\tilde{R}_n,\omega}(h_{R_n}|_{\tilde{R}_n}) + C_\omega|R_n \setminus \tilde{R}_n|\,,$$

so

$$Z_\omega(B(R_n, h_R, \varepsilon)) \le 2^{|R_n\setminus\tilde{R}_n|} Z_\omega(B(\tilde{R}_n, h_R, \varepsilon)) \exp\big(C_\omega|R_n \setminus \tilde{R}_n|\big)$$

and

$$\begin{aligned}
&\mathrm{Ent}_{R_n}\big(B(R_n, h_R, \varepsilon), \omega\big)\\
&\quad\ge \frac{|\tilde{R}_n|}{|R_n|}\,\mathrm{Ent}_{\tilde{R}_n}\big(B(\tilde{R}_n, h_R, \varepsilon), \omega\big)\\
&\qquad\quad - \log(2)\,\frac{|R_n \setminus \tilde{R}_n|}{|R_n|} - C_\omega|R_n \setminus \tilde{R}_n|\\
&\quad= \mathrm{Ent}_{\tilde{R}_n}\big(B(\tilde{R}_n, h_R, \varepsilon), \omega\big) - C_\omega\theta_R(\varepsilon) - C_\omega\theta_{R,\varepsilon}\big(\tfrac{1}{n}\big)\,.
\end{aligned}$$

To prove the second inequality in (4.15), we first note that there exists an injection from $B(\tilde{R}_n, h_R, \frac{c}{3}\varepsilon^2)$ into $B(R_n, h_R, \varepsilon)$. A height function $h_{\tilde{R}_n} \in B(\tilde{R}_n, h_R, \frac{c}{3}\varepsilon^2)$ is extended to $h_{R_n} \in B(R_n, h_R, \varepsilon)$ in such a way that $\big|h_{R_n}(z) - nh_R\big(\frac{1}{n}z\big)\big| \le 1$ when z is in R_n and sufficiently far away from $\tilde{R}_n$; the parameter value $\frac{c}{3}\varepsilon^2$ is chosen so that such an extension is admissible by the Kirszbraun theorem. For details, see the proof of [31, Lemma 37]. For this injection $h_{\tilde{R}_n} \mapsto h_{R_n}$,

$$H_{\tilde{R}_n,\omega}(h_{\tilde{R}_n}) \leq H_{R_n,\omega}(h_{R_n}) + C_\omega |R_n \setminus \tilde{R}_n| ,$$

so

$$Z_\omega\big(B(\tilde{R}_n, h_R, \tfrac{c}{3}\varepsilon^2)\big) \leq Z_\omega\big(B(R_n, h_R, \varepsilon)\big) \exp\big(C_\omega |R_n \setminus \tilde{R}_n|\big)$$

and

$$\begin{aligned} &\operatorname{Ent}_{\tilde{R}_n}\big(B(\tilde{R}_n, h_R, \tfrac{c}{3}\varepsilon), \omega\big) \\ &\quad \geq \frac{|R_n|}{|\tilde{R}_n|} \operatorname{Ent}_{R_n}\big(B(R_n, h_R, \varepsilon), \omega\big) \\ &\qquad - \log(2) \frac{|R_n \setminus \tilde{R}_n|}{|R_n|} - C_\omega |R_n \setminus \tilde{R}_n| \\ &\quad = \operatorname{Ent}_{R_n}\big(B(R_n, h_R, \varepsilon), \omega\big) - C_\omega \theta_R(\varepsilon) - C_\omega \theta_{R,\varepsilon}\big(\tfrac{1}{n}\big) . \end{aligned}$$

To prove the profile theorem, we reduce to the special case of Lemma 4.3, where the domain is a collection of simplices and the asymptotic height function is piecewise affine. Before that, in order to apply Lemma 4.7, we reduce to the case where h_R has Lipschitz constant strictly less than 1. Both reductions are simple applications of the robustness results above.

Proof of the profile theorem (Theorem 2.22). For the reader's convenience we recall the conclusion of the theorem that we are about to prove, namely:

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{P}\bigg(\Big| &\operatorname{Ent}_{R_n}\big(B(R_n, h_R, \delta), \omega\big) - \operatorname{Ent}_{\mathsf{an}}(R, h_R) \Big| \\ &\geq \eta + C_\omega \theta_{h_R}(\delta) + C_\omega \theta_{h_R,\delta}\big(\tfrac{1}{n}\big) \bigg) = 0 . \end{aligned} \tag{4.16}$$

For the first step of the proof, we reduce from the case of an arbitrary asymptotic height function $h_R \in M(R, h_{\partial R})$, i.e. a continuous function $h_R : R \to \mathbb{R}$ with Lipschitz constant at most 1 (with respect to the ℓ^1 norm on R), to an asymptotic height function with Lipschitz constant strictly less than 1. Indeed, let $c := (2 \operatorname{diam}_1 R)^{-1} \wedge 1$, where $\operatorname{diam}_1 R$ denotes the diameter of R under the ℓ^1 norm. By translation invariance of the random field ω, we assume that there exists $x_0 \in R$ with $h_R(x_0) = 0$. Define

$$\tilde{h}_R := (1 - c\delta) h_R .$$

We make the following observations. First,

$$\operatorname{Lip}(\tilde{h}_R) \;=\; (1 - c\delta) \operatorname{Lip}(h_R) \;\leq\; 1 - c\delta .$$

Second, for any $x \in R$,

$$|h_R(x) - \tilde{h}_R(x)| \;\leq\; c\delta |h_R(x)| \;\leq\; c\delta |x - x_0|_1 \;\leq\; \tfrac{\delta}{2} . \tag{4.17}$$

Third, for any $x \in R$,

$$|\nabla h_R(x) - \nabla \tilde{h}_R(x)| \leq c\delta . \tag{4.18}$$

Lemma 4.5, together with (4.18) and the choice of constant $c = c(R)$, yields

$$\operatorname{Ent}_{R,\mathsf{an}}(h_R) = \operatorname{Ent}_{R,\mathsf{an}}(\tilde{h}_R) + \theta_R(\delta) . \tag{4.19}$$

Similarly, Lemma 4.6 and (4.17) imply that almost surely,

$$\begin{aligned}&\mathrm{Ent}_{R_n}\big(B(R_n,\tilde{h}_R,2\delta),\omega\big)\\&\quad\le \mathrm{Ent}_{R_n}\big(B(R_n,h_R,\delta),\omega\big)\\&\quad\le \mathrm{Ent}_{R_n}\big(B(R_n,\tilde{h}_R,\tfrac{1}{2}\delta),\omega\big)\,.\end{aligned}$$

Assume for the sake of the proof that (4.16) holds for $\tilde{h}_R$. Then almost surely,

$$\begin{aligned}\mathrm{Ent}_{R_n}\big(B(R_n,h_R,\delta),\omega\big)&\le \mathrm{Ent}_{R_n}\big(B(R_n,\tilde{h}_R,\tfrac{\delta}{2}),\omega\big)\\&\le \mathrm{Ent}_{R,\mathsf{an}}(\tilde{h}_R)\,+\,\eta\,+\,C_\omega\theta_{\tilde{h}_R}\big(\tfrac{\delta}{2}\big)\,+\,C_\omega\theta_{\tilde{h}_R,\delta/2}\big(\tfrac{1}{n}\big)\\&=\mathrm{Ent}_{R,\mathsf{an}}(h_R)\,+\,\eta\,+\,C_\omega\theta_{h_R}(\delta)\,+\,C_\omega\theta_{h_R,\delta}\big(\tfrac{1}{n}\big),\end{aligned}$$

where in the last line, we combine the $\theta_R(\delta)$ term from (4.19) together with the $C_\omega\theta_{\tilde{h}_R}(\frac{\delta}{2})$ term above; this is admissible since $C_\omega\ge 1$ by definition (recall that $C_\omega := 1\vee\sup_{e\in E(\mathbb{Z})}|\omega_e|$) and since the various factors of $\frac{1}{2}$ do not affect the asymptotics. The reverse inequality is similar, and so we have reduced to the problem of proving (4.16) with the added assumption that $\mathrm{Lip}(h_R)\le 1-c\delta$ for $c=c(R)\in(0,1)$.

We reduce further to the special case from Lemma 4.3, i.e. a piecewise affine asymptotic height function defined on a collection of simplices. First, we choose parameter values $\varepsilon=\varepsilon(\delta)$ and $\ell=\ell(\varepsilon,\delta)$ satisfying three criteria:

1. $\varepsilon\to 0$ as $\delta\to 0$,
2. $\delta=\varepsilon\ell$,
3. ℓ is sufficiently small so that the simplicial Rademacher theorem (Theorem 4.4) applies.

The choices of ε and ℓ may be realized as follows, from [31]: Choose a sequence $\varepsilon_k\searrow 0$ arbitrarily, e.g. $\varepsilon_k=\frac{1}{k}$. Let ℓ_k be the largest admissible ℓ value based on ε_k, but not larger than 1. For any given δ choose the smallest ε_k such that $\varepsilon_k\ell_k>\delta$; this ensures the first criterion. Set $\varepsilon=\varepsilon_k$ and $\ell=\frac{\delta}{\varepsilon_k}\le\ell_k$; this ensures the last two criteria.

For the remainder of the argument, fix $\delta>0$. Let ε and ℓ satisfy the above criteria, and let $K\subseteq R\subset\mathbb{R}^m$ be a simplicial domain and $h_K\in M(K)$ an asymptotic height function satisfying the conclusions of the simplicial Rademacher theorem (Theorem 4.4). Since $\nabla h_K\approx\nabla h_R$ (cf. conclusion (3) of Theorem 4.4) and since the macroscopic entropy is robust (Lemma 4.5),

$$\Big|\mathrm{Ent}_{R,\mathsf{an}}(h_R)-\mathrm{Ent}_{K,\mathsf{an}}(h_K)\Big|\;\le\;\theta_R(\varepsilon)\;=\;\theta_R(\delta)\,,\tag{4.20}$$

where we use the fact that $\varepsilon\to 0$ as $\delta\to 0$ in order to replace ε by δ in the θ error term.

Similarly, by conclusions 1 and 2 of Theorem 4.4 and the microscopic entropy robustness,

$$\begin{aligned}&\mathrm{Ent}_{R_n}\big(B(R_n,h_R,\varepsilon\ell),\omega\big)\\&\quad\overset{(Lemma\ 4.7)}{\le}\mathrm{Ent}_{K_n}\big(B(K_n,h_R|_K,\tfrac{c}{3}(\varepsilon\ell)^2),\omega\big)+C_\omega\theta(\varepsilon)+C_\omega\theta_\varepsilon\big(\tfrac{1}{n}\big)\\&\quad\overset{(Lemma\ 4.6)}{\le}\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\tfrac{c}{6}(\varepsilon\ell)^2),\omega\big)+C_\omega\theta(\varepsilon)+C_\omega\theta_\varepsilon\big(\tfrac{1}{n}\big)\end{aligned}\tag{4.21}$$

and

$$\begin{aligned}&\mathrm{Ent}_{R_n}\big(B(R_n,h_R,\varepsilon\ell),\omega\big)\\&\quad\overset{(Lemma\ 4.7)}{\ge}\mathrm{Ent}_{K_n}\big(B(K_n,h_R|_K,\varepsilon\ell),\omega\big)-C_\omega\theta(\varepsilon)-C_\omega\theta_\varepsilon\big(\tfrac{1}{n}\big)\\&\quad\overset{(Lemma\ 4.6)}{\ge}\mathrm{Ent}_{K_n}\big(B(K_n,h_K,\tfrac{1}{2}\varepsilon\ell),\omega\big)-C_\omega\theta(\varepsilon)-C_\omega\theta_\varepsilon\big(\tfrac{1}{n}\big)\,.\end{aligned}\tag{4.22}$$

Combining (4.20), (4.21), (4.22), and the special case of the profile theorem proved in Lemma 4.3 completes the proof. □

5 Variational principle

In this section we prove the variational principle (Theorem 2.23). The proof follows the steps of the corresponding proof for the uniform case in [31]. The main difference and the step that needs attention is that the deterministic convergence needs to be lifted to a convergence in probability. The two main inequalities in the proof follow from first comparing the set of height functions $M(R_n, h_{\partial R_n}, \delta)$ to the subset $B(R_n, h^*_R, \delta)$ for a well-chosen asymptotic height function h^*_R, and second from comparing to a superset $\bigcup_{i=1}^k B(R_n, h^{(i)}_R, \delta_i)$ for a collection of asymptotic height functions $h^{(1)}_R, \ldots, h^{(k)}_R$. Especially in the second part of the argument, some care is needed in regards to the asymptotic parameters. In particular:

- The choice (and number) of height functions $h^{(i)}_R$ depends on δ,
- the radii δ_i of the balls around these height functions depends on η,
- the probability that the profile theorem fails (i.e. the probability that the quantities $\mathrm{Ent}_{R_n}(B(R_n, h^{(i)}_R, \delta_i), \omega)$ and $\mathrm{Ent}_{R,\mathsf{an}}(h^{(i)}_R)$ differ by a large amount due to the exact configuration ω of the random potential) depends not just on the error tolerance η but also on the number of height functions $h^{(i)}_R$.

Proof of Theorem 2.23. Let $\eta > 0$ and $p_{\mathsf{max}} > 0$. First we will establish that

$$\limsup_{\delta\to 0}\ \limsup_{n\to\infty}\ \mathbb{P}\Big(\mathrm{Ent}_{R_n}\big(M(R_n, h_{\partial R_n}, \delta), \omega\big) > \inf_{h_R \in M(R, h_{\partial R})} \mathrm{Ent}_{R,\mathsf{an}}(h_R) + \eta\Big) \le p_{\mathsf{max}}\,. \tag{5.1}$$

We choose $h^*_R \in M(R, h_{\partial R})$ such that

$$\mathrm{Ent}_{R,\mathsf{an}}(h^*_R) \le \inf_{h_R \in M(R, h_{\partial R})} \mathrm{Ent}_{R,\mathsf{an}}(h_R) + \tfrac{\eta}{4}\,. \tag{5.2}$$

For any $\delta > 0$ and $n \in \mathbb{N}$, $B(R_n, h^*_R, \delta) \subseteq M(R_n, h_{\partial R_n}, \delta)$. Hence almost surely,

$$\mathrm{Ent}_{R_n}\big(M(R_n, h_{\partial R_n}, \delta), \omega\big) \le \mathrm{Ent}_{R_n}\big(B(R_n, h^*_R, \delta), \omega\big)\,. \tag{5.3}$$

By the profile theorem (applied to h^*_R),

$$\mathbb{P}\Big(\big|\mathrm{Ent}_{R_n}\left(B(R_n, h^*_R, \delta), \omega\right) - \mathrm{Ent}_{R,\mathsf{an}}(h^*_R)\big| > \tfrac{\eta}{4} + C_\omega \theta_{h^*_R}(\delta) + C_\omega \theta_{h^*_R,\delta}\big(\tfrac{1}{n}\big)\Big) \underset{n\to\infty}{\to} 0\,. \tag{5.4}$$

Let us spend a part of the available probability p_{max} to establish a bound on C_ω. Specifically, since $C_\omega \in L^1$, Markov's inequality implies that

$$\mathbb{P}\big(C_\omega > \tfrac{2\|C_\omega\|_1}{p_{\mathsf{max}}}\big) \le \tfrac{1}{2} p_{\mathsf{max}}\,.$$

Therefore as long as δ is small enough so that the $\theta_{h^*_R}(\delta)$ term is less than $\frac{\eta}{4} \cdot \frac{p_{\mathsf{max}}}{2\|C_\omega\|_1}$, and as long as n is large enough that the $\theta_{h^*_R,\delta}(\frac{1}{n})$ term is less than $\frac{\eta}{4} \cdot \frac{p_{\mathsf{max}}}{2\|C_\omega\|_1}$ and the probability in (5.4) is less than $\frac{1}{2}p_{\mathsf{max}}$, we have

$$\mathbb{P}\Big(\mathrm{Ent}_{R_n}\big(B(R_n, h^*_R, \delta), \omega\big) > \mathrm{Ent}_{R,\mathsf{an}}(h^*_R) + \tfrac{3\eta}{4}\Big) < p_{\mathsf{max}}\,. \tag{5.5}$$

The first desired inequality (5.1) follows immediately from (5.3), (5.5), and (5.2).

Now we turn to the second half of the variational principle, namely:

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\Big(\operatorname{Ent}_{R_n}\big(M(R_n,h_{\partial R_n},\delta),\omega\big) < \inf_{h_R\in M(R,h_{\partial R})}\operatorname{Ent}_{R,\mathsf{an}}(h_R)-\eta\Big)\le p_{\mathsf{max}}\,. \tag{5.6}$$

In order to establish (5.6), we overcount the set $M(R_n,h_{\partial R},\delta)$ using compactness of the space of asymptotic height functions $M(R,h_{\partial R},\delta)$ (with respect to the topology of uniform convergence). Indeed, choose asymptotic height functions $h_R^{(1)},\dots,h_R^{(k)}$ such that

$$M(R,h_{\partial R},\delta)\subset\bigcup_{i=1}^{k}B(R,h_R^{(i)},\delta_i)\,, \tag{5.7}$$

where the values $\delta_i>0$ are such that the $\theta_{h_R^{(i)}}(\delta_i)$ terms from the profile theorem (Theorem 2.22) are each less than $\frac{\eta}{4}\cdot\frac{p_{\mathsf{max}}}{2\|C_\omega\|_1}$.

As in the first part of the proof, we restrict to the event

$$\Omega' := \left\{C_\omega<\frac{2\|C_\omega\|_1}{p_{\mathsf{max}}}\right\},$$

which has $\mathbb{P}(\Omega')\ge 1-\frac{p_{\mathsf{max}}}{2}$. Furthermore, we assume implicitly that n is large enough that:

- each of the $\theta_{h_R^{(i)},\delta_i}(\frac{1}{n})$ terms from the profile theorem is less than $\frac{\eta}{4}\cdot\frac{p_{\mathsf{max}}}{2\|C_\omega\|_1}$, and
- the exceptional events

$$E_{i,n}:=\Omega'\cap\left\{\left|\operatorname{Ent}_{R_n}\big(B(R_n,h_R^{(i)},\delta_i),\omega\big)-\operatorname{Ent}_{R,\mathsf{an}}(h_R^{(i)})\right|>\tfrac{3\eta}{4}\right\}$$

  satisfy $\mathbb{P}(E_{i,n})<\frac{p_{\mathsf{max}}}{2k}$ for $i=1,\dots,k$.

Then for sufficiently small δ and sufficiently large n, the "good" event

$$\Omega_{\delta,n}:=\Omega'\cap E_{1,n}^c\cap\cdots\cap E_{k,n}^c$$

satisfies $\mathbb{P}(\Omega_{\delta,n})\ge 1-p_{\mathsf{max}}$ and, for $\omega\in\Omega_{\delta,n}$,

$$\left|\operatorname{Ent}_{R_n}\big(B(R_n,h_R^{(i)},\delta_i),\omega\big)-\operatorname{Ent}_{R,\mathsf{an}}(h_R^{(i)})\right|\le\tfrac{3\eta}{4}\,. \tag{5.8}$$

Assume in the sequel that $\omega\in\Omega_{\delta,n}$. By the set inclusion (5.7),

$$\operatorname{Ent}_{R_n}\big(M(R_n,h_{\partial R_n},\delta),\omega\big)\ge-\frac{1}{|R_n|}\log\left(\sum_{i=1}^{k}Z_\omega\big(B(R_n,h_R^{(i)},\delta_i)\big)\right). \tag{5.9}$$

To handle the sum inside the logarithm, we compare each summand $Z_\omega(B(R_n,h_R^{(i)},\delta_i))$ against $\inf_{h_R}\operatorname{Ent}_{R,\mathsf{an}}(h_R)$. Indeed,

$$\begin{aligned}\operatorname{Ent}_{R_n}\big(B(R_n,h_R^{(i)},\delta),\omega\big)&\overset{(5.8)}{\ge}\operatorname{Ent}_{R,\mathsf{an}}\big(h_R^{(i)}\big)-\frac{3\eta}{4}\\&\ge\inf_{h_R\in M(R,h_{\partial R})}\operatorname{Ent}_{R,\mathsf{an}}(h_R)-\frac{3\eta}{4}\,,\end{aligned}$$

and so

$$Z_\omega\big(B(R_n,h_R^{(i)},\delta_i)\big)\le\exp\Big[|R_n|\Big(-\inf_{h_R\in M(R,h_{\partial R})}\operatorname{Ent}_{R,\mathsf{an}}(h_R)+\tfrac{3\eta}{4}\Big)\Big]$$

and

$$\sum_{i=1}^{k} Z_\omega\big(B(R_n, h_R^{(i)}, \delta_i)\big) \le k \exp\Big[|R_n|\Big(-\inf_{h_R\in M(R,h_{\partial R})} \mathrm{Ent}_{R,\mathsf{an}}(h_R) + \tfrac{3\eta}{4}\Big)\Big]\,.$$

Returning to (5.9), this yields

$$\begin{aligned}&\mathrm{Ent}_{R_n}\big(M(R_n, h_{\partial R_n}, \delta), \omega\big)\\ &\qquad \ge \inf_{h_R\in M(R,h_{\partial R})} \mathrm{Ent}_{R,\mathsf{an}}(h_R) - \frac{\log k}{|R_n|} - \frac{3\eta}{4}\,.\end{aligned}$$

As long as n is large enough (depending on k, which in turn depends on δ), we have $\frac{\log k}{|R_n|} < \frac{\eta}{4}$, and so

$$\mathrm{Ent}_{R_n}\big(M(R_n, h_{\partial R_n}, \delta), \omega\big) \ge \inf_{h_R\in M(R,h_{\partial R})} \mathrm{Ent}_{R,\mathsf{an}}(h_R) - \eta\,,$$

for any $\omega \in \Omega_{\delta,n}$. This establishes (5.6) and thereby proves the variational principle (Theorem 2.23). □

6 Open problems

- A natural question is whether in the profile theorem (Theorem 2.22) and the variational principle (Theorem 2.23), the mode of convergence can be improved from convergence in probability to almost-sure convergence. The obstacle to achieving almost-sure convergence via the method of proof above is the shifted environments $\tau_{i,n}\omega$ in (4.7). Without the shifts $\tau_{i,n}$, almost sure convergence would follow from the ergodic theorem, applied individually for each index i with $n \to \infty$. It is possible that the ergodic theorem can be modified to account for such shifts, or that another method of proof can be used to improve the convergence result.
- The proofs in this article assume that the random potential ω is almost surely bounded, and simulations provide evidence that the model does not homogenize for some distributions of ω that are unbounded. We conjecture that the model fails to homogenize when additionally to our Assumption 2.9, $\sup_{e\in E(\mathbb{Z})}|\omega_e| = \infty$ almost surely. Alternatively, find the correct conditions on ω that ensure homogenization.
- We prove that the local surface tension is convex (cf. Lemma 3.16). This is sufficient to conclude that the infima in the variational principle and large deviations principle are attained (as long as the set of height functions A in the large deviations principle (1.6) is closed). It would be useful to prove that the local surface tension is, moreover, strictly convex. Indeed, if the local surface tension is strictly convex, then it follows that the minimizing height function in the variational principle (1.5) is unique, and hence is a limit shape. Many random surface models are known to have a strictly convex local surface tension, e.g. domino tilings [13] and SAP models [44]. For other models it is known that the local surface tension is not strictly convex, e.g. the asymmetric five vertex model (a degenerate case of the six-vertex model) [15].
- Characterize the fluctuations of the perturbed probability measure μ_ω. This is likely a complex problem. By analogy to the dimer model studied in [28], we expect that fluctuations may exhibit different asymptotics in different parts of the domain (even asymptotically away from the boundary), and by analogy to the random bridge model of [23], we expect non-trivial influences from the random potential.
- Simulations suggest that the arctic circle phenomenon is universal, i.e. that the shape of the boundary between the frozen and non-frozen regions does not depend on the realization of the random field or on the statistics of the random field. This

universality may even extend to unbounded random fields; cf. Figure 2. A promising method for studying the arctic circle is the tangent method described in [14].

- We conjecture that concentration of measure holds, at least in an appropriate asymptotic sense, e.g. with high probability in the realization of the random field ω. It might be possible prove concentration by adapting the idea of the harmonic embedding and corrector from the study of random walks in random environment, as explained in e.g. [5].

A Ergodic theorem

Ergodic theory is a rich field of modern mathematics with an extensive literature. This includes several variants of the superadditive (or subadditive) multidimensional ergodic theorem, such as [45, 38, 2], which all propose technically different definitions of superadditivity in the multidimensional setting. The definition of superadditivity in [2] is a close match for our application (i.e. establishing that the limit almost surely exists in our definition the quenched local surface tension). However we actually need a version of the ergodic theorem with weaker hypotheses, to allow for asymptotically negligible errors in the superadditivity inequality (3.11) and in the translation property (3.12). These differences are not major or novel, but neither are they so trivial that we are comfortable with omitting the proof of the ergodic theorem under these weaker hypotheses. At the time of writing we have not been able to find this version of the ergodic theorem (or a stronger version) in the literature, so we include a proof here. The proof follows [2] closely; for each step in the argument below, we cite the corresponding step in [2].

Definition A.1 (Boxes in $\mathbb{Z}^m$)**.** *For $n \in \mathbb{N}$, let S_n denote the box*

$$S_n := [0, n)^m \cap \mathbb{Z}^m .$$

Let $\mathcal{B}$ denote the set of boxes

$$\mathcal{B} = \big\{ ([a_1, b_1) \times \cdots \times [a_m, b_m)) \cap \mathbb{Z}^m \;\big|\; a_i < b_i \text{ for all } i, \text{ where } a_i, b_i \in \mathbb{Z} \big\},$$

and for $k \in \mathbb{N}$, let $\mathcal{B}_k$ denote the set of boxes

$$\mathcal{B}_k = \big\{ ([a_1, b_1) \times \cdots \times [a_m, b_m)) \cap \mathbb{Z}^m \in \mathcal{B} \;\big|\; \text{all } a_i \text{ and } b_i \text{ are divisible by } k \big\}.$$

Lemma A.2 (A covering lemma; cf. [2, Lemma 3.1])**.** *Let Z be a finite subset of $\mathbb{Z}^m$. For each $z \in Z$ let $n(z) \geq 1$ be an integer. Then there is a set $Z' \subseteq Z$ such that $\{z + S_{n(z)} \mid z \in Z'\}$ is a family of disjoint sets and such that*

$$3^m \sum_{z \in Z'} |S_{n(z)}| \geq |Z|.$$

This is a modification of a common covering lemma due to Wiener. The proof is standard.

Theorem A.3 (A maximal inequality; cf. [2, Theorem 3.2])**.** *For $\alpha > 0$, let E_α denote the event*

$$E_\alpha := \left\{ \limsup_{n \geq 1} \frac{1}{|S_n|} F_{S_n} > \alpha \right\} .$$

Then

$$\mathbb{P}(E_\alpha) \leq \frac{3^m}{\alpha/2}\, \gamma(F).$$

Proof. For $N < M \in \mathbb{N}$, set

$$E_{N,M,\alpha} := \left\{ \sup_{N \le n \le M} \frac{1}{|S_n|} F_{S_n} > \alpha \right\}.$$

Clearly $E_\alpha = \cap_{N>0} \cup_{M>N} E_{N,M,\alpha}$, so it suffices to prove that

$$\mathbb{P}(E_{N,M,\alpha}) \le \frac{3^m}{\alpha/2}\,\gamma(F) + o(N). \tag{A.1}$$

Fix for now a larger integer $K > M$. We will soon take $K \to \infty$. But first, consider a single $\omega \in \Omega$. Define the set Z as follows:

$$Z = Z(\omega) := \{z \in S_{K-M} \mid \tau_z \omega \in E_{N,M,\alpha}\}.$$

We make two claims about Z: first, that $\frac{1}{|S_K|}\mathbb{E}|Z|$ is less than or equal to the right hand side of (A.1) in the limit (see (A.3) for the precise inequality), and second, that $\frac{1}{|S_K|}\mathbb{E}|Z| \ge \mathbb{P}(E_{N,M,\alpha})$ in the limit (see (A.4) for the precise inequality). After establishing these two claims, the result will follow quickly.

Towards the first claim, consider any $z \in Z$. There is an integer $n(z)$ (implicitly depending on ω) such that $N \le n(z) \le M$ and

$$\frac{1}{|S_{n(z)}|}\, F_{S_{n(z)}}(\tau_z \omega) > \alpha.$$

By (A.7), there exists $N_0 \in \mathbb{N}$ (independent of z and ω) such that, whenever $N \ge N_0$,

$$\frac{1}{|S_{n(z)}|}\, F_{z+S_{n(z)}}(\omega) > \frac{\alpha}{2}. \tag{A.2}$$

Apply the covering lemma (Lemma A.2), to pick $z_1, \dots, z_l \in Z$ (again, implicitly depending on ω) such that the boxes $z_i + S_{n(z_i)}$ are disjoint but $3^m \sum_{i=1}^{l} |S_{n(z_i)}| \ge |Z|$. Combining this with (A.2) we get

$$|Z| \le 3^m \sum_{i=1}^{l} |S_{n(z_i)}| \le \frac{3^m}{\alpha/2} \sum_{i=1}^{l} F_{z_i + S_{n(z_i)}},$$

and since $F \ge 0$ is almost superadditive,

$$|Z| \le \frac{3^m}{\alpha/2}\, F_{S_K} + A(\omega) \sum_{i=1}^{l} |\partial S_{n(z_i)}|.$$

Let $\varepsilon(N) = \sup_{n \ge N} \frac{|\partial S_n|}{|S_n|}$. Note that $\varepsilon(N) \to 0$ as $N \to \infty$. Since the boxes $z_i + S_{n(z_i)}$ are disjoint and contained in S_K,

$$|Z| \le \frac{3^m}{\alpha/2}\, F_{S_K} + A(\omega)\,\varepsilon(N)\,|S_K|.$$

Taking expectations and dividing by $|S_K|$ yields the first claim, namely

$$\frac{1}{|S_K|}\mathbb{E}|Z| \le \frac{3^m}{\alpha/2} \cdot \frac{\mathbb{E}[F_{S_K}]}{|S_K|} + \|A\|_{L^1}\,\varepsilon(N). \tag{A.3}$$

Towards the second claim, observe that as random variables,

$$|Z| = \sum_{z \in S_{K-M}} \mathbf{1}_{E_{N,M,\alpha}} \circ \tau_z.$$

By translation invariance of the measure $\mathbb{P}$ on the random potential (cf. Assumption 2.9),

$$\begin{aligned}\mathbb{E}|Z| &= \sum_{z\in S_{K-M}} \mathbb{P}(\tau_z\omega \in E_{N,M,\alpha})\\ &= \sum_{z\in S_{K-M}} \mathbb{P}(\omega \in E_{N,M,\alpha})\\ &= |S_{K-M}|\,\mathbb{P}(E_{N,M,\alpha}).\end{aligned}$$

In other words,

$$\frac{1}{|S_K|}\mathbb{E}|Z| \geq \frac{|S_{K-M}|}{|S_K|}\mathbb{P}(E_{N,M,\alpha}). \tag{A.4}$$

Combining the two claims that were just established, namely (A.3) and (A.4), we have

$$\frac{|S_{K-M}|}{|S_K|}\,\mathbb{P}(E_{N,M,\alpha}) \;\leq\; \frac{3^m}{\alpha/2}\,\frac{\mathbb{E}[F_{S_K}]}{|S_K|} \;+\; \|A\|_{L^1}\,\varepsilon(N).$$

Send K to infinity:

$$\mathbb{P}(E_{N,M,\alpha}) \;\leq\; \frac{3^m}{\alpha/2}\,\gamma(F) \;+\; \|A\|_{L^1}\,\varepsilon(N).$$

This proves the desired inequality (A.1) and completes the proof of Theorem A.3. □

Lemma A.4 (Convergence of expectations; cf. [2, Lemma 3.4])**.**

$$\gamma(F) = \lim_{n\to\infty} \frac{1}{|S_n|}\,\mathbb{E}[F_{S_n}]. \tag{A.5}$$

Moreover, if $H = (H_B)_{B\in\mathcal{B}_k}$ is almost superadditive but defined only on boxes in $\mathcal{B}_k$, the same equality holds (except that both in the definition of $\gamma(H)$ and in the right-hand side above, we only consider values of n that are divisible by k as we take $n \to \infty$).

Proof. By definition $\gamma = \limsup_{n\to\infty} \frac{1}{|S_n|}\mathbb{E}[F_{S_n}]$, so it suffices to show that $\liminf_{n\to\infty} \frac{1}{|S_n|}\mathbb{E}[F_{S_n}] \geq \gamma$. Let $k \in \mathbb{N}$. For $n \geq k$, we can subdivide the large box S_n into $r \geq 1$ translates of S_k and $s \geq 0$ translates of S_1, say $S_n = \bigcup_{i=1}^r (u_i + S_k) \cup \bigcup_{j=1}^s (v_j + S_1)$. By the superadditivity property (A.6),

$$F_{S_n} \geq \sum_{i=1}^r F_{u_i+S_k} + \sum_{j=1}^s F_{v_j+S_1} - A\big(r|\partial S_k| + s|\partial S_1|\big).$$

Taking expectations and dividing by $|S_n|$, we have

$$\begin{aligned}\frac{1}{|S_n|}\mathbb{E}[F_{S_n}] \geq{}& \frac{r}{|S_n|}\mathbb{E}[F_{S_k}] + \frac{s}{|S_n|}\mathbb{E}[F_{S_1}]\\ &- \mathbb{E}[A]\left(\frac{r|\partial S_k|}{|S_n|} + \frac{s|\partial S_1|}{|S_n|}\right)\\ &- \sup_{z\in\mathbb{Z}^m} \frac{1}{|S_n|}\big(r\mathbb{E}|F_{S_k} - F_{z+S_k}| + s\mathbb{E}|F_{S_1} - F_{z+S_1}|\big)\\ ={}& \frac{1}{|S_k|}\mathbb{E}[F_{S_k}] - o(n) - o(k).\end{aligned}$$

Thus for every $k \geq 1$,

$$\liminf_{n\to\infty} \frac{1}{|S_n|}\mathbb{E}[F_{S_n}] \geq \frac{1}{|S_k|}\mathbb{E}[F_{S_k}] - o(k),$$

and (A.5) follows by taking $k \to \infty$.

Let us deal quickly with the case where the almost superadditive process $H = (H_B)_{B \in \mathcal{B}_k}$ is defined only on boxes in $\mathcal{B}_k$, i.e. only on boxes whose vertices lie on points of $\mathbb{Z}^m$ whose every coordinate is divisible by k. We may define a process $F = (F_B)_{B \in \mathcal{B}}$ by scaling, i.e. $F_B = \frac{1}{|S_k|} H_{kB}$, where $kB = \{ku \,|\, u \in B\}$ is the k-fold rescaling of B. Then

$$\frac{1}{|S_n|} F_{S_n} = \frac{1}{|S_n||S_k|} H_{S_{kn}} = \frac{1}{|S_{kn}|} F_{S_{kn}}$$

so that $\gamma(F) = \gamma(H)$, and the result just proven for F also carries over (via linearity of the limit) to H. □

Theorem A.5 (Ergodic theorem for almost superadditive random families)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\tau = (\tau_u)_{u \in \mathbb{Z}^m}$ be a family of measure-preserving transformations on Ω, and let $F = (F_B)_{B \in \mathcal{B}}$ be a family of L^1 random variables satisfying the following three conditions:*

- *F is almost superadditive, i.e.*

$$F_B \geq \sum_{i=1}^{n} F_{B_i} - A \sum_{i=1}^{n} |\partial B_i| \quad \text{(a.s.)}\,, \tag{A.6}$$

  where $A = A(\omega) : \Omega \to [0, \infty)$ is an L^1 random variable.
- *For all $u \in \mathbb{Z}^m$,*

$$\lim_{n\to\infty} \sup_{u \in \mathbb{Z}^m} \frac{1}{|S_n|} \left\| F_{u+S_n} - F_{S_n} \circ \tau_u \right\|_{L^\infty(\omega)} = 0\,, \tag{A.7}$$

  where $u + B = \{u + x \,|\, x \in B\}$ is the translation of B by u.
- *The quantity $\gamma(F) = \limsup_{n\to\infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}]$ is finite.*

Then the limit $\lim_{n\to\infty} \frac{1}{|S_n|} F_{S_n}$ exists almost surely and in L^1. If moreover $\{\tau_u\}_{u \in \mathbb{Z}^m}$ is ergodic, then the limit is

$$\lim_{n\to\infty} \frac{1}{|S_n|} F_{S_n} = \gamma(F).$$

Proof of Theorem A.5. The proof is in four steps.

Step 1 (reduction to $F \geq 0$) Consider the to the additive process

$$G_B(\omega) := \sum_{u \in B} F_{u+S_1}(\omega) - A(\omega)\,|B|.$$

By the superadditivity property (A.6), $F' = F - G \geq 0$. The desired convergence result is known for additive processes, so it suffices to prove that $\frac{1}{|S_n|} F'_{S_n}$ converges almost surely. So, from this point on we shall assume that the process F is non-negative.

Step 2 (alternate rates of convergence) Let $\overline{f} = \overline{f}(\omega)$ and $\underline{f} = \underline{f}(\omega)$ denote respectively the pointwise $\limsup$ and $\liminf$ of $\frac{1}{|S_n|} F_{S_n}$. We shall show that, for m fixed, these two functions are also the pointwise $\limsup$ and $\liminf$ of $\frac{1}{|S_{km}|} F_{S_{km}}$ as $k \to \infty$.

For convenience, we write $\overline{f}^{(m)}$ for the pointwise $\limsup$ of the sequence $\frac{1}{|S_{km}|} F_{S_{km}}$ as $k \to \infty$. Clearly $\overline{f}^{(m)} \leq \overline{f}$. We must prove the opposite inequality. Consider first any two boxes $B \subseteq B'$. Since F is almost superadditive and non-negative, we have $F_{B'} \geq F_B - O(|B'|)$. In particular, when $k = \lceil \frac{n}{m} \rceil$,

$$\frac{1}{|S_n|} F_{S_{m\lceil n/m \rceil}} \geq \frac{1}{|S_n|} F_{S_n} - \frac{O(|S_{m\lceil n/m \rceil}|)}{|S_n|}\,.$$

Since $|S_{m\lceil n/m \rceil}|/|S_n| \to 1$, the left-hand side converges to $\overline{f}^{(m)}$ as $n \to \infty$, and the right-hand side converges to $\overline{f}$. The corresponding result for $\underline{f}$ is proved similarly.

Step 3 (approximating F) Fix $\alpha > 0$. Let $E = \{\omega : \overline{f}(\omega) - \underline{f}(\omega) > \alpha\}$. In order to show that $\mathbb{P}(E) = 0$, let $\varepsilon > 0$. By Lemma A.4, there exist k arbitrarily large such that $\frac{1}{|S_k|}\mathbb{E}[F_{S_k}] > \gamma - \frac{\varepsilon}{2}$. Define an additive family H on $\mathcal{B}_k$ (which, we recall, is the set of boxes whose vertices all lie in the sub-lattice $k\mathbb{Z}^m \subset \mathbb{Z}^m$) by

$$H_B = \sum_{u \in B \cap k\mathbb{Z}^m} F_{u+S_k} - A\,|B \cap k\mathbb{Z}^m|\,|\partial S_k|.$$

By almost superadditivity (A.6),

$$F_B \geq \sum_{u \in B \cap k\mathbb{Z}^m} F_{u+S_k} - |A| \sum_{u \in B \cap k\mathbb{Z}^m} |\partial(u + S_k)| = H_B.$$

Now let $F' = F - H$, so that F' is a non-negative random family defined on $\mathcal{B}_k$. It holds that

$$\begin{aligned} \overline{f}(\omega) - \underline{f}(\omega) &:= \limsup_{n\to\infty} \frac{1}{|S_{kn}|} F_{S_{kn}} - \liminf_{n\to\infty} \frac{1}{|S_{kn}|} F_{S_{kn}} \\ &\overset{(*)}{=} \limsup_{n\to\infty} \frac{1}{|S_{kn}|} F'_{S_{kn}} - \liminf_{n\to\infty} \frac{1}{|S_{kn}|} F'_{S_{kn}} \\ &\overset{(**)}{\leq} \sup_{n\to\infty} \frac{1}{|S_{kn}|} F'_{S_{kn}}. \end{aligned} \tag{A.8}$$

In particular, $(*)$ holds because H is additive, so it converges pointwise almost surely, and $(**)$ holds because $F' \geq 0$.

Next, we compute $\gamma(H)$ and $\gamma(F')$. Applying Lemma A.4:

$$\begin{aligned} \gamma(H) &= \lim_{n\to\infty} \left(\frac{1}{|S_{kn}|} \mathbb{E}[H_{S_{kn}}] \right) \\ &= \lim_{n\to\infty} \left(\frac{\sum_{u \in S_{kn} \cap k\mathbb{Z}^m} \mathbb{E}[F_{u+S_k}]}{k^m n^m} - \frac{\mathbb{E}[A]\,|S_{kn} \cap k\mathbb{Z}^m|\,|\partial S_k|}{k^m n^m} \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{|S_k|} \mathbb{E}[F_{S_k}] - \mathbb{E}[A] \frac{|\partial S_k|}{|S_k|} - o(k) \right). \end{aligned}$$

Note that n no longer appears in the final expression. Taking $k \to \infty$, we conclude that $\gamma(H) > \gamma(F) - \varepsilon$. Additionally from Lemma A.4, we can write $\gamma(F')$ as a limit. Importantly, γ is linear, so $\gamma(F') = \gamma(F) - \gamma(H) < \varepsilon$.

By (A.8) the event $E := \{\overline{f} - \underline{f} > \alpha\}$ is contained in $\{\sup_{n\geq 1} \frac{1}{|S_{kn}|} F'_{S_{kn}} > \alpha\}$. By Lemma A.3,

$$\mathbb{P}(E) \leq \frac{3^m \gamma(F')}{\alpha/2} \leq \frac{3^m \varepsilon}{\alpha/2}.$$

Taking $\varepsilon \to 0$, we see that $\mathbb{P}(E) = 0$. Since $\alpha > 0$ was arbitrary, we conclude that $\overline{f} = \underline{f}$ almost surely, and thus that $\frac{1}{|S_n|} F_{S_n}$ converges pointwise almost surely. $\square$

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