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On the number of components of random polynomial lemniscates

Subhajit Ghosh*

Abstract

A lemniscate of a complex polynomial Q_n of degree n is a sublevel set of its modulus, i.e., of the form $\{z \in \mathbb{C} : |Q_n(z)| < t\}$ for some $t > 0$. In general, the number of connected components of a unit lemniscate (i.e. for $t = 1$) can vary anywhere between 1 and n. In this paper, we study the expected number of connected components for two models of random lemniscates. First, we show that the expected number of connected components of lemniscates whose defining polynomial has i.i.d. roots chosen uniformly from $\mathbb{D}$, is bounded above and below by a constant multiple $\sqrt{n}$. On the other hand, if the i.i.d. roots are chosen uniformly from $\mathbb{S}^1$, we show that the expected number of connected components, divided by n, converges to $\frac{1}{2}$.

Keywords: random lemniscates; concentration inequalities; critical points pairing; fluctuation of CLT.
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1 Introduction

Let $Q_n(z)$ be a monic polynomial of degree n in the complex plane such that all its roots are contained within the closed unit disk $\overline{\mathbb{D}}$. That is,

$$Q_n(z) := \prod_{i=1}^{n} (z - z_i), \tag{1.1}$$

where $|z_j| \leq 1$, for $1 \leq j \leq n$. We denote the unit lemniscate of $Q_n(z)$ by

$$\Lambda(Q_n) := \{z \in \mathbb{C} : |Q_n(z)| < 1\}.$$

The quantity of interest is the number of connected components of $\Lambda(Q_n)$. The maximum principle implies that each connected component of the lemniscate must contain a zero of the polynomial; therefore, there are at most n components. In this paper, we investigate

*Tata Institute of Fundamental Research, Centre for Applicable Mathematics, Bengaluru 560065, India.
E-mail: subhajitg@tifrbng.res.in

the number of components of a *typical* lemniscate. Numerical simulations for random polynomials with roots chosen from the uniform probability measure on the unit disk $\mathbb{D}$, and on the circle $\mathbb{S}^1$ show a giant component alongside some tiny components (see Figures- 1, 2). In this paper, we quantify this numerical observation.

1.1 Motivation and previous results

The study of the metric and topological properties of polynomial lemniscates serves two main purposes. Firstly, it is the simplest curve with an algebraic boundary that is relevant to many problems in mathematical physics [19, 5, 2]. Secondly, polynomial lemniscates are used as a tool for approximating and analyzing complex geometric objects due to implications of Hilbert's lemniscate Theorem and its generalizations [29, 25]. For a more detailed exposition, please refer to [20] and the corresponding references therein. Taking all these into account, in 1958, Erdős, Herzog, and Piranian in [9] studied the geometric and topological properties of polynomial lemniscates and posed a long list of open problems. One of the key motivations behind the work related to random polynomial lemniscates is to offer a probabilistic approach to the problems in [9]. Krishnapur, Lundberg, and Ramachandran recently showed that the inradius of a random lemniscate whose defining polynomial has roots chosen from a measure μ depends on the negative set of the logarithmic potential U_μ. Lundberg, Epstein, and Hanin conducted a study on the lemniscate tree that encodes the nesting structure of the level sets of a random polynomial in [8]. Lundberg and Ramachandran in [23] conducted a study on the *Kac ensemble* and found that the expected number of connected components is asymptotically n. Lerario and Lundberg [21] proved that for random rational lemniscates, which are defined as the quotient of two *spherical random polynomials*, the average number of connected components is $\mathcal{O}(n)$. Later, Kabluchko and Wigman [18] discovered the exact asymptotics. Fyodorov, Lerario, and Lundberg studied the number of connected components of random algebraic hypersurfaces in [10]. In this article, we examine random polynomials with random roots, in contrast to random coefficients. Another stream of research on random polynomials includes studying the roots and critical points of random polynomials. In this work, we have made use of one such *pairing* result due to Kabluchko and Seidel [17], which states that for random polynomials whose roots are sampled from an appropriate probability measure ν supported within the unit disk, each root is associated with a critical point in close proximity. For more background, details and generalizations consult [13], [27], [16], [31], [28], [4], [24], [1], [14], [15], [26]. To find related research on meromorphic functions and Gaussian polynomials, please refer to [12], [11]. We emphasize the fact that such pairing phenomena are exclusive to random polynomials. A somewhat related result in the deterministic setting is the Sendov's conjecture [30], which was recently proven by Tao in [32] for all polynomials of sufficiently large degree.

1.2 Main results

In all the theorems we have the following setting.
Setting and notations: Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with law μ, supported in the closed unit disk. Consider the sequence of random polynomials defined by

$$P_n(z) := \prod_{i=1}^{n}(z - X_i), \tag{1.2}$$

and its lemniscate

$$\Lambda_n := \Lambda(P_n) = \{z \in \mathbb{C} : |P_n(z)| < 1\}.$$

We denote by $C(\Lambda_n)$ the number of connected components of the lemniscate Λ_n. Throughout the paper, we denote by C a positive numerical constant whose value may vary from line to line. For a set $S \subset \mathbb{C}$, we denote by $|S|$ the cardinality of the set S. The following are the main theorems of this paper.

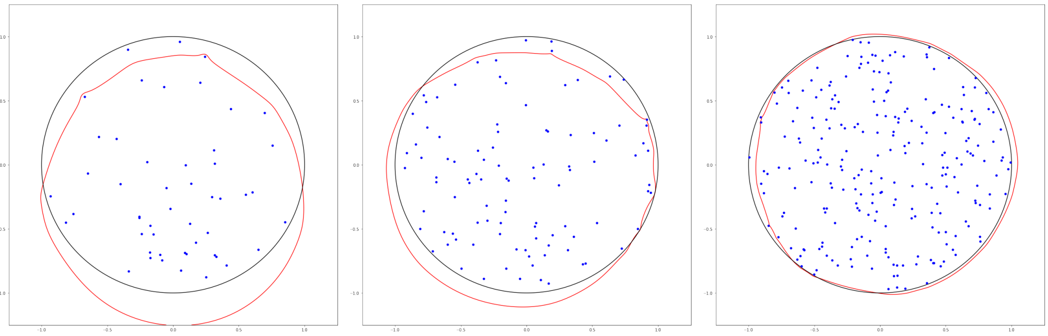

Figure 1: Lemniscates of degree n = 50, 100, 250 with zeros sampled uniformly from the open unit disk.

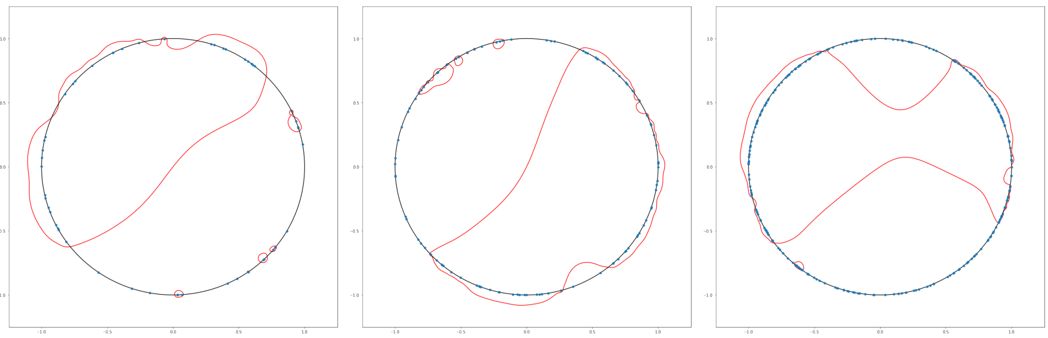

Figure 2: Lemniscates of degree n = 50, 100, 250 with zeros sampled uniformly from the unit circle. A unit circle is also plotted for reference in each case.

Theorem 1.1. *Let μ be the probability measure distributed uniformly in the unit disk $\mathbb{D}$. Then there exist absolute constants $C_1, C_2 > 0$ such that for all large n we have*

$$C_1\sqrt{n} \leq \mathbb{E}[C(\Lambda_n)] \leq C_2\sqrt{n}.$$

Theorem 1.2. *Let μ be the probability measure distributed uniformly in the unit circle $\mathbb{S}^1$. Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}[C(\Lambda_n)]}{n} = \frac{1}{2}.$$

1.3 Remarks

What happens if we choose μ to be the uniform measure on $r\mathbb{D}$ or $r\mathbb{S}^1$? Let us consider the uniform probability measure on $r\mathbb{S}^1$, say μ_r. Then it is easy to show that the logarithmic potential is

$$U_{\mu_r}(z) = \begin{cases} \log|z| & \textit{if } |z| \geq r, \\ \log r & \textit{if } |z| < r. \end{cases} \tag{1.3}$$

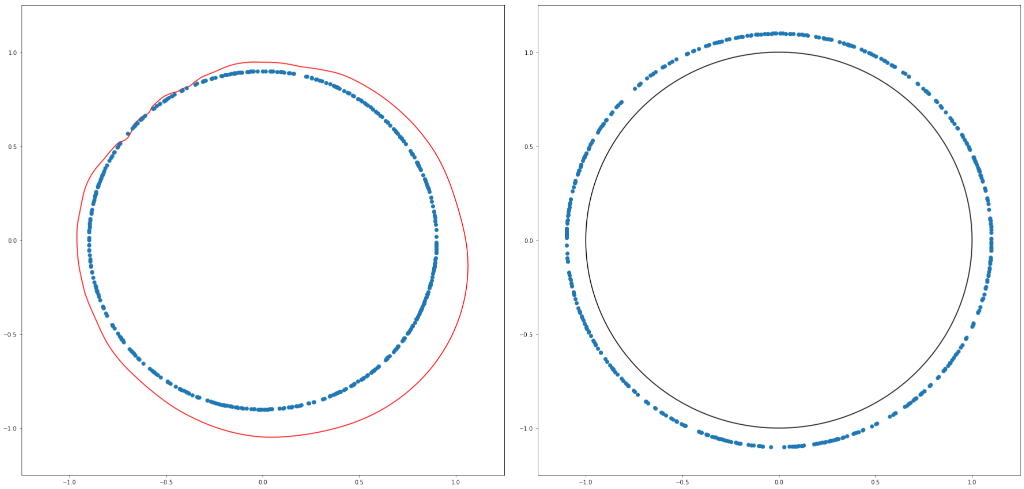

Figure 3: Lemniscates of degree n $= 500$ with zeros sampled uniformly from $r\mathbb{S}^1$, for $r = 0.9$ and 1.1 respectively.

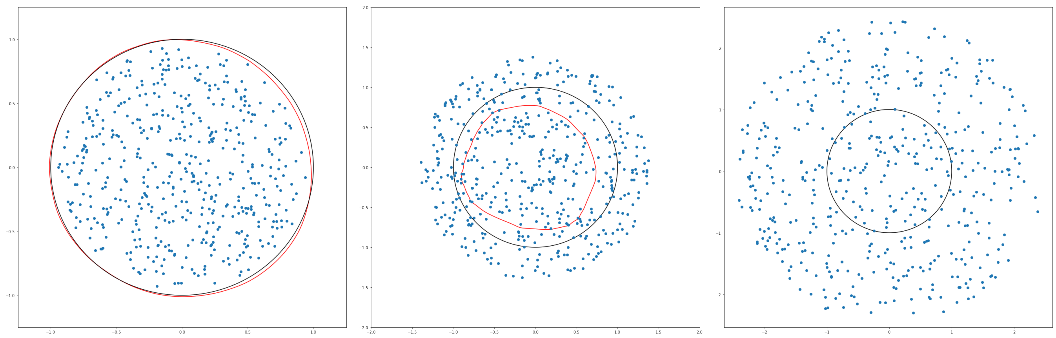

Figure 4: Lemniscates of degree n $= 500$ with zeros sampled uniformly from $r\mathbb{D}$ for $r = 0.95, 0.85\sqrt{e}$, and $1.5\sqrt{e}$ respectively.

Case 1 $(r < 1)$ In this case, the potential (1.3) is negative in the whole unit disk. Therefore the set $r\mathbb{D}$ is enclosed within the lemniscate by Theorem 1.1 in [20], resulting in a single connected component with overwhelming probability.

Case 2 $(r > 1)$ In this case, the potential (1.3) is positive in the entire complex plane therefore we get with overwhelming probability, n components for the lemniscate, by the implications of Theorem 1.3 of [20].

So in some sense, $r = 1$ is the critical case in this model. A similar analysis for the uniform probability measure on $r\mathbb{D}$ is done in [20], example-1.7. (where the interval $r > 1$ may be split at the point $r = \sqrt{e}$ for a more detailed description of the outcomes. Nevertheless, we observe asymptotically linear behavior throughout the entire interval $r > 1$). See Figure 3, 4. The above results and the results in this paper are summarized schematically in Table 1.

1.4 Heuristics and ideas of proof

We will now provide an overview of the underlying heuristics behind our results. In the first model, which involves random polynomials with uniformly chosen roots from $\mathbb{D}$, the potential $U_\mu(z)$ is negative throughout the unit disk. By writing $\log|P_n(z)| =$

Table 1: Asymptotics of expected no. of components for different values of r.

	$\boldsymbol{\mu}$	$\mathbf{r<1}$	$\mathbf{r=1}$	$\sqrt{\mathbf{e}} \geq \mathbf{r} > \mathbf{1}$	$\mathbf{r} > \sqrt{\mathbf{e}}$
$\mathbb{E}[\boldsymbol{C(\Lambda_n)}]$	Uniform probability on $r\mathbb{D}$	1	$\Theta(\sqrt{n})$	$C_r n$	n
$\mathbb{E}[\boldsymbol{C(\Lambda_n)}]$	Uniform probability in $r\mathbb{S}^1$	1	$\frac{n}{2}$	n	n

$\sum_{i=1}^{n} \log|z - X_i|$ as the sum of independent random variables with mean $U_\mu(z)$, we employ various concentration estimates to analyze the behavior of $|P_n(z)|$. Since the sum of i.i.d. random variables concentrate near its mean which is negative, most of the region within the disk, away from the boundary, lies inside the lemniscate. It is only near the boundary, where the potential approaches zero, isolated components are formed due to the fluctuations governed by the *Central Limit Theorem*, resulting in $\Theta(\sqrt{n})$ many components. In the other model, i.e., random polynomial with roots chosen uniformly on the circle, the potential is zero in the whole disk. The probability of any point on $\mathbb{S}^1$ being inside the lemniscate is close to $\frac{1}{2}$. Therefore, if we start with P_n and introduce a new root X_{n+1} to build P_{n+1}, X_{n+1} will land outside Λ_n with probability approximately $\frac{1}{2}$, forming an isolated component. Therefore, on average, we get approximately $\frac{n}{2}$ components. In both models, we establish the lower bound by estimating the number of isolated components. To determine the upper bound for the disk case, we utilize an analytical characterization for the number of components (see Lemma 2.8), which asserts that the number of components is one more than the number of critical points whose critical value is larger or equal to 1. To determine the number of such critical points, we employ a *pairing* result from [17] to associate critical points with roots with some desired properties. The number of such roots yields the desired upper bound. However, in the other case, the pairing phenomena do not occur. There we establish the upper bound by showing that the number of components possessing fewer than n^ϵ roots, when divided by n, tends towards $\frac{1}{2}$, for sufficiently small ϵ. This is established by estimating from below the probability of a moderately small disk (which shrinks appropriately with n) being contained inside the lemniscate.

2 Preliminary lemmas

Before delving into the proofs of the main theorems, we gather preliminary theorems and lemmas that are utilized repeatedly in both theorems.

Theorem 2.1. ***(Berry-Esseen)*** *Let $X_1, X_2, \cdots$ be i.i.d. random variables with $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = \sigma^2$, and $\mathbb{E}|X_i|^3 = \rho < \infty$. If $F_n(x)$ is the distribution function of $\frac{(X_1+\cdots+X_n)}{\sigma\sqrt{n}}$ and $\Phi(x)$ is the standard normal distribution, then*

$$|F_n(x) - \Phi(x)| \leq \frac{3\rho}{\sigma^3\sqrt{n}}, \tag{2.1}$$

for all $x \in \mathbb{R}$ and every natural number n.

The proof of Theorem 2.1 can be found in [7] Theorem 3.4.17.

Theorem 2.2. ***[Bennett's inequality]*** *Let $Y_1, Y_2, ..., Y_n$ be independent random variables with finite variances such that $\forall\ i \leq n$, $Y_i \leq b$, for some $b > 0$ almost surely. Let*

$$S = \sum_{i=1}^{n} (Y_i - \mathbb{E}[Y_i]),$$

and $\nu = \sum_{i=1}^n \mathbb{E}[{Y_i}^2]$. *Then for any* $t > 0$*, we have*

$$\mathbb{P}(S > t) \leq \exp\left(-\frac{\nu}{b^2} h\Big(\frac{bt}{\nu}\Big)\right),$$

where $h(u) = (1+u)\log(1+u) - u$*, for* $u > 0$*.*

For the proof of this concentration inequality and other similar results see [3].

Lemma 2.3. *Let* X *be a random variable taking values in* $\overline{\mathbb{D}}$ *with law* μ*. Assume that there exist constants* $\epsilon, M_1, M_2 \in (0, \infty)$*, such that*

$$M_1 r^{\epsilon} \leq \mu(B(z,r)) \leq M_2 r^{\epsilon}, \tag{2.2}$$

hold uniformly for all $z \in \operatorname{supp}(\mu)$*, and all* $0 < r \leq 2$*. Fix p, and define the function* $F_p(z) := \mathbb{E}\left[|\log|z-X||^p\right] : \operatorname{supp}(\mu) \to \mathbb{R}$*. Then, there exist positive real constants* C_1, C_2 *depending only on* p, ϵ, M_1, M_2*, such that*

$$C_1 \leq \inf_{z \in \operatorname{supp}(\mu)} F_p(z) \leq \sup_{z \in \operatorname{supp}(\mu)} F_p(z) \leq C_2. \tag{2.3}$$

Proof of Lemma 2.3. We will utilize the layer cake representation (c.f. [22], Theorem 1.13) and write

$$\begin{aligned}
\mathbb{E}\left[\big|\log|z-X|\big|^p\right] &= p\int_0^{\infty} t^{p-1}\mathbb{P}\left(\big|\log|z-X|\big| > t\right)dt \\
&= p\int_0^{2} t^{p-1}\mathbb{P}\left(\big|\log|z-X|\big| > t\right)dt + p\int_2^{\infty} t^{p-1}\mathbb{P}\left(\big|\log|z-X|\big| > t\right)dt.
\end{aligned}$$

In the second integral, notice that $(\log|z-X|)^+ < 2$, therefore, probability is non zero when $\log|z-X|$ is negative. Taking this into account and using the upper bound in (2.2)

$$\begin{aligned}
\mathbb{E}\left[\big|\log|z-X|\big|^p\right] &\leq p\int_0^2 t^{p-1}dt + pM_2\int_2^{\infty} t^{p-1}e^{-t\epsilon}dt \\
&\leq p2^{p+1}\left(1 + C\left(\epsilon\right)M_2\right).
\end{aligned}$$

The lower bound follows similarly using the left inequality in (2.2) along with the layer cake representation. □

Lemma 2.4. *Let* X *be a uniform random variable on the open unit disk* $\mathbb{D}$*. For* $p < 2$*, and* $z \in \mathbb{D}$ *there exists a constant* C_p *independent of* z*, such that*

$$\mathbb{E}\left[\frac{1}{|z-X|^p}\right] \leq C_p. \tag{2.4}$$

Proof of Lemma 2.4. This proof is again based on the layer cake representation(c.f. [22], Theorem 1.13).

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{|z-X|^p}\right] &= \int_0^{\infty}\mathbb{P}\left(\frac{1}{|z-X|^p} > t\right)dt \\
&= \int_0^{\infty}\mathbb{P}\left(|z-X| < \frac{1}{t^{1/p}}\right)dt \\
&= \int_0^{2}\mathbb{P}\left(|z-X| < \frac{1}{t^{1/p}}\right)dt + \int_2^{\infty}\mathbb{P}\left(|z-X| < \frac{1}{t^{1/p}}\right)dt \\
&\leq \int_0^2 dt + \int_2^{\infty} t^{-2/p}dt \\
&\leq \left(2 + \frac{p}{2-p}2^{\frac{p-2}{p}}\right).
\end{aligned}$$

□

Lemma 2.5. *(**Distance between the roots**) Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with law μ, supported in the closed unit disk. Suppose that there exists a real-valued function f such that*

$$\mathbb{P}\left(|z - X_j| > t\right) \geq 1 - f(t), \tag{2.5}$$

for all $z \in \mathbb{D}$ and $t > 0$ small. Then for any set $B \subset \mathbb{D}$, with $\mu(B) > 0$ we have

$$\mathbb{P}\left(\min_{2\leq j\leq n} |X_1 - X_j| > t \middle| X_1 \in B\right) \geq \left(1 - f(t)\right)^n. \tag{2.6}$$

Proof of Lemma 2.5. We condition on random variable X_1 to write

$$\begin{aligned}
&\mathbb{P}\left(\min_{2\leq j\leq n} |X_1 - X_j| > t \middle| X_1 \in B\right) \\
&= \frac{1}{\mathbb{P}(X_1 \in B)} \int_{\mathbb{D}} \mathbb{P}\left(\min_{2\leq j\leq n} |X_1 - X_j| > t, X_1 \in B \middle| X_1 = z\right) d\mu(z) \\
&= \frac{1}{\mathbb{P}(X_1 \in B)} \int_{B} \mathbb{P}\left(\min_{2\leq j\leq n} |z - X_j| > t\right) d\mu(z).
\end{aligned}$$

Notice that, the event $\{\min_{2\leq j\leq n} |z - X_j| > t\}$ can be viewed as $\{|z - X_2| > t\} \cap \{|z - X_3| > t\} \cap \cdots \cap \{|z - X_n| > t\}$. Now Independence of the random variables along with (2.5) gives,

$$\begin{aligned}
\mathbb{P}\left(\min_{2\leq j\leq n} |X_1 - X_j| > t \middle| X_1 \in B\right) &= \frac{1}{\mathbb{P}(X_1 \in B)} \int_B \mathbb{P}\left(|z - X_j| > t\right)^{(n-1)} d\mu(z) \\
&\geq \frac{1}{\mathbb{P}(X_1 \in B)} \int_B \left(1 - f(t)\right)^{(n-1)} d\mu(z) \\
&\geq \left(1 - f(t)\right)^{(n-1)}.
\end{aligned}$$ □

Lemma 2.6. *(**Lower bound on first derivative**) Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with law μ, supported in the closed unit disk. Assume that for every $1 \leq p \leq 3$, there exists positive constants $C_1(p), C_2(p)$, depending only on p such that*

$$C_1(p) < \mathbb{E}\left[|\log|z - X_1||^p\right] < C_2(p),$$

uniformly for all $z \in \operatorname{supp}(\mu)$. Let $U_n \subset \mathbb{D}$ be such that for some $M \geq 0$ and $\forall z \in U_n$, we have $\mathbb{E}\left[|\log|z - X_1||\right] \geq -\frac{M}{\sqrt{n}}$. Then for n large, there exists a constant $\hat{C}(M) > 0$, depending on M such that,

$$\mathbb{P}\left(\left|P_n'(X_1)\right| \geq e^{\sqrt{n}} \middle| X_1 \in U_n\right) \geq \hat{C}(M). \tag{2.7}$$

Proof of Lemma 2.6. We start by taking the logarithm to write

$$
\begin{aligned}
\mathbb{P}\left(\left|P_n'(X_1)\right| \geq e^{\sqrt{n}}\middle| X_1 \in U_n\right) &= \mathbb{P}\left(\prod_{j=2}^{n}|X_1 - X_j| \geq e^{\sqrt{n}}\middle| X_1 \in U_n\right) \\
&= \mathbb{P}\left(\sum_{j=2}^{n}\log|X_1 - X_j| \geq \sqrt{n}\middle| X_1 \in U_n\right) \\
&= \frac{\mathbb{P}\left(\sum_{j=2}^{n}\log|X_1 - X_j| \geq \sqrt{n}, X_1 \in U_n\right)}{\mathbb{P}\left(X_1 \in U_n\right)} \\
&= \frac{1}{\mathbb{P}(X_1 \in U_n)}\int_{U_n}\mathbb{P}\left(\sum_{j=2}^{n}\log|z - X_j| \geq \sqrt{n}\right)d\mu(z).
\end{aligned}
\tag{2.8}
$$

We estimate the probability inside the integral in (2.8) using Berry–Esseen theorem (2.1) to arrive at

$$
\begin{aligned}
&\int_{U_n}\mathbb{P}\left(\sum_{j=2}^{n}(\log|z - X_j| - \mathbb{E}[\log|z - X_j|]) \geq \sqrt{n} - (n-1)\mathbb{E}[\log|z - X_j|]\right)\frac{d\mu(z)}{\mu(U_n)} \\
&\geq \frac{1}{\mathbb{P}(X_1 \in U_n)}\int_{U_n}\mathbb{P}\left(\frac{1}{\sqrt{n}}\sum_{j=2}^{n}(\log|z - X_j| - \mathbb{E}[\log|z - X_j|]) \geq (M+1)\right)d\mu(z) \\
&\geq \frac{1}{\mathbb{P}(X_1 \in U_n)}\int_{U_n}\left(\Phi\left(\frac{M+1}{\sigma(z)}\right) - \frac{C\rho(z)}{\sigma^3(z)\sqrt{n}}\right)d\mu(z),
\end{aligned}
\tag{2.9}
$$

where $\sigma^2(z) = \mathbb{E}\left[(\log|z - X_j|)^2\right]$, $\rho(z) = \mathbb{E}\left[|\log|z - X_j||^3\right]$ and Φ is the distribution function of standard normal. From the hypothesis, we have uniform upper and lower bounds on $\sigma^2(z)$ and $\rho(z)$ using which we can bound the integrand in (2.9) as

$$
\Phi\left(\frac{(M+1)}{\sigma(z)}\right) - \frac{C\rho(z)}{\sigma^3(z)\sqrt{n}} \geq \left(C_1(M) - \frac{C_2}{\sqrt{n}}\right). \tag{2.10}
$$

Putting the bound (2.10) in the estimate (2.9) we get the required probability (2.7) for some absolute constant $\hat{C}$.

$$
\mathbb{P}\left(|P_n'(X_1)| \geq e^{\sqrt{n}}|X_1 \in U_n\right) = \frac{1}{\mathbb{P}(X_1 \in U_n)}\int_{U_n}\left(C_1(M) - \frac{C_2}{\sqrt{n}}\right)d\mu(z) \geq \hat{C}(M). \qquad \square
$$

Lemma 2.7. ***(Bound on higher derivatives)*** *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with law μ, supported on the closed unit disk and absolutely continuous with respect to 2- dimensional Lebesgue measure. If there exists a constant $C > 0$, such that $\mathbb{E}\left[\frac{1}{|z-X_1|}\right] < C$ for all $z \in \mathbb{D}$, then for any $K \subset \mathbb{D}$ with $\mu(K) > 0$*

$$
\mathbb{E}\left[\frac{1}{k!}\left|\frac{P_n^{(k)}(X_1)}{P_n'(X_1)}\right| \middle| X_1 \in K\right] \leq \binom{n-1}{k-1}C^{k-1}, \quad \text{for } k = 2, \cdots, n. \tag{2.11}
$$

Proof of Lemma 2.7. We write $P_n(z)$ as $P_n(z) = (z - X_1)Q_n(z)$, where $Q_n(z) := \prod_2^n (z - X_j)$, then differentiation yields,

$$P_n^{(k)}(z) = kQ_n^{(k-1)}(z) + (z - X_1)Q_n^{(k)}(z).$$

Putting $z = X_1$ in the above equation, we get $\frac{P_n^{(k)}(X_1)}{P'_n(X_1)} = \frac{kQ_n^{(k-1)}(X_1)}{Q_n(X_1)}$. Since μ cannot have atoms X_1 is not a root of $Q_n(z)$ almost surely, therefore $\frac{Q_n^{(k-1)}(X_1)}{Q_n(X_1)}$ will have $(n-1)(n-2)...\big(n-(k-1)\big)$ many summands of the form $\left[\frac{1}{(X_1-X_2)...(X_1-X_k)}\right]$. Here, we only care about the number of summands because after conditioning on X_1, all of them will have the same expected value.

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{k!}\left|\frac{P_n^{(k)}(X_1)}{P_n'(X_1)}\right| \big| X_1 \in K\right] \\
&\leq \int_K \frac{1}{k!}\mathbb{E}\left(\left|\frac{P_n^{(k)}(X_1)}{P_n'(X_1)}\right| \big| X_1 = z\right)\frac{d\mu(z)}{\mu(K)} \\
&\leq \int_K \frac{k(n-1)(n-2)...(n-k+1)}{k!}\mathbb{E}\left(\left|\frac{1}{(z-X_2)...(z-X_k)}\right|\right)\frac{d\mu(z)}{\mu(K)} \\
&\leq \binom{n-1}{k-1}\int_K \left(\mathbb{E}\left|\frac{1}{(z-X_2)}\right|\right)^{k-1}\frac{d\mu(z)}{\mu(K)} \\
&\leq \binom{n-1}{k-1}C^{k-1},
\end{aligned}$$

where we got the last estimate using the hypothesis of the lemma. □

We will need one last lemma from complex analysis which relates the number of components of a polynomial lemniscate with the number of critical points with critical value bigger or equal to 1.

Lemma 2.8. *Let $Q_n(z)$, $\Lambda(Q_n)$ be as in (1.1), and $\{\beta_j\}_{j=1}^{n-1}$ be the set of critical points of Q_n. Then,*

$$C(\Lambda(Q)) = 1 + |\{j : |Q(\beta_j)| \geq 1\}|.$$

Proof of Lemma 2.8. Let us assume that $C(\Lambda) = m$, i.e. there are m many components of the lemniscate. Let $n_1, ..., n_m$ be the number of zeroes in each of the components. We know that for a simple closed level curve of $f(z)$, say $\mathcal{C}$ if $f(z)$ is analytic up to the boundary of $\mathcal{C}$ and has n zeroes inside $\mathcal{C}$, then $f'(z)$ has $(n-1)$ zeros inside it. The proof of this result can be found in [33], Proposition 3.55. Then the component containing n_i many zeroes will have $(n_i - 1)$ many critical points inside the component. Since all these critical points are inside the lemniscate, all the critical values are strictly less than 1. Therefore, the following algebraic manipulations yield the required equality.

$$\begin{aligned}
|\{j : |Q(\beta_j)| \geq 1\}| &= (n-1) - |\{j : |Q(\beta_j)| < 1\}| \\
&= (n-1) - \sum_{i=1}^{m}(n_i - 1) = (m-1).
\end{aligned}$$ □

3 Proof of Theorem 1.1

Lower bound: The proof of the lower bound in both theorems relies on estimating the number of a specific type of isolated component called *lonely component*. We start by defining what we mean by a *lonely component* of a polynomial lemniscate. Let $Q_n(z)$ be defined as in (1.1), then we say that a root z_j forms a *lonely component* if there exists a ball $\mathcal{B}$ containing z_j such that,

$$\begin{cases} z_k \notin \mathcal{B}, & \forall k \neq j \\ |Q_n(z)| \geq 1, & \forall z \in \partial\mathcal{B}. \end{cases} \tag{3.1}$$

The key observation here is that bounds on the derivatives at the root provide a sufficient condition for a lonely component. Suppose for the root z_1 there exists some $r > 0$, such that the following holds,

$$\begin{cases} |Q_n^{'}(z_1)\frac{r}{2}| \geq 1, \\ \left|\frac{Q_n^{(k)}(z_1)\frac{r^k}{k!}}{Q_n^{'}(z_1)\frac{r}{1!}}\right| < \frac{1}{2n^2}, & \textit{for } k = 2, ..., n \\ \min\limits_{2\leq j\leq n} |z_1 - z_j| > r. \end{cases} \tag{3.2}$$

Then using Taylor series expansion of $Q_n(z)$ for $z \in \partial B(z_1, r)$ we get,

$$\begin{aligned} |Q_n(z)| &\geq \left|Q_n^{'}(z_1)r\right| - \sum_{k=2}^{n}\left|Q_n^{(k)}(z_1)\frac{r^k}{k!}\right| \\ &\geq \left|Q_n^{'}(z_1)r\right|\left(1 - \sum_{k=2}^{n}\frac{|Q_n^{(k)}(z_1)\frac{r^k}{k!}|}{|Q_n^{'}(z_1)\frac{r}{1!}|}\right) \\ &\geq \left|Q_n^{'}(z_1)r\right|\left(1 - \sum_{k=2}^{n}\frac{1}{2n^2}\right), \end{aligned}$$

where we have used the second condition of (3.2) in the last step. Now using the fact $\left(1 - \sum_{k=2}^{n}\frac{1}{2n^2}\right) \geq \frac{1}{2}$, along with the first condition of (3.2) we get

$$|Q_n(z)| \geq \left|Q_n^{'}(z_1)\frac{r}{2}\right| \geq 1, \tag{3.3}$$

which ensures that there is a connected component of the lemniscate inside the disk $B(z_1, r)$.

We now define for each $1 \leq i \leq n$, the event $L_i = \{X_i \textit{ forms a lonely component}\}$. Then it immediately follows that

$$\mathbb{E}\left[C(\Lambda_n)\right] \geq \mathbb{E}\left[\sum_{i=1}^{n}\mathbb{1}_{L_i}\right] \geq n\mathbb{E}\left[\mathbb{1}_{L_i}\right] \geq n\mathbb{P}\left(L_1\right).$$

Since the *lonely* roots are near the unit circle with high probability we only consider roots lying in $A_n := \left\{z : 1 - \frac{1}{\sqrt{n}} < |z| < 1\right\}$.

$$\mathbb{E}[C(\Lambda_n)] \geq n\mathbb{P}\left(L_1|X_1 \in A_n\right)\mathbb{P}(X_1 \in A_n) \geq \sqrt{n}\mathbb{P}\left(L_1|X_1 \in A_n\right). \tag{3.4}$$

We now define the following events with $r_n = \frac{1}{n^6}$,

$$\begin{cases} G_1 := \left\{|P_n^{'}(X_1)| \geq e^{\sqrt{n}}\right\} \\ G_k := \left\{\left|\frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n^{'}(X_1)\frac{r_n}{1!}}\right| < \frac{1}{2n^2}\right|\right\}, \textit{ for } k = 2, ..., n. \\ G_{n+1} := \left\{\min\limits_{2\leq j\leq n}|X_1 - X_j| > \frac{1}{n^6}\right\}. \end{cases} \tag{3.5}$$

In the setting of (3.2), occurrence of the events (3.5) implies that X_1 forms a *lonely component*. Hence,

$$\mathbb{P}\left(L_1|X_1 \in A_n\right) \geq \mathbb{P}\left(\cap_{j=1}^{n+1} G_j \middle| X_1 \in A_n\right). \tag{3.6}$$

Now we will estimate the conditional probabilities of $G_1, G_2, ..., G_{n+1}$ one by one. The uniform probability measure on the unit disk satisfies the conditions (2.2) with $M_1 = \frac{1}{8}$, $M_2 = 1$, and $\epsilon = 2$. Therefore, using Lemma 2.3 along with Lemma 2.6, we have

$$\mathbb{P}\left(G_1|X_1 \in A_n\right) \geq C_1. \tag{3.7}$$

Applying Lemma 2.7 with $K = A_n$, and the uniform moment bound from Lemma 2.4, we get for $k = 2, ..., n$

$$\mathbb{E}\left(\left|\frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n'(X_1)\frac{r_n}{1!}}\right| \middle| X_1 \in A_n\right) \leq \frac{C^{k-1}}{n^{4(k-1)}}. \tag{3.8}$$

With (3.8), using conditional Markov inequality we get,

$$\mathbb{P}\left(G_k^c|X_1 \in A_n\right) = \mathbb{P}\left(\left|\frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n'(X_1)\frac{r_n}{1!}}\right| \geq \frac{1}{2n^2} \middle| X_1 \in A_n\right) \leq \frac{1}{n^{2(k-1)}}. \tag{3.9}$$

Lastly, Lemma 2.5 with $t = \frac{1}{n^6}, f(x) = x^2$, and $C = 1$ gives,

$$\mathbb{P}\left(G_{n+1}|X_1 \in A_n\right) \geq \left(1 - \frac{1}{n^{12}}\right)^{n-1} \geq 1 - \frac{1}{n^{10}}. \tag{3.10}$$

Combining the estimates (3.7), (3.9), (3.10), we arrive at

$$\begin{aligned}
\mathbb{P}\Big(\cap_{j=1}^{n+1} G_j|X_1 \in A_n\Big) &\geq \mathbb{P}(G_1|X_1 \in A_n) - \mathbb{P}\Big(G_1 \cap \big(\cap_{k=2}^{n+1} G_k\big)^c|X_1 \in A_n\Big) \\
&\geq C_1 - \mathbb{P}\Big(\cup_{k=2}^{n+1}\left(G_1 \cap {G_k}^c|X_1 \in A_n\right)\Big) \\
&\geq C_1 - \sum_{k=2}^{n+1}\mathbb{P}\Big({G_k}^c|X_1 \in A_n\Big) \\
&\geq C_1 - \frac{1}{n},
\end{aligned} \tag{3.11}$$

where we have used $\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c)$ in the first step and the union bound in the third step. Finally, putting (3.11) in (3.4) the required bound is obtained.

$$\mathbb{E}[C(\Lambda_n)] \geq \sqrt{n}\mathbb{P}\left(L_1|X_1 \in A_n\right) \geq \sqrt{n}\mathbb{P}\Big(\cap_{j=1}^{n+1} G_j|X_1 \in A_n\Big) \geq C\sqrt{n},$$

for some constant $C > 0$.

Upper bound: The proof of the upper bound uses Lemma 2.8 to relate the number of components to certain critical points. We will take an indirect route to estimate the number of such critical points via the roots. We say a root z_1 of the polynomial $Q_n(z)$ is *good*, if there exists $r > 0$ such that,

$$\begin{cases}
B\left(z_1, r\right) \subset \Lambda_n, \\
\min\limits_{2 \leq j \leq n} |z_1 - z_j| > 3r, \\
\exists \textit{ a unique critical point } \xi \in B\left(z_1, r\right).
\end{cases} \tag{3.12}$$

Resembling the proof of lower bound, we first give a sufficient condition for the ball of radius $r_n := \frac{1}{n^{3/4}}$ around z_1 to be inside the lemniscate. Assume the following holds,

$$\begin{cases} 0 < |Q_n^{'}(z_1)| < e^{-\sqrt{n}}, \\ \left| \frac{Q_n^{(k)}(z_1)\frac{r_n^k}{k!}}{Q_n^{'}(z_1)\frac{r_n}{1!}} \right| < n^2 \binom{n-1}{k-1} \left(\frac{C}{n^{3/4}}\right)^{k-1}, \quad 2 \le k \le n. \end{cases} \tag{3.13}$$

Then for $z \in \partial B(z_1, r_n)$ and n large enough, using (3.13) we have,

$$\begin{aligned} |Q_n(z)| &\le \left| Q_n^{'}(z_1)\frac{r_n}{1!} \right| + \left| Q_n^{''}(z_1)\frac{r_n^2}{2!} \right| + ... + \left| Q_n^{(k)}(z_1)\frac{r_n^k}{k!} \right| + ... + \left| Q_n^{(n)}(z_1)\frac{r_n^n}{n!} \right| \\ &\le \left| Q_n^{'}(z_1) r_n \right| \left(1 + \sum_{k=2}^{n} \frac{|Q_n^{(k)}(z_1)\frac{r_n^k}{k!}|}{|Q_n^{'}(z_1)\frac{r_n}{1!}|} \right) \\ &\le \left| Q_n^{'}(z_1) r_n \right| \left(1 + \sum_{k=2}^{n} n^2 \binom{n-1}{k-1} \left(\frac{C}{n^{3/4}}\right)^{k-1} \right) \\ &\le n^2 e^{-\sqrt{n}} \left(1 + \frac{C}{n^{3/4}}\right)^{n-1} \\ &\le n^2 e^{-\sqrt{n}} e^{Cn^{1/4}} < 1. \end{aligned}$$

The maximum principle then ensures that the disk $B(z_1, r)$ is inside the lemniscate. Let us now go back to the random setting and define the events $T_i := \left\{ X_i \textit{ is a } \text{good} \textit{ root with } r = \frac{1}{n^{3/4}} \right\}$. The conditions in (3.12) immediately imply that the number of *good* roots is less than or equal to the number of critical points with a critical value less than 1, therefore,

$$\begin{aligned} \mathbb{E}[C(\Lambda_n)] &= n - \mathbb{E}\left[\{\textit{Number of critical points with critical value less than } 1\}\right] + 1 \\ &\le n - \mathbb{E}\left[\sum_{1}^{n} \mathbb{1}_{T_i}\right] + 1 \le n\left(1 - \mathbb{P}(T_1)\right) + 1. \end{aligned}$$

By concentration estimates, we expect that most of the *good* roots are within the region $\{|z| < 1 - \frac{1}{\sqrt{n}}\}$. Near the origin the Cauchy transform vanishes so pairing phenomena is not strong enough (see [17]) so excluding a small ball around the origin we consider the annulus $D_n := \{z : \frac{3}{n^{1/4}} < |z| \le 1 - \frac{1}{\sqrt{n}}\}$ as the location of good roots. So we estimate

$$\mathbb{E}[C(\Lambda_n)] \le n\left(1 - \mathbb{P}\left(T_1 | X_1 \in D_n\right) \mathbb{P}(X_1 \in D_n)\right) + 1. \tag{3.14}$$

Now let us define the events $H_1, ..., H_{n+1}$ with $r_n := \frac{1}{n^{3/4}}$.

$$\begin{cases} H_1 := \left\{ \left| P_n^{'}(X_1) \right| < e^{-\frac{\sqrt{n}}{2}} \right\}, \\ H_k := \left\{ \left| \frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n^{'}(X_1)\frac{r_n}{1!}} \right| < n^2 \binom{n-1}{k-1} \left(\frac{C}{n^{3/4}}\right)^{k-1} \right\}, \textit{ for } k = 2, ..., n. \\ H_{n+1} := \left\{ \min_{2 \le j \le n} |X_1 - X_j| > 3r_n \right\}, \\ H_{n+2} := \left\{ \exists \textit{ a unique critical point } \xi \in B\left(X_1, r_n\right) \right\}. \end{cases} \tag{3.15}$$

Notice that on the events (3.15), we have a good root. Therefore

$$\mathbb{P}\left(T_1|X_1 \in D_n\right) \geq \mathbb{P}\left(\cap_{j=1}^{n+2} H_j|X_1 \in D_n\right). \tag{3.16}$$

Next, we estimate the conditional probabilities of each of the events $H_1, ..., H_{n+2}$. To estimate the probability of the event H_1 we require the following lemma.

Lemma 3.1. ***(Upper bound on the first derivative)*** *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. uniform random variables in the open unit disk. Let $D_n := \left\{z : \frac{3}{n^{1/4}} < |z| \leq 1 - \frac{1}{\sqrt{n}}\right\}$. Then there exists a constant $C > 0$ such that,*

$$\mathbb{P}\left(|P_n^{'}(X_1)| \leq e^{-\frac{\sqrt{n}}{2}} \big| X_1 \in D_n\right) \geq 1 - \frac{C}{\sqrt{n}}. \tag{3.17}$$

Proof of Lemma 3.1. This proof adopts a methodology similar to Lemma 2.6, but with a slight variation. Instead of using a uniform bound for the integrand, we actually perform the integration to achieve the desired inequality. We have

$$\begin{aligned}\mathbb{P}\left(\left|P_n^{'}(X_1)\right| \geq e^{-\frac{\sqrt{n}}{2}} \big| X_1 \in D_n\right) &= \mathbb{P}\left(\prod_{j=2}^{n} |X_1 - X_j| \geq e^{-\frac{\sqrt{n}}{2}} \Big| X_1 \in D_n\right) \\ &= \mathbb{P}\left(\sum_{j=2}^{n} \log|X_1 - X_j| \geq -\frac{\sqrt{n}}{2} \Big| X_1 \in D_n\right) \\ &= \frac{1}{\mathbb{P}(X_1 \in D_n)} \int_{D_n} \mathbb{P}\left(\sum_{j=2}^{n} \log|z - X_j| \geq -\frac{\sqrt{n}}{2}\right) d\mu(z).\end{aligned} \tag{3.18}$$

Notice that, for $z \in \mathbb{D}$ the random variables $\{\log|z - X_j|\}_2^n$ are i.i.d. with mean $\mathbb{E}[\log|z - X_j|] = \frac{1-|z|^2}{2}$, which is nothing but the logarithmic potential of μ at z. We use Bennett's inequality (2.2) after subtracting the mean in (3.18), with the uniform upper and lower bounds of $\mathbb{E}\left[\log|z - X_j|^2\right]$ from Lemma 2.3 to obtain,

$$\begin{aligned}&\frac{1}{\mathbb{P}(X_1 \in D_n)} \int_{D_n} \mathbb{P}\left(\sum_{j=2}^{n} \left(\log|z - X_j| - \mathbb{E}\left[\log|z - X_j|\right]\right) \geq \frac{(n-1)(1-|z|^2)}{2} - \frac{\sqrt{n}}{2}\right) d\mu(z) \\ &\leq \frac{1}{\mathbb{P}(X_1 \in D_n)} \int_{D_n} \exp\left(-C_1 nh\left(\frac{(n-1)(1-|z|^2) - \sqrt{n}}{2C_2(n-1)}\right)\right) d\mu(z) \\ &\leq \frac{1}{\pi\mathbb{P}(X_1 \in D_n)} \int_0^{2\pi} \int_{\frac{3}{n^{1/4}}}^{1-\frac{1}{\sqrt{n}}} \exp\left(-C_1 nh\left(\frac{(n-1)(1-r^2) - \sqrt{n}}{2C_2(n-1)}\right)\right) r dr d\theta \\ &\leq \frac{2}{\mathbb{P}(X_1 \in D_n)} \int_{\frac{3}{n^{1/4}}}^{1-\frac{1}{\sqrt{n}}} \exp\left(-C_1 nh\left(\frac{(n-1)(1-r^2) - \sqrt{n}}{2C_2(n-1)}\right)\right) r dr.\end{aligned} \tag{3.19}$$

To estimate the integral we do a change of variables of $(1 - r^2) = s$ in (3.19) and use the fact that $C_3 u^2 \leq h(u) \leq C_4 u^2$, for all $u \in [0, 1]$, for some constants $C_3, C_4 > 0$. Then (3.19) becomes

$$\begin{aligned}
&\frac{2}{\mathbb{P}(X_1 \in D_n)} \int_{\frac{2}{\sqrt{n}}-\frac{1}{n}}^{1-\frac{9}{\sqrt{n}}} \exp\left(-C_1 n h\left(\frac{(n-1)s-\sqrt{n}}{2C_2(n-1)}\right)\right) ds \\
&\leq \frac{2}{\mathbb{P}(X_1 \in D_n)} \int_0^1 \exp\left(-C_1 n\left(s-\frac{1}{2\sqrt{n}}\right)^2\right) ds \\
&\leq \frac{2}{\mathbb{P}(X_1 \in D_n)} \int_0^\infty \exp\left(-x^2\right) \frac{dx}{C_1\sqrt{n}} \\
&\leq \frac{C}{\sqrt{n}}.
\end{aligned}$$

We finish the proof by taking the probability of the complementary event. □

Using Lemma 3.1 above we deduce that,

$$\mathbb{P}\left(H_1 \middle| X_1 \in D_n\right) = \mathbb{P}\left(\left|P_n'(X_1)\right| < e^{-\frac{\sqrt{n}}{2}} \middle| X_1 \in D_n\right) \geq 1 - \frac{C_1}{\sqrt{n}}. \tag{3.20}$$

Now we estimate $\mathbb{P}\left(H_k \middle| X_1 \in D_n\right)$ for $2 \leq k \leq n$. By taking $K = D_n$ in Lemma 2.7 and the uniform bound from Lemma 2.4, we arrive at

$$\mathbb{E}\left[\left|\frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n'(X_1)\frac{r_n}{1!}}\right| \middle| X_1 \in D_n\right] \leq \binom{n-1}{k-1}\left(\frac{C}{n^{3/4}}\right)^{k-1}. \tag{3.21}$$

Now conditional Markov inequality along with (3.21) gives,

$$\mathbb{P}\left(H_k \middle| X_1 \in D_n\right) = \mathbb{P}\left(\left|\frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n'(X_1)\frac{r_n}{1!}}\right| \geq n^2\binom{n-1}{k-1}\left(\frac{C}{n^{3/4}}\right)^{k-1} \middle| X_1 \in D_n\right) \leq \frac{1}{n^2}. \tag{3.22}$$

Using Lemma 2.5 with $t = \frac{1}{n^{3/4}}$ and $f(x) = x^2$ we obtain,

$$\begin{aligned}
\mathbb{P}\left(H_{n+1} \middle| X_1 \in D_n\right) &= \mathbb{P}\left(\min_{2 \leq j \leq n} |X_1 - X_j| > \frac{3}{n^{3/4}} \middle| X_1 \in D_n\right) \\
&\geq \left(1 - \frac{1}{n^{3/2}}\right)^{n-1} \geq 1 - \frac{2}{\sqrt{n}}.
\end{aligned} \tag{3.23}$$

Lastly, the probability bound for the event H_{n+2} follows from the following lemma.

Lemma 3.2. ***(Distance between roots and critical points)*** *Let $\{X_i\}_{i=1}^\infty$ be a sequence of i.i.d. uniform random variables in the open unit disk. We define the random polynomial P_n as in* (1.2)*. Let $D_n := \{z : \frac{3}{n^{1/4}} < |z| < 1 - \frac{1}{n^{1/2}}\}$, and $r_n = \frac{1}{n^{3/4}}$. Then*

$$\mathbb{P}\left(\left\{\exists \text{ a unique critical point } \xi \in B\left(X_1, r_n\right)\right\} \middle| X_1 \in D_n\right) \geq 1 - \frac{C}{\sqrt{n}}. \tag{3.24}$$

Proof of Lemma 3.2. The proof can essentially be deduced from ideas in [17]. We first condition on the location of X_1 and rewrite the probability as

$$\begin{aligned}
&\mathbb{P}\left(\left\{\exists \text{ a unique critical point } \xi \in B\left(X_1, r_n\right)\right\} \middle| X_1 \in D_n\right) \\
&\qquad = \int_{D_n} \mathbb{P}\left(\left\{\exists \text{ a unique critical point } \xi \in B\left(X_1, r_n\right)\right\} \middle| X_1 = u\right) d\mu(u).
\end{aligned} \tag{3.25}$$

Fixing $u \in D_n$ we define the event

$$\mathcal{E}_n(u) := \left\{ \sup_{z \in \partial B(u,r_n)} \left| \frac{1}{n} \frac{P_n'(z)}{P_n(z)} - f(u) \right| < |f(u)| \right\}, \tag{3.26}$$

where $r_n = \frac{1}{n^{3/4}}$ and $f(z) := \mathbb{E}\left[\frac{1}{z-X_2}\right] = \bar{z}$ is the Cauchy transform of the uniform probability measure on $\mathbb{D}$. On the event $\mathcal{E}_n(u)$, by Rouche's theorem (c.f. [6], pp.125-126) the difference between the number of zeros and critical points of $\frac{P_n'(z)}{P_n(z)}$ on $B(u, r_n)$ is same as the difference between the number of zeros and poles of the constant function $z \mapsto f(u)$, which is zero. Now we define another event $\mathcal{F}_n(u) := \{|X_2 - u| > 3r_n, ..., |X_n - u| > 3r_n\}$ which guarantees that there is only one root of P_n inside $B(u, r_n)$, hence only one critical point inside $B(u, r_n)$. Following the idea of proof of Lemma 2.5 one can show that $\mathbb{P}(\mathcal{F}_n(u)) \geq 1 - \frac{C}{\sqrt{n}}$, therefore,

$$\mathbb{P}\left(\{\exists \textit{ a unique critical point } \xi \in B(u, r_n)\}\right) \geq \mathbb{P}(\mathcal{E}_n(u) \cap \mathcal{F}_n(u)) \geq \mathbb{P}(\mathcal{E}_n(u)) - \frac{C}{\sqrt{n}}. \tag{3.27}$$

Next, writing $\frac{P_n'(z)}{P_n(z)}$ as sum of i.i.d random variables with mean $f(z)$ in (3.26) we get,

$$\mathbb{P}(\mathcal{E}_n(u)) = \mathbb{P}\left\{ \sup_{z \in \partial B(u,r_n)} \left| \frac{1}{n(z-u)} + \frac{1}{n} \sum_2^n \frac{1}{z - X_j} - f(u) \right| < |f(u)| \right\}.$$

Let $\tilde{z}_n$ be a sequence of complex numbers in $B(u, r_n)$ converging to u. Now adding and subtracting $\frac{1}{n}\sum_2^n \frac{1}{\tilde{z}_n - X_j}$ and $f(\tilde{z}_n)$ we bound the probability from below as

$$\mathbb{P}(\mathcal{E}_n(u)) \geq \mathbb{P}\left(\sup_{z \in \partial B(u,r_n)} \left| \frac{1}{n(z-u)} \right| + \sup_{z, \tilde{z}_n \in B(u,r_n)} \left| \frac{1}{n} \sum_2^n \left(\frac{1}{z - X_j} - \frac{1}{\tilde{z}_n - X_j} \right) \right| \right.$$
$$\left. + \left| \frac{1}{n} \sum_2^n \frac{1}{\tilde{z}_n - X_j} - f(\tilde{z}_n) \right| + |\overline{\tilde{z}_n - u}| < |f(u)| \right). \tag{3.28}$$

Notice that, $\max\left\{ \sup_{z \in \partial B(u,r_n)} \left| \frac{1}{n(z-u)} \right|, |\overline{\tilde{z}_n - u}| \right\} \leq \frac{1}{n^{1/4}}$ i.e. the maximum of the first and last term in (3.28) can be controlled by $|f(u)|$, where $|f(u)| \geq \frac{3}{n^{1/4}}$. Therefore by triangle inequality, we get

$$|f(u)| - \sup_{z \in \partial B(u,r_n)} \left| \frac{1}{n(z-u)} \right| - |\overline{\tilde{z}_n - u}| \geq \frac{|f(u)|}{3}. \tag{3.29}$$

Plugging the estimate (3.29) in (3.28) we arrive at,

$$\mathbb{P}(\mathcal{E}_n(u)) \geq$$
$$\mathbb{P}\left(\sup_{z, \tilde{z}_n \in B(u,r_n)} \left| \frac{1}{n} \sum_2^n \left(\frac{1}{z - X_j} - \frac{1}{\tilde{z}_n - X_j} \right) \right| + \left| \frac{1}{n} \sum_2^n \frac{1}{\tilde{z}_n - X_j} - f(\tilde{z}_n) \right| < \frac{|f(u)|}{3} \right). \tag{3.30}$$

Now taking complimentary events and using the fact that $\mathbb{P}(a+b > 2) \leq \mathbb{P}(a > 1) + \mathbb{P}(b >$

1) we obtain,

$$\mathbb{P}\left(\mathcal{E}_n(u)\right) \geq 1 - \underbrace{\mathbb{P}\left(\sup_{z,\tilde{z}_n \in B(u,r_n)} \left|\frac{1}{n}\sum_2^n \left(\frac{1}{z-X_j} - \frac{1}{\tilde{z}_n - X_j}\right)\right| \geq \frac{|f(u)|}{6}\right)}_{\text{(I)}}$$
$$- \underbrace{\mathbb{P}\left(\left|\frac{1}{n}\sum_2^n \frac{1}{\tilde{z}_n - X_j} - f(\tilde{z}_n)\right| \geq \frac{|f(u)|}{6}\right)}_{\text{(II)}}. \quad (3.31)$$

To estimate (I), we first simplify it using the following change of variables $z' = z-u, z_n^{''} = \tilde{z}_n - u, X_j^{'} = X_j - u$ to get

$$(\mathrm{I}) = \mathbb{P}\left(\sup_{z',z_n'' \in B(0,r_n)} \left|\frac{1}{n}\sum_2^n \left(\frac{(z'-z_n^{''})}{(z'-X_j')(z_n''-X_j')}\right)\right| \geq \frac{|f(u)|}{6}\right)$$
$$\leq \mathbb{P}\left(\sup_{z',z_n'' \in B(0,r_n)} \frac{2r_n}{n}\left|\sum_2^n \left(\frac{1}{(z'-X_j')(z_n''-X_j')}\right)\right| \geq \frac{|f(u)|}{6}\right). \quad (3.32)$$

Now using Markov inequality in (3.32) along with the estimates from the proof of Lemma 5.9 in [17], (see in particular equation 5.29 and the paragraph above it), with $r_n = s_n = \frac{1}{n^{3/4}}$ and $a_n = \frac{2r_n}{n}$, we get

$$(\mathrm{I}) \leq \frac{6}{|f(u)|}\left[4na_n\left(-2\pi C_1 \log(2s_n)\right) + C_2 4na_n + 4n\pi s_n^2 C_3\right] \leq \frac{C}{|f(u)|\sqrt{n}}. \quad (3.33)$$

We next use the bound (5.40) in Lemma 5.11 from [17] with $p = 1.5, \epsilon = \frac{|f(u)|}{6}$ and uniform bounds on the moments from Lemma (2.4) to estimate (II).

$$(\mathrm{II}) \leq \frac{C}{|f(u)|^{3/2}\sqrt{n}}\left(\mathbb{E}\left|\frac{1}{\tilde{z}_n - X_1}\right|^{1.5} + |f(\tilde{z}_n)|^{1.5}\right) \leq \frac{C}{|f(u)|^{3/2}\sqrt{n}}. \quad (3.34)$$

Now inserting (3.33), and (3.34), in (3.27) we obtain,

$$\begin{aligned}
&\mathbb{P}\left(\left\{\exists \text{ a unique critical point } \xi \in B\left(X_1, r_n\right)\right\} \middle| X_1 \in D_n\right) \\
&\geq \int_{D_n}\left(1 - \frac{C}{|f(u)|^{3/2}\sqrt{n}} - \frac{C}{|f(u)|\sqrt{n}} - \frac{C}{\sqrt{n}}\right) d\mu(u) \\
&\geq 1 - \frac{C}{\sqrt{n}} - C_1\int_{\frac{1}{n^{3/4}}}^{1-\frac{1}{\sqrt{n}}}\left(\frac{C}{r^{3/2}\sqrt{n}} - \frac{C}{r\sqrt{n}}\right) r\,dr \\
&\geq 1 - \frac{C}{\sqrt{n}}.
\end{aligned}$$

□

Applying the union bound along with the estimates (3.20), (3.22), (3.23), and (3.25) leads to

$$\mathbb{P}\left(T_1 | X_1 \in D_n\right) \geq \mathbb{P}\left(\cap_{k=1}^{n+2} H_k \middle| X_1 \in D_n\right) \geq 1 - \sum_{k=1}^{n+2}\mathbb{P}\left({H_k}^c \middle| X_1 \in D_n\right) \geq 1 - \frac{C}{\sqrt{n}}. \quad (3.35)$$

Feeding (3.35) into (3.14) the required upper bound is obtained.

$$\begin{aligned}
\mathbb{E}[C(\Lambda_n)] &\leq n\left(1 - \mathbb{P}\left(T_1 | X_1 \in D_n\right)\mathbb{P}(X_1 \in D_n)\right) + 1 \\
&\leq n\left(1 - \left(1 - \frac{C}{\sqrt{n}}\right)\left(1 - \frac{2}{\sqrt{n}}\right)\right) + 1 \\
&\leq C_2\sqrt{n}.
\end{aligned}$$

□

4 Proof of Theorem 1.2

In a recent paper [20], Krishnapur, Lundberg, and Ramachandran have shown that the polynomial lemniscate for roots chosen uniformly from the unit circle is a genuinely random quantity in the sense that it converges in distribution to a sub-level set of a certain Gaussian function. In contrast, the lemniscates associated to some other models approaches a deterministic limit. For instance, lemniscates for polynomials with roots chosen uniformly from the unit disk approaches the unit disk in Hausdorff distance.

Lower limit: The proof of the lower bound in this case follows the same strategy as in the previous theorem. The definition of a *lonely component* remains unchanged, and our focus lies on determining the number of such components. However, we cannot follow the proof verbatim because in this case, $\mathbb{E}\left[\frac{1}{|z-X_j|}\right] = \infty$. Therefore we condition on the following event to bypass this problem. Let us define the event $A := \left\{\min_{2\le j\le n} |X_1 - X_j| > \frac{1}{n^3}\right\}$, then by Lemma 2.5 with $t = \frac{1}{n^3}$ and $f(x) = 2x$, the probability of the event A is

$$\mathbb{P}(A) = \mathbb{P}\left(\min_{2\le j\le n} |X_1 - X_j| > \frac{1}{n^3}\right) \ge 1 - \frac{2}{n^2}. \tag{4.1}$$

For $1 \le i \le n$, let us define the events $S_i := \{X_i$ forms a *lonely component* $\}$. Then it immediately follows that

$$\mathbb{E}\left[C(\Lambda_n)\right] \ge \mathbb{E}\left[\sum_{i=1}^{n} \mathbb{1}_{S_i}\right] \ge n\mathbb{E}\left[\mathbb{1}_{S_i}\right] \ge n\mathbb{P}\left(S_1\right) \ge n\mathbb{P}\left(S_1 \cap A\right) \ge n\mathbb{P}\left(S_1|A\right) - \frac{2}{n}. \tag{4.2}$$

Next, we define events $F_1, ...F_{n+1}$ as follows.

$$\begin{cases} F_1 := \left\{\left|P_n^{'}(X_1)\right| \ge e^{n^{1/2-\epsilon}}\right\}, \\ F_k := \left\{\left|\frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n^{'}(X_1)\frac{r_n}{1!}}\right| < \frac{1}{2n^2}\right\} \text{, for } k = 2, ..., n, \\ F_{n+1} := \left\{\min_{2\le j\le n} |X_1 - X_j| > \frac{1}{n^6}\right\}. \end{cases} \tag{4.3}$$

From the calculations of (3.2) and (3.3) it follows that on the events (4.3), we have a *lonely component*. Hence

$$\mathbb{P}\left(S_1|A\right) \ge \mathbb{P}\left(\cap_{j=1}^{n+1} F_j|A\right). \tag{4.4}$$

As before we will calculate the conditional probability of the events F_j. Taking logarithms and using Theorem 2.1 (Berry-Esseen) as done in Lemma 2.6, along with uniform bounds on the moments from Lemma 2.3 one can show

$$\mathbb{P}\left(F_1\right) = \mathbb{P}\left(|P_n^{'}(X_1)| \ge e^{n^{1/2-\epsilon}}\right) \ge \frac{1}{2} - \frac{\hat{C}}{n^{\epsilon}}. \tag{4.5}$$

Now using (4.1) and (4.5) with the fact that $\mathbb{P}(A \cap B) \ge \mathbb{P}(A) - \mathbb{P}(B^c)$ we get

$$\mathbb{P}\left(F_1|A\right) = \frac{\mathbb{P}\left(F_1 \cap A\right)}{\mathbb{P}(A)} \ge \mathbb{P}\left(F_1 \cap A\right) \ge \mathbb{P}\left(|P_n^{'}(X_1)| \ge e^{n^{1/2-\epsilon}}\right) - \frac{1}{n^2} \ge \frac{1}{2} - \frac{\hat{C}}{n^{\epsilon}}.$$

Notice that, for $2 \le k \le n$, on the event A, we have

$$\begin{aligned}\left|\frac{P_n^{(k)}(X_1)\frac{r_n^k}{k!}}{P_n'(X_1)\frac{r_n}{1!}}\right| &\le \frac{1}{n^{6(k-1)}k!}\sum_{i_1,\dots,i_{k-1}}\frac{1}{|X_1-X_{i_1}|\dots|X_1-X_{k-1}|}\\ &\le \frac{1}{n^{6(k-1)}k!}\sum_{i_1,\dots,i_{k-1}} n^{3(k-1)}\\ &\le \frac{k(n-1)(n-2)\dots(n-k+1)n^{3(k-1)}}{n^{6(k-1)}k!}\\ &\le \frac{1}{2n^2}.\end{aligned}$$

Therefore, $\mathbb{P}(F_k \cap A) = \mathbb{P}(A)$ and as a result for $2 \le k \le n$, we have $\mathbb{P}(F_k|A) = 1$. Since $A \subset F_{n+1}$, we get the conditional probability $\mathbb{P}(F_{n+1}|A) = 1$. Now using these bounds in (4.2) we obtain,

$$\begin{aligned}&\mathbb{E}\left[C(\Lambda_n)\right] \ge n\mathbb{P}\left(S_1|A\right) - \frac{2}{n} \ge \frac{n}{2} - Cn^{1-\epsilon}\\ \implies &\liminf_{n\to\infty}\ \frac{\mathbb{E}\left[C(\Lambda_n)\right]}{n} \ge \frac{1}{2}.\end{aligned}$$

Upper limit: The pairing of zeros and critical points does not occur in general if the law of the random variable does not have a density. Therefore when μ is the uniform probability measure on $\mathbb{S}^1$, we can not proceed by exploiting the pairing result. We prove the upper limit by showing the number of components having less than n^ϵ roots is approximately $\frac{n}{2}$. Let $C_k(\Lambda_n)$ denote the number of components containing exactly k roots. Then it immediately follows that

$$\sum_1^n C_k(\Lambda_n) = C(\Lambda_n), \tag{4.6}$$

$$\sum_1^n kC_k(\Lambda_n) = n. \tag{4.7}$$

For $i = 1, ..., n$, fix an $\epsilon > 0$ small and define the events $D_i := \big\{$ There are at least $n^{\epsilon/2}$ many roots inside the component containing the root $X_i\big\}$. Now we claim that,

$$C(\Lambda_n) \le n - \sum_1^n \mathbb{1}_{D_i} + \sum_{k\ge n^{\epsilon/2}} C_k(\Lambda_n). \tag{4.8}$$

Substituting (4.6) and (4.7) in (4.8), we see that to establish the claim it suffices to show the following

$$\sum_1^n \mathbb{1}_{D_i} \le \sum_{k<n^{\epsilon/2}} (k-1)C_k(\Lambda_n) + \sum_{k\ge n^{\epsilon/2}} kC_k(\Lambda_n) \tag{4.9}$$

Since the first sum on the R.H.S. of (4.9) is non-negative, the claim would be proved if we show

$$\sum_1^n \mathbb{1}_{D_i} \le \sum_{k\ge n^{\epsilon/2}} kC_k(\Lambda_n). \tag{4.10}$$

Now a moment's thought convinces us that (4.10) follows simply from the definitions of D_i and $C_k(\Lambda_n)$.

Since the total number of roots is n, we can obtain a bound on the rightmost term of (4.8) in the following way.

$$n = \sum_{k<n^{\epsilon/2}} kC_k(\Lambda_n) + \sum_{k\geq n^{\epsilon/2}} kC_k(\Lambda_n) \geq \sum_{k\geq n^{\epsilon/2}} kC_k(\Lambda_n)$$

$$\implies n^{1-\epsilon/2} \geq \sum_{k\geq n^{\epsilon/2}} C_k(\Lambda_n). \tag{4.11}$$

After putting the estimate (4.10) and taking expectation in both sides of (4.8) we arrive at,

$$\mathbb{E}[C(\Lambda_n)] \leq n - n\mathbb{P}(D_1) + n^{1-\epsilon/2}. \tag{4.12}$$

To calculate the probability of the event D_1, let us first calculate the probability of having at least $n^{\epsilon/2}$ roots in the ball $B(rX_1, \tilde{r})$, where $r := 1 - \frac{1}{n^{1-\epsilon}}$, $\tilde{r} := \frac{2}{n^{1-\epsilon}}$. For $i = 2, ..., n$, define the events $\mathcal{T}_i := \{X_i \in B(rX_1, \tilde{r})\}$, then by the Paley-Zygmund inequality,

$$\mathbb{P}\left(\sum_2^n \mathbb{1}_{\mathcal{T}_i} \geq n^{\epsilon/2}\right) \geq \left(1 - \frac{n^{\epsilon/2}}{\mathbb{E}\left[\sum_2^n \mathbb{1}_{\mathcal{T}_i}\right]}\right)^2 \frac{\mathbb{E}\left[\sum_2^n \mathbb{1}_{\mathcal{T}_i}\right]^2}{\mathbb{E}\left[|\sum_2^n \mathbb{1}_{\mathcal{T}_i}|^2\right]}. \tag{4.13}$$

Using the rotation invariance of the measure we get,

$$\mathbb{P}\left(X_i \in B\left(rX_1, \tilde{r}\right)\right) = \int_0^{2\pi} \mathbb{P}\left(X_i \in B(rX_1, \tilde{r}) \middle| X_1 = e^{i\phi}\right) \frac{d\phi}{2\pi} = \mathbb{P}\left(X_i \in B(r, \tilde{r})\right).$$

Let us assume that $B(r, \tilde{r})$ intersects the unit circle at points W_1, W_2 and the angle subtended at the origin divided by 2π is θ. That is, $\frac{1}{2\pi}\angle OW_1W_2 = \theta$, where O is the origin. Then it is easy to see that $\mathbb{P}(X_j \in B(rX_1, \tilde{r})) = \theta$ and $\mathbb{P}(X_j, X_k \in B(rX_1, \tilde{r})) = \theta^2$, for all $j, k \neq 1$. Then

$$\mathbb{E}\left[\sum_2^n \mathbb{1}_{\mathcal{T}_i}\right] = (n-1)\theta, \tag{4.14}$$

$$\mathbb{E}\left[\left(\sum_2^n \mathbb{1}_{\mathcal{T}_i}\right)^2\right] = (n-1)\theta + (n-2)(n-1)\theta^2. \tag{4.15}$$

Equation (4.14) and (4.15) along with Bernoulli's inequality yields,

$$\frac{\mathbb{E}\left[\sum_2^n \mathbb{1}_{\mathcal{T}_i}\right]^2}{\mathbb{E}\left[|\sum_2^n \mathbb{1}_{\mathcal{T}_i}|^2\right]} = \frac{(n-1)^2\theta^2}{(n-1)\theta + (n-2)(n-1)\theta^2} \geq 1 - \frac{1}{\theta(n-1)} \geq 1 - \frac{C}{n^{\epsilon}}, \tag{4.16}$$

where we got the last inequality using elementary geometry in the following way. For n large, $\sin\left(\frac{\theta}{4}\right) \geq \frac{C}{n^{1-\epsilon}}$, utilizing this, we bound $(n-1)\theta$ as

$$(n-1)\theta \geq 4(n-1)\sin\left(\frac{\theta}{4}\right) \geq Cn^{\epsilon}.$$

Plugging the bound (4.16) in (4.13) we have,

$$\mathbb{P}\left(\sum_2^n \mathbb{1}_{\mathcal{T}_i} \geq n^{\epsilon/2}\right) \geq \left(1 - \frac{n^{\epsilon/2}}{\mathbb{E}\left[\sum_2^n \mathbb{1}_{\mathcal{T}_i}\right]}\right)^2 \left(1 - \frac{C}{n^{\epsilon}}\right) \geq \left(1 - \frac{C}{n^{\epsilon/2}}\right). \tag{4.17}$$

If the ball $B(rX_1, \tilde{r})$ is inside the lemniscate and there are at least $n^{\epsilon/2}$ roots inside the ball $B(rX_1, \tilde{r})$, then the connected component containing X_1 must have at least $n^{\epsilon/2}$ roots inside it. Now all we need is to estimate the probability that the ball $B(rX_1, \tilde{r})$ is inside the lemniscate, which follows from the next lemma.

Lemma 4.1. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables which are uniformly distributed on the unit circle. Fix $\epsilon \in (0, \frac{1}{4})$ and define $r := 1 - \frac{1}{n^{1-\epsilon}}$, $\tilde{r} := \frac{2}{n^{1-\epsilon}}$. Then there exists a constant $C > 0$, such that,*

$$\mathbb{P}\left(B(rX_1, \tilde{r}) \subset \Lambda_n\right) \geq \frac{1}{2} - \frac{C}{n^{\epsilon}} \tag{4.18}$$

Proof of Lemma 4.1. Define $\tilde{Q}_n(z) := \frac{Q_n(z)}{(z-z_1)}$ and assume that for some $r_1, \tilde{r}_1$ the following is satisfied.

$$\begin{cases} 4\tilde{r_1} < 1, \\ |\tilde{Q}_n(r_1 z_1)| \leq \exp\left(-n^{\frac{1}{2}-\epsilon}\right), \\ \left|\frac{\tilde{Q}_n^{(k)}(r_1 z_1)\tilde{r_1}^k}{\tilde{Q}_n(r_1 z_1)}\right| \leq n\sqrt{(n-1)...(n-k)}\left(\frac{4}{n^{1-\epsilon}}\right)^{k/2}, \quad k \geq 1. \end{cases} \tag{4.19}$$

Then for $z \in \partial B(r_1 z_1, \tilde{r})$ and n large, we have,

$$\begin{aligned} |Q_n(z)| &= |z - z_1||\tilde{Q}_n(z)| \\ &\leq 2\tilde{r}\left(|\tilde{Q}_n(r_1 z_1)| + \left|\tilde{Q}_n^{'}(r_1 z_1)\frac{\tilde{r_1}}{1!}\right| + \dots \left|\frac{\tilde{Q}_n^{(k)}(r_1 z_1)\tilde{r_1}^k}{k!}\right| \dots + \left|\frac{\tilde{Q}_n^{(n-1)}(r_1 z_1)\tilde{r_1}^{(n-1)}}{(n-1)!}\right|\right) \\ &\leq |\tilde{Q}_n(r_1 z_1)|\left(1 + \sum_{k=1}^{n-1}\left|\frac{\tilde{Q}_n^k(r_1 z_1)}{\tilde{Q}_n(r_1 z_1)}\frac{\tilde{r}^k}{k!}\right|\right) \\ &\leq \exp\left(-n^{\frac{1}{2}-\epsilon}\right)\left(1 + \sum_{k=1}^{n-1}\frac{n\sqrt{(n-1)...(n-k)}}{k!}\left(\frac{4}{n^{1-\epsilon}}\right)^{k/2}\right), \end{aligned}$$

where we got the last line using (4.19). Now taking n common in the parentheses above and using the Cauchy–Schwarz inequality one has,

$$\begin{aligned} &\leq n\exp\left(-n^{\frac{1}{2}-\epsilon}\right)\left(1 + \sum_{k=1}^{n-1}\binom{n-1}{k}\left(\frac{1}{n^{1/2+\epsilon/2}}\right)^k\right)^{1/2}\left(1 + \sum_{k=1}^{n-1}\frac{1}{k!}\left(\frac{4}{n^{1/2-3/2\epsilon}}\right)^k\right)^{1/2} \\ &\leq n\exp\left(-n^{\frac{1}{2}-\epsilon}\right)\left(1 + \sum_{k=1}^{n-1}\binom{n-1}{k}\left(\frac{1}{n^{1/2+\epsilon/2}}\right)^k\right)^{1/2}\left(1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{4}{n^{1/2-3/2\epsilon}}\right)^k\right)^{1/2} \\ &\leq n\exp\left(-n^{\frac{1}{2}-\epsilon}\right)\left(1 + \left(\frac{1}{n^{1/2+\epsilon/2}}\right)\right)^{\frac{(n-1)}{2}}\exp\left(2n^{-1/2+3/2\epsilon}\right) \\ &\leq n\exp\left(-n^{\frac{1}{2}-\epsilon}\right)\exp\left(n^{\frac{1}{2}-\epsilon/2}\right)\exp\left(2n^{-1/2+3/2\epsilon}\right) < 1. \end{aligned}$$

This ensures that the disk $B(z_1, r)$ is inside the lemniscate. Now with $r := 1 - \frac{1}{n^{1-\epsilon}}$, $\tilde{r} := \frac{2}{n^{1-\epsilon}}$ and defining $\tilde{P}_n$ similarly to $\tilde{Q}_n$, we define the following events,

$$\begin{cases} \mathcal{G}_1 & := |\tilde{P}_n(rX_1)| \leq \exp\left(-n^{\frac{1}{2}-\epsilon}\right) \\ \mathcal{G}_k & := \left|\frac{\tilde{P}_n^{(k)}(rX_1)\tilde{r}^k}{\tilde{P}_n(rX_1)}\right| \leq n\sqrt{(n-1)...(n-k)}\left(\frac{4}{n^{1-\epsilon}}\right)^{k/2}, \qquad \textit{for } k = 2, ..., n. \end{cases} \tag{4.20}$$

By the conditions in (4.19) it immediately follows that,

$$\mathbb{P}\left(B(rX_1,\tilde{r})\subset\Lambda_n\right)\geq\mathbb{P}\left(\cap_1^n\mathcal{G}_k\right). \tag{4.21}$$

Let us calculate the probabilities of the events $\mathcal{G}_1,...,\mathcal{G}_n$ individually. To estimate $\mathbb{P}(\mathcal{G}_1)$, we take logarithm, use the fact that the mean of this random variable is 0, and apply the Berry-Esseen Theorem (2.1) as done in Lemma 2.6 (note that, one can not use Lemma 2.3 directly in this case. However, using similar ideas one can show uniform upper and lower bounds on log moments). Then it follows that for some constant C_1,

$$\mathbb{P}(\mathcal{G}_1)\geq\frac{1}{2}-\frac{C_1}{n^{\epsilon}}. \tag{4.22}$$

For the events $\mathcal{G}_k$, we use Chebyshev's inequality to obtain,

$$\begin{aligned}\mathbb{P}&\left(\left|\frac{\tilde{P}_n^{(k)}(rX_1)\tilde{r}^k}{\tilde{P}_n(rX_1)}\right|\geq n\sqrt{(n-1)...(n-k)}\left(\frac{4}{n^{1-\epsilon}}\right)^{k/2}\right)\\ &\qquad\leq\frac{1}{n^2(n-1)...(n-k)}\left(\frac{n^{1-\epsilon}}{4}\right)^k\tilde{r}^{2k}\mathbb{E}\left[\left|\frac{\tilde{P}_n^{(k)}(rX_1)}{\tilde{P}_n(rX_1)}\right|^2\right].\end{aligned} \tag{4.23}$$

We estimate $\mathbb{E}\left[\left|\frac{\tilde{P}_n^{(k)}(rX_1)}{\tilde{P}_n(rX_1)}\right|^2\right]$ using the following facts

$$\mathbb{E}\left[\frac{1}{z-X_1}\right]=0,\quad\forall z\in\mathbb{D}, \tag{4.24}$$

$$\mathbb{E}\left[\frac{1}{|r-X_1|^2}\right]=\frac{1}{1-r^2}. \tag{4.25}$$

The identity (4.24) follows from the Cauchy integral formula, and (4.25) follows using standard integration techniques.

$$\begin{aligned}\mathbb{E}\left[\left|\frac{\tilde{P}_n^{(k)}(rX_1)}{\tilde{P}_n(rX_1)}\right|^2\right]&=\mathbb{E}\left[\left|\sum_{2\leq i_1<i_2<...<i_k\leq n}\frac{1}{(rX_1-X_{i_1})...(rX_1-X_{i_k})}\right|^2\right]\\ =&\mathbb{E}\left[\sum_{2\leq i_1<...<i_k\leq n}\frac{1}{(rX_1-X_{i_1})...(rX_1-X_{i_k})}\sum_{2\leq j_1<...<j_k\leq n}\frac{1}{\overline{(rX_1-X_{j_1})}...\overline{(rX_1-X_{j_k})}}\right]\\ =&\frac{1}{2\pi}\int_0^{2\pi}\mathbb{E}\left[\sum_{2\leq i_1<...<i_k\leq n}\frac{1}{(re^{i\theta}-X_{i_1})...(re^{i\theta}-X_{i_k})}\right.\\ &\qquad\left.\times\sum_{2\leq j_1<...<j_k\leq n}\frac{1}{\overline{(re^{i\theta}-X_{j_1})}...\overline{(re^{i\theta}-X_{j_k})}}\right]d\theta\\ =&\mathbb{E}\left[\sum_{2\leq i_1<...<i_k\leq n}\frac{1}{(r-X_{i_1})...(r-X_{i_k})}\sum_{2\leq j_1<...<j_k\leq n}\frac{1}{\overline{(r-X_{j_1})}...\overline{(r-X_{j_k})}}\right]\end{aligned}$$

Notice that by the independence of the random variables, and identity (4.24), the cross terms will vanish. We estimate the remaining terms using (4.25) in the following way.

$$\begin{aligned}\mathbb{E}\left[\left|\frac{\tilde{P}_n^{(k)}(rX_1)}{\tilde{P}_n(rX_1)}\right|^2\right] =&\mathbb{E}\left[\sum_{2\le i_1<\ldots<i_k\le n}\frac{1}{|r-X_{i_1}|^2\ldots|r-X_{i_k}|^2}\right]\\ =&(n-1)\ldots(n-k)\mathbb{E}\left[\frac{1}{|r-X_1|^2}\right]^k\\ \le&(n-1)\ldots(n-k)(1-r^2)^{-k}\le(n-1)\ldots(n-k)n^{k(1-\epsilon)}, \qquad (4.26)\end{aligned}$$

where we got the last line using the fact that $1-r^2\ge\frac{1}{n^{1-\epsilon}}$, by our choice of r. Now plugging the bound (4.26) in (4.23) and taking the complementary events we get,

$$\mathbb{P}\left(\left|\frac{\tilde{P}_n^{(k)}(rX_1)\tilde{r}^k}{\tilde{P}_n(rX_1)}\right|\le n\sqrt{(n-1)\ldots(n-k)}\left(\frac{4}{n^{1-\epsilon}}\right)^{k/2}\right)\ge 1-\frac{1}{n^2}. \qquad (4.27)$$

Making use of (4.27) and (4.22) in (4.21) we arrive at the required probability.

$$\begin{aligned}\mathbb{P}\left(B(rX_1,\tilde{r})\subset\Lambda_n\right)\ge&\,\mathbb{P}\left(\mathcal{G}_1\right)-\mathbb{P}\left(\mathcal{G}_1\cap\left(\cap_2^n\mathcal{G}_k\right)^c\right)\\ \ge&\,\frac{1}{2}-\frac{C_1}{n^\epsilon}-\sum_2^n\frac{1}{n^2}\\ \ge&\,\frac{1}{2}-\frac{C}{n^\epsilon}.\end{aligned}$$

□

Then using the bound (4.18) in Lemma 4.1 and (4.17) we get the required probability.

$$\begin{aligned}\mathbb{P}(D_1)\ge&\,\mathbb{P}\left(\left\{\sum_2^n\mathbb{1}_{\mathcal{T}_i}\ge n^{\epsilon/2}\right\}\bigcap\left\{B(rX_1,\tilde{r})\subset\Lambda_n\right\}\right)\\ \ge&\,\frac{1}{2}-\frac{C}{n^\epsilon}-\frac{2C}{n^{\epsilon/2}}\ge\frac{1}{2}-\frac{C}{n^{\epsilon/2}}. \qquad (4.28)\end{aligned}$$

Now setting (4.28) in (4.12) and taking the limsup we get the asymptotic upper bound, i.e,

$$\limsup_{n\to\infty}\frac{\mathbb{E}[C(\Lambda_n)]}{n}\le\limsup_{n\to\infty}\,\frac{1}{n}\left[n-n\left(\frac{1}{2}-\frac{C}{n^{\epsilon/2}}\right)+n^{1-\epsilon/2}\right]\le\frac{1}{2}.$$

□

References

[1] J. Angst, D. Malicet, and G. Poly, *Almost sure behavior of the critical points of random polynomials*, 2023. MR4711583

[2] I. Bauer and F. Catanese, *Generic lemniscates of algebraic functions*, Math. Ann., 307 (1997), pp. 417–444. MR1437047

[3] S. Boucheron, G. Lugosi, and P. Massart, *Concentration inequalities*, Oxford University Press, Oxford, 2013. A nonasymptotic theory of independence, With a foreword by Michel Ledoux. MR3185193

[4] S.-S. Byun, J. Lee, and T. R. Reddy, *Zeros of random polynomials and their higher derivatives*, Trans. Amer. Math. Soc., 375 (2022), pp. 6311–6335. MR4474893

[5] F. Catanese and M. Paluszny, *Polynomial-lemniscates, trees and braids*, Topology, 30 (1991), pp. 623–640. MR1133876

[6] J. B. Conway, *Functions of one complex variable. II*, vol. 159 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995. MR1344449

[7] R. Durrett, *Probability: Theory and Examples*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2019. MR3930614

[8] M. Epstein, B. Hanin, and E. Lundberg, *The lemniscate tree of a random polynomial*, Ann. Inst. Fourier (Grenoble), 70 (2020), pp. 1663–1687. MR4245584

[9] P. Erdős, F. Herzog, and G. Piranian, *Metric properties of polynomials*, J. Analyse Math., 6 (1958), pp. 125–148. MR0101311

[10] Y. V. Fyodorov, A. Lerario, and E. Lundberg, *On the number of connected components of random algebraic hypersurfaces*, J. Geom. Phys., 95 (2015), pp. 1–20. MR3357820

[11] B. Hanin, *Correlations and pairing between zeros and critical points of Gaussian random polynomials*, Int. Math. Res. Not. IMRN, (2015), pp. 381–421. MR3340325

[12] B. Hanin, *Pairing of zeros and critical points for random meromorphic functions on Riemann surfaces*, Math. Res. Lett., 22 (2015), pp. 111–140. MR3342181

[13] B. Hanin, *Pairing of zeros and critical points for random polynomials*, Ann. Inst. Henri Poincaré Probab. Stat., 53 (2017), pp. 1498–1511. MR3689975

[14] J. Hoskins and Z. Kabluchko, *Dynamics of zeroes under repeated differentiation*, 2021. MR4669281

[15] I.-S. Hu and C.-C. Chang, *The common limit of the linear statistics of zeros of random polynomials and their derivatives*, 2017.

[16] Z. Kabluchko, *Critical points of random polynomials with independent identically distributed roots*, Proc. Amer. Math. Soc., 143 (2015), pp. 695–702. MR3283656

[17] Z. Kabluchko and H. Seidel, *Distances between zeroes and critical points for random polynomials with i.i.d. zeroes*, Electronic Journal of Probability, 24 (2019), pp. 1 – 25. MR3940764

[18] Z. Kabluchko and I. Wigman, *Asymptotics for the expected number of nodal components for random lemniscates*, Int. Math. Res. Not. IMRN, (2022), pp. 2337–2375. MR4373237

[19] V. Kharlamov, A. Korchagin, G. Polotovskiĭ, and O. Viro, eds., *Topology of real algebraic varieties and related topics*, vol. 173 of American Mathematical Society Translations, Series 2, American Mathematical Society, Providence, RI, 1996. Dedicated to the memory of Dmitriĭ Andreevich Gudkov, Advances in the Mathematical Sciences, 29. MR1384303

[20] M. Krishnapur, E. Lundberg, and K. Ramachandran, *Inradius of random lemniscates*, 2023. MR4703355

[21] A. Lerario and E. Lundberg, *On the geometry of random lemniscates*, Proc. Lond. Math. Soc. (3), 113 (2016), pp. 649–673. MR3570241

[22] E. H. Lieb and M. Loss, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001. MR1817225

[23] E. Lundberg and K. Ramachandran, *The arc length and topology of a random lemniscate*, J. Lond. Math. Soc. (2), 96 (2017), pp. 621–641. MR3742436

[24] M. Michelen and X.-T. Vu, *Zeros of a growing number of derivatives of random polynomials with independent roots*, 2022. MR4741259

[25] B. Nagy and V. Totik, *Sharpening of Hilbert's lemniscate theorem*, J. Anal. Math., 96 (2005), pp. 191–223. MR2177185

[26] S. O'Rourke, *Critical points of random polynomials and characteristic polynomials of random matrices*, Int. Math. Res. Not. IMRN, (2016), pp. 5616–5651. MR3567254

[27] S. O'Rourke and N. Williams, *Pairing between zeros and critical points of random polynomials with independent roots*, Trans. Amer. Math. Soc., 371 (2019), pp. 2343–2381. MR3896083

[28] R. Pemantle and I. Rivin, *The distribution of zeros of the derivative of a random polynomial*, in Advances in combinatorics, Springer, Heidelberg, 2013, pp. 259–273. MR3363974

[29] T. Ransford, *Potential theory in the complex plane*, vol. 28 of London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 1995. MR1334766

[30] B. Sendov, *On the Hausdorff geometry of polynomials, a generalized conjecture*, in Discrete mathematics and applications (Bansko, 2001), vol. 6 of Res. Math. Comput. Sci., South-West Univ., Blagoevgrad, 2002, pp. 1–12. MR1928406

[31] S. D. Subramanian, *On the distribution of critical points of a polynomial*, Electron. Commun. Probab., 17 (2012), pp. no. 37, 9. MR2970701

[32] T. Tao, *Sendov's conjecture for sufficiently-high-degree polynomials*, Acta Math., 229 (2022), pp. 347–392. MR4554225

[33] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, Oxford, 1958. Reprint of the second (1939) edition. MR3155290

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