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Analytic aspects of the dilation inequality for symmetric convex sets in Euclidean spaces*

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Abstract

We discuss an analytic form of the dilation inequality with respect to a probability measure for symmetric convex sets in Euclidean spaces, which is a counterpart of analytic aspects of Cheeger's isoperimetric inequality. We show that the dilation inequality for symmetric convex sets is equivalent to a certain bound of the relative entropy for even quasi-convex functions, which is close to the logarithmic Sobolev inequality or Cramér–Rao inequality. As corollaries, we investigate the reverse Shannon inequality, logarithmic Sobolev inequality, Kahane–Khintchine inequality, deviation inequality and isoperimetry. We also give new probability measures satisfying the dilation inequality for symmetric convex sets via bounded perturbations and tensorization.

Keywords: dilation; relative entropy; log-Sobolev inequality; Cramér–Rao inequality; Kahane–Khintchine inequality; deviation inequality; isoperimetry.
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1 Introduction

Cheeger's isoperimetric inequality with respect to a probability measure μ on $\mathbb{R}^n$ is one of the most important geometric inequalities in geometry and geometric analysis. Cheeger [15] and Maz'ya [28, 29] showed that Cheeger's isoperimetric inequality gives the spectral gap of the Laplace–Beltrami operator induced by μ. Conversely, Buser [13] (see also Ledoux [26]) also proved that the spectral gap, or equivalently the Poincaré inequality, gives Cheeger's isoperimetric inequality. Hence we can naturally regard the Poincaré inequality as an analytic form of Cheeger's isoperimetric inequality. Moreover, Bobkov–Houdré [8] also gave an equivalence between Cheeger's isoperimetric inequality and the $(1,1)$-Poincaré inequality, and thus the $(1,1)$-Poincaré inequality is another analytic aspect of Cheeger's isoperimetric inequality. Further developments in this direction are investigated in the work by Milman [30] in which the equivalence of

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Cheeger's isoperimetric inequality, the Poincaré inequality, exponential concentration and First-Moment concentration are discussed.

On the other hand, Nazarov–Sodin–Volberg [33] showed a new sharp isoperimetric-type inequality for a log-concave probability measure on $\mathbb{R}^n$, which we call the dilation inequality in this paper. This inequality is originally given by Borell [10] and investigated by many researchers in [27, 19, 33, 5, 6, 9, 17, 22, 35] where the sharpness and generalization of the dilation inequality are discussed. Here, a measure μ on $\mathbb{R}^n$ is log-concave if for any compact subsets $A, B \subset \mathbb{R}^n$, it holds

$$\mu((1-t)A + tB) \geq \mu(A)^{1-t}\mu(B)^t, \quad \forall t \in (0,1),$$

where $(1-t)A + tB \coloneqq \{(1-t)a + tb \mid a \in A, b \in B\}$ is the Minkowski average. For a Borel subset $A \subset \mathbb{R}^n$ and $\varepsilon \in (0,1)$, we define the ε-dilation of A by

$$A_\varepsilon \coloneqq A \cup \left\{ x \in \mathbb{R}^n \,\middle|\, \exists y \in \mathbb{R}^n, \int_0^1 \mathbf{1}_A((1-t)x + ty)\, dt > 1 - \varepsilon \right\} \tag{1.1}$$

and define the dilation area of A by

$$\mu^*(A) \coloneqq \liminf_{\varepsilon \downarrow 0} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}. \tag{1.2}$$

The ε-dilation A_ε is a counterpart of the ε-neighborhood $[A]_\varepsilon \coloneqq \{x \in \mathbb{R}^n \mid \exists a \in A, |x-a| < \varepsilon\}$, where $|\cdot|$ is the standard Euclidean norm, and the dilation area $\mu^*(A)$ is a counterpart of the μ-perimeter of A (or the Minkowski content of A with respect to μ) given by

$$\mu^+(A) \coloneqq \liminf_{\varepsilon \downarrow 0} \frac{\mu([A]_\varepsilon) - \mu(A)}{\varepsilon}. \tag{1.3}$$

Then, from the work by Nazarov–Sodin–Volberg [33] (see also [35]), we see that any log-concave probability measure μ on $\mathbb{R}^n$ satisfies

$$\mu^*(A) \geq -(1 - \mu(A)) \log(1 - \mu(A)), \tag{1.4}$$

for any Borel subset $A \subset \mathbb{R}^n$.

We note that it is natural to consider (1.4) as a counterpart of Cheeger's isoperimetric inequality, namely

$$\mu^+(A) \geq \kappa \min\{\mu(A), 1 - \mu(A)\}, \tag{1.5}$$

for any Borel subset $A \subset \mathbb{R}^n$ with some $\kappa > 0$. In fact, Kannan–Lovász–Simonovits [21] (see also [4, 6]) showed that every log-concave probability measure μ on $\mathbb{R}^n$ also satisfies (1.5) with some positive constant depending on μ. Moreover, on one hand, we see that the two-sided exponential measure $d\nu_2(x) = \frac{1}{2}e^{-|x|}\, dx$ on $\mathbb{R}$ satisfies $\nu_2^+((-\infty, x)) = \min\{\nu_2((-\infty, x)), \nu_2((x, \infty))\}$ for any $x \in \mathbb{R}$, on the other hand, the one-sided exponential measure $d\nu_1(x) = e^{-x}\, dx$ on $(0, \infty)$ satisfies (1.4) with equality for $A = (0, x)$ for any $x > 0$. Thus the both inequalities (1.4) and (1.5) are sharp in the class of log-concave probability measures.

Our main goal in this paper is to investigate an analytic aspect of the dilation inequality (1.4) as the Poincaré inequality and $(1,1)$-Poincaré inequality are analytic forms of Cheeger's isoperimetric inequality. To this end, however the definition of the dilation (1.1) is complicated, and thus as first step, we focus only on symmetric open convex sets $K \subset \mathbb{R}^n$ (we say that K is symmetric if $K = -K$). In this case, it is known (see [17]) that the ε-dilation of K can be represented simply as

$$K_\varepsilon = \frac{1+\varepsilon}{1-\varepsilon} K.$$

From this property, we consider (1.1) as a generalization of the dilation. We also remark that on one hand, we have

$$K_\varepsilon = K + \frac{2\varepsilon}{1-\varepsilon}K, \tag{1.6}$$

on the other hand, the ε-neighborhood of K can be rewritten as

$$[K]_\varepsilon = K + \varepsilon \mathrm{B}_2^n,$$

where $\mathrm{B}_2^n := \{x \in \mathbb{R}^n \mid |x| < 1\}$ is the standard Euclidean open unit ball. Therefore the difference between the ε-neighborhood and ε-dilation is clear.

We also note that our restriction to symmetric open convex sets is enough to develop theory related to the dilation inequality. In fact, the dilation inequality has its origin in Borell's lemma which states that

$$\mu(\mathbb{R}^n \setminus tK) \leq \left(\frac{1-\mu(K)}{\mu(K)}\right)^{\frac{t+1}{2}} \mu(K), \quad t \geq 1$$

for any log-concave probability measure μ on $\mathbb{R}^n$ and symmetric convex set $K \subset \mathbb{R}^n$. Moreover Lovász–Simonovits [27] strengthened Borell's lemma as follows:

$$\mu(\mathbb{R}^n \setminus tK) \leq (1-\mu(K))^{\frac{t+1}{2}}, \quad t \geq 1. \tag{1.7}$$

Replacing t by $\frac{1+\varepsilon}{1-\varepsilon}$, (1.7) may be rewritten as

$$\mu(K_\varepsilon) \geq 1 - (1-\mu(K))^{\frac{1}{1-\varepsilon}}.$$

Hence we may actually check that (1.4) follows as $A = K$ by the definition of the dilation area. On the other hand, it follows from Borell's lemma that

$$\mu(\mathbb{R}^n \setminus tK) \leq ce^{-Ct}, \quad t \geq 1 \tag{1.8}$$

whenever $\mu(K) \geq 1/2 + \varepsilon$, where $c, C > 0$ are constants depending only on $\varepsilon > 0$. This inequality is a concentration of measure with respect to dilations, and we can observe that the same inequality follows from (1.4) for log-concave probability measures (see [35, Theorem 4.1]). By using (1.8), various geometric and analytic inequalities are deduced like the Kahane–Khintchine inequality in [10, 19] (see also [32]) and Cheeger's isoperimetric inequality in [4]. We can see other applications of (1.8) in [12].

To describe our results in this paper, we introduce some notions. Let $\Omega \subset \mathbb{R}^n$ be a symmetric convex domain, and let $\mathcal{K}_s^n(\Omega)$ be the set of all nonempty, symmetric open convex sets in Ω.

Definition 1.1. *A probability measure μ supported on a symmetric convex domain $\Omega \subset \mathbb{R}^n$ satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega)$ with $\kappa > 0$ if for any $K \in \mathcal{K}_s^n(\Omega)$, it holds that*

$$\mu^*(K) \geq -\kappa(1-\mu(K))\log(1-\mu(K)). \tag{1.9}$$

We may replace $\mathcal{K}_s^n(\Omega)$ by $\mathcal{K}_s^n(\mathbb{R}^n)$ in (1.9). Indeed, by the definition (1.1), it holds that $(K \cap \Omega)_\varepsilon \subset K_\varepsilon$ for any $K \in \mathcal{K}_s^n(\mathbb{R}^n)$ and $\varepsilon \in (0,1)$, and thus $\mu^*(K \cap \Omega) \leq \mu^*(K)$ by $\mu(K) = \mu(K \cap \Omega)$. In addition, $K \cap \Omega \in \mathcal{K}_s^n(\Omega)$ follows since K and Ω are symmetric convex domains. Thus the fact that (1.9) holds true for all $K \in \mathcal{K}_s^n(\Omega)$ is equivalent to one for all $K \in \mathcal{K}_s^n(\mathbb{R}^n)$. We also remark that μ might not be symmetric even if its support is symmetric. As we have already mentioned, all log-concave probability measures on Ω (and thus on $\mathbb{R}^n$) satisfy the dilation inequality for $\mathcal{K}_s^n(\Omega)$ with $\kappa = 1$. In particular, important examples are symmetric log-concave probability measures on $\mathbb{R}$

and the standard Gaussian measure $d\gamma_n \coloneqq (2\pi)^{-n/2}e^{-|x|^2/2}\,dx$ on $\mathbb{R}^n$. We can observe that these measures satisfy (1.9) with $\kappa = 2$ (see Appendix).

Next, we introduce the relative entropy. For a nonnegative Borel function f and a probability measure μ on Ω with $\int_\Omega f\,d\mu < +\infty$, we define the relative entropy of f with respect to μ by

$$\mathrm{Ent}_\mu(f) \coloneqq \int_\Omega f\log f\,d\mu - \int_\Omega f\,d\mu \log\int_\Omega f\,d\mu,$$

where we put $0\log 0 \coloneqq 0$. Jensen's inequality implies that the relative entropy is nonnegative, and is 0 if and only if f is constant μ-a.e., on Ω.

The following functional inequalities, which are special cases of Theorem 2.5, follow from (1.9).

Theorem 1.2. *Let μ be a probability measure supported on a symmetric convex domain Ω and let $f\colon \Omega \to [0,\infty)$ be a continuous and even quasi-convex function with $f \in L^1(\mu)$. We assume that μ satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega)$ for some $\kappa > 0$.*

(1) If f is locally Lipschitz on Ω, then it holds that

$$\mathrm{Ent}_\mu(f) \le \frac{2}{\kappa}\int_\Omega \langle x, \nabla f(x)\rangle\,d\mu(x). \tag{1.10}$$

(2) If f is locally Lipschitz on $\{x\in\Omega \mid f(x) > f(0)\}$, then it holds that

$$\mathrm{Ent}_\mu(f) \le \frac{2}{\kappa}\int_{\{f>f(0)\}} \langle x, \nabla f(x)\rangle\,d\mu(x). \tag{1.11}$$

Here we say that a function $f\colon\Omega\to\mathbb{R}$ is quasi-convex if $\{x\in\Omega \mid f(x) < \lambda\}$ is a convex set for any $\lambda\in\mathbb{R}$. In particular, quasi-convexity is a generalization of convexity. An important example is $|\cdot|^p$ for $p>0$, which is continuous and even quasi-convex on $\mathbb{R}^n$ and locally Lipschitz on $\mathbb{R}^n\setminus\{0\}$. In addition, $|\cdot|^p$ is locally Lipschitz and convex on $\mathbb{R}^n$ when $p\ge 1$. Therefore (1.11) is meaningful for $f=|\cdot|^p$ even if $p\in(0,1)$. See Section 2 for more details and other examples of quasi-convex functions.

We also remark that f is differentiable almost everywhere on Ω in (1.10) (or $\{f > f(0)\}$ in (1.11)) by Rademacher's theorem since f is locally Lipschitz.

We emphasize that Theorem 1.2 is the special case of Theorem 2.5 where we will show a more general inequality for functions in a more wider class. Moreover, we will actually confirm that Theorem 2.5 can recover the dilation inequality (1.9) in Theorem 4.1. In this sense, our theorem gives the optimal estimate.

As the first application of Theorem 1.2, we obtain the following reverse Shannon entropy inequality.

Corollary 1.3. *Let h be a nonnegative differentiable function such that h/γ_n is even quasi-convex function with $\int_{\mathbb{R}^n} h(x)\,dx = 1$ and $\lim_{|x|\to+\infty}|x|h(x) = 0$. Then it holds that*

$$\int_{\mathbb{R}^n} h\log h\,dx \le \frac{1}{2}\int_{\mathbb{R}^n}|x|^2h(x)\,dx - \frac{n}{2}\log(2\pi e^2). \tag{1.12}$$

We remark that the assumption $\lim_{|x|\to+\infty}|x|h(x) = 0$ is used to ensure integrating by parts, and thus this assumption might not be essential.

The classical Shannon entropy inequality (for instance, see [16]) implies the lower bound of the Shannon entropy such that

$$\int_{\mathbb{R}^n} h\log h\,dx \ge -\frac{n}{2}\log\left(\frac{2\pi e}{n}\int_{\mathbb{R}^n}|x|^2h(x)\,dx\right)$$

for any nonnegative function h on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} h\,dx = 1$ and $\int_{\mathbb{R}^n} |x|^2 h(x)\,dx < +\infty$. Furthermore, certain modified Shannon entropy inequality for log-concave probability measures is also discussed by Artstein-Avidan–Klartag–Schütt–Werner [1] and Caglar–Fradelizi–Guédon–Lehec–Schütt–Werner [14]. To see this, let us denote the Shannon entropy of a density h by

$$S(h) := -\mathrm{Ent}_{dx}(h) = -\int_{\mathbb{R}^n} h\log h\,dx.$$

Then in [1, 14], they showed that if $h = e^{-\psi}$ is a log-concave density, then

$$\int_{\mathbb{R}^n} \log\det\nabla^2\psi\,dx \le 2(S(\gamma_n) - S(h)). \tag{1.13}$$

On the other hand, (1.12) gives the upper bound of the Shannon entropy. Specifically, we can check that (1.12) is equivalent to

$$2(S(\gamma_n) - S(h)) \le \int_{\mathbb{R}^n} |x|^2\,d\gamma_n - n.$$

Hence (1.12) may be also regard as the reverse inequality of (1.13) for specific functions.

We remark that as we will see in Subsection 3.1, one has $\int_{\mathbb{R}^n} |x|^2 h(x)\,dx \ge n$ in our settings, and thus it always holds that

$$-\frac{n}{2}\log\left(\frac{2\pi e}{n}\int_{\mathbb{R}^n}|x|^2 h(x)\,dx\right) \le \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 h(x)\,dx - \frac{n}{2}\log(2\pi e^2).$$

In addition, we can check that when $h = \gamma_n$, then equality in (1.12) holds.

As another application of Theorem 1.2, we can observe the logarithmic Sobolev type or Cramér–Rao type inequality in the special case, which will be investigated in Subsection 3.2. In general, we say that a probability measure μ satisfies the logarithmic Sobolev inequality with $\rho > 0$ if

$$\mathrm{Ent}_\mu(f) \le \frac{1}{2\rho} I_\mu(f) \tag{1.14}$$

for any nonnegative locally Lipschitz function f on $\mathbb{R}^n$, where $I_\mu(f)$ is the Fisher information of f with respect to μ given by

$$I_\mu(f) := \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\,d\mu.$$

It is known that if $d\mu = e^{-\varphi}\,dx$ with $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies $\nabla^2\varphi \ge \rho$ for some $\rho > 0$, then μ satisfies the logarithmic Sobolev inequality with ρ. However, when $\nabla^2\varphi \ge 0$ (which means that φ is convex), μ may not satisfy (1.14) for any $\rho > 0$. Indeed, if μ satisfies the logarithmic Sobolev inequality, μ should satisfy the normal concentration, or equivalently $\int_{\mathbb{R}^n} e^{\varepsilon|x|^2}\,d\mu(x) < +\infty$ for some $\varepsilon > 0$. In particular, since $\nabla^2\varphi \ge 0$ is equivalent to the log-concavity of μ by [11], we can observe that a log-concave probability measure may not satisfy (1.14) for any $\rho > 0$ in general. We refer the reader to [3] for details of the logarithmic Sobolev inequality. Nevertheless, we obtain the relation between the relative entropy and the Fisher information from Theorem 1.2 by the Cauchy–Schwarz inequality immediately.

Proposition 1.4. *Let μ, Ω and f be as in Theorem 1.2. If f is a locally Lipschitz and quasi-convex function on Ω, then it holds*

$$\mathrm{Ent}_\mu(f) \le \frac{2}{\kappa}\left(\int_\Omega |x|^2 f(x)\,d\mu(x)\right)^{1/2}\sqrt{I_\mu(f)}. \tag{1.15}$$

We note that (1.15) is also close to the Cramér–Rao inequality. The classical Cramér–Rao inequality (or the Heisenberg–Pauli–Weyl inequality) implies that for any nonnegative locally Lipschitz function h with $\int_{\mathbb{R}^n} h\,dx = 1$ and $\int_{\mathbb{R}^n} |x|^2 h(x)\,dx < +\infty$, it holds

$$n \leq \left(\int_{\mathbb{R}^n} |x|^2 h(x)\,dx \right)^{\frac{1}{2}} \sqrt{I_{dx}(h)}. \tag{1.16}$$

Our result (1.15) does not induce (1.16) since the relative entropy can take the value 0, and in this sense (1.15) is different from the uncertainty principle. However, this difference is natural since on one hand, we cannot take any constant function in (1.16), on the other hand, we can take one in (1.15) due to the finite mass of μ. Nevertheless, the behavior of the relative entropy is closely related to the dimension. In fact, given probability measures μ_1, μ_2 and nonnegative functions f_1, f_2 on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ with $\int_{\mathbb{R}^{n_1}} f_1(x)\,dx = \int_{\mathbb{R}^{n_2}} f_2(x)\,dx = 1$, we can check that

$$\mathrm{Ent}_{\mu_1 \otimes \mu_2}(f_{12}) = \mathrm{Ent}_{\mu_1}(f_1) + \mathrm{Ent}_{\mu_2}(f_2),$$

where $f_{12}(x_1, x_2) \coloneqq f_1(x_1) f_2(x_2)$ for $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. This implies that the relative entropy can be linear increasing in the dimension n, and in this sense, the bound (1.15) is similar to (1.16).

As we will see in Subsection 3.3, we will also discuss Kahane–Khintchine inequalities with positive and negative exponents for symmetric quasi-convex functions via Theorem 1.2 (and Theorem 2.5 and Proposition 2.4), and discuss deviation inequalities as their application. Similar inequalities for general functions have been already investigated in [33, 6, 17, 35] where we need to assume the Remez type inequality. On the other hand, we can obtain Kahane–Khintchine inequalities and deviation inequalities without the Remez type inequality. We enumerate our results only on deviation inequalities in special cases.

Corollary 1.5. *Let μ and Ω be as in Theorem 1.2.*

(1) *Let f be a positive, differentiable and even quasi-convex function on Ω satisfying*

$$f \in \bigcap_{p \geq 1} L^p(\mu).$$

We set

$$\alpha \coloneqq \frac{\kappa}{2\|\langle \cdot, \nabla \log f(\cdot) \rangle\|_{L^\infty}}.$$

If $1 \leq \alpha < +\infty$, then it holds that

$$\mu(\{x \in \Omega \mid f(x) \geq C t \alpha^{-1/\alpha} \|f\|_{L^\alpha(\mu)}\}) \leq 2\exp(-t^\alpha), \quad \forall t \geq 1, \tag{1.17}$$

where $C > 0$ is an absolute constant.

(2) *Suppose that Ω is bounded, and let f be a positive, differentiable and even quasi-convex function on some neighborhood of $\overline{\Omega}$ and set*

$$\beta \coloneqq \frac{2}{\kappa \log 2} \|\langle \cdot, \nabla \log f(\cdot) \rangle\|_{L^\infty}.$$

Suppose that $0 < \beta < +\infty$ with $f^{-1/\beta} \in L^1(\mu)$. Then for any small enough $\varepsilon > 0$, it holds that

$$\mu(\{x \in \Omega \mid f(x) \leq t\,\mathrm{med}(f)\}) \leq \left(\frac{e}{\varepsilon \beta} \right)^{1 - \varepsilon\beta} t^{\frac{1}{\beta} - \varepsilon}, \quad \forall t \in (0, 1],$$

where $\mathrm{med}(f) \in \mathbb{R}$ is a Lévy mean of f, which means that

$$\mu(\{x \in \Omega \mid f(x) \geq \mathrm{med}(f)\}) \geq \frac{1}{2}, \quad \mu(\{x \in \Omega \mid f(x) \leq \mathrm{med}(f)\}) \geq \frac{1}{2}.$$

As the final application, we can also obtain the following result on isoperimetry.

Corollary 1.6. *Let $\mu = e^{-\varphi(x)}\,dx$ be a probability measure supported on a symmetric convex domain $\Omega \subset \mathbb{R}^n$. Suppose that φ is smooth on some neighborhood of $\overline{\Omega}$ and μ satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega)$ with $\kappa > 0$. Then for any bounded $K \in \mathcal{K}_s^n(\Omega)$ with smooth boundary and $p \in (1, 2]$, we have*

$$\mu^+(K) \geq \left(\frac{r(K)}{\int_{\partial K} \langle x, \eta(x)\rangle |x|^{p'} e^{-\varphi(x)}\,d\sigma_K(x)}\right)^{p-1} \left[-\frac{\kappa}{2}(1-\mu(K))\log(1-\mu(K))\right]^p, \quad (1.18)$$

where p' is the conjugate of p, $r(K)$ is the maximal constant $c > 0$ such that $c\mathrm{B}_2^n \subset K$, η is the outer unit normal vector along ∂K and σ_K is the surface measure on ∂K.

Corollary 1.6 reminds us of Cheeger's isoperimetric inequality for log-concave probability measures by Kannan–Lovász–Simonovits [21] and Bobkov [4, 7] where the first or second moment appears as isoperimetric constants. Kannan–Lovász–Simonovits also conjecture that the isoperimetric constant of every log-concave probability measure is controlled by the covariance matrix, which is called the KLS conjecture. We refer the reader to [12] for its history and related works and to [23] for the recent development.

We remark that the dilation inequality (1.9) can give an estimate of the μ-perimeter directly. Indeed, if $R(K) > 0$ is the minimal constant $C > 0$ such that $K \subset C\mathrm{B}_2^n$ for $K \in \mathcal{K}_s^n(\mathbb{R}^n)$, then it follows from (1.6) that

$$K_\varepsilon \subset K + \frac{2\varepsilon}{1-\varepsilon}R(K)\mathrm{B}_2^n = [K]_{\frac{2\varepsilon}{1-\varepsilon}R(K)},$$

which implies that

$$\mu^*(K) \leq 2R(K)\mu^+(K).$$

Combining this inequality with (1.9), we conclude

$$\mu^+(K) \geq -\frac{\kappa}{2R(K)}(1-\mu(K))\log(1-\mu(K)). \quad (1.19)$$

We can find a similar estimate in [4] for log-concave probability measures. However, (1.18) seems different from (1.19) since (1.18) requires not only the geometric structure of K, but also the distribution. In particular, we can recover (1.19) from (1.18). In fact, by the definition of $R(K)$, we have $|x| \leq R(K)$ for $x \in \partial K$, which implies that

$$\int_{\partial K} \langle x, \eta(x)\rangle |x|^{p'} e^{-\varphi(x)}\,d\sigma_K(x) \leq R(K)^{p'+1} \int_{\partial K} e^{-\varphi}\,d\sigma_K.$$

Hence (1.18) yields that

$$\mu^+(K) \geq \frac{1}{R(K)^{2p-1}} \left(\frac{r(K)}{\int_{\partial K} e^{-\varphi}\,d\sigma_K}\right)^{p-1} \left[-\frac{\kappa}{2}(1-\mu(K))\log(1-\mu(K))\right]^p,$$

and thus letting $p \downarrow 1$, we obtain (1.19).

In Section 4, we will show the equivalence between (1.9) and Theorem 2.5 which generalizes Theorem 1.2, and as its corollaries, we will give new classes satisfying the dilation inequality (1.9). More precisely, we will discuss the stability under bounded perturbations and tensor products.

Corollary 1.7. *Let μ be a probability measure supported on a symmetric convex domain $\Omega \subset \mathbb{R}^n$ with $\int_\Omega |x|\,d\mu(x) < +\infty$ and let h be a positive Borel function on Ω such that $b^{-1} \leq h \leq b$ for some $b > 1$ and $\int_\Omega h\,d\mu = 1$. Let ν be a probability measure on Ω given by $d\nu = h\,d\mu$. If μ satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega)$ with $\kappa > 0$, then ν satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega)$ with the constant $b^{-2}\kappa$.*

Corollary 1.8. *Let μ_1, μ_2 be probability measures supported on symmetric convex domains $\Omega_1 \subset \mathbb{R}$ and $\Omega_2 \subset \mathbb{R}^n$, respectively, with $\int_{\Omega_1} |x|\, d\mu_1, \int_{\Omega_2} |x|\, d\mu_2 < +\infty$. We suppose that μ_1, μ_2 satisfy the dilation inequality for $\mathcal{K}_s^1(\Omega_1), \mathcal{K}_s^n(\Omega_2)$ with some $\kappa_1, \kappa_2 > 0$, respectively. Let $K \subset \mathbb{R} \times \mathbb{R}^n$ be an open convex set such that if $(x, y) \in K \subset \mathbb{R} \times \mathbb{R}^n$, then it holds that $(-x, y), (x, -y), (-x, -y) \in K$. Then $\mu_1 \otimes \mu_2$ satisfies* (1.9) *for K with the constant $\kappa = (\kappa_1^{-1} + \kappa_2^{-1})^{-1}$.*

The structure of the rest of this paper is as follows. In Section 2, we introduce the class of functions including good enough even quasi-convex functions and define certain derivative as a counterpart of the gradient. After that, we show the functional form of the dilation inequality which leads to Theorem 1.2. In Section 3, we give some applications which follow from Theorems 1.2 and 2.5. More precisely, we show the reverse Shannon inequality, logarithmic Sobolev inequality, Kahane–Khintchine inequality, deviation inequality and the estimate of isoperimetry. In the final section, we show the dilation inequality from the functional inequality constructed in Section 2, and confirm the equivalence between the dilation inequality and the functional form. As corollaries, we give stability results of the dilation inequality via bounded perturbations and tensorization.

2 Functional inequality derived from the dilation inequality

Our main result in this section is Theorem 2.5, which is a functional form of the dilation inequality (1.9) and generalizes Theorem 1.2.

In what follows, let $\Omega \subset \mathbb{R}^n$ be a symmetric convex domain. Recall that a function $f\colon \Omega \to \mathbb{R}$ is quasi-convex if a set $\{x \in \Omega \mid f(x) < \lambda\}$ is convex for any $\lambda \in \mathbb{R}$, or equivalently it holds that

$$f((1-t)x + ty) \le \max\{f(x), f(y)\}, \quad \forall x, y \in \Omega, \forall t \in [0,1].$$

For instance, all convex functions are quasi-convex. Another example is $|\cdot|^p$ on $\mathbb{R}^n$ for $p > 0$ which is quasi-convex, but not convex when $p \in (0,1)$. This example also implies that quasi-convexity does not yield convexity. It is also known that a continuous function f on $\mathbb{R}$ is quasi-convex if and only if f is either monotone on $\mathbb{R}$ or there exists some point $x_0 \in \mathbb{R}$ such that f is non-increasing on $(-\infty, x_0]$ and non-decreasing on $[x_0, \infty)$ (see [2, Proposition 3.8 and Proposition 3.9]). Moreover, if a function f on $\Omega \subset \mathbb{R}^n$ is differentiable, then quasi-convexity of f is characterized by

$$\langle x - y, \nabla f(x) \rangle \ge 0$$

for any $x, y \in \Omega$ with $f(x) \ge f(y)$ (see [2, Theorem 3.1]). In particular, when f is even (which means that $f(x) = f(-x)$ for any $x \in \Omega$), then we have

$$\langle x, \nabla f(x) \rangle \ge 0, \quad \forall x \in \Omega$$

since $f(0) = \min_{x \in \Omega} f(x)$ by quasi-convexity and evenness of f. The reader is referred to [2] for more information on quasi-convexity.

Given an even quasi-convex function $f\colon \Omega \to [0, \infty)$, we define the function $\Phi_f\colon \Omega \to [0, \infty]$ by

$$\Phi_f(x) := \limsup_{\varepsilon \downarrow 0} \frac{f(x) - f(\frac{1-\varepsilon}{1+\varepsilon} x)}{\varepsilon}, \quad x \in \Omega. \tag{2.1}$$

Since f is a nonnegative and even quasi-convex function, for any $\varepsilon \in (0,1)$ and $x \in \Omega$, it holds that $f(x) \ge f(\frac{1-\varepsilon}{1+\varepsilon} x)$, and thus that Φ_f is always nonnegative. In particular, when f is differentiable at $x \in \Omega$, we see that

$$\Phi_f(x) = 2\langle x, \nabla f(x) \rangle. \tag{2.2}$$

An important example is a norm. Let $\|\cdot\|_K$ for $K \in \mathcal{K}_s^n(\mathbb{R}^n)$ be a nonnegative function on $\mathbb{R}^n$ defined by

$$\|x\|_K \coloneqq \inf\{\lambda > 0 \mid x \in \lambda K\}, \quad x \in \mathbb{R}^n. \tag{2.3}$$

We call it the gauge function of K. If $\overline{K}$ is a convex body, then the gauge function $\|\cdot\|_K$ is exactly a norm whose closed unit ball is $\overline{K}$. By the definition, we can immediately check that

$$\Phi_{\|\cdot\|_K} = 2\|\cdot\|_K \tag{2.4}$$

since $\|\cdot\|_K$ is 1-homogeneous. We remark that $\|\cdot\|_K$ is not differentiable at least at the origin. An advantage of the definition of (2.1) is that we may not suppose certain regularities of f and thus we can consider function which is non-differentiable on the whole space like a norm. We also note that by the definition, Φ_f is Borel measurable when Φ_f is finite and f is continuous on Ω.

Next, let μ be a probability measure supported on Ω. We denote by $\mathrm{QC}(\Omega, \mu)$ all nonnegative, continuous and even quasi-convex functions f on Ω such that there exists a nonnegative Borel function $g\colon \Omega \to [0, \infty)$ in $L^1(\mu)$ and small enough $\varepsilon_0 \in (0, 1]$ satisfying

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \frac{f(x) - f(\frac{1-\varepsilon}{1+\varepsilon}x)}{\varepsilon} \le g(x), \quad \forall x \in \Omega. \tag{2.5}$$

We may replace ε_0 with 1 in (2.5) when $f \in L^1(\mu)$. To see this, let f, g and ε_0 be as above. Then we can observe that for any $\varepsilon \in (0, 1)$,

$$\frac{f(x) - f(\frac{1-\varepsilon}{1+\varepsilon}x)}{\varepsilon} \le g(x) + \frac{1}{\varepsilon_0} f(x), \quad \forall x \in \Omega.$$

This fact implies that we can take a function $\widetilde{g} \in L^1(\mu)$ satisfying

$$\sup_{\varepsilon \in (0,1)} \frac{f(x) - f(\frac{1-\varepsilon}{1+\varepsilon}x)}{\varepsilon} \le \widetilde{g}(x), \quad \forall x \in \Omega.$$

We remark that by the definition, if $f \in \mathrm{QC}(\Omega, \mu)$, then af and $f + \alpha$ for any $a > 0$ and $\alpha \ge -\inf_{x\in\Omega} f(x)$ also belong to $\mathrm{QC}(\Omega, \mu)$, and in particular, we have $\Phi_{af} = a\Phi_f$ and $\Phi_{f+\alpha} = \Phi_f$.

An important example belonging to $\mathrm{QC}(\Omega, \mu)$ is a norm. Indeed, we can easily check that the gauge function $\|\cdot\|_K$ for $K \in \mathcal{K}_s^n(\Omega)$ is in $\mathrm{QC}(\Omega, \mu)$ when μ has finite first moment, namely $\int_\Omega |x|\, d\mu(x) < +\infty$. More generally, we can ensure that $\mathrm{QC}(\Omega, \mu)$ includes good locally Lipschitz and even quasi-convex functions. Here a function $f\colon \Omega \to \mathbb{R}$ is locally Lipschitz if for any $x \in \Omega$, there exists some $r > 0$ such that f is Lipschitz on $B(x; r) \coloneqq \{y \in \Omega \mid |x - y| < r\}$, or equivalently

$$|\nabla f(z)| \coloneqq \limsup_{y\to x} \frac{|f(y) - f(z)|}{|y - z|}$$

is finite on $B(x; r)$.

Proposition 2.1. *Let μ be a probability measure supported on a bounded symmetric convex domain $\Omega \subset \mathbb{R}^n$. Let f be a nonnegative, continuous and even quasi-convex function on some neighborhood of $\overline{\Omega}$. If f is locally Lipschitz on $\overline{\Omega}$, then it holds that $f \in \mathrm{QC}(\Omega, \mu)$ and $\Phi_f(x) \le 2|x||\nabla f(x)|$ for any $x \in \Omega$.*

Proof. Since f is locally Lipschitz, for any $x \in \overline{\Omega}$, there exist some $\varepsilon(x) \in (0, 1)$, $r(x) > 0$ and $M(x) > 0$ such that

$$\frac{|f(\frac{1-\varepsilon}{1+\varepsilon}y) - f(y)|}{|\frac{1-\varepsilon}{1+\varepsilon}y - y|} \le M(x), \quad \forall \varepsilon \in (0, \varepsilon(x)), \forall y \in B(x, r(x)). \tag{2.6}$$

Since $\overline{\Omega}$ is compact, we can take finite points $\{x_k\}_{k=1}^N \subset \overline{\Omega}$ ($N \in \mathbb{N}$) such that

$$\frac{|f(\frac{1-\varepsilon}{1+\varepsilon}y) - f(y)|}{|\frac{1-\varepsilon}{1+\varepsilon}y - y|} \leq \max_{k=1,2,\dots,N} M(x_k), \quad \forall \varepsilon \in (0, \overline{\varepsilon}), \forall y \in \Omega,$$

where we set $\overline{\varepsilon} \coloneqq \min_{k=1,2,\dots,N} \varepsilon(x_k) > 0$. In particular, we obtain

$$\frac{|f(\frac{1-\varepsilon}{1+\varepsilon}y) - f(y)|}{\varepsilon} \leq \frac{2}{1+\varepsilon}|y| \max_{k=1,2,\dots,N} M(x_k) \leq 2\mathrm{diam}\,\Omega \max_{k=1,2,\dots,N} M(x_k)$$

for any $\varepsilon \in (0, \overline{\varepsilon})$ and $y \in \Omega$, which ensures (2.5). Hence we enjoy $f \in \mathrm{QC}(\Omega, \mu)$.

In particular, by the definition, for any $\delta > 0$ and $x \in \Omega$, we can take some $\varepsilon_0 \in (0,1)$ depending on δ and x such that

$$\frac{|f(\frac{1-\varepsilon}{1+\varepsilon}x) - f(x)|}{|\frac{1-\varepsilon}{1+\varepsilon}x - x|} \leq (1+\delta)|\nabla f(x)|, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

from which we see that

$$\frac{f(x) - f(\frac{1-\varepsilon}{1+\varepsilon}x)}{\varepsilon} \leq \frac{2(1+\delta)}{1+\varepsilon}|x||\nabla f(x)| \leq 2(1+\delta)|x||\nabla f(x)|.$$

Letting $\varepsilon \downarrow 0$, we have

$$\Phi_f(x) \leq 2(1+\delta)|x||\nabla f(x)|.$$

Since $\delta > 0$ is arbitrary, we obtain $\Phi_f(x) \leq 2|x||\nabla f(x)|$ for any $x \in \Omega$. □

Moreover, when f is a convex function instead of a quasi-convex function in Proposition 2.1, we can specify Φ_f.

Proposition 2.2. *Let μ be a probability measure supported on a symmetric convex domain $\Omega \subset \mathbb{R}^n$. If an even function $f\colon \Omega \to [0,\infty)$ is convex, then it holds that $\Phi_f(x) = 2\inf_{y\in\partial f(x)}\langle x, y\rangle$ for any $x \in \Omega$. Moreover, if we have $\int_\Omega \inf_{y\in\partial f(x)}\langle x,y\rangle \, d\mu(x) < +\infty$, then $f \in \mathrm{QC}(\Omega,\mu)$. Here $\partial f(x) \subset \mathbb{R}^n$ is the subdifferential of f at $x \in \Omega$, namely*

$$y \in \partial f(x) \iff [f(z) \geq f(x) + \langle y, z - x\rangle, \quad \forall z \in \Omega].$$

We note that when f is convex, the subdifferential of f is always nonempty on Ω, and in particular $\partial f(x) = \{\nabla f(x)\}$ if f is differentiable at $x \in \Omega$ (see [36]). Especially, since any convex function is differentiable Legesgue-almost everywhere by Rademacher's theorem, we see that $\Phi_f = 2\langle x, \nabla f(x)\rangle$ for dx-a.e., $x \in \Omega$ in Proposition 2.2.

Remark 2.3. The first assertion in Proposition 2.2 implies that $\inf_{y\in\partial f(x)}\langle x,y\rangle$ is Borel measurable since Φ_f is Borel measurable.

Proof. We fix $x \in \Omega$, and firstly show $\Phi_f(x) = 2\inf_{y\in\partial f(x)}\langle x,y\rangle$. By the definition of the subdifferential of f, we have

$$f\left(\frac{1-\varepsilon}{1+\varepsilon}x\right) \geq f(x) + \left\langle y, \frac{1-\varepsilon}{1+\varepsilon}x - x\right\rangle$$

for any $x \in \Omega$, $\varepsilon \in (0,1)$ and $y \in \partial f(x)$, which is equivalent to

$$\frac{f(x) - f(\frac{1-\varepsilon}{1+\varepsilon}x)}{\varepsilon} \leq \frac{2}{1+\varepsilon}\langle x, y\rangle.$$

Hence we obtain

$$\frac{f(x) - f(\frac{1-\varepsilon}{1+\varepsilon}x)}{\varepsilon} \leq 2\inf_{y\in\partial f(x)}\langle x,y\rangle. \tag{2.7}$$

Letting $\varepsilon \downarrow 0$, we conclude $\Phi_f(x) \le 2\inf_{y\in\partial f(x)}\langle x,y\rangle$.

Next we show $\Phi_f(x) \ge 2\inf_{y\in\partial f(x)}\langle x,y\rangle$. Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,1)$ be a monotone decreasing sequence satisfying $\lim_{k\to+\infty}\varepsilon_k = 0$ and

$$\lim_{k\to+\infty}\frac{f(x)-f(x_k)}{\varepsilon_k} = \Phi_f(x),$$

where we set $x_k := \frac{1-\varepsilon_k}{1+\varepsilon_k}x$, and take $y_k \in \partial f(x_k)$ for each $k\in\mathbb{N}$. If we can take a subsequence $\{y_{k_\ell}\}_{\ell\in\mathbb{N}}$ of $\{y_k\}_{k\in\mathbb{N}}$ such that $y_{k_\ell}=0$ for all $\ell\in\mathbb{N}$, then for any $z\in\Omega$ and $\ell\in\mathbb{N}$, we have $f(z)\ge f(x_{k_\ell})$. Letting $\ell\to+\infty$, we get $f(z)\ge f(x)$ for any $z\in\Omega$, which yields $0\in\partial f(x)$. Hence we have $\inf_{y\in\partial f(x)}\langle x,y\rangle = 0$. On the other hand, we also see that $f(x_{k_\ell}) = f(0)$ since f is even and convex. Thus it follows that $f(tx)=f(0)$ for any $t\in[0,1]$, which implies $\Phi_f(x)=0$. Hence we have $\Phi_f(x) = 0 = 2\inf_{y\in\partial f(x)}\langle x,y\rangle$.

Therefore we may suppose that $y_k\neq 0$ for all $k\in\mathbb{N}$. In addition, without loss of generality, we may suppose that $\{x_k\}_{k\in\mathbb{N}}\subset B(x;r/2)$ and $\overline{B(x;r)}\subset\Omega$ for some $r>0$, where $B(x;r) := \{w\in\mathbb{R}^n \mid |x-w|<r\}$. By the definition of the subdifferential, it holds that

$$f(z) \ge f(x_k) + \langle y_k, z-x_k\rangle, \quad \forall z\in\Omega \tag{2.8}$$

for each $k\in\mathbb{N}$. Inserting $z_k := \frac{ry_k}{2|y_k|} + x_k$ in z, we obtain

$$f(z_k) \ge f(x_k) + \frac{r}{2}|y_k|.$$

Here we remark that $|x-z_k| \le r/2 + |x_k - x| < r$, and thus $z_k\in\Omega$. Moreover we have $z_k, x_k \in B(x;r)$. Hence since $\overline{B(x;r)}\subset\Omega$ and f is continuous, we have

$$|y_k| \le \frac{1}{r}\max_{w\in\overline{B(x;r)}} f(w), \quad \forall k\in\mathbb{N}.$$

Hence we can take a subsequence of $\{y_k\}_{k\in\mathbb{N}}$ converging to some $\widetilde{y}\in\mathbb{R}^n$. Without loss of generality, we may suppose that $\lim_{k\to+\infty} y_k = \widetilde{y}$. Letting $k\to+\infty$ in (2.8), we have

$$f(z) \ge f(x) + \langle \widetilde{y}, z-x\rangle, \quad \forall z\in\Omega,$$

which implies that $\widetilde{y}\in\partial f(x)$. Moreover, it follows from (2.8) that

$$f(x) \ge f(x_k) + \langle y_k, x-x_k\rangle,$$

which yields that

$$\frac{f(x)-f(x_k)}{\varepsilon_k} \ge \frac{2}{1+\varepsilon_k}\langle y_k, x\rangle.$$

Letting $k\to+\infty$, we obtain that

$$\Phi_f(x) \ge 2\langle\widetilde{y},x\rangle \ge 2\inf_{y\in\partial f(x)}\langle x,y\rangle.$$

This is the desired assertion.

Finally, (2.7) and $\int_\Omega \inf_{y\in\partial f(x)}\langle x,y\rangle\, d\mu(x) < +\infty$ imply that $f\in \mathrm{QC}(\Omega,\mu)$. ☐

To show our main result, we firstly give the following co-area formula associated with the dilation area, which has appeared in [35] with a more weaker form.

Proposition 2.4. *Let μ be a probability measure supported on a symmetric convex domain $\Omega\subset\mathbb{R}^n$ and let $p>0$. Then for any nonnegative function f with $f^p\in\mathrm{QC}(\Omega,\mu)$, we have*

$$\int_0^\infty t^{p-1}\mu^*(\{x\in\mathbb{R}^n \mid f(x)<t\})\,dt \le \int_\Omega f^{p-1}\Phi_f\,d\mu. \tag{2.9}$$

Moreover, for any positive function f with $f^p \in \mathrm{QC}(\Omega,\mu)$, we also have

$$\int_0^\infty t^{p-1}\mu^*\left(\left\{x\in\mathbb{R}^n \,\middle|\, \frac{1}{f(x)}>t\right\}\right)dt \le \int_\Omega f^{-p-1}\Phi_f\,d\mu. \tag{2.10}$$

We remark that $\mu^*(\{x\in\mathbb{R}^n \mid f(x)<t\})$ is Borel measurable in t since $\{x\in\mathbb{R}^n \mid f(x)<t\}$ is monotone in t and μ has the finite mass.

Proof. Since $f^p \in \mathrm{QC}(\Omega,\mu)$ and $p>0$, f is a nonnegative, continuous and even quasi-convex function, and thus $\{x\in\mathbb{R}^n \mid f(x)<\lambda\}$ is a symmetric open convex set for any $\lambda>0$, from which it holds

$$\{x\in\mathbb{R}^n \mid f(x)<\lambda\}_\varepsilon = \frac{1+\varepsilon}{1-\varepsilon}\{x\in\mathbb{R}^n \mid f(x)<\lambda\} = \left\{x\in\mathbb{R}^n \,\middle|\, f\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)<\lambda\right\}$$

for any $\lambda>0$. Hence it follows from Fatou's lemma and $\mu(\mathbb{R}^n)=1$ that

$$\begin{aligned}
&\int_0^\infty t^{p-1}\mu^*(\{x\in\mathbb{R}^n \mid f(x)<t\})\,dt\\
=&\int_0^\infty t^{p-1}\liminf_{\varepsilon\downarrow 0}\frac{\mu(\{x\in\mathbb{R}^n \mid f(x)<t\}_\varepsilon)-\mu(\{x\in\mathbb{R}^n \mid f(x)<t\})}{\varepsilon}\,dt\\
\le&\liminf_{\varepsilon\downarrow 0}\int_0^\infty t^{p-1}\frac{\mu(\{x\in\mathbb{R}^n \mid f(x)<t\}_\varepsilon)-\mu(\{x\in\mathbb{R}^n \mid f(x)<t\})}{\varepsilon}\,dt\\
=&\liminf_{\varepsilon\downarrow 0}\int_0^\infty \frac{1}{\varepsilon}t^{p-1}\left[\mu\left(\left\{x\in\mathbb{R}^n \,\middle|\, f\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)<t\right\}\right)-\mu(\{x\in\mathbb{R}^n \mid f(x)<t\})\right]dt\\
=&\liminf_{\varepsilon\downarrow 0}\int_0^\infty \frac{1}{\varepsilon}t^{p-1}\left[\mu(\{x\in\mathbb{R}^n \mid f(x)\ge t\})-\mu\left(\left\{x\in\mathbb{R}^n \,\middle|\, f\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)\ge t\right\}\right)\right]dt\\
=&\frac{1}{p}\liminf_{\varepsilon\downarrow 0}\int_\Omega \frac{1}{\varepsilon}\left(f^p(x)-f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)\right)d\mu(x).
\end{aligned}$$

Moreover, by (2.5) and $f^p \in \mathrm{QC}(\Omega,\mu)$, we can justify

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0}\int_\Omega \frac{1}{\varepsilon}\left(f^p(x)-f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)\right)d\mu(x) &\le \limsup_{\varepsilon\downarrow 0}\int_\Omega \frac{1}{\varepsilon}\left(f^p(x)-f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)\right)d\mu(x)\\
&\le \int_\Omega \Phi_{f^p}(x)\,d\mu(x),
\end{aligned}$$

where we used Fatou's lemma again. Hence we obtain

$$\int_0^\infty t^{p-1}\mu^*(\{x\in\mathbb{R}^n \mid f(x)<t\})\,dt \le \frac{1}{p}\int_\Omega \Phi_{f^p}(x)\,d\mu(x).$$

Since we see that $\Phi_{f^p}=pf^{p-1}\Phi_f$ by the definition of Φ_f and continuity of f, we can conclude (2.9).

Next, we show (2.10). We remark that $a := \inf_{x\in\Omega} f(x) = f(0) > 0$ since $f>0$ on Ω and f is an even quasi-convex function by $p>0$. Moreover, $\{x\in\mathbb{R}^n \mid f(x)^{-1}>t\}$ is a symmetric open convex set for any $t>0$ since f is a continuous and symmetric quasi-convex function. As in the above argument, Fatou's lemma yields that

$$\begin{aligned}
&\int_0^\infty t^{p-1}\mu^*\left(\left\{x\in\mathbb{R}^n \,\middle|\, \frac{1}{f(x)}>t\right\}\right)dt\\
\leq&\liminf_{\varepsilon\downarrow 0}\int_0^\infty \frac{1}{\varepsilon}t^{p-1}\left[\mu\left(\left\{x\in\mathbb{R}^n \,\middle|\, \frac{1}{f(\frac{1-\varepsilon}{1+\varepsilon}x)}>t\right\}\right)-\mu\left(\left\{x\in\mathbb{R}^n \,\middle|\, \frac{1}{f(x)}>t\right\}\right)\right]dt\\
=&\frac{1}{p}\liminf_{\varepsilon\downarrow 0}\int_\Omega \frac{1}{\varepsilon}\left(\frac{1}{f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)}-\frac{1}{f^p(x)}\right)d\mu(x).
\end{aligned}$$

Since we see that

$$\frac{1}{\varepsilon}\left(\frac{1}{f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)}-\frac{1}{f^p(x)}\right)\leq a^{-2p}\frac{1}{\varepsilon}\left(f^p(x)-f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)\right)$$

and since $f^p\in \mathrm{QC}(\Omega,\mu)$, we can apply Fatou's lemma to see that

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0}\int_\Omega \frac{1}{\varepsilon}\left(\frac{1}{f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)}-\frac{1}{f^p(x)}\right)d\mu(x)&\leq\limsup_{\varepsilon\downarrow 0}\int_\Omega \frac{1}{\varepsilon}\left(\frac{1}{f^p\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)}-\frac{1}{f^p(x)}\right)d\mu(x)\\
&\leq\int_\Omega\frac{1}{f(x)^{2p}}\Phi_{f^p}(x)\,d\mu(x).
\end{aligned}$$

Finally using $\Phi_{f^p}=pf^{p-1}\Phi_f$, we obtain (2.10). ☐

Let $\mathrm{QC}^p(\Omega,\mu)$ for $p>0$ be the set of all functions f on Ω such that $f^p\in\mathrm{QC}(\Omega,\mu)\cap L^1(\mu)$. Our main theorem in this section is the following.

Theorem 2.5. *Let μ be a probability measure supported on a symmetric convex domain $\Omega\subset\mathbb{R}^n$. We assume that μ satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega)$ for some $\kappa>0$. If f is in $\mathrm{QC}^1(\Omega,\mu)$ and differentiable on Ω, then* (1.10) *holds.*

More generally, if $f\in\mathrm{QC}^1(\Omega,\mu)$, then it holds that

$$\mathrm{Ent}_\mu(f)\leq\frac{1}{\kappa}\int_\Omega\Phi_f\,d\mu. \tag{2.11}$$

Our proof of this claim is almost same as the proof of [35, Theorem 5.3]. For the completeness, we give the proof of Theorem 2.5 here.

Proof. First assertion follows from (2.11). Hence let us show (2.11). Since $f\in\mathrm{QC}(\Omega,\mu)$, we can apply (2.9) with $p=1$ for sublevel sets of f. It follows from (1.9), $\mu(\Omega)=1$ and (2.9) that we have

$$-\int_0^\infty \mu(A_f(t))\log\mu(A_f(t))\,dt\leq\frac{1}{\kappa}\int_\Omega\Phi_f\,d\mu, \tag{2.12}$$

where we defined

$$A_f(t):=\{x\in\Omega\mid f(x)\geq t\},\quad t\geq 0.$$

To prove (2.11), without loss of generality, we may assume that $\int_\Omega f\,d\mu=1$. In fact, we know that $f\in L^1(\mu)$ and

$$\Phi_{af}=a\Phi_f,\quad \mathrm{Ent}_\mu(af)=a\mathrm{Ent}_\mu(f)$$

for any $a > 0$ and $f \in \mathrm{QC}^1(\Omega, \mu)$ from which we can add the condition $\int_\Omega f\, d\mu = 1$. Now, recall the dual formula of the relative entropy: for any continuous function $h\colon \Omega \to [0, \infty)$ with $\int_\Omega h\, d\mu = 1$, it holds that

$$\mathrm{Ent}_\mu(h) = \sup_{\varphi \in C_b(\Omega)} \left[\int_\Omega h\varphi\, d\mu - \log \int_\Omega e^\varphi\, d\mu \right], \tag{2.13}$$

where $C_b(\Omega)$ is the set of all bounded continuous functions on Ω (for instance, see [34, 18] and their proofs). Hence, since $\mathrm{Ent}_\mu(\mu(A)^{-1}\mathbf{1}_A) = -\log \mu(A)$ and $\int_\Omega \mu(A)^{-1}\mathbf{1}_A\, d\mu = 1$ for any Borel subset $A \subset \Omega$ with $\mu(A) > 0$, it holds that

$$\begin{aligned}
&- \int_0^\infty \mu(A_f(t)) \log \mu(A_f(t))\, dt \\
=& \int_0^\infty \mu(A_f(t)) \mathrm{Ent}_\mu(\mu(A_f(t))^{-1}\mathbf{1}_{A_f(t)})\, dt \\
=& \int_0^\infty \sup_{\varphi \in C_b(\Omega)} \left[\int_\Omega \varphi \mathbf{1}_{A_f(t)}\, d\mu - \mu(A_f(t)) \log \int_\Omega e^\varphi\, d\mu \right] dt \\
\geq& \sup_{\varphi \in C_b(\Omega)} \left[\int_\Omega \int_0^\infty \varphi \mathbf{1}_{A_f(t)}\, dt d\mu - \int_0^\infty \mu(A_f(t))\, dt \log \int_\Omega e^\varphi\, d\mu \right] \\
=& \sup_{\varphi \in C_b(\Omega)} \left[\int_\Omega \varphi f\, d\mu - \log \int_\Omega e^\varphi\, d\mu \right] \\
=& \mathrm{Ent}_\mu(f),
\end{aligned}$$

where we used $\int_0^\infty \mathbf{1}_{A_f(t)}(x)\, dt = f(x)$ for every $x \in \Omega$ and $\int_0^\infty \mu(A_f(t))\, dt = \int_\Omega f\, d\mu = 1$. Combining this with (2.12), we conclude the desired assertion. □

We conclude this section by giving the proof of Theorem 1.2.

Proof of Theorem 1.2. First, let us show (2). Since the proof of (1) is almost same as (2), we will give a comment after the proof of (2).

Take an increasing sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ such that Ω_k is an open, bounded symmetric convex set with $\overline{\Omega_k} \subset \Omega$ and $\lim_{k\to+\infty} \Omega_k = \Omega$. We can take such a sequence, for instance, by considering $\{x \in \Omega \mid \|x\|_\Omega < 1 - 1/k\} \cap (k\mathrm{B}_2^n)$. Let μ_k be a normalized probability measure of μ on Ω_k, namely $d\mu_k := \mu(\Omega_k)^{-1}\mathbf{1}_{\Omega_k}\, d\mu$. Next, let f be a function given in Theorem 1.2 (2), and for $\ell, m \in \mathbb{N}$, we set $f_{\ell,m}(x) := \max\{\min\{f(x), \ell\}, f(0) + 1/m\}$ for $x \in \Omega$. Then since f is locally Lipschitz on $\{x \in \Omega \mid f(x) > f(0)\}$, $f_{\ell,m}$ is locally Lipschitz on Ω for any $\ell, m \in \mathbb{N}$. In addition, $f_{\ell,m}$ is even quasi-convex with $f_{\ell,m} \in L^1(\mu_k)$ for any $k, \ell, m \in \mathbb{N}$ by its construction. Hence applying Proposition 2.1, we have $f_{\ell,m} \in \mathrm{QC}^1(\Omega_k, \mu_k)$ for any $k, \ell, m \in \mathbb{N}$. Moreover, μ_k satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega_k)$ with κ. To see this, let $K \in \mathcal{K}_s^n(\Omega_k)$. Then by the definition, we see that

$$\mu_k^*(K) = \frac{1}{\mu(\Omega_k)} \mu^*(K).$$

Hence (1.9) yields that

$$\begin{aligned}
\mu_k^*(K) &\geq -\frac{\kappa}{\mu(\Omega_k)} (1 - \mu(K)) \log(1 - \mu(K)) \\
&= -\frac{\kappa}{\mu(\Omega_k)} (1 - \mu(\Omega_k)\mu_k(K)) \log(1 - \mu(\Omega_k)\mu_k(K)).
\end{aligned}$$

By the elementary inequality $-\theta^{-1}(1 - \theta x) \log(1 - \theta x) \geq -(1 - x) \log(1 - x)$ for $\theta \in (0, 1)$ and $x \in (0, 1)$, we can obtain

$$\mu_k^*(K) \geq -\kappa(1 - \mu_k(K)) \log(1 - \mu_k(K)),$$

which is the desired assertion.

Thus we can apply Theorem 2.5 to see that

$$\mathrm{Ent}_{\mu_k}(f_{\ell,m}) \le \frac{1}{\kappa}\int_{\Omega_k} \Phi_{f_{\ell,m}}\, d\mu_k.$$

Since $f_{\ell,m}$ is a bounded continuous function, the lower semi-continuity of the relative entropy (which follows from (2.13)) and the monotone convergence theorem as $k \to +\infty$ imply that

$$\mathrm{Ent}_{\mu}(f_{\ell,m}) \le \frac{1}{\kappa}\int_{\Omega} \Phi_{f_{\ell,m}}\, d\mu.$$

Since f is locally Lipschitz, we have

$$\Phi_{f_{\ell,m}}(x) = \begin{cases} 2\langle x, \nabla f(x)\rangle & \text{if } f(0)+1/m < f(x) \le \ell \\ 0 & \text{otherwise} \end{cases}$$

for dx-a.e., $x \in \Omega$. Moreover, we see that

$$\begin{cases} 2\langle x, \nabla f(x)\rangle & \text{if } f(0)+1/m < f(x) \le \ell \\ 0 & \text{otherwise} \end{cases} \le \begin{cases} 2\langle x, \nabla f(x)\rangle & \text{if } f(x) > f(0) \\ 0 & \text{otherwise} \end{cases}$$

for dx-a.e., $x \in \Omega$. Hence we have

$$\mathrm{Ent}_{\mu}(f_{\ell,m}) \le \frac{2}{\kappa}\int_{\{f>f(0)\}} \langle x, \nabla f(x)\rangle\, d\mu(x).$$

Thus by $\lim_{\ell,m\to+\infty} f_{\ell,m} = f$ and the lower semi-continuity of the relative entropy, we obtain (1.11).

If f is a locally Lipschitz on Ω, then we may also obtain

$$\Phi_{f_{\ell,m}}(x) \le 2\langle x, \nabla f(x)\rangle$$

for dx-a.e., $x \in \Omega$ from the same argument above, and thus we have

$$\mathrm{Ent}_{\mu}(f_{\ell,m}) \le \frac{2}{\kappa}\int_{\Omega} \langle x, \nabla f(x)\rangle\, d\mu(x).$$

Hence as $\ell, m \to +\infty$, we also conclude (1.10). □

3 Some applications of Theorem 2.5

3.1 Comparisons of the relative entropy, Wasserstein distance and variance

As the first application of Theorem 2.5, we give comparisons of the relative entropy, Wasserstein distance and the variance in the case of the Gaussian measure. We denote the standard Gaussian measure on $\mathbb{R}^n$ by $d\gamma_n = (2\pi)^{-n/2}e^{-\frac{1}{2}|x|^2}\,dx$. To state our results, we introduce the L^2-Wasserstein distance, which appears in optimal transport theory.

Let μ, ν be probability measures on $\mathbb{R}^n$ with finite second moment. Then the L^2-Wasserstein distance of μ and ν is given by

$$W_2(\mu,\nu) := \left\{ \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\, d\pi(x,y) \right\}^{1/2},$$

where $\Pi(\mu,\nu)$ is the set of all couplings π between μ and ν, namely π is a probability measure on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\pi(A\times\mathbb{R}^n) = \mu(A)$ and $\pi(\mathbb{R}^n \times A) = \nu(A)$ for any Borel subset $A \subset \mathbb{R}^n$. It is known that W_2 is a distance function on the set of all probability measures on $\mathbb{R}^n$ with finite second moment. We refer the reader to [36, 37] for optimal transport theory and its related topics.

Proposition 3.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}_+$ be a differentiable even quasi-convex function with $\int_{\mathbb{R}^n} f\, d\gamma_n = 1$ and*

$$\lim_{|x|\to+\infty} |x|f(x)\gamma_n(x) = 0. \tag{3.1}$$

Then we have

$$\mathrm{Ent}_{\gamma_n}(f) \le \int_{\mathbb{R}^n} |x|^2 f(x)\, d\gamma_n(x) - n \tag{3.2}$$

and

$$\frac{1}{2}W_2^2(\gamma_n,\nu) \le \int_{\mathbb{R}^n} |x|^2 f(x)\, d\gamma_n(x) - n, \tag{3.3}$$

where $d\nu \coloneqq f\, d\gamma_n$.

We remark that (3.2) in particular implies that every differentiable even quasi-convex function $f\colon \mathbb{R}^n \to [0,\infty)$ as in Proposition 3.1 satisfies

$$\int_{\mathbb{R}^n} |x|^2 f(x)\, d\gamma_n(x) \ge n, \tag{3.4}$$

and equality holds if and only if $f \equiv 1$ on Ω. Moreover, when the deficit of (3.4) is small such that

$$\int_{\mathbb{R}^n} |x|^2 f(x)\, d\gamma_n(x) - n \le \varepsilon$$

for small enough $\epsilon > 0$, then (3.2) and (3.3) imply that f is close to the constant 1 in the both senses of the relative entropy and the L^2-Wasserstein distance.

We also note that we have the trivial upper bound of $W_2(\gamma_n,\nu)$ such as

$$\frac{1}{2}W_2^2(\gamma_n,\nu) \le \int_{\mathbb{R}^n} |x|^2 f(x)\, d\gamma_n(x) + n.$$

Thus (3.3) strengthen this trivial bound.

Proof. We first note that the standard Gaussian measure satisfies (1.9) for $\mathcal{K}_s^n(\mathbb{R}^n)$ with $\kappa = 2$ (see Appendix). Hence by Theorem 1.2, we have

$$\mathrm{Ent}_{\gamma_n}(f) \le \int_{\mathbb{R}^n} \langle x, \nabla f(x)\rangle\, d\gamma_n(x).$$

Moreover, by (3.1), integrating by parts yields that

$$\int_{\mathbb{R}^n} \langle x, \nabla f(x)\rangle\, d\gamma_n(x) = \int_{\mathbb{R}^n} |x|^2 f(x)\, d\gamma_n(x) - n.$$

Combining these facts, we obtain (3.2).

The second assertion (3.3) immediately follows from (3.2) and Gaussian Talagrand's transportation inequality (see [36, 37])

$$\frac{1}{2}W_2^2(\gamma_n,\nu) \le \mathrm{Ent}_{\gamma_n}(f).$$

□

Proof of Corollary 1.3. We set $f \coloneqq h/\gamma_n$, then we can check that f satisfies the assumptions in Proposition 3.1. Hence applying Proposition 3.1 to f, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} h\log h\, dx \le& \int_{\mathbb{R}^n} h\log\gamma_n\, dx + \int_{\mathbb{R}^n} |x|^2 h(x)\, dx - n\\ =&\frac{1}{2}\int_{\mathbb{R}^n} |x|^2 h(x)\, dx - \frac{n}{2}\log(2\pi e^2),\end{aligned}$$

which is the desired assertion. □

3.2 Cramér–Rao inequality and logarithmic Sobolev inequality

From Theorem 1.2, we can obtain the following logarithmic Sobolev type inequality or Cramér–Rao type inequality, which includes Proposition 1.4.

Proposition 3.2. *Let μ and Ω be as in Theorem 2.5 and let $f \in L^1(\mu)$ be a nonnegative, locally Lipschitz and even quasi-convex function. Then it holds that*

$$\mathrm{Ent}_\mu(f) \leq \frac{2}{\kappa}\left(\int_\Omega |x|^2 f(x)\,d\mu(x)\right)^{1/2}\sqrt{I_\mu(f)} \tag{3.5}$$

and

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{\kappa}I_\mu(f) + \frac{1}{\kappa}\int_\Omega |x|^2 f(x)\,d\mu(x). \tag{3.6}$$

Proof. Let f be a function satisfying our assumptions. Then by Theorem 1.2, we see that

$$\mathrm{Ent}_\mu(f) \leq \frac{2}{\kappa}\int_\Omega |x||\nabla f(x)|\,d\mu.$$

Then (3.5) follows by combining this with the Cauchy–Schwarz inequality, and (3.6) follows from (3.5) and the arithmetic-geometric mean inequality. □

As we described in our introduction, (3.5) is close to the logarithmic Sobolev inequality, and exactly gives

$$\mathrm{Ent}_\mu(f) \leq \frac{2}{\kappa}\left(\int_\Omega |x|^2 f(x)\,d\mu(x)\right)^{1/2} I_\mu(f)$$

if $I_\mu(f) \geq 1$, and

$$\mathrm{Ent}_\mu(f) \leq \frac{4}{\kappa^2}\int_\Omega |x|^2 f(x)\,d\mu(x) I_\mu(f)$$

if $\mathrm{Ent}_\mu(f) \geq 1$. We emphasize that the constant of (3.5) depends only on $\int_\Omega |x|^2 f(x)\,d\mu(x)$ and κ. The logarithmic Sobolev inequality also appears in [35] where we need the Poincaré constant of μ.

On the other hand, (3.6) is close to the defective logarithmic Sobolev inequality. Here we say that a probability measure μ on $\mathbb{R}^n$ satisfies the defective logarithmic Sobolev inequality with constants $\rho > 0$ and $\tau \geq 0$ if

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{2\rho}I_\mu(f) + \tau\int_\Omega f\,d\mu$$

for any nonnegative locally Lipschitz function f on Ω. We refer the reader to [3] for details of the defective logarithmic Sobolev inequality. In our case, when Ω is bounded, since Ω is symmetric, (3.6) implies that

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{\kappa}I_\mu(f) + \frac{1}{4\kappa}(\mathrm{diam}\,\Omega)^2\int_\Omega f\,d\mu. \tag{3.7}$$

In particular, when Ω is an interval in $\mathbb{R}$, we also obtain the logarithmic Sobolev inequality associated with the Poincaré constant of μ. Here we say that μ satisfies the Poincaré inequality with constant $C_\mu > 0$ if

$$C_\mu\int_\Omega f^2\,d\mu \leq \int_\Omega |\nabla f|^2\,d\mu$$

for any locally Lipschitz function f on Ω with $\int_\Omega f\,d\mu = 0$.

Corollary 3.3. *Let μ be a symmetric probability measure on a bounded symmetric open interval $I \subset \mathbb{R}$ and $f\colon I \to \mathbb{R}$ be a locally Lipschitz function. Suppose that μ satisfies the dilation inequality for $\mathcal{K}_s^1(I)$ with $\kappa > 0$ and the Poincaré inequality with $C_\mu > 0$. In addition, we suppose that f is odd and monotone function with $f \in L^2(\mu)$. Then it holds that*

$$\mathrm{Ent}_\mu(f^2) \leq \frac{1}{\kappa}\left(4 + \frac{1}{4C_\mu}(\mathrm{diam}\, I)^2\right)\int_I |f'|^2\, d\mu. \tag{3.8}$$

In particular, when $d\mu = e^{-\varphi(x)}\, dx$ is log-concave, then we have

$$\mathrm{Ent}_\mu(f^2) \leq \left(2 + \frac{1}{8e^{-2\varphi(0)}}(\mathrm{diam}\, I)^2\right)\int_I |f'|^2\, d\mu. \tag{3.9}$$

Proof. First we remark that f^2 is a nonnegative, locally Lipschitz and even quasi-convex function. Indeed, since $|f|^2$ is decreasing on $I\cap(-\infty, 0]$ and increasing on $I\cap[0,\infty)$ by the monotonicity of f, $|f|^2$ is quasi-convex. Moreover $|f|^2$ is locally Lipschitz and symmetric since f is locally Lipschitz and odd. Applying Proposition 3.2, in particular (3.7), to f^2, we obtain

$$\mathrm{Ent}_\mu(f^2) \leq \frac{4}{\kappa}\int_I |f'|^2\, d\mu + \frac{1}{4\kappa}(\mathrm{diam}\, I)^2\int_I f^2\, d\mu.$$

On the other hand, since f is odd and μ is symmetric from which we have $\int_I f\, d\mu = 0$, we can apply the Poincaré inequality to f to see that

$$C_\mu \int_I f^2\, d\mu \leq \int_I |f'|^2\, d\mu.$$

Therefore we enjoy

$$\mathrm{Ent}_\mu(f^2) \leq \frac{4}{\kappa}\int_I |f'|^2\, d\mu + \frac{1}{4\kappa C_\mu}(\mathrm{diam}\, I)^2\int_I |f'|^2\, d\mu,$$

which implies (3.8).

For the second assertion, we employ the result by Bobkov [4] where it is shown that every log-concave probability measure $\mu = e^{-\varphi}\, dx$ on $\mathbb{R}$ satisfies Cheeger's isoperimetric inequality with the constant $2e^{-\varphi(m)}$, where $m \in I$ is the median of μ. In our case, since μ is symmetric, we can take m as 0. Hence, Cheeger's inequality [15] (see also [12, Theorem 14.1.6]) implies that $C_\mu \geq e^{-2\varphi(0)}$. (3.9) follows from combining (3.8) with the bound of the Poincaré constant and $\kappa = 2$. The latter follows from every symmetric log-concave probability measure satisfying (1.9) with $\kappa = 2$ (see Appendix). ☐

3.3 Kahane–Khintchine inequalities and deviation inequalities

In this subsection, we consider deviation inequalities as described in Corollary 1.5. To see this, we give the following moment estimate for positive exponent which is a generalization of the comparison result of moments for log-concave probability measures, firstly discussed by Borell [10] (see also [32, 20, 31]).

Proposition 3.4. *Let μ and Ω be as in Theorem 2.5 and $p_0 > 1$. If a nonnegative function f on Ω satisfies*

$$f \in \bigcap_{1\leq p\leq p_0} \mathrm{QC}^p(\Omega, \mu),$$

it holds that

$$\|f\|_{L^q(\mu)} \leq \left(\frac{q}{p}\right)^{\frac{1}{\kappa}\|\frac{\Phi_f}{f}\|_{L^\infty(\{f>0\})}}\|f\|_{L^p(\mu)} \tag{3.10}$$

for any $1 \leq p \leq q < p_0$*, where*

$$\|\frac{\Phi_f}{f}\|_{L^\infty(\{f>0\})} \coloneqq \operatorname{ess\,sup}\{\frac{\Phi_f(x)}{f(x)} \mid x \in \Omega, f(x) > 0\}.$$

In particular, if f *is differentiable on* Ω *and satisfies*

$$f \in \bigcap_{p \geq 1} \mathrm{QC}^p(\Omega, \mu), \tag{3.11}$$

then we have

$$\|f\|_{L^q(\mu)} \leq \left(\frac{q}{p}\right)^{\frac{2}{\kappa}\|\langle \cdot, \nabla \log f(\cdot)\rangle\|_{L^\infty(\{f>0\})}} \|f\|_{L^p(\mu)} \tag{3.12}$$

for any $1 \leq p \leq q$*.*

Proof. Set

$$\Lambda(t) \coloneqq \frac{1}{t} \log \int_\Omega f^t \, d\mu,$$

then we see that

$$\begin{aligned}\Lambda'(t) =& -\frac{1}{t^2} \log \int_\Omega f^t \, d\mu + \frac{1}{t} \frac{\int_\Omega f^t \log f \, d\mu}{\int_\Omega f^t \, d\mu} \\ =& \frac{1}{t^2} \frac{1}{\int_\Omega f^t \, d\mu} \mathrm{Ent}_\mu(f^t).\end{aligned}$$

Since $f \in \mathrm{QC}^t(\Omega, \mu)$ for any $1 \leq t < p_0$, it follows from Theorem 2.5 that

$$\begin{aligned}\Lambda'(t) \leq& \frac{1}{t^2} \frac{1}{\kappa} \frac{1}{\int_\Omega f^t \, d\mu} \int_\Omega \Phi_{f^t} \, d\mu \\ =& \frac{1}{t} \frac{1}{\kappa} \frac{1}{\int_\Omega f^t \, d\mu} \int_{\{x \in \Omega | f(x) > 0\}} f^{t-1} \Phi_f \, d\mu \\ \leq& \frac{1}{\kappa t} \|\frac{\Phi_f}{f}\|_{L^\infty(\{f>0\})}.\end{aligned}$$

Here we used the fact that $\Phi_{f^t}(x) = 0$ for any $t > 0$ if $f(x) = 0$ at $x \in \Omega$, which follows from $f^t \in \mathrm{QC}(\Omega, \mu)$. Hence integrating the above inequality from p to q with $1 \leq p \leq q < p_0$ yields the desired assertion (3.10).

(3.12) also follows by the same proof above and by (2.2). □

For instance, when μ is log-concave, the gauge function $\|\cdot\|_K$ for $K \in \mathcal{K}_s^n(\mathbb{R}^n)$ satisfies (3.11), and we can check $\|\frac{\Phi_{\|\cdot\|_K}}{\|\cdot\|_K}\|_{L^\infty(\{\|\cdot\|_K>0\})} = 2$. Hence, (3.10) yields

$$\|\|\cdot\|_K\|_{L^q(\mu)} \leq \left(\frac{q}{p}\right)^2 \|\|\cdot\|_K\|_{L^p(\mu)}$$

for any $1 \leq p \leq q$ since all log-concave probability measures satisfy the dilation inequality with $\kappa = 1$. In particular, when μ is symmetric on $\mathbb{R}$, since we can take $\kappa = 2$ as we see in Appendix, we also obtain

$$\||\cdot|\|_{L^q(\mu)} \leq \frac{q}{p} \||\cdot|\|_{L^p(\mu)}$$

for any $1 \leq p \leq q$. It is known that the order of q/p above is optimal (for instance, see [20]). On the other hand, it is known that all log-concave probability measures on $\mathbb{R}^n$ satisfy

$$\|\|\cdot\|_K\|_{L^q(\mu)} \leq C\frac{q}{p} \|\|\cdot\|_K\|_{L^p(\mu)}$$

for any $1 \le p \le q$, where $C > 0$ is an absolute constant (see [32, 12, 20]). We remark that the similar inequality for general functions has already appeared in [35] (see also [6, 17]). More precisely, in [35], we need the Remez function to construct the moment comparison like (3.10). For $s \ge 1$, we define $u_f(s) \ge 1$ by the best constant $C \ge 1$ such that

$$\{x \in \Omega \mid f(x) \le \lambda\}_{1-1/s} \subset \{x \in \Omega \mid f(x) \le \lambda u_f(s)\}, \quad \forall \lambda > 0.$$

We call the function $u_f \colon [1,\infty) \to [1,\infty)$ the Remez function of f, and set

$$u_f'(1) := \limsup_{s \downarrow 1} \frac{u(s) - 1}{s - 1}.$$

Then it follows from [35, Corollary 5.7] that

$$\|f\|_{L^q(\mu)} \le \left(\frac{q}{p}\right)^{u_f'(1)} \|f\|_{L^p(\mu)}$$

for any nonnegative integrable function f with $u_f'(1) < +\infty$ and for any $1 \le p \le q$. We remark that $\|\frac{\Phi_f}{f}\|_{L^\infty(\{f>0\})} \le u_f'(1)$ holds when f is a continuous and even quasi-convex function. Indeed by the definition of u_f, we have

$$f(x) \le f\left(\frac{1-\varepsilon}{1+\varepsilon}x\right) u_f\left(\frac{1}{1-\varepsilon}\right), \quad \forall \varepsilon(0,1), \forall x \in \Omega.$$

Hence it holds that $\Phi_f(x) \le f(x) u_f'(1)$ for $x \in \Omega$, which yields $\|\frac{\Phi_f}{f}\|_{L^\infty(\{f>0\})} \le u_f'(1)$.

As a corollary of Proposition 3.4, we give a tail estimate of a measure. To see this, we introduce the Orlicz norm $\|\cdot\|_{\psi_\alpha}$ for $\alpha \ge 1$. Given any $\alpha \ge 1$ and Borel function $f \colon \Omega \to \mathbb{R}$, we set

$$\|f\|_{\psi_\alpha} := \inf\left\{ t > 0 \,\middle|\, \int_\Omega \exp\left(\left(\frac{|f(x)|}{t}\right)^\alpha\right) d\mu \le 2 \right\}.$$

It is known that the Orlicz norm $\|\cdot\|_{\psi_\alpha}$ is also given by L^p-norms for $p \ge \alpha$ (see [12, Lemma 2.4.2]).

Lemma 3.5. *Let $\alpha \ge 1$ and $f \colon \Omega \to \mathbb{R}$ be a Borel function. Then*

$$\|f\|_{\psi_\alpha} \simeq \sup_{p \ge \alpha} \frac{\|f\|_{L^p(\mu)}}{p^{1/\alpha}}.$$

Here $A \simeq B$ means that there exist some absolute constants $c, C > 0$ such that $cB \le A \le CB$.

By Proposition 3.4, we obtain the following estimate of some Orlicz norm and the deviation inequality.

Corollary 3.6. *Let μ and Ω be as in Theorem 2.5 and let f be a nonnegative function on Ω satisfying*

$$f \in \bigcap_{p \ge 1} \mathrm{QC}^p(\Omega, \mu).$$

We set

$$\alpha := \frac{\kappa}{\|\frac{\Phi_f}{f}\|_{L^\infty(\{f>0\})}}.$$

If $1 \le \alpha < +\infty$, then it holds that

$$\|f\|_{\psi_\alpha} \simeq \alpha^{-\frac{1}{\alpha}} \|f\|_{L^\alpha(\mu)}. \tag{3.13}$$

In addition, we have

$$\mu(\{x \in \Omega \mid f(x) \ge Ct\alpha^{-1/\alpha}\|f\|_{L^\alpha(\mu)}\}) \le 2\exp(-t^\alpha), \quad \forall t \ge 1, \tag{3.14}$$

where $C > 0$ is an absolute constant.

Proof. (3.13) is a direct consequence of Proposition 3.4 and Lemma 3.5. (3.14) also follows from (3.13) and Markov's inequality. □

Proof of Corollary 1.5 (1). Let f be a function given in Corollary 1.5 (1). Then for any $t \geq 1$, f^t is a positive, differentiable and even quasi-convex function on Ω with $f^t \in L^1(\mu)$. Hence by Theorem 1.2, we have

$$\mathrm{Ent}_\mu(f^t) \leq \frac{2t}{\kappa} \int_\Omega f^{t-1} \langle x, \nabla f(x) \rangle \, d\mu(x).$$

Applying this inequality instead of Theorem 2.5 in the proof of Proposition 3.4, we obtain

$$\|f\|_{L^q(\mu)} \leq \left(\frac{q}{p}\right)^{\frac{2}{\kappa}\|\langle x, \nabla \log f \rangle\|_{L^\infty}} \|f\|_{L^p(\mu)}$$

for any $1 \leq p \leq q$. Finally by the same argument as in the proof of Corollary 3.6, we conclude the desired assertion. □

Next, we consider the Kahane–Khintchine inequality for negative exponent via Proposition 2.4.

Proposition 3.7. *Let μ and Ω be as in Theorem 2.5. Let f be a positive, continuous and even quasi-convex function with*

$$0 < \beta := \frac{1}{\kappa \log 2} \|\frac{\Phi_f}{f}\|_{L^\infty} < +\infty.$$

Suppose that f also satisfies $f^p \in \mathrm{QC}(\Omega, \mu)$ and $f^{-p} \in L^1(\mu)$ for some $0 < p < 1/\beta$. Then it holds that

$$\mathrm{med}(f) \leq \left(\frac{e}{1-\beta p}\right)^\beta \|f\|_{L^{-p}(\mu)}.$$

Proof. We may suppose that $\mathrm{med}(f) > 0$, otherwise our assertion is obvious.

We firstly note that we have

$$-(1-\theta)\log(1-\theta) \geq \log 2 \min\{\theta, 1-\theta\}, \quad \forall \theta \in [0,1].$$

Hence, by (1.9) and (2.10), we have

$$\begin{aligned} &\kappa \log 2 \int_0^\infty t^{p-1} \min\{\mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\}), 1 - \mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\})\} \, dt \\ \leq &\int_\Omega f^{-p-1} \Phi_f \, d\mu. \end{aligned} \tag{3.15}$$

Since the definition of the Lévy mean implies that

$$\mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\}) \leq \mu(\{x \in \mathbb{R}^n \mid f(x) < \mathrm{med}(f)\}) < \frac{1}{2}, \quad \forall t \geq \frac{1}{\mathrm{med}(f)},$$

we enjoy

$$\begin{aligned} &\int_0^\infty t^{p-1} \min\{\mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\}), 1 - \mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\})\} \, dt \\ \geq &\int_{\frac{1}{\mathrm{med}(f)}}^\infty t^{p-1} \mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\}) \, dt \\ = &\int_0^\infty t^{p-1} \mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\}) \, dt - \int_0^{\frac{1}{\mathrm{med}(f)}} t^{p-1} \mu(\{x \in \mathbb{R}^n \mid f(x) < t^{-1}\}) \, dt \\ \geq &\frac{1}{p} \int_\Omega f^{-p} \, d\mu - \frac{1}{p} \left(\frac{1}{\mathrm{med}(f)}\right)^p. \end{aligned} \tag{3.16}$$

On the other hand, it holds that

$$\int_\Omega f^{-p-1}\Phi_f \, d\mu \leq \|\frac{\Phi_f}{f}\|_{L^\infty} \int_\Omega f^{-p} \, d\mu. \tag{3.17}$$

Since $f^{-p} \in L^1(\mu)$ and $\|\frac{\Phi_f}{f}\|_{L^\infty} < +\infty$, combining (3.15) with (3.16) and (3.17), if $\frac{\kappa \log 2}{p} > \|\frac{\Phi_f}{f}\|_{L^\infty}$, we obtain

$$\left(\frac{\kappa \log 2}{p} - \|\frac{\Phi_f}{f}\|_{L^\infty}\right) \int_\Omega f^{-p} \, d\mu \leq \frac{\kappa \log 2}{p} \left(\frac{1}{\mathrm{med}(f)}\right)^p.$$

Therefore it holds that

$$\mathrm{med}(f) \leq \left(1 - \frac{p}{\kappa \log 2}\|\frac{\Phi_f}{f}\|_{L^\infty}\right)^{-\frac{1}{p}} \|f\|_{L^{-p}(\mu)}.$$

Since direct calculations yield $(1 - t_0 p)^{-\frac{1}{p}+t_0} \leq e^{t_0}$ for $t_0 > 0$ and any $0 < p < t_0^{-1}$, we can obtain the desired assertion. □

Guédon [19] (see also [12, Theorem 2.4.9]) showed that every log-concave probability measure and norm $\|\cdot\|$ on $\mathbb{R}^n$ satisfy

$$\int_{\mathbb{R}^n} \|x\| \, d\mu \leq \frac{C}{1+q} \left(\int_{\mathbb{R}^n} \|x\|^q \, d\mu\right)^{1/q},$$

for any $-1 < q < 0$, where $C > 0$ is an absolute constant. Hence Proposition 3.7 is a generalization of Guédon's result in some sense. An extension of Guédon's result for general functions is also discussed in [9, 17]. We also remark that Guédon's result follows from the small ball estimate,

$$\mu\left(\left\{x \in \mathbb{R}^n \,\middle|\, \|x\| \leq t \int_{\mathbb{R}^n} \|x\| \, d\mu\right\}\right) \leq Ct, \quad \forall t \geq 1,$$

which is shown by Latała [24]. Similarly, we can show the deviation inequality around the origin.

Corollary 3.8. *Let μ and Ω be as in Theorem 2.5. Let f be a positive, continuous and even quasi-convex function with*

$$0 < \beta := \frac{1}{\kappa \log 2}\|\frac{\Phi_f}{f}\|_{L^\infty} < +\infty.$$

Suppose that f also satisfies $f^p \in \mathrm{QC}(\Omega, \mu)$ and $f^{-p} \in L^1(\mu)$ for any $0 < p < 1/\beta$. Then for any small enough $\varepsilon > 0$, it holds that

$$\mu(\{x \in \Omega \mid f(x) \leq t \, \mathrm{med}(f)\}) \leq \left(\frac{e}{\varepsilon\beta}\right)^{1-\varepsilon\beta} t^{\frac{1}{\beta}-\varepsilon}, \quad \forall t \in (0,1].$$

Proof. Let $p := \frac{1}{\beta} - \varepsilon \in (0, \frac{1}{\beta})$ for small enough $\varepsilon > 0$. It follows from Proposition 3.7 that

$$\int_\Omega f^{-p} \, d\mu \leq \left(\frac{e^\beta}{\mathrm{med}(f)(1-\beta p)^\beta}\right)^p.$$

Hence Markov's inequality implies that

$$\mu(\{x \in \Omega \mid f(x) \leq t \, \mathrm{med}(f)\}) \leq \left(\frac{e}{1-\beta p}\right)^{\beta p} t^p, \quad \forall t \in (0,1].$$

This implies the desired assertion by $p = \frac{1}{\beta} - \varepsilon$. □

Proof of Corollary 1.5 (2). Let f be a positive, differentiable and even quasi-convex function on some neighborhood of $\overline{\Omega}$ with $0 < \beta < +\infty$ and $f^{-1/\beta} \in L^1(\mu)$. Then for any $p > 0$, f^p is also a positive, differentiable and even quasi-convex function on some neighborhood of $\overline{\Omega}$, and thus $f^p \in \mathrm{QC}(\mu, \Omega)$ by Proposition 2.1 since Ω is bounded. Moreover, for any $0 < p < 1/\beta$, we have $f^{-p} \in L^1(\mu)$ by $f^{-1/\beta} \in L^1(\mu)$ and Hölder's inequality. Hence we see that f satisfies the assumptions in Corollary 3.8. Applying Corollary 3.8 to f, we can conclude the desired assertion. $\square$

3.4 μ-perimeter

Our goal in this subsection is to give the estimate of the μ-perimeter of $K \in \mathcal{K}_s^n(\Omega)$ described in Corollary 1.6.

Proof of Corollary 1.6. Without loss of generality, we may suppose that $\mu(\overline{K}) = \mu(K)$. We set for $\varepsilon > 0$,

$$f_\varepsilon(x) := \min\left\{1, \frac{1}{\varepsilon} d(x, K)\right\}, \quad x \in \mathbb{R}^n.$$

Then f_ε is a locally Lipschitz function on $\{x \in \mathbb{R}^n \mid f_\varepsilon(x) > 0\}$. Moreover, we can check that f_ε is a nonnegative and even quasi-convex function with $f_\varepsilon \in L^1(\mu)$. Hence it follows from Theorem 1.2 that

$$\mathrm{Ent}_\mu(f_\varepsilon) \le \frac{2}{\kappa} \int_\Omega |x||\nabla f_\varepsilon(x)| \, d\mu(x). \tag{3.18}$$

Since we see that $\nabla f_\varepsilon(x) = 0$ if $x \in K \cup (\mathbb{R}^n \setminus \overline{[K]_\varepsilon})$ and $|\nabla f_\varepsilon| \le \varepsilon^{-1}$ on $\mathbb{R}^n$, it holds that

$$\begin{aligned}
&\int_\Omega |x||\nabla f_\varepsilon(x)| \, d\mu(x) \\
\le &\frac{1}{\varepsilon} \int_\Omega |x| \mathbf{1}_{[K]_\varepsilon \setminus K} \, d\mu(x) \\
\le &\left(\frac{1}{\varepsilon} \int_\Omega |x|^{p'} \mathbf{1}_{[K]_\varepsilon \setminus K}(x) \, d\mu(x) \right)^{1/p'} \left(\frac{1}{\varepsilon} \int_\Omega \mathbf{1}_{[K]_\varepsilon \setminus K}(x) \, d\mu(x) \right)^{1/p} \\
= &\left(\frac{1}{\varepsilon} \int_\Omega |x|^{p'} \mathbf{1}_{[K]_\varepsilon \setminus K}(x) \, d\mu(x) \right)^{1/p'} \left(\frac{1}{\varepsilon} (\mu([K]_\varepsilon) - \mu(K)) \right)^{1/p},
\end{aligned}$$

where we used Hölder's inequality. Furthermore, since we have by $r(K)\mathrm{B}_2^n \subset K$,

$$[K]_\varepsilon = K + \varepsilon \mathrm{B}_2^n \subset \left(1 + \frac{\varepsilon}{r(K)}\right) K,$$

it holds that

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_\Omega |x|^{p'} \mathbf{1}_{[K]_\varepsilon \setminus K}(x) \, d\mu(x) \\
= &\frac{1}{\varepsilon} \left(\int_{[K]_\varepsilon} |x|^{p'} \, d\mu(x) - \int_K |x|^{p'} \, d\mu(x) \right) \\
\le &\frac{1}{\varepsilon} \left(\int_{(1+\frac{\varepsilon}{r(K)})K} |x|^{p'} \, d\mu(x) - \int_K |x|^{p'} \, d\mu(x) \right) \\
= &\int_K \frac{1}{\varepsilon} \left[\left(1 + \frac{\varepsilon}{r(K)}\right)^{p'+n} e^{-\varphi((1+\frac{\varepsilon}{r(K)})x)} - e^{-\varphi(x)} \right] |x|^{p'} \, dx.
\end{aligned}$$

Since K is bounded and φ is smooth, Fatou's lemma yields that

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_\Omega |x|^{p'} \mathbf{1}_{[K]_\varepsilon \setminus K}(x) \, d\mu(x) \le \frac{1}{r(K)} \int_K (p' + n - \langle x, \nabla \varphi(x) \rangle) |x|^{p'} e^{-\varphi(x)} \, dx.$$

Moreover, since we enjoy

$$\operatorname{div}(x|x|^{p'}e^{-\varphi(x)}) = (p' + n - \langle x, \nabla\varphi(x)\rangle)|x|^{p'}e^{-\varphi(x)},$$

the divergence theorem implies that

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_\Omega |x|^{p'}\mathbf{1}_{[K]_\varepsilon\setminus K}(x)\,d\mu(x) \le \frac{1}{r(K)}\int_{\partial K}\langle x, \eta(x)\rangle|x|^{p'}e^{-\varphi(x)}\,d\sigma_K(x).$$

Therefore, letting $\varepsilon \downarrow 0$ in (3.18), we obtain

$$\liminf_{\varepsilon\downarrow 0} \operatorname{Ent}_\mu(f_\varepsilon) \le \frac{2}{\kappa}\left(\frac{1}{r(K)}\int_{\partial K}\langle x, \eta(x)\rangle|x|^{p'}e^{-\varphi(x)}\,d\sigma_K(x)\right)^{1/p'}\mu^+(K)^{1/p}.$$

Since $\lim_{\varepsilon\downarrow 0} f_\varepsilon = \mathbf{1}_{\mathbb{R}^n\setminus\overline{K}}$ and $\mu(\overline{K}) = \mu(K)$, the lower semi-continuity of the relative entropy yields that

$$-(1-\mu(K))\log(1-\mu(K)) \le \frac{2}{\kappa}\left(\frac{1}{r(K)}\int_{\partial K}\langle x, \eta(x)\rangle|x|^{p'}e^{-\varphi(x)}\,d\sigma_K(x)\right)^{1/p'}\mu^+(K)^{1/p},$$

which implies the desired assertion. □

4 Revisit to the dilation inequality

4.1 Reconstruction

In Section 2, we investigated the functional form of the dilation inequality. In this section, conversely, we will confirm that the dilation inequality can be recovered from the functional inequality (2.11).

For $K \in \mathcal{K}_s^n(\mathbb{R}^n)$, we define a function $\mathcal{N}_K\colon \mathbb{R}^n \to [0,\infty)$ by

$$\mathcal{N}_K(x) := \begin{cases} \|x\|_K & \text{if } x \notin K \\ 1 & \text{if } x \in K. \end{cases}$$

Then we can easily check that $\mathcal{N}_K$ is a continuous, even and quasi-convex function on $\mathbb{R}^n$.

Theorem 4.1. *Let μ be a probability measure supported on a symmetric convex domain $\Omega \subset \mathbb{R}^n$ with $\int_\Omega |x|\,d\mu(x) < +\infty$. We suppose that* (2.11) *holds for any $f \in \mathrm{QC}^1(\Omega,\mu)$ with some $\kappa > 0$. Then μ satisfies the dilation inequality* (1.9) *for $\mathcal{K}_s^n(\Omega)$ with the constant κ.*

Proof. Let us fix $K \in \mathcal{K}_s^n(\Omega)$. We first remark that we may assume $\mu(\overline{K}) = \mu(K)$, otherwise we have $\mu^*(K) = +\infty$ since

$$\frac{\mu(K_\varepsilon) - \mu(K)}{\varepsilon} \ge \frac{\mu(\overline{K}) - \mu(K)}{\varepsilon}$$

and thus nothing to prove.

Let $\sigma \in (0,1)$ and set $\delta := 2\sigma/(1-\sigma)^2$. We define

$$\begin{aligned} f_\sigma(x) &:= \min\left\{\frac{1}{\delta}(\mathcal{N}_K(x)-1), 1-\sigma\right\} \\ &= \begin{cases} \frac{1}{\delta}(\mathcal{N}_K(x)-1) & \text{if } x \in K_\sigma \\ 1-\sigma & \text{if } x \notin K_\sigma \end{cases} \\ &= \begin{cases} 0 & \text{if } x \in K \\ \frac{1}{\delta}(\|x\|_K - 1) & \text{if } x \in K_\sigma \setminus K \\ 1-\sigma & \text{if } x \notin K_\sigma \end{cases} \end{aligned}$$

for $x \in \mathbb{R}^n$. Note that f_σ is a nonnegative, even, continuous and quasi-convex function. We can also check that $f_\sigma \in L^1(\mu)$ since $f_\sigma \leq 1-\sigma$ on $\mathbb{R}^n$. Moreover, we can obtain $f_\sigma \in \mathrm{QC}^1(\Omega,\mu)$. To see this, we shall justify (2.5) for f_σ. For any $\varepsilon \in (0,1)$ and $x \in \mathbb{R}^n$, it holds

$$f_\sigma\left(\frac{1-\varepsilon}{1+\varepsilon}x\right) = \begin{cases} 0 & \text{if } x \in K_\varepsilon \\ \frac{1}{\delta}(\frac{1-\varepsilon}{1+\varepsilon}\|x\|_K - 1) & \text{if } x \in (K_\sigma)_\varepsilon \setminus K_\varepsilon \\ 1-\sigma & \text{if } x \notin (K_\sigma)_\varepsilon, \end{cases}$$

where we used $\frac{1+\varepsilon}{1-\varepsilon}K = K_\varepsilon$. Hence for any $\varepsilon \in (0,\sigma)$, we have

$$\frac{1}{\varepsilon}\left(f_\sigma(x) - f_\sigma\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)\right) = \begin{cases} \frac{1}{\varepsilon}\frac{1}{\delta}(\|x\|_K - 1) & \text{if } x \in K_\varepsilon \setminus K \\ \frac{1}{\varepsilon}(\frac{1}{\delta}(\|x\|_K - 1) - \frac{1}{\delta}(\frac{1-\varepsilon}{1+\varepsilon}\|x\|_K - 1)) & \text{if } x \in K_\sigma \setminus K_\varepsilon \\ \frac{1}{\varepsilon}(1-\sigma - \frac{1}{\delta}(\frac{1-\varepsilon}{1+\varepsilon}\|x\|_K - 1)) & \text{if } x \in (K_\sigma)_\varepsilon \setminus K_\sigma \\ 0 & \text{otherwise.} \end{cases}$$

For any $x \in K_\varepsilon$, since $\frac{1-\varepsilon}{1+\varepsilon}\|x\|_K = \|x\|_{K_\varepsilon} \leq 1$, we have

$$\frac{1}{\varepsilon}\frac{1}{\delta}(\|x\|_K - 1) \leq \frac{1}{\varepsilon}\frac{1}{\delta}\left(\|x\|_K - \frac{1-\varepsilon}{1+\varepsilon}\|x\|_K\right) = \frac{2}{\delta(1+\varepsilon)}\|x\|_K \leq \frac{2}{\delta}\|x\|_K.$$

Next for $x \notin K_\sigma$, we have

$$1-\sigma \leq \frac{1}{\delta}(\|x\|_K - 1)$$

by $\|x\|_K \geq \frac{1+\sigma}{1-\sigma}$. Thus it holds that

$$\begin{aligned}\frac{1}{\varepsilon}\left(1-\sigma-\frac{1}{\delta}\left(\frac{1-\varepsilon}{1+\varepsilon}\|x\|_K - 1\right)\right) \leq & \frac{1}{\varepsilon}\left(\frac{1}{\delta}(\|x\|_K - 1) - \frac{1}{\delta}\left(\frac{1-\varepsilon}{1+\varepsilon}\|x\|_K - 1\right)\right) \\ = & \frac{2}{\delta(1+\varepsilon)}\|x\|_K \\ \leq & \frac{2}{\delta}\|x\|_K.\end{aligned}$$

In particular, for any $x \in \mathbb{R}^n$, we have

$$\frac{1}{\varepsilon}\left(\frac{1}{\delta}(\|x\|_K - 1) - \frac{1}{\delta}\left(\frac{1-\varepsilon}{1+\varepsilon}\|x\|_K - 1\right)\right) \leq \frac{2}{\delta}\|x\|_K.$$

Therefore summarizing our arguments above, we can conclude that

$$\frac{1}{\varepsilon}\left(f_\sigma(x) - f_\sigma\left(\frac{1-\varepsilon}{1+\varepsilon}x\right)\right) \leq \frac{2}{\delta}\|x\|_K$$

for any $x \in \mathbb{R}^n$ and $\varepsilon \in (0,\sigma)$, which ensures (2.5) for f_σ since we can take some constant $C > 0$ such that $\|\cdot\|_K \leq C|\cdot|$ and since we have $\int_\Omega |x|\,d\mu < +\infty$. Hence we could check $f_\sigma \in \mathrm{QC}^1(\Omega,\mu)$. Moreover, we can get

$$\Phi_{f_\sigma}(x) = \frac{2}{\delta}\|x\|_K \mathbf{1}_{\overline{K_\sigma}\setminus\overline{K}}(x), \quad x \in \mathbb{R}^n.$$

Hence we can apply (2.11) to f_σ to see

$$\mathrm{Ent}_\mu(f_\sigma) \leq \frac{1}{\kappa}\int_\Omega \Phi_{f_\sigma}\,d\mu.$$

In the right hand side, since it holds that

$$\|x\|_K \leq \frac{1+\sigma}{1-\sigma}, \quad x \in \overline{K_\sigma},$$

we obtain

$$\int_\Omega \Phi_{f_\sigma}\, d\mu \leq \frac{2}{\delta}\frac{1+\sigma}{1-\sigma}(\mu(\overline{K_\sigma}) - \mu(\overline{K})) \leq (1-\sigma^2)\frac{1}{\sigma}(\mu(K_{(1+\tau)\sigma}) - \mu(K))$$

for any small enough $\tau > 0$, where we used $\overline{K_\sigma} \subset K_{(1+\tau)\sigma}$ and $\mu(\overline{K}) = \mu(K)$ in the last inequality. Hence we have

$$\liminf_{\sigma \downarrow 0} \int_\Omega \Phi_{f_\sigma}\, d\mu \leq (1+\tau)\mu^*(K)$$

for any small enough $\tau > 0$, and thus

$$\liminf_{\sigma \downarrow 0} \int_\Omega \Phi_{f_\sigma}\, d\mu \leq \mu^*(K).$$

On the other hand, since it holds that $f_\sigma \to \mathbf{1}_{\mathbb{R}^n \setminus \overline{K}}$ as $\sigma \downarrow 0$, it follows from the lower semi-continuity of Ent_μ that

$$\liminf_{\sigma \downarrow 0} \mathrm{Ent}_\mu(f_\sigma) \geq \mathrm{Ent}_\mu(\mathbf{1}_{\mathbb{R}^n \setminus \overline{K}}) = -(1-\mu(\overline{K}))\log(1-\mu(\overline{K})) = -(1-\mu(K))\log(1-\mu(K))$$

by $\mu(\overline{K}) = \mu(K)$. Eventually, we have

$$-(1-\mu(K))\log(1-\mu(K)) \leq \frac{1}{\kappa}\mu^*(K),$$

which is the desired assertion. □

4.2 Applications

As a corollary of Theorem 4.1, we obtain the stability of the dilation inequality for bounded perturbations, which is described in Corollary 1.7. To show this corollary, we employ the following lemma.

Lemma 4.2 ([3, Lemma 5.1.7])**.** *For any nonnegative function $f \in L^1(\mathbb{R}^n, \mu)$ satisfying $\int_{\mathbb{R}^n} f\, d\mu > 0$, it holds*

$$\mathrm{Ent}_\mu(f) = \inf_{r \in (0,\infty)} \int_{\mathbb{R}^n} [\phi(f) - \phi(r) - \phi'(r)(f-r)]\, d\mu.$$

Here we set $\phi(r) \coloneqq r\log r$ for $r > 0$.

Proof of Corollary 1.7. We firstly note that since h is bounded from above and below by a positive constant, we have $\mathrm{QC}^1(\Omega,\mu) = \mathrm{QC}^1(\Omega,\nu)$. Moreover it follows from $h \leq b$ and Lemma 4.2 that $\mathrm{Ent}_\nu(f) \leq b\mathrm{Ent}_\mu(f)$ for any $f \in \mathrm{QC}^1(\Omega,\nu)$. Since μ satisfies the dilation inequality for $\mathcal{K}_s^n(\Omega)$ with $\kappa > 0$, we can apply Theorem 2.5 to see

$$\mathrm{Ent}_\nu(f) \leq \frac{b}{\kappa}\int_\Omega \Phi_f\, d\mu$$

for any $f \in \mathrm{QC}^1(\Omega,\nu)$. By $h \geq b^{-1}$, we conclude

$$\mathrm{Ent}_\nu(f) \leq \frac{b^2}{\kappa}\int_\Omega \Phi_f\, d\nu$$

for any $f \in \mathrm{QC}^1(\Omega,\nu)$.

Now we can check $\int_\Omega |x|\, d\nu < +\infty$ since $\int_\Omega |x|\, d\mu < +\infty$ and h is bounded from above. Hence we obtain the desired assertion by applying Theorem 4.1 to ν. □

As another consequence of Theorem 4.1, we can observe the tensorization property in the special case, which is described in Corollary 1.8. We remark that $\Omega_1 \times \Omega_2$ is also a symmetric convex domain in $\mathbb{R}^{n+1}$.

Proof of Corollary 1.8. Let $\mathrm{QC}^{1,*}(\Omega_1\times\Omega_2, \mu_1\otimes\mu_2)$ be the set of all functions $f \in \mathrm{QC}^1(\Omega_1\times\Omega_2, \mu_1 \otimes \mu_2)$ such that f is bounded on $\Omega_1 \times \Omega_2$ and satisfies

$$f(x, y) = f(-x, y) = f(x, -y) = f(-x, -y), \quad \forall (x, y) \in \Omega_1 \times \Omega_2. \tag{4.1}$$

Now, fix $f \in \mathrm{QC}^{1,*}(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$, and set

$$g(x) := \int_{\Omega_2} f(x, y)\, d\mu_2(y).$$

Then we see that

$$\begin{aligned}&\mathrm{Ent}_{\mu_1\otimes\mu_2}(f)\\ =&\mathrm{Ent}_{\mu_1}(g)\\ &+\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)\log f(x,y)\,d\mu_2(y) - \int_{\Omega_2} f(x,y)\,d\mu_2(y)\log\int_{\Omega_2} f(x,y)\,d\mu_2(y)\right) d\mu_1(x).\end{aligned}$$

If we have $g \in \mathrm{QC}^1(\Omega_1, \mu_1)$ and $f(x, \cdot) \in \mathrm{QC}^1(\Omega_2, \mu_2)$ for μ_1-a.e., $x \in \Omega_1$, then we can apply Theorem 2.5 to see that

$$\mathrm{Ent}_{\mu_1\otimes\mu_2}(f) \le \frac{1}{\kappa_1}\int_{\Omega_1} \Phi_g(x)\, d\mu_1(x) + \frac{1}{\kappa_2}\int_{\Omega_1}\int_{\Omega_2} \Phi_{f(x,\cdot)}(y)\, d\mu_2(y)d\mu_1(x). \tag{4.2}$$

By the definition, we have

$$\Phi_g(x) = \limsup_{\varepsilon\downarrow 0}\int_{\Omega_2} \frac{1}{\varepsilon}\left(f(x, y) - f\left(\frac{1-\varepsilon}{1+\varepsilon}x, y\right)\right) d\mu_2(y).$$

Now we remark that $f(\frac{1-\varepsilon}{1+\varepsilon}x, y) \ge f(\frac{1-\varepsilon}{1+\varepsilon}x, \frac{1-\varepsilon}{1+\varepsilon}y)$ for any $(x, y) \in \Omega_1 \times \Omega_2$. Indeed, since f is even quasi-convex function on $\Omega_1 \times \Omega_2$ and satisfies (4.1), $f(z, \cdot)$ is even quasi-convex on Ω_2 for each $z \in \Omega_1$. In particular, $f(z, ty)$ is monotone increasing in $t \ge 0$ for any $y \in \Omega_2$, and hence we conclude $f(\frac{1-\varepsilon}{1+\varepsilon}x, y) \ge f(\frac{1-\varepsilon}{1+\varepsilon}x, \frac{1-\varepsilon}{1+\varepsilon}y)$. From this and $f \in \mathrm{QC}^1(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$, we obtain

$$\frac{1}{\varepsilon}\left(f(x, y) - f\left(\frac{1-\varepsilon}{1+\varepsilon}x, y\right)\right) \le \frac{1}{\varepsilon}\left(f(x, y) - f\left(\frac{1-\varepsilon}{1+\varepsilon}x, \frac{1-\varepsilon}{1+\varepsilon}y\right)\right) \le h(x, y)$$

for all small enough $\varepsilon > 0$ and $(x, y) \in \Omega_1 \times \Omega_2$, where h is a Borel function in $L^1(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$. In particular, $h(x, \cdot) \in L^1(\Omega_2, \mu_2)$ holds for μ_1-a.e., $x \in \Omega_1$. Hence it follows from Fatou's lemma that

$$\Phi_g(x) \le \int_{\Omega_2} \Phi_f(x, y)\, d\mu_2(y)$$

for μ_1-a.e., $x \in \Omega_1$. Similarly, we can observe that $\Phi_{f(x,\cdot)}(y) \le \Phi_f(x, y)$. Therefore we can conclude

$$\mathrm{Ent}_{\mu_1\otimes\mu_2}(f) \le \left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)\int_{\Omega_1\times\Omega_2} \Phi_f(x, y)\, d\mu_1 \otimes \mu_2(x, y) \tag{4.3}$$

for any $f \in \mathrm{QC}^{1,*}(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$.

Now let $K \subset \mathbb{R} \times \mathbb{R}^n$ be a symmetric open convex set such that if $(x, y) \in \mathbb{R} \times \mathbb{R}^n$ belongs to K, then $(-x, y), (x, -y), (-x, -y)$ also belong to K. Let us consider the function f_σ given in the proof of Theorem 4.1 for $\sigma \in (0, 1)$ and K. Then by the property of K and the definition of f_σ, we see that $f_\sigma \in \mathrm{QC}^{1,*}(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$. We also remark that it holds that $\int_{\mathbb{R}\times\mathbb{R}^n} |(x,y)| \, d\mu_1 \otimes \mu_2(x,y) < +\infty$ by assumptions. Hence, by iterating the same arguments for f_σ as in the proof of Theorem 4.1 via (4.3), we can obtain

$$\mu^*(K) \geq -\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)^{-1} (1 - \mu(K)) \log(1 - \mu(K)),$$

which is the desired assertion.

Finally, to justify the above argument, we prove $g \in \mathrm{QC}^1(\Omega_1, \mu_1)$ and $f(x, \cdot) \in \mathrm{QC}^1(\Omega_2, \mu_2)$ for μ_1-a.e., $x \in \Omega_1$. Since f is bounded and satisfies (4.1), we can easily check that $f(x, \cdot)$ and g are nonnegative, continuous and even functions, and $f(x, \cdot)$ is quasi-convex. In addition, since f is integrable, we see that $g \in L^1(\Omega_1, \mu_1)$ and $f(x, \cdot) \in L^1(\Omega_2, \mu_2)$ for μ_1-a.e., $x \in \Omega_1$. Since we have already shown that g and $f(x, \cdot)$ satisfy the condition (2.5) by the above argument, we obtain $f(x, \cdot) \in \mathrm{QC}^1(\Omega_2, \mu_2)$ for μ_1-a.e., $x \in \Omega_1$. Therefore it suffices to show that g is quasi-convex from which we can conclude $g \in \mathrm{QC}^1(\Omega_1, \mu_1)$.

Let $\lambda > 0$ and consider $A(\lambda) := \{x \in \mathbb{R} \mid g(x) < \lambda\}$. Without loss of generality, we may assume that $A(\lambda) \neq \emptyset$. We remark that $A(\lambda)$ is symmetric since g is even. Now take $z \in A(\lambda)$ with $z \geq 0$. Since f is quasi-convex, $f(\cdot, y)$ is also quasi-convex for any $y \in \Omega_2$. In particular, since $f(\cdot, y)$ is even by (4.1), we see that $f(tz, y)$ is monotone increasing in $t \geq 0$. Thus for any $z' \in [0, z]$ and $y \in \Omega_2$, we have $f(z', y) \leq f(z, y)$ which implies that $g(z') \leq g(z) < \lambda$. Since g is even, we obtain $[-z, z] \subset A(\lambda)$ from which $A(\lambda)$ should be an interval. Hence g is quasi-convex, and our proof is complete. □

Remark 4.3. It is natural to expect the tensorization property for high dimensional spaces. More precisely, if μ_1 and μ_2 are probability measures on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ satisfying dilation inequalities for $\mathcal{K}_s^{n_1}(\mathbb{R}^{n_1})$ and $\mathcal{K}_s^{n_2}(\mathbb{R}^{n_2})$, respectively, then does $\mu_1 \otimes \mu_2$ also satisfy the dilation inequality for $\mathcal{K}_s^{n_1+n_2}(\mathbb{R}^{n_1+n_2})$? Corollary 1.8 gives a partial answer affirmatively when either n_1 or n_2 is 1, but it is open when $n_1, n_2 \geq 2$. In our argument, this difficulty comes from quasi-convexity. In fact, let f_1 and f_2 be nonnegative even quasi-convex functions on $\mathbb{R}^n$. Then in our proof of Corollary 1.8, we used the fact that $f_1 + f_2$ is also an even quasi-convex function when $n = 1$. However, when $n \geq 2$, the same phenomenon fails. For instance, let us consider functions $f_1(x_1, x_2) = |x_1|^{2/3}$ and $f_2(x_1, x_2) = |x_2|^{2/3}$ for $(x_1, x_2) \in \mathbb{R}^2$. Then we can check that both functions are even and quasi-convex, but $f_1 + f_2$ is not quasi-convex on $\mathbb{R}^2$ (the curve $\{(x_1, x_2) \in \mathbb{R}^2 \mid f_1(x_1, x_2) + f_2(x_1, x_2) = 1\}$ is the astroid).

In our proof of Corollary 1.8, we also showed the tensorization property of the functional dilation inequality (2.11). If we focus on this tensorization, we can improve the functional version of Corollary 1.8 in the following special case.

Corollary 4.4. *Let μ_1, μ_2 be probability measures supported on symmetric convex domains $\Omega_1 \subset \mathbb{R}$ and $\Omega_2 \subset \mathbb{R}^n$, respectively. We suppose that μ_1, μ_2 satisfy* (2.11) *with some $\kappa_1, \kappa_2 > 0$, respectively. Then $\mu_1 \otimes \mu_2$ satisfies* (2.11) *with the constant $\kappa = \min\{\kappa_1, \kappa_2\}$ for any bounded function $f \in \mathrm{QC}^1(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2) \cap C^1(\Omega_1 \times \Omega_2)$ satisfying* (4.1).

Proof. Almost all arguments are the same as in the proof of Corollary 1.8, but we remark that since $f \in C^1(\Omega_1 \times \Omega_2)$, we have

$$\Phi_g(x) = 2 \int_{\Omega_2} \langle x, \nabla_x f(x,y) \rangle \, d\mu_2(y), \quad x \in \Omega_1$$

and

$$\Phi_{f(x,\cdot)}(y) = 2 \langle y, \nabla_y f(x,y) \rangle, \quad (x,y) \in \Omega_1 \times \Omega_2.$$

Hence since we also see that

$$\Phi_f(x,y) = 2\langle (x,y), \nabla f(x,y)\rangle, \quad (x,y) \in \Omega_1 \times \Omega_2,$$

it follows from (4.2) that

$$\begin{aligned}
&\mathrm{Ent}_{\mu_1\otimes\mu_2}(f)\\
\leq &\max\left\{\frac{1}{\kappa_1}, \frac{1}{\kappa_2}\right\}\\
&\times\left(2\int_{\Omega_1}\int_{\Omega_2}\langle x, \nabla_x f(x,y)\rangle\, d\mu_2(y)\, d\mu_1(x) + 2\int_{\Omega_1}\int_{\Omega_2}\langle y, \nabla_y f(x,y)\rangle\, d\mu_2(y) d\mu_1(x)\right)\\
= &\frac{2}{\min\{\kappa_1,\kappa_2\}}\int_{\Omega_1\times\Omega_2}\langle (x,y), \nabla f(x,y)\rangle\, d\mu_1\otimes\mu_2(x,y)\\
= &\frac{1}{\min\{\kappa_1,\kappa_2\}}\int_{\Omega_1\times\Omega_2}\Phi_f(x,y)\, d\mu_1\otimes\mu_2(x,y),
\end{aligned}$$

which is the desired assertion. □

A Appendix

Here we will investigate the dilation inequality for symmetric log-concave probability measures on $\mathbb{R}$ and the standard Gaussian measure γ_n on $\mathbb{R}^n$.

Proposition A.1. *Every symmetric log-concave probability measure on $\mathbb{R}$ satisfies the dilation inequality for $\mathcal{K}_s^1(\mathbb{R})$ with $\kappa = 2$.*

Proof. Let $\mu = e^{-\varphi(x)}\, dx$ be a symmetric probability measure on $\mathbb{R}$. Then we see that $\mu^*((-t,t)) = 4e^{-\varphi(t)}t$ for any $t > 0$ (see [35]). We set $d\nu := 2e^{-\varphi(x)}\mathbf{1}_{(0,\infty)}\, dx$, then ν is a log-concave probability measure since μ is symmetric. Moreover, we get $\nu^*((0,t)) = 2e^{-\varphi(t)}t$ for any $t > 0$. Hence, we obtain $\mu^*((-t,t)) = 2\nu^*((0,t))$. On the other hand, since every log-concave probability measure satisfies the dilation inequality (1.4), we can conclude

$$\mu^*((-t,t)) \geq -2(1-\nu((0,t)))\log(1-\nu((0,t))).$$

Finally, since we have $\nu((0,t)) = \mu((-t,t))$ by symmetry of μ, we obtain the desired assertion. □

Proposition A.2. *The standard Gaussian measure γ_n satisfies the dilation inequality for $\mathcal{K}_s^n(\mathbb{R}^n)$ with $\kappa = 2$.*

Proof. We employ the result by Latała–Oleszkiewicz [25] where they showed that for any $K \in \mathcal{K}_s^n(\mathbb{R}^n)$, it holds that

$$\gamma_n(tK) \geq \gamma_n(tP), \quad \forall t \geq 1,$$

where $P \subset \mathbb{R}^n$ is a symmetric strip such that $\gamma_n(P) = \gamma_n(K)$. In particular, we can take $\theta_K > 0$ satisfying $\gamma_n(P) = \gamma_n(\mathbb{R}^{n-1}\times(-\theta_K,\theta_K))$, $\gamma_n(tK) \geq \gamma_1((-t\theta_K, t\theta_K))$ for any $t \geq 1$ and $\gamma_n(K) = \gamma_1((-\theta_K,\theta_K))$. Hence, by the definition of the dilation area, we can conclude $\gamma_n^*(K) \geq \gamma_1^*((-\theta_K,\theta_K))$. Finally, since γ_1 is log-concave, Proposition A.1 implies the desired assertion. □

We also remark that $\kappa = 2$ is optimal in Proposition A.2. Indeed, we can check that for $t > 0$,

$$\gamma_1^*((-t,t)) = \frac{4}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}t$$

and

$$-(1-\gamma_1((-t,t)))\log(1-\gamma_1((-t,t))) = \frac{2}{\sqrt{2\pi}}t + o(t)$$

as $t \to +0$. Hence if γ_1 satisfies (1.9) for $\mathcal{K}_s^1(\mathbb{R})$ with $\kappa > 0$, then κ should satisfy $\kappa \leq 2$.

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