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Self-similar solution for fractional Laplacian in cones*

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Abstract

We construct a self-similar solution of the heat equation for the fractional Laplacian with Dirichlet boundary conditions in every fat cone. Furthermore, we give the entrance law from the vertex and the Yaglom limit for the corresponding killed isotropic stable Lévy process and precise large-time asymptotics for solutions of the Cauchy problem in the cone.

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1 Introduction

Let $d \in \mathbb{N} := \{1, 2, \ldots\}$. Consider an arbitrary nonempty open cone $\Gamma \subseteq \mathbb{R}^d$. Thus, $ry \in \Gamma$ whenever $y \in \Gamma$ and $r > 0$. Let $p_t^\Gamma(x, y)$, $t > 0$, $x, y \in \Gamma$, be the Dirichlet heat kernel of the cone for the fractional Laplacian. More specifically, for $\alpha \in (0, 2)$, p^Γ is the transition density of the isotropic α-stable Lévy process in $\mathbb{R}^d$ killed upon leaving Γ; see, e.g., Bogdan and Grzywny [9]. Let $M_\Gamma \colon \mathbb{R}^d \to [0, \infty)$ be the Martin kernel of Γ with the pole at infinity (for definitions, see Section 2). The function is homogeneous (or self-similar) of some degree $\beta \in [0, \alpha)$. Our first result captures the asymptotics of p_t^Γ at the vertex 0 of Γ, as follows.

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Theorem 1.1. *If the cone Γ is fat, then for $s,t>0$, $x\in\Gamma$, we have*

$$\Psi_t(x) := \lim_{\Gamma\ni y\to 0} p_t^\Gamma(x,y)/M_\Gamma(y) \in (0,\infty),$$

$$\Psi_t(x) = t^{-(d+\beta)/\alpha}\Psi_1\big(t^{-1/\alpha}x\big), \tag{1.1}$$

and

$$\int_\Gamma \Psi_s(y) p_t^\Gamma(x,y)\, dy = \Psi_{s+t}(x). \tag{1.2}$$

The proof of Theorem 1.1 is given in Section 3. In view of (1.1) and (1.2), $\Psi_t(x)$ is a self-similar semigroup solution of the heat equation for the fractional Laplacian with Dirichlet conditions. By (1.2), Ψ_t is also an entrance law for p^Γ at the origin, see, e.g., Blumenthal [7, p. 104], Haas and Rivero [23] or Bañuelos et al. [2].

The result is the next step in the program for the boundary potential theory sketched in the Introduction of Bogdan at al. [10], building on the boundary Harnack principle, Green function, and Dirichlet heat kernel estimates. In Theorems 3.12 and 3.13 below, we make further applications to Probability by obtaining the so-called Yaglom limit for Γ. We also give applications to Functional Analysis and Partial Differential Equations, as follows.

Let $1\leqslant q\leqslant\infty$ and $L^q(\Gamma) := L^q(\Gamma, \mathrm{d}x)$. For a weight function $w>0$, we denote $L^q(w) := L^q(\Gamma, w(x)\,\mathrm{d}x)$. For instance, $L^1(M_\Gamma) = \{f/M_\Gamma\colon f\in L^1\}$. Then, for $1\leqslant q<\infty$, we define

$$\|f\|_{q,M_\Gamma} := \|f/M_\Gamma\|_{L^q(M_\Gamma^2)} = \left(\int_\Gamma |f(x)|^q M_\Gamma^{2-q}(x)\,\mathrm{d}x\right)^{\frac{1}{q}} = \|f\|_{L^q(M_\Gamma^{2-q})},$$

and, for $q=\infty$, we let

$$\|f\|_{\infty,M_\Gamma} := \operatorname*{ess\,sup}_{x\in\Gamma} |f(x)|/M_\Gamma(x).$$

Of course, $\|f\|_{1,M_\Gamma} = \|f\|_{L^1(M_\Gamma)}$. For a nonnegative or integrable function f we let

$$P_t^\Gamma f(x) := \int_\Gamma p_t^\Gamma(x,y) f(y)\,\mathrm{d}y, \quad t>0,\ x\in\Gamma.$$

We say that the cone Γ is *smooth* if its boundary is $C^{1,1}$ outside of origin, to wit, there is $r>0$ such that at every boundary point of Γ on the unit sphere $\mathbb{S}^{d-1}$, there exist inner and outer tangent balls for Γ, with radii r. Put differently, for $d\geqslant 2$, the *spherical cap*, $\Gamma\cap\mathbb{S}^{d-1}$, is a $C^{1,1}$ subset of $\mathbb{S}^{d-1}$. For instance, the right-circular cones (see Section 2) are smooth.

The second result describes the large-time asymptotic behavior of the semigroup P_t^Γ.

Theorem 1.2. *Let $q\in[1,\infty)$. Assume that the cone Γ is smooth with $\beta\geqslant\alpha/2$. Then for every $f\in L^1(M_\Gamma)$ and $A=\int_\Gamma f(x)M_\Gamma(x)\,\mathrm{d}x$ we have*

$$\lim_{t\to\infty} t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f - A\Psi_t\|_{q,M_\Gamma} = 0. \tag{1.3}$$

As stated in Lemma 4.9, the condition $\beta\geqslant\alpha/2$ is sharp for smooth cones and $d\geqslant 2$; see also Example 4.12. Theorem 1.2 follows from the more general Theorem 4.6, by means of Corollary 4.10.

Let us comment on previous developments in the literature and our methods. If $\Gamma=\mathbb{R}^d$, then $\beta=0$, $M_\Gamma=1$ and $p_t^\Gamma(x,y) = p_t(x,y)$ is the transition density of the fractional Laplacian on $\mathbb{R}^d$ (see below). In this case, $\Psi_t(x) = p_t(0,y)$ and Theorem 1.2 was resolved by Vázquez [35], see also Bogdan et al. [13] with $\kappa=0$ in [13, Eq. (1.1)]; see Example 4.4 below, too.

For general cones Γ, the behavior of p_t^Γ is intrinsically connected to properties of M_Γ, see, e.g., [9], [15], or Kyprianou et al. [27]. The identification of the Martin kernel M_Γ was accomplished by Bañuelos and Bogdan [3]. Its crucial property is the homogeneity of order $\beta \in [0, \alpha)$, which is also reflected in the behavior of the Green function studied by Kulczycki [24] and Michalik [28, Lemma 3.3], at least when Γ is a right-circular cone. As we see in Theorems 1.1 and 1.2, the exponent β determines the self-similarity of the semigroup solution and the asymptotic behavior of the semigroup P_t^Γ, too. For more information on β we refer to [3] and Bogdan et al. [17].

If Γ is a Lipschitz cone, then Theorem 1.1 follows from [15, Corollary 3.2 and Theorem 3.3]. However, the method presented in [15] does not apply to general fat cones, in particular, to $\Gamma = \mathbb{R} \setminus \{0\}$ or $\Gamma = \mathbb{R}^2 \setminus ([0, \infty) \times \{0\})$, which are of interest for $\alpha \in (1, 2)$.

Instead, in this work we develop an approach suggested by [13], where the authors employ a stationary density of an Ornstein-Uhlenbeck type semigroup corresponding to a homogeneous (self-similar) heat kernel in a different setting (on the whole of $\mathbb{R}^d$). Another key tool in their analysis is the so-called Doob conditioning using an invariant function for the heat kernel. Due to Theorem 3.1 below, the Martin kernel M_Γ is invariant with respect to P_t^Γ, which indeed allows for Doob conditioning. Then we form the corresponding Ornstein-Uhlenbeck semigroup and prove existence of a stationary density φ in Theorem 3.4 by using the Schauder-Tychonoff fixed-point theorem. As we shall see in the proof of Theorem 1.1, the self-similar semigroup solution Ψ_t is directly expressed by φ and M_Γ.

The remaining developments in our paper are as follows. In Subsection 3.3 we obtain an asymptotic relation between the Martin kernel and the survival probability near the vertex of the cone (see Corollary 3.11). We also obtain a Yaglom limit (quasi-stationary distribution) in Theorem 3.12, which describes the behavior of the stable process starting from a fixed point $x \in \Gamma$ and conditioned to stay in a cone, generalizing Theorem 1.1 of [15]. In Theorem 3.13 we considerably extend both results by allowing arbitrary initial distributions with finite moment of order α.

We note that the Yaglom limit for random walks in cones is discussed by Denisov and Wachtel [26], but generally there are very few results in literature on unbounded sets. For a broad survey on quasi-stationary distributions, we refer to van Doorn and Pollet [34]; see also Champagnat and Villemonais [19]. Self-similar solutions for general homogeneous semigroups are discussed in Cholewa and Rodriguez-Bernal [21]. Patie and Savov [29] discuss generalized Ornstein-Uhlenbeck semigroups, which they call generalized Laguerre semigroups. Results related to Theorem 1.2, but for fractal Burgers equation and fractional p-Laplacian can be found in Biler et al. [5, Theorem 2.2] and Vázquez [36, Theorem 1.2], respectively. For an approach to entrance laws based on fluctuation theory of Markov additive processes, we refer to [26], see also Chaumont et al. [20].

This brief review shows that the asymptotics of the heat kernel is in a busy intersection between Probability, Partial Differential Equations and Functional Analysis even though the fields do not always communicate. Our approach should apply to rather general self-similar operator semigroup kernels, at least when they enjoy an invariant function and suitable upper and lower bounds.

Here is a simple example to illustrate our findings.

Example 1.3. If $d = 1$, $\alpha \in (0, 2)$, and $\Gamma = (0, \infty)$, then $\beta = \alpha/2$, $M_\Gamma(x) = (0 \vee x)^{\alpha/2}$ for $x \in \mathbb{R}$, so, by Theorem 4.6, $\lim_{t\to\infty} t^{(1+\alpha)/(2\alpha)} \|P_t^\Gamma f\|_2 = 0$ if $\int_0^\infty f(x) x^{\alpha/2}\,\mathrm{d}x = 0$; see Example 4.11. Furthermore, by Remark (3.10) and (4.10), the survival probability is $\mathbb{P}_x(\tau_\Gamma > t) \approx (1 \wedge t^{-1/\alpha} x)^{\alpha/2}$ and the heat kernel satisfies $p_1^\Gamma(x, y) \approx (1 + |x - y|)^{-1-\alpha}(1 \wedge x)^{\alpha/2}(1 \wedge y)^{\alpha/2}$, so $\Psi_t(x) \approx (t^{1/\alpha} \vee x)^{-1-\alpha}(t^{1/\alpha} \wedge x)^{\alpha/2}$. Here and below $x, y, t > 0$. By Lemma 3.6, the stationary density of the corresponding Ornstein-Uhlenbeck semigroup is $\varphi(x) \approx (1+x)^{-1-3\alpha/2}$ and the Yaglom limit has density $\varphi(x)x^{\alpha/2}/\int_0^\infty \varphi(y)y^{\alpha/2}\,\mathrm{d}y \approx x^{\alpha/2}(1+x)^{-1-3\alpha/2} \approx \Psi_1(x)$; see Theorem 3.12. We refer to [23, Example 5] for an exact but less explicit expression for the Yaglom limit by means of exponential functionals. See also Example 4.11 for the case of $\Gamma = \mathbb{R}\setminus\{0\}$ and Example 4.12 for d-dimensional extensions.

2 Preliminaries

For $x, z \in \mathbb{R}^d$, the standard scalar product of is denoted by $x \cdot z$ and $|z|$ is the Euclidean norm. For $x \in \mathbb{R}^d$ and $r \in (0,\infty)$, we let $B(x,r) = \{y \in \mathbb{R}^d \colon |x-y| < r\}$, the ball centered at x with radius r, and write $B_r := B(0,r)$. All the considered sets, functions and measures are Borel. For nonnegative functions f, g, we write $f \approx g$ if there is a number $c \in (0,\infty)$, i.e., a *constant*, such that $c^{-1}f \leqslant g \leqslant cf$, and write $f \lesssim g$ if there is a constant c such that $f \leqslant cg$. As usual, for two real numbers $a, b \in \mathbb{R}$, we write $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$.

Recall that $\alpha \in (0,2)$ and let

$$\nu(z) = c_{d,\alpha}|z|^{-d-\alpha}, \quad z \in \mathbb{R}^d,$$

where the constant $c_{d,\alpha}$ is such that

$$\int_{\mathbb{R}^d} \big(1-\cos(\xi\cdot z)\big)\nu(z)\,\mathrm{d}z = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d.$$

For $t > 0$ we let

$$p_t(x) := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} e^{-i\xi\cdot x}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^d. \tag{2.1}$$

By the Lévy-Khintchine formula, p_t is a probability density function and

$$\int_{\mathbb{R}^d} e^{i\xi\cdot x}\,p_t(x)\,\mathrm{d}x = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d,\ t > 0.$$

We consider the isotropic α-stable Lévy process $\mathbf{X} = (X_t, t \geqslant 0)$ in $\mathbb{R}^d$, with

$$p_t(x,y) := p_t(y-x), \quad x, y \in \mathbb{R}^d,\ t > 0,$$

as transition density. Thus,

$$\mathbb{E}_x e^{i\xi\cdot X_t} = \int_{\mathbb{R}^d} e^{i\xi\cdot y}\,p_t(x,y)\,\mathrm{d}y = e^{i\xi\cdot x - t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d,\, x \in \mathbb{R}^d,\ t > 0.$$

The Lévy-Khintchine exponent of $\mathbf{X}$ is, of course, $|\xi|^\alpha$ and ν is the intensity of jumps. By (2.1),

$$p_t(x,y) = t^{-d/\alpha}p_1\big(t^{-1/\alpha}x, t^{-1/\alpha}y\big), \quad x, y \in \mathbb{R}^d,\ t > 0, \tag{2.2}$$

and

$$p_t\,(Tx, Ty) = p_t(x,y), \quad x, y \in \mathbb{R}^d,\ t > 0, \tag{2.3}$$

for every isometry T on $\mathbb{R}^d$. It is well known that

$$p_t(x,y) \approx t^{-d/\alpha} \wedge t|y-x|^{-d-\alpha}, \quad x, y \in \mathbb{R}^d,\ t > 0, \tag{2.4}$$

see, e.g., [6]. We then consider the time of the first exit of $\mathbf{X}$ from an open set $D \subseteq \mathbb{R}^d$,

$$\tau_D := \inf\{t \geqslant 0 \colon X_t \notin D\},$$

and we define the Dirichlet heat kernel for D,

$$p_t^D(x,y) := p_t(x,y) - \mathbb{E}_x\left[\tau_D < t; p_{t-\tau_D}\left(X_{\tau_D}, y\right)\right], \quad x, y \in D,\ t > 0,$$

see [9, 14, 22]. It immediately follows that $p_t^D(x,y) \leqslant p_t(x,y)$ for all $x, y \in D$ and $t > 0$. The Dirichlet heat kernel is nonnegative, and symmetric: $p_t^D(x,y) = p_t^D(y,x)$ for $x, y \in D$, $t > 0$. It satisfies the Chapman-Kolmogorov equations:

$$p_{t+s}^D(x,y) = \int_D p_t^D(x,z) p_s^D(z,y)\,\mathrm{d}z, \quad x, y \in D,\ s, t > 0. \tag{2.5}$$

For nonnegative or integrable functions f we define the *killed semigroup* by

$$P_t^D f(x) := \mathbb{E}_x\left[\tau_D > t; f(X_t)\right] = \int_D p_t^D(x,y) f(y)\,\mathrm{d}y, \quad x \in D,\ t > 0.$$

In particular, for $f \equiv 1$ we obtain the *survival probability*:

$$\mathbb{P}_x(\tau_D > t) = \int_D p_t^D(x,y)\,\mathrm{d}y, \quad x \in D,\, t > 0, \tag{2.6}$$

see [11, Remark 1.9].

From this moment on, we concentrate on $D = \Gamma$. Since $t^{-1/\alpha}\Gamma = \Gamma$, the scaling (2.2) extends to the Dirichlet heat kernel:

$$p_t^\Gamma(x,y) = t^{-d/\alpha} p_1^\Gamma\big(t^{-1/\alpha}x, t^{-1/\alpha}y\big), \quad x, y \in \Gamma,\ t > 0.$$

As a consequence,

$$\mathbb{P}_x(\tau_\Gamma > t) = \mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1), \quad x \in \Gamma,\ t > 0. \tag{2.7}$$

Furthermore, by (2.3),

$$p_t^{T\Gamma}(Tx, Ty) = p_t^\Gamma(x,y), \quad x, y \in \Gamma,\ t > 0. \tag{2.8}$$

The operators P_t^Γ and the kernel $p_t^\Gamma(x,y)$ are the main subject of the paper. In view of (2.8), without loss of generality we may assume that $\mathbf{1} := (0, \ldots, 0, 1) \in \Gamma$. By [3, Theorem 3.2], there is a unique nonnegative function M_Γ on $\mathbb{R}^d$ such that $M_\Gamma(\mathbf{1}) = 1$, $M_\Gamma = 0$ on Γ^c, and for every open bounded set $B \subseteq \Gamma$,

$$M_\Gamma(x) = \mathbb{E}_x M_\Gamma(X_{\tau_B}), \quad x \in \mathbb{R}^d. \tag{2.9}$$

Moreover, M_Γ is locally bounded on $\mathbb{R}^d$ and homogeneous of some order $\beta \in [0, \alpha)$, i.e.,

$$M_\Gamma(x) = |x|^\beta M_\Gamma(x/|x|), \quad x \in \Gamma. \tag{2.10}$$

We call M_Γ the Martin kernel of Γ with the pole at infinity.

Example 2.1. By [3], $\beta = \alpha/2$ if Γ is a half-space and $\beta = \alpha - 1$ if $\Gamma = \mathbb{R} \setminus \{0\}$ and $1 < \alpha < 2$. By [12], $\beta = (\alpha - 1)/2$ if $\Gamma = \mathbb{R}^2 \setminus ([0, \infty) \times \{0\})$ and $1 < \alpha < 2$.

Below we often assume that Γ is *fat*, more precisely, that $\kappa \in (0,1)$ exists such that for all $Q \in \overline{\Gamma}$ and $r \in (0, \infty)$, there is a point $A = A_r(Q) \in \Gamma \cap B(Q, r)$ with $B(A, \kappa r) \subseteq \Gamma \cap B(Q, r)$; compare, for example, Song and Wu [33, Definition 3.1] and [10, Definition 1]. Recall that Γ is *smooth* if $d = 1$ or $d \geqslant 2$ and $\Gamma \cap \mathbb{S}^{d-1}$ is a $C^{1,1}$ subset of $\mathbb{S}^{d-1}$. Furthermore, a cone Γ is called *right-circular*, if $\Gamma = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus \{0\} : x_d > |x| \cos \eta\}$, with $\eta \in (0, \pi)$ called the angle of the cone. Of course, every right-circular cone is smooth, and every smooth cone is fat.

By [10, Theorem 1], the following approximate factorization holds true for fat cones:

$$p_t^\Gamma(x,y) \approx \mathbb{P}_x(\tau_\Gamma > t) p_t(x,y) \mathbb{P}_y(\tau_\Gamma > t), \quad x, y \in \Gamma,\ t > 0. \tag{2.11}$$

For $R \in (0, \infty)$, we let $\Gamma_R := \Gamma \cap B_R$, the *truncated cone*.

3 Doob conditioning

The Martin kernel M_Γ is invariant for the semigroup P_t^Γ, as follows.

Theorem 3.1. *For all $x \in \Gamma$ and $t > 0$, we have $P_t^\Gamma M_\Gamma(x) = M_\Gamma(x)$.*

Proof. Fix $t > 0$ and $x \in \Gamma$. We have

$$P_t^\Gamma M_\Gamma(x) = \mathbb{E}_x\Big[\tau_\Gamma > t;\, M_\Gamma(X_t)\Big]. \tag{3.1}$$

Let $R > 0$ and $\tau_R := \tau_{\Gamma_R}$. By (2.9) and the strong Markov property,

$$M_\Gamma(x) = \mathbb{E}_x M_\Gamma(X_{\tau_{\Gamma_R}}) = \mathbb{E}_x M_\Gamma\big(X_{t\wedge\tau_R}\big) = \mathbb{E}_x\Big[X_{t\wedge\tau_R} \in \Gamma;\, M_\Gamma\big(X_{t\wedge\tau_R}\big)\Big], \tag{3.2}$$

where the last equality follows from the fact that $M_\Gamma = 0$ outside Γ. We note that $\mathbb{P}_x - a.s.$, $\tau_R \to \tau_\Gamma$ as $R \to \infty$ (see, e.g., [1, proof of Proposition A.1]). We consider two scenarios. On $\{\tau_\Gamma = \infty\}$, for R large enough, we have: $\tau_R > t$, $\mathbb{1}_{X_{t\wedge\tau_R}\in\Gamma} = 1 = \mathbb{1}_{t<\tau_\Gamma}$, and

$$M_\Gamma\big(X_{t\wedge\tau_R}\big)\mathbb{1}_{X_{t\wedge\tau_R}\in\Gamma} = M_\Gamma(X_t) = M_\Gamma(X_t)\mathbb{1}_{t<\tau_\Gamma}.$$

On $\{\tau_\Gamma < \infty\}$, for R large enough we have: $\tau_R = \tau_\Gamma$, $\mathbb{1}_{X_{t\wedge\tau_R}\in\Gamma} = \mathbb{1}_{t<\tau_\Gamma}$, and

$$M_\Gamma\big(X_{t\wedge\tau_R}\big)\mathbb{1}_{X_{t\wedge\tau_R}\in\Gamma} = M_\Gamma(X_t)\mathbb{1}_{t<\tau_\Gamma},$$

too. In both cases, the integrand on the right-hand side of (3.2) converges $a.s.$ to the integrand on the right-hand side of (3.1) as $R \to \infty$. By the local boundedness of M_Γ and (2.10),

$$\Big|M_\Gamma\big(X_{t\wedge\tau_R}\big)\mathbb{1}_{X_{t\wedge\tau_R}\in\Gamma}\Big| \leqslant c\big|X_{t\wedge\tau_R}\big|^\beta \leqslant c(X_t^*)^\beta,$$

where

$$X_t^* := \sup_{0\leqslant s\leqslant t} |X_s|.$$

Using [4, Theorem 2.1] and the fact that $\beta \in [0, \alpha)$, we conclude that $\mathbb{E}_x(X_t^*)^\beta < \infty$. An application of the dominated convergence theorem ends the proof. □

3.1 Renormalized kernel

We define the renormalized (Doob-conditioned) kernel

$$\rho_t(x, y) = \frac{p_t^\Gamma(x, y)}{M_\Gamma(x)M_\Gamma(y)}, \quad x, y \in \Gamma,\ t > 0. \tag{3.3}$$

Note that ρ is jointly continuous. By Theorem 3.1,

$$\int_\Gamma \rho_t(x, y)M_\Gamma^2(y)\,\mathrm{d}y = 1, \quad x \in \Gamma,\ t > 0, \tag{3.4}$$

and by (2.5),

$$\int_\Gamma \rho_t(x, y)\rho_s(y, z)M_\Gamma^2(y)\,\mathrm{d}y = \rho_{t+s}(x, z), \quad x, y \in \Gamma,\ s, t > 0. \tag{3.5}$$

In other words, ρ_t is a symmetric transition probability density on Γ with respect to the measure $M_\Gamma^2(y)\,\mathrm{d}y$. Furthermore, the following scaling property holds true: for all $x, y \in \Gamma$ and all $t > 0$,

$$\rho_t(x, y) = \frac{t^{-d/\alpha}p_1^\Gamma(t^{-1/\alpha}x, t^{-1/\alpha}y)}{t^{2\beta/\alpha}M_\Gamma(t^{-1/\alpha}x)M_\Gamma(t^{-1/\alpha}y)} = t^{-(d+2\beta)/\alpha}\rho_1(t^{-1/\alpha}x, t^{-1/\alpha}y). \tag{3.6}$$

Therefore,

$$\rho_{st}(t^{1/\alpha}x, t^{1/\alpha}y) = t^{-(d+2\beta)/\alpha}\rho_s(x,y), \quad x,y \in \Gamma,\ s,t>0. \tag{3.7}$$

By (2.11), for fat cones we have

$$\rho_t(x,y) \approx \frac{\mathbb{P}_x(\tau_\Gamma > t)}{M_\Gamma(x)} p_t(x,y) \frac{\mathbb{P}_y(\tau_\Gamma > t)}{M_\Gamma(y)}, \quad x,y \in \Gamma,\ t>0. \tag{3.8}$$

The boundary behavior of $\mathbb{P}_x(\tau_\Gamma > t)/M_\Gamma(x)$ is important due to (3.8), but it is rather elusive. The next lemma strengthens the upper bound from [3, Lemma 4.2].

Lemma 3.2. *There exists a constant c depending only on α and Γ, such that*

$$\mathbb{P}_x(\tau_\Gamma > t) \leqslant c\Big(t^{-\beta/\alpha} + t^{-1}|x|^{\alpha-\beta}\Big) M_\Gamma(x), \quad t>0,\ x \in \Gamma. \tag{3.9}$$

Remark 3.3. (1) For $t=1$, (3.9) reads as follows,

$$\mathbb{P}_x(\tau_\Gamma > 1) \leqslant c(1+|x|^{\alpha-\beta}) M_\Gamma(x), \quad x \in \Gamma. \tag{3.10}$$

(2) The estimate (3.9) applies to arbitrary cones and arguments t, x, however, it is not optimal. For example, for the right-circular cones, we can confront (3.8) with

$$M_\Gamma(x) \approx \delta_\Gamma(x)^{\alpha/2}|x|^{\beta-\alpha/2}, \quad x \in \Gamma,$$

and

$$\mathbb{P}_x(\tau_\Gamma > 1) \approx \big(1 \wedge \delta_\Gamma(x)\big)^{\alpha/2}\big(1 \wedge |x|\big)^{\beta-\alpha/2}, \quad x \in \Gamma,$$

as provided by [28, Lemma 3.3] and [10, Example 7].
(3) For the right-circular cones, the ratio

$$\frac{\mathbb{P}_x(\tau_\Gamma > 1)}{M_\Gamma(x)} \approx \frac{\big(1+\delta_\Gamma(x)\big)^{-\alpha/2}}{(1+|x|)^{\beta-\alpha/2}}, \quad x \in \Gamma,$$

is bounded if and only if $\beta \geqslant \alpha/2$.

Proof of Lemma 3.2. We slightly modify the proof of [3, Lemma 4.2]. First, suppose that $t=1$. The case $x \in \Gamma_1$ in (3.10) is resolved by [3, Lemma 4.2], so we assume that $x \in \Gamma \setminus \Gamma_1$. For every $z \in \mathbb{R}^d \setminus \{0\}$ we define its projection on the unit sphere $\tilde{z} := z/|z|$. By (2.7),

$$\mathbb{P}_x(\tau_\Gamma > 1) = \mathbb{P}_{\tilde{x}}(\tau_\Gamma > |x|^{-\alpha}).$$

Then we have

$$\mathbb{P}_{\tilde{x}}(\tau_\Gamma > |x|^{-\alpha}) \leqslant \mathbb{P}_{\tilde{x}}(\tau_{\Gamma_8} > |x|^{-\alpha}) + \mathbb{P}_{\tilde{x}}(\tau_{\Gamma_8} < \tau_\Gamma).$$

By the boundary Harnack principle (BHP), see [33, Theorem 3.1], and the homogeneity (2.10) of M_Γ,

$$\mathbb{P}_{\tilde{x}}(\tau_{\Gamma_8} < \tau_\Gamma) \leqslant c_1 M_\Gamma(\tilde{x}) = c_1|x|^{-\beta}M_\Gamma(x). \tag{3.11}$$

We let

$$c_2 = \inf_{y \in \Gamma_8} \int_{\Gamma \setminus \Gamma_8} \nu(y-z)\,\mathrm{d}z.$$

Clearly, $c_2 > 0$. We recall the Ikeda-Watanabe formula: for every open set $D \subseteq \mathbb{R}^d$,

$$\mathbb{P}_x[\tau_D \in I,\ Y_{\tau_D -} \in A,\ Y_{\tau_D} \in B] = \int_I \int_A \int_B p_s^D(x,v)\nu(v,z)\,\mathrm{d}z\,\mathrm{d}v\,\mathrm{d}s,$$

where $x \in D$, $I \subseteq [0, \infty)$, $A \subseteq D$ and $B \subseteq D^c$, which follows, e.g., from the proof of Bogdan et al. [16, Section 4.2]. By Markov inequality and BHP,

$$\begin{aligned}
\mathbb{P}_{\tilde{x}}\big(\tau_{\Gamma_8} > |x|^{-\alpha}\big) &\leqslant |x|^\alpha \mathbb{E}_{\tilde{x}} \tau_{\Gamma_8} = |x|^\alpha \int_{\Gamma_8} G_{\Gamma_8}(\tilde{x}, y)\, \mathrm{d}y \\
&\leqslant c_2^{-1} |x|^\alpha \int_{\Gamma \setminus \Gamma_8} \int_{\Gamma_8} G_{\Gamma_8}(\tilde{x}, y) \nu(y - z)\, \mathrm{d}y\, \mathrm{d}z \\
&\leqslant c_2^{-1} |x|^\alpha \mathbb{P}_{\tilde{x}}\big(X_{\tau_{\Gamma_8}} \in \Gamma\big) \\
&\leqslant c_1 c_2^{-1} |x|^\alpha \mathbb{P}_{\mathbf{1}}\big(X_{\tau_{\Gamma_8}} \in \Gamma\big) M_\Gamma(\tilde{x}) \\
&= c_1 c_2^{-1} c |x|^{\alpha - \beta} M_\Gamma(x).
\end{aligned}$$

By (3.11), we get (3.10) when $x \in \Gamma \setminus \Gamma_1$. For arbitrary $t > 0$, we use (2.7) and (3.10):

$$\begin{aligned}
\mathbb{P}_x(\tau_\Gamma > t) &= \mathbb{P}_{t^{-1/\alpha} x}(\tau_\Gamma > 1) \\
&\leqslant c \left(1 + \left(t^{-1/\alpha} |x|\right)^{\alpha - \beta}\right) M_\Gamma(t^{-1/\alpha} x) \\
&= c \left(t^{-\beta/\alpha} + t^{-1} |x|^{\alpha - \beta}\right) M_\Gamma(x). \qquad \square
\end{aligned}$$

By the proof of [3, Lemma 4.2], for every $R \in (0, \infty)$ there exists a constant c, depending only on α, Γ and R, such that

$$c^{-1} M_\Gamma(x) t^{-\beta/\alpha} \leqslant \mathbb{P}_x(\tau_\Gamma > t) \leqslant c M_\Gamma(x) t^{-\beta/\alpha}, \quad x \in \Gamma_{R t^{1/\alpha}},\ t > 0.$$

In particular, for fat cones, in view of (2.4) and (3.8),

$$\rho_1(x, y) \approx (1 + |y|)^{-d-\alpha} \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{M_\Gamma(y)}, \quad x \in \Gamma_R,\ y \in \Gamma, \tag{3.12}$$

with comparability constant depending only on α, Γ and R. Using Lemma 3.2 we also conclude that for every $R \geqslant 1$ there is a constant c depending only on R, α and Γ, such that

$$\rho_1(x, y) \leqslant c(1 + |y|)^{-d-\beta}, \quad x \in \Gamma_R,\ y \in \Gamma. \tag{3.13}$$

3.2 Ornstein-Uhlenbeck kernel

Encouraged by [13], we let

$$\ell_t(x, y) := \rho_{1 - e^{-t}}(e^{-t/\alpha} x, y), \quad x, y \in \Gamma,\ t > 0, \tag{3.14}$$

and, by (3.5) and (3.7), we get the Chapman-Kolmogorov property for ℓ_t:

$$\begin{aligned}
\int_\Gamma \ell_t(x, y) \ell_s(y, z) M_\Gamma^2(y)\, \mathrm{d}y &= e^{s(d + 2\beta)/\alpha} \int_\Gamma \rho_{1 - e^{-t}}\left(e^{-t/\alpha} x, y\right) \rho_{e^s - 1}\left(y, e^{s/\alpha} z\right) M_\Gamma^2(y)\, \mathrm{d}y \\
&= e^{s(d + 2\beta)/\alpha} \rho_{e^s - e^{-t}}\left(e^{-t/\alpha} x, e^{s/\alpha} z\right) \\
&= \rho_{1 - e^{-(t+s)}}\left(e^{-(t+s)/\alpha} x, z\right) \\
&= \ell_{t+s}(x, z), \quad x, z \in \Gamma,\ s, t > 0.
\end{aligned}$$

By (3.4),

$$\int_\Gamma \ell_t(x, y) M_\Gamma^2(y)\, \mathrm{d}y = 1, \quad x \in \Gamma,\ t > 0.$$

Thus, ℓ_t is a transition probability density on Γ with respect to $M_\Gamma^2(y)\, \mathrm{d}y$. We define the corresponding Ornstein-Uhlenbeck semigroup:

$$L_t f(y) = \int_\Gamma \ell_t(x, y) f(x) M_\Gamma^2(x)\, \mathrm{d}x, \quad x \in \Gamma, t > 0.$$

We easily see that the operators are bounded on $L^1(M_\Gamma^2(y)\,\mathrm{d}y)$. In fact, for every $f \geqslant 0$,

$$\int_\Gamma L_t f(y) M_\Gamma^2(y)\,\mathrm{d}y = \int_\Gamma f(x) M_\Gamma^2(x)\,\mathrm{d}x. \tag{3.15}$$

In particular, L_t preserve densities, i.e., functions $f \geqslant 0$ such that $\int_\Gamma f(x)M_\Gamma^2(x)\,\mathrm{d}x = 1$.

Before we immerse into details, let us note that the relations (3.12) and (3.13) will be crucial in what follows. Both of them rely on the factorization of the Dirichlet heat kernel (2.11), which is valid for fat sets. So, below in this section we assume (sometimes tacitly) that Γ is a fat cone.

Let us first derive some preliminary properties of ℓ_t, which will be useful later on. First, observe that in view of (3.14), (3.7) and (3.12), for any $R > 0$ and all $x, z \in B_R$,

$$\ell_1(x,y) \approx \ell_1(z,y) \approx (1+|y|)^{-d-\alpha} \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{M_\Gamma(y)}, \quad y \in \Gamma, \tag{3.16}$$

with the implied comparability constant depending on R, but not on x and z. Then,

$$\ell_1(x,y) \approx \frac{\mathbb{P}_{e^{-1/\alpha}x}\big(\tau_\Gamma > 1-e^{-1}\big)}{M_\Gamma\big(e^{-1/\alpha}x\big)} p_{1-e^{-1}}\big(e^{-1/\alpha}x, y\big) \frac{\mathbb{P}_y\big(\tau_\Gamma > 1-e^{-1}\big)}{M_\Gamma(y)}, \quad x, y \in \Gamma,$$

by (3.14) and (3.8). Applying (2.7), [10, Remark 3] and the homogeneity (2.10) of M_Γ to the first component, (2.4) to the second and (3.9) to the third, we get

$$\ell_1(x,y) \lesssim (1+|x|)^{-d-\alpha} \frac{\mathbb{P}_x(\tau_\Gamma > 1)}{M_\Gamma(x)}, \quad x \in \Gamma, y \in \Gamma_R, \tag{3.17}$$

with the implied comparability constant depending on R, but not on x or y.

Theorem 3.4. *Assume Γ is a fat cone. Then there is a unique stationary density φ for the operators L_t, $t > 0$.*

Proof. Fix $t > 0$ and consider the family F of nonnegative functions on Γ that have the form

$$f(y) = \int_{\Gamma_1} \rho_1(x,y)\,\mu(\mathrm{d}x), \quad y \in \Gamma,$$

for some probability measure μ concentrated on Γ_1. Note that by (3.4), F is a convex set of densities on $L^1(M_\Gamma^2(y)\,\mathrm{d}y)$. We claim that $L_t F \subseteq F$. Indeed, by symmetry, Chapman-Kolmogorov property (3.5) and the scaling property (3.7), for every $f \in F$ given by a probability measure μ on Γ_1, we have

$$\begin{aligned}
L_t f(u) &= \int_\Gamma \int_{\Gamma_1} \rho_{1-e^{-t}}\Big(e^{-t/\alpha}x, u\Big) \rho_1(z,x)\,\mu(\mathrm{d}z) M_\Gamma^2(x)\,\mathrm{d}x \\
&= \int_{\Gamma_1} e^{t(d+2\beta)/\alpha} \int_\Gamma \rho_{e^t-1}\Big(e^{t/\alpha}u, x\Big) \rho_1(x,z) M_\Gamma^2(x)\,\mathrm{d}x\,\mu(\mathrm{d}z) \\
&= \int_{\Gamma_1} e^{t(d+2\beta)/\alpha} \rho_{e^t}(e^{t/\alpha}u, z)\,\mu(\mathrm{d}z) \\
&= \int_{\Gamma_1} \rho_1(e^{-t/\alpha}z, u)\,\mu(\mathrm{d}z) \\
&= \int_{\Gamma_1} \rho_1(z,u)\,\tilde{\mu}(\mathrm{d}z),
\end{aligned}$$

where $\tilde{\mu}(A) = \mu(e^{t/\alpha}A)$. In particular, $\tilde{\mu}$ is a probability measure concentrated on $e^{-t/\alpha}\Gamma_1 \subseteq \Gamma_1$, so $L_t f \in F$, as claimed. Since L_t is continuous, we also have $L_t\overline{F} \subseteq \overline{F}$, where $\overline{F}$ is the closure of F in the norm topology of $L^1(\Gamma, M_\Gamma^2(y)\,\mathrm{d}y)$. By [31, Theorem 3.12], $\overline{F}$ is the same as the closure of F in the weak topology. In view of (3.12),

$$f(y) \approx (1+|y|)^{-d-\alpha}\frac{\mathbb{P}_y(\tau_\Gamma > 1)}{M_\Gamma(y)}, \quad y \in \Gamma, \tag{3.18}$$

uniformly for $f \in F$. Moreover, (3.4) and (3.12) show that the right-hand side of (3.18) is integrable with respect to $M_\Gamma^2(y)\,\mathrm{d}y$. Therefore, the family F is uniformly integrable with respect to $M_\Gamma^2(y)\,\mathrm{d}y$. By [8, Theorem 4.7.20], F is weakly pre-compact in $L^1(M_\Gamma^2(y))$, so $\overline{F}$ is weakly compact. Furthermore, we invoke [18, Theorem 3.10] to conclude that L_t is weakly continuous. By the Schauder-Tychonoff fixed point theorem [31, Theorem 5.28], there is a density $\varphi \in \overline{F}$ satisfying $L_t\varphi = \varphi$ a.e. in Γ.

Let us prove the uniqueness. Assume that there is another density ψ satisfying $L_t\psi = \psi$ a.e. in Γ. Then it is easy to see that $r := \varphi - \psi$ satisfies $L_t r = r$, too. Suppose that $r \neq 0$ on the set of positive Lebesgue measure, so

$$\int_\Gamma r_+(x) M_\Gamma^2(x)\,\mathrm{d}x = \int_\Gamma r_-(x) M_\Gamma^2(x)\,\mathrm{d}x > 0.$$

We have

$$L_t r(x) = L_t r_+(x) - L_t r_-(x) = \int_\Gamma \ell_t(y,x) r_+(y) M_\Gamma^2(y)\,\mathrm{d}y - \int_\Gamma \ell_t(y,x) r_-(y) M_\Gamma^2(y)\,\mathrm{d}y,$$

and both terms on the right-hand side are strictly positive by the strict positivity of ℓ_t. It follows that $|L_t r(x)| < L_t|r|(x)$ a.e. in Γ. Therefore, using (3.15), we get

$$\int_\Gamma |r(x)| M_\Gamma^2(x)\,\mathrm{d}x = \int_\Gamma \big|L_t r(x)\big| M_\Gamma^2(x)\,\mathrm{d}x < \int_\Gamma L_t|r|(x) M_\Gamma^2(x)\,\mathrm{d}x = \int_\Gamma |r(x)| M_\Gamma^2(x)\,\mathrm{d}x,$$

which is a contradiction. Thus, $r = 0$ a.e. in Γ and, consequently, $\varphi = \psi$ a.e. in Γ. The operators $(L_t \colon t > 0)$ commute, therefore for every $s, t > 0$,

$$L_t(L_s\varphi) = L_s(L_t\varphi) = L_s\varphi.$$

So, by uniqueness, $L_s\varphi = \varphi$ a.e. in Γ. □

In the first version of the paper, the measures μ in the definition of the convex set F above were sub-probabilities (rather than probabilities), so the fix-point φ could be zero. We thank the referee for pointing out the problem to us. A similar glitch remains in [13, the proof of Theorem 3.2], but can be resolved in the same way.

Let us note that by Theorem 3.4 and [25, Theorem 1 and Remark 2], the following stability result for kernels l_t in $L^1(M_\Gamma^2(y)\,\mathrm{d}y)$ holds true for every $x \in \Gamma$:

$$\int_\Gamma \big|\ell_t(x,y) - \varphi(y)\big| M_\Gamma^2(y)\,\mathrm{d}y \to 0 \quad \text{as} \quad t \to \infty. \tag{3.19}$$

We claim that the convergence in (3.19) is in fact uniform for x in any bounded subset $A \subseteq \Gamma$. Indeed, let $x, x_0 \in A$. In view of (3.16), we may write

$$\begin{aligned}\int_\Gamma \big|\ell_{1+t}(x,y) - \varphi(y)\big| M_\Gamma^2(y)\,\mathrm{d}y &= \int_\Gamma \left| \int_\Gamma \ell_1(x,z)\big(\ell_t(z,y) - \varphi(y)\big) M_\Gamma^2(z)\,\mathrm{d}z \right| M_\Gamma^2(y)\,\mathrm{d}y \\ &\leqslant c \int_\Gamma \ell_1(x_0,z) \int_\Gamma \big|\ell_t(z,y) - \varphi(y)\big| M_\Gamma^2(y)\,\mathrm{d}y\, M_\Gamma^2(z)\,\mathrm{d}z.\end{aligned} \tag{3.20}$$

By (3.19), for every $z \in \Gamma$,

$$I_t(z) := \int_\Gamma \left|\ell_t(z,y) - \varphi(y)\right| M_\Gamma^2(y)\,\mathrm{d}y \to 0 \quad \text{as} \quad t \to \infty.$$

Moreover, $I_t(z) \leqslant \int_\Gamma \big(\ell_t(z,y) + \varphi(y)\big) M_\Gamma^2(y)\,\mathrm{d}y = 2$. Since

$$\int_\Gamma 2\ell_1(x_0,z) M_\Gamma^2(z)\,\mathrm{d}z = 2 < \infty,$$

by the dominated convergence theorem the iterated integral in (3.20) tends to 0 as $t \to \infty$, so the convergence in (3.19) is uniform for all $x \in A$, as claimed. By rewriting (3.19) in terms of ρ, we get that, uniformly for $x \in A$,

$$\int_\Gamma \left|\rho_{1-e^{-t}}\big(e^{-t/\alpha}x, y\big) - \varphi(y)\right| M_\Gamma^2(y)\,\mathrm{d}y \to 0 \quad \text{as} \quad t \to \infty. \tag{3.21}$$

This leads to the following spatial asymptotics for ρ_1.

Corollary 3.5. *Let Γ be a fat cone. If $\Gamma \ni x \to 0$ then $\int_\Gamma \left|\rho_1(x,y) - \varphi(y)\right| M_\Gamma^2(y)\,dy \to 0$.*

Proof. By the scalings (3.6) and (3.7),

$$\rho_{1-e^{-t}}\big(e^{-t/\alpha}x, y\big) = \big(1-e^{-t}\big)^{-(d+2\beta)/\alpha} \rho_1\Big(\big(e^t - 1\big)^{-1/\alpha}x, \big(1-e^{-t}\big)^{-1/\alpha}y\Big),$$

thus, in view of (3.21),

$$\int_\Gamma \left|\rho_1\Big(\big(e^t - 1\big)^{-1/\alpha}x, \big(1-e^{-t}\big)^{-1/\alpha}y\Big) - \varphi(y)\right| M_\Gamma^2(y)\,\mathrm{d}y \to 0 \quad \text{as} \quad t \to \infty. \tag{3.22}$$

By the continuity of dilations $(0,\infty) \ni r \mapsto f(r\,\cdot\,) \in L^1(\mathbb{R}^d)$ at $r = 1$,

$$\int_\Gamma \left|\varphi\Big(\big(1-e^{-t}\big)^{1/\alpha}y\Big) M_\Gamma^2(y) - \varphi(y) M_\Gamma^2(y)\right| \mathrm{d}y \to 0 \quad \text{as} \quad t \to \infty.$$

Thus, by a change of variables in (3.22) and the triangle inequality, we conclude that

$$\int_\Gamma \left|\rho_1\Big(\big(e^t - 1\big)^{-1/\alpha}z, y\Big) - \varphi(y)\right| M_\Gamma^2(y)\,\mathrm{d}y \to 0 \quad \text{as} \quad t \to \infty$$

uniformly for all $z \in A$. To end the proof, we take $A = B_1$ and $x = \big(e^t - 1\big)^{-1/\alpha}z$, where $t = \ln\big(1 + |x|^{-\alpha}\big)$ and $z = x/|x| \in A$. □

Lemma 3.6. *After a modification on set of Lebesgue measure 0, φ is continuous on Γ and*

$$\varphi(y) \approx (1+|y|)^{-d-\alpha} \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{M_\Gamma(y)}, \quad y \in \Gamma.$$

Proof. By Corollary 3.5 and (3.12),

$$\varphi(y) \approx (1+|y|)^{-d-\alpha} \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{M_\Gamma(y)} \tag{3.23}$$

on Γ less a set of Lebesgue measure zero. Theorem 3.4 entails that $\varphi = L_1\varphi$ *a.e.*, so it suffices to verify that $L_1\varphi$ is continuous on Γ. To this end we note that $\ell_1(x,y)$ is continuous in $x, y \in \Gamma$. Therefore, for every $R > 0$ we may use (3.17) and the dominated convergence theorem to conclude that $L_1\varphi$ is continuous on Γ_R. □

Remark 3.7. From the proof of Lemma 3.6, $L_1 f$ is continuous on Γ provided f satisfies $f(y) \leqslant c(1+|y|)^{-d-\alpha}\mathbb{P}_y(\tau_\Gamma > 1)/M_\Gamma(y)$ for $y \in \Gamma$.

In what follows, φ denotes the continuous modification from Lemma 3.6.

Theorem 3.8. *Let* Γ *be a fat cone. For every* $t>0$*, uniformly in* $y\in\Gamma$ *we have*

$$\rho_t(0,y) := \lim_{\Gamma\ni x\to 0}\rho_t(x,y) = t^{-(d+2\beta)/\alpha}\varphi\big(t^{-1/\alpha}y\big).$$

Proof. If $\beta=0$ then $\rho_t(x,y)=p_t(x,y)$ and the claim is simply the continuity property of the heat kernel p_t. Thus, we assume that $\beta>0$.

We only prove the claim for $t=1$; the extension to arbitrary t is a consequence of the scaling (3.6). By (3.7) and the Chapman-Kolmogorov property, for $x,y\in\Gamma$,

$$\begin{aligned}\rho_1(x,y) &= 2^{(d+2\beta)/\alpha}\rho_2\big(2^{1/\alpha}x,2^{1/\alpha}y\big)\\ &= 2^{(d+2\beta)/\alpha}\int_\Gamma \rho_1\big(2^{1/\alpha}x,z\big)\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z.\end{aligned}$$

We will prove that, uniformly in $y\in\Gamma$,

$$\int_\Gamma \rho_1\big(2^{1/\alpha}x,z\big)\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z \to \int_\Gamma \varphi(z)\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z, \tag{3.24}$$

as $\Gamma\ni x\to 0$. To this end we first claim that there is $c\in(0,\infty)$ dependent only on α and Γ, such that for all $x\in\Gamma_1$ and $y\in\Gamma$,

$$\int_\Gamma \big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z \leqslant c(1+|y|)^{-\beta}. \tag{3.25}$$

Indeed, denote $\tilde y = 2^{1/\alpha}y$. By (3.8), Lemma 3.6 and (3.10), there is $c>0$ such that for all $z,y\in\Gamma$ and $x\in\Gamma_1$,

$$\big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|\rho_1\big(z,\widetilde y\big)M_\Gamma^2(z) \lesssim (1+|z|)^{-d-\alpha}(1+|z-\widetilde y|)^{-d-\alpha}(1+|y|)^{\alpha-\beta}.$$

We split the integral in (3.25) into two integrals. For $z\in A:=\Gamma\cap B(\widetilde y,|\widetilde y|/2)$ we use the fact that $|z|\approx|\widetilde y|\approx|y|$ and $1+|z-\widetilde y|\geqslant 1$, therefore

$$\int_A \big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|\rho_1\big(z,\widetilde y\big)M_\Gamma^2(z)\,\mathrm{d}z \lesssim |y|^d(1+|y|)^{-d-\beta} \leqslant (1+|y|)^{-\beta}. \tag{3.26}$$

For $z\in\Gamma\setminus A$ we simply have $1+|z|\geqslant 1$, thus,

$$\begin{aligned}\int_{\Gamma\setminus A} \big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|\rho_1\big(z,\widetilde y\big)M_\Gamma^2(z)\,\mathrm{d}z &\lesssim (1+|y|)^{\alpha-\beta}\int_{\Gamma\setminus A}(1+|z-\widetilde y|)^{-d-\alpha}\,\mathrm{d}z\\ &\lesssim (1+|y|)^{\alpha-\beta}\int_{|\widetilde y|/2}^\infty (1+r)^{-1-\alpha}\,\mathrm{d}r\\ &\lesssim (1+|y|)^{-\beta}.\end{aligned}$$

Combining it with (3.26), we arrive at (3.25), as claimed.

Let $\varepsilon>0$. In view of (3.25) and the fact that $\beta>0$, there is $R\in(0,\infty)$ depending only on α, β, Γ and ε such that

$$\int_\Gamma \big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z < \varepsilon, \tag{3.27}$$

provided that $y\in\Gamma\setminus\Gamma_R$. For $y\in\Gamma_R$, by (3.13) we get

$$\int_\Gamma \big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z \lesssim \int_\Gamma \big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|M_\Gamma^2(z)\,\mathrm{d}z,$$

with the implied constant dependent only on α, β, Γ and R, but not otherwise dependent of y. Thus, by Corollary 3.5,

$$\int_\Gamma \big|\rho_1\big(2^{1/\alpha}x,z\big)-\varphi(z)\big|\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z<\varepsilon \tag{3.28}$$

for all $y\in\Gamma_R$ and $x\in\Gamma_1$ small enough. Putting (3.28) together with (3.27) we arrive at (3.24). Using the scaling property (3.7) and Theorem 3.4,

$$\begin{aligned}\lim_{\Gamma\ni x\to 0}\rho_1(x,y)&=2^{(d+2\beta)/\alpha}\int_\Gamma\varphi(z)\rho_1\big(z,2^{1/\alpha}y\big)M_\Gamma^2(z)\,\mathrm{d}z\\&=\int_\Gamma\varphi(z)\rho_{1/2}\big(2^{-1/\alpha}z,y\big)M_\Gamma^2(z)\,\mathrm{d}z=L_{\ln 2}\varphi(y)=\varphi(y).\end{aligned}$$

The proof is complete. □

Note that by the symmetry of ρ_t, for $x\in\Gamma$,

$$\rho_t(x,0):=\lim_{\Gamma\ni y\to 0}\rho_t(x,y)=\rho_t(0,x)=t^{-(d+2\beta)/\alpha}\varphi\big(t^{-1/\alpha}x\big).$$

Recall also that by (3.12) and (3.4),

$$\rho_1(x,y)\approx(1+|y|)^{-d-\alpha}\frac{\mathbb{P}_y(\tau_\Gamma>1)}{M_\Gamma(y)}\in L^1(M_\Gamma^2(y)\,\mathrm{d}y).$$

Thus, by Theorem 3.8 and the dominated convergence theorem,

$$\int_\Gamma\varphi(x)M_\Gamma^2(x)\ \mathrm{d}x=1. \tag{3.29}$$

Let us summarize the results of this section in one statement.

Theorem 3.9. *Assume Γ is a fat cone. Then the function ρ has a continuous extension to $(0,\infty)\times(\Gamma\cup\{0\})\times(\Gamma\cup\{0\})$ and*

$$\rho_t(0,y):=\lim_{\Gamma\ni x\to 0}\rho_t(x,y)\in(0,\infty),\quad t>0,\ y\in\Gamma, \tag{3.30}$$

satisfies

$$\rho_t(0,y)=t^{-(d+2\beta)/\alpha}\rho_1(0,t^{-1/\alpha}y),\quad t>0,\ y\in\Gamma, \tag{3.31}$$

and

$$\int_\Gamma\rho_t(0,y)\rho_s(y,z)M_\Gamma^2(y)\,dy=\rho_{t+s}(0,z),\quad s,t>0,\ z\in\Gamma. \tag{3.32}$$

Proof. The existence of the limit (3.30) and the scaling property (3.31) are proved in Theorem 3.8. For the proof of (3.32) we employ (3.5) to write

$$\rho_{t+s}(0,z)=\lim_{\Gamma\ni y\to 0}\rho_{t+s}(y,z)=\lim_{\Gamma\ni y\to 0}\int_\Gamma\rho_t(y,w)\rho_s(w,z)M_\Gamma^2(w)\,\mathrm{d}w,$$

and use (3.4), (3.6), (3.13), and the dominated convergence theorem.

Thus, it remains to prove the continuity of ρ on $(0,\infty)\times(\Gamma\cup\{0\})\times(\Gamma\cup\{0\})$. By symmetry and the Chapman-Kolmogorov property (3.5) of ρ_1,

$$\rho_1(x,y)=\int_\Gamma\rho_{1/2}(x,z)\rho_{1/2}(y,z)M_\Gamma^2(z)\,\mathrm{d}z,\quad x,y\in\Gamma. \tag{3.33}$$

By the continuity of ρ_1 on $\Gamma\times\Gamma$ together with Theorem 3.8, for every $x_0,y_0\in\Gamma\cup\{0\}$ we have $\rho_{1/2}(x,z)\to\rho_{1/2}(x_0,z)$ and $\rho_{1/2}(y,z)\to\rho_{1/2}(y_0,z)$ as $x\to x_0$ and $y\to y_0$. Moreover, (3.12) entails that

$$\rho_{1/2}(x,z)\rho_{1/2}(y,z)M_\Gamma^2(z)\leqslant c(1+|z|)^{-2d-2\alpha},$$

with the constant c possibly dependent on x_0 and y_0. It follows by the dominated convergence theorem that

$$\rho_1(x,y) \to \int_\Gamma \rho_{1/2}(x_0,z)\rho_{1/2}(y_0,z)M_\Gamma^2(z)\,\mathrm{d}z,$$

as $x \to x_0$ and $y \to y_0$ and in view of (3.33), it is an extension of ρ_1 to $(\Gamma \cup \{0\}) \times (\Gamma \cup \{0\})$, which will be denoted by the same symbol. It follows now from (3.6) that

$$t^{-(d+2\beta)/\alpha}\rho_1\big(t^{-1/\alpha}x, t^{-1/\alpha}y\big), \quad x, y \in \Gamma \cup \{0\},\, t > 0,$$

is a finite continuous extension of ρ_t for every $t > 0$. It remains to observe that the extension is unique and jointly continuous in $(t,x,y) \in \mathbb{R}_+ \times (\Gamma \cup \{0\}) \times (\Gamma \cup \{0\}$. □

Corollary 3.10. *We have* $\rho_1(0,0) = \lim_{\Gamma \ni x,y \to 0} \rho_1(x,y) \in (0,\infty)$.

Proof. By Theorem 3.9, $\rho_1(0,0) = \lim_{\Gamma \ni y \to 0} \varphi(y) =: \varphi(0)$. Thus, the claim follows by Lemma 3.6 and [3, Lemma 4.2]. □

Proof of Theorem 1.1. By (3.30), $\Psi_t(x) = \rho_t(0,x)M_\Gamma(x)$, $t > 0$, $x \in \Gamma$. Thus, the existence of Ψ_t is just a reformulation of (3.30). The scaling property (1.1) follows immediately from (3.31) and the homogeneity of the Martin kernel (2.10), and (1.2) is equivalent to (3.32). □

We conclude this part by rephrasing (3.29) in terms of Ψ_t:

$$\int_\Gamma \Psi_t(x)M_\Gamma(x)\,\mathrm{d}x = 1, \quad t > 0. \tag{3.34}$$

3.3 Yaglom limit

The above results quickly lead to calculation of the Yaglom limit for the stable process in a cone. Note that our proof is different from that in [15]. We also cover more general cones, e.g., $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2 \setminus ([0,\infty) \times \{0\})$. First, we obtain the following extension of [15, Theorem 3.1].

Corollary 3.11. *Let Γ be a fat cone. For every $t > 0$,*

$$\lim_{\Gamma \ni x \to 0} \frac{\mathbb{P}_x(\tau_\Gamma > t)}{M_\Gamma(x)} = C_1 t^{-\beta/\alpha} \quad \textit{where} \quad C_1 = \int_\Gamma \varphi(z)M_\Gamma(z)\,dz \in (0,\infty).$$

Proof. It is enough to prove the claim for $t = 1$; the general case follows by the scalings (2.7) and (2.10). We have

$$\frac{\mathbb{P}_x(\tau_\Gamma > 1)}{M_\Gamma(x)} = \int_\Gamma \frac{p_1^\Gamma(x,y)}{M_\Gamma(x)}\,\mathrm{d}y = \int_\Gamma \rho_1(x,y)M_\Gamma(y)\,\mathrm{d}y, \quad x \in \Gamma.$$

We use (3.12), the dominated convergence theorem, and Theorem 3.8 to get the conclusion. □

The first identity below is the Yaglom limit.

Theorem 3.12. *Assume Γ is a fat cone and B is a bounded subset of Γ. Then,*

$$\lim_{t\to\infty} \mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) = \mu(A), \quad A \subseteq \Gamma,$$

uniformly in $x \in B$, where

$$\mu(A) := \frac{1}{C_1}\int_A \varphi(y)M_\Gamma(y)\,dy, \quad A \subseteq \Gamma.$$

Proof. By (2.6) and the scaling property (2.7),

$$\begin{aligned}\mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) &= \frac{\mathbb{P}_x\left(\tau_\Gamma > t, t^{-1/\alpha}X_t \in A\right)}{\mathbb{P}_x(\tau_\Gamma > t)} \\ &= \frac{\mathbb{P}_{t^{-1/\alpha}x}\left(\tau_\Gamma > 1, X_1 \in A\right)}{\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1)} \\ &= \int_A \frac{p_1^\Gamma\big(t^{-1/\alpha}x, y\big)}{M_\Gamma\big(t^{-1/\alpha}x\big)}\,\mathrm{d}y \cdot \frac{M_\Gamma(t^{-1/\alpha}x)}{\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1)}.\end{aligned}$$

The claim follows by Theorem 3.8, Corollary 3.11, (3.12), and the dominated convergence theorem. □

Theorem 3.13. *If Γ is a fat cone and γ is a probability measure on Γ with $\int_\Gamma (1 + |y|)^\alpha\, \gamma(\mathrm{d}y) < \infty$, then*

$$\lim_{t\to\infty} \mathbb{P}_\gamma\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) = \mu(A), \quad A \subseteq \Gamma.$$

Proof. Let $t \geqslant 1$. We have

$$\begin{aligned}\mathbb{P}_\gamma\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) &= \frac{\mathbb{P}_\gamma\left(t^{-1/\alpha}X_t \in A, \tau_\Gamma > t\right)}{\mathbb{P}_\gamma\left(\tau_\Gamma > t\right)} \\ &= \int_\Gamma \mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) \frac{\mathbb{P}_x\left(\tau_\Gamma > t\right)}{\mathbb{P}_\gamma\left(\tau_\Gamma > t\right)}\,\gamma(\mathrm{d}x).\end{aligned}$$

We first prove that for all $x \in \Gamma$,

$$\frac{\mathbb{P}_\gamma\left(\tau_\Gamma > t\right)}{\mathbb{P}_x\left(\tau_\Gamma > t\right)} = \int_\Gamma \frac{\mathbb{P}_y\left(\tau_\Gamma > t\right)}{\mathbb{P}_x\left(\tau_\Gamma > t\right)}\,\gamma(\,\mathrm{d}y) \to \int_\Gamma \frac{M_\Gamma(y)}{M_\Gamma(x)}\,\gamma(\mathrm{d}y), \tag{3.35}$$

as $t \to \infty$. Indeed, fix $x \in \Gamma$. Note that by local boundedness of M_Γ and (2.10),

$$\int_\Gamma M_\Gamma(y)\,\gamma(\mathrm{d}y) \leqslant c\int_\Gamma (1 + |y|)^\beta\,\gamma(\mathrm{d}y) < \infty,$$

so the right-hand side of (3.35) is finite. Next, by Corollary 3.11, (2.7), and (2.10),

$$\lim_{t\to\infty} \frac{\mathbb{P}_y\left(\tau_\Gamma > t\right)}{\mathbb{P}_x\left(\tau_\Gamma > t\right)} = \lim_{t\to\infty} \frac{\mathbb{P}_{t^{-1/\alpha}y}\left(\tau_\Gamma > 1\right) M_\Gamma(t^{-1/\alpha}x)}{\mathbb{P}_{t^{-1/\alpha}x}\left(\tau_\Gamma > 1\right) M_\Gamma(t^{-1/\alpha}y)} \frac{M_\Gamma(y)}{M_\Gamma(x)} = \frac{M_\Gamma(y)}{M_\Gamma(x)}, \quad x, y \in \Gamma.$$

Moreover, since x is fixed, we may assume that $t \geqslant 1 \vee |x|^\alpha$. Thus, by [3, Lemma 4.2], Lemma 3.2, the local boundedness of M_Γ and (2.10),

$$\frac{\mathbb{P}_y\left(\tau_\Gamma > t\right)}{\mathbb{P}_x\left(\tau_\Gamma > t\right)} \leqslant c\frac{\big(t^{-\beta/\alpha} + t^{-1}|y|^{\alpha-\beta}\big)M_\Gamma(y)}{t^{-\beta/\alpha}M_\Gamma(x)} \leqslant c\frac{\big(1 + |y|^{\alpha-\beta}\big)M_\Gamma(y)}{M_\Gamma(x)} \leqslant c\frac{(1 + |y|)^\alpha}{M_\Gamma(x)}.$$

Thus, the dominated convergence theorem yields (3.35), as desired.

Next, we consider a family $\mathcal{F}_1$ of functions f_t of the form

$$f_t(x) = \frac{\mathbb{P}_x(\tau_\Gamma > t)}{\mathbb{P}_\gamma(\tau_\Gamma > t)}, \quad x \in \Gamma,\ t \geqslant 1.$$

Denote

$$f(x) = \frac{M_\Gamma(x)}{\int_\Gamma M_\Gamma(y)\,\gamma(\mathrm{d}y)}, \quad x \in \Gamma.$$

By virtue of (3.35), $f_t \to f$ everywhere in Γ as $t \to \infty$. Thus, $f_t \to f$ in measure γ as $t \to \infty$, see [32, Definition 22.2]. Moreover, we have

$$\int_\Gamma f(x)\,\gamma(\mathrm{d}x) = 1 = \lim_{t\to\infty} 1 = \lim_{t\to\infty} \int_\Gamma f_t(x)\,\gamma(\mathrm{d}x).$$

Therefore, by Vitali's theorem [32, Theorem 22.7], the family $\mathcal{F}_1$ is uniformly integrable. If we now consider the family $\mathcal{F}_2$ of functions $\tilde{f}_t$ of the form

$$\tilde{f}_t(x) = \mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) f_t(x), \quad x \in \Gamma,\, t \geqslant 1,$$

then a trivial bound $\mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) \leqslant 1$ shows that $\mathcal{F}_2$ is uniformly integrable as well (see, e.g., [32, Theorem 22.9]). By Theorem 3.12, (3.35) and [32, Theorem 22.7],

$$\lim_{t\to\infty} \int_\Gamma \mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,\middle|\, \tau_\Gamma > t\right) \frac{\mathbb{P}_x(\tau_\Gamma > t)}{\mathbb{P}_\gamma(\tau_\Gamma > t)}\,\gamma(\mathrm{d}x) = \int_\Gamma \mu(A) \frac{M_\Gamma(x)}{\int_\Gamma M_\Gamma(y)\,\gamma(\mathrm{d}y)}\,\gamma(\mathrm{d}x) = \mu(A).$$

The proof is complete. □

Example 3.14. Note that $\beta = 0$ if and only if Γ^c is a polar set and then $M_\Gamma(x) = 1$ for all $x \in \Gamma$, see [3, Theorem 3.2]. Consequently, we have $p_t^\Gamma(x,y) = p_t(x,y)$ and $\mathbb{P}_x(\tau_\Gamma > t) = 1$ for all $x, y \in \Gamma$ and all $t > 0$. It follows that $\rho_t(x,y) = p_t(x,y)$ and a direct calculation using the Chapman-Kolmogorov property entails that $\varphi(y) = p_1(0,y)$ is the stationary density for the (classical) α-stable Ornstein-Uhlenbeck semigroup, see (3.14) and Theorem 3.4. The statement of Theorem 3.8 thus reduces to continuity of the heat kernel of the isotropic α-stable Lévy process. Theorems 3.12 and 3.13 trivialize in a similar way. Incidentally, in this case the moment condition on γ in Theorem 3.13 is superfluous. Further examples are given in Section 4.

4 Asymptotic behavior for the killed semigroup

This section is devoted to examples and applications in Functional Analysis and Partial Differential Equations. Note that in Lemmas 4.1 and 4.2 we do not assume that Γ is fat.

Lemma 4.1. *$\{P_t^\Gamma\}_{t>0}$ is a strongly continuous contraction semigroup on $L^1(M_\Gamma)$ and*

$$\int_\Gamma P_t^\Gamma f(x) M_\Gamma(x)\,dx = \int_\Gamma f(x) M_\Gamma(x)\,dx, \quad t > 0, \quad f \in L^1(M_\Gamma). \tag{4.1}$$

Proof. Let $f \geqslant 0$. By the Fubini-Tonelli theorem, the symmetry of p_t^Γ and Theorem 3.1,

$$\int_\Gamma P_t^\Gamma f(x) M_\Gamma(x)\,\mathrm{d}x = \int_\Gamma \int_\Gamma p_t^\Gamma(x,y) f(y) M_\Gamma(y)\,\mathrm{d}y\,\mathrm{d}x = \int_\Gamma f(y) M_\Gamma(y)\,\mathrm{d}y. \tag{4.2}$$

Since $|P_t^\Gamma f| \leqslant P_t^\Gamma |f|$, the contractivity follows. Furthermore, for arbitrary $f \in L^1(M_\Gamma)$ we write $f = f_+ - f_-$ and use (4.2) to prove (4.1). The semigroup property follows from (2.5).

To prove the strong continuity, we fix $f \in L^1(M_\Gamma)$ and let $G := f M_\Gamma \in L^1(\Gamma)$. There is a sequence $g_n \in C_c^\infty(\Gamma)$ such that $\|g_n - G\|_{L^1(\Gamma)} \to 0$ as $n \to \infty$. For $f_n := g_n / M_\Gamma$ we get $f_n \in C_c^\infty(\Gamma)$ and $\|f_n - f\|_{L^1(M_\Gamma)} = \|g_n - G\|_{L^1(\Gamma)} \to 0$. By the first part of the proof,

$$\begin{aligned}\|P_t^\Gamma f - f\|_{L^1(M_\Gamma)} &\leqslant \|P_t^\Gamma f - P_t^\Gamma f_n\|_{L^1(M_\Gamma)} + \|P_t^\Gamma f_n - f_n\|_{L^1(M_\Gamma)} + \|f_n - f\|_{L^1(M_\Gamma)} \\ &\leqslant 2\|f_n - f\|_{L^1(M_\Gamma)} + \|P_t^\Gamma f_n - f_n\|_{L^1(M_\Gamma)}.\end{aligned}$$

It remains to prove that $\|P_t^\Gamma f - f\|_{L^1(M_\Gamma)} \to 0$ as $t \to 0^+$ for every $f \in C_c^\infty(\Gamma)$. To this end we let $\varepsilon > 0$ and choose $R > 0$ such that $\operatorname{supp} f \in B_R$ and $\int_{\Gamma\setminus\Gamma_R} P_t^\Gamma |f|(x) M_\Gamma(x)\,\mathrm{d}x < \varepsilon$. Then,

$$\|P_t^\Gamma f - f\|_{L^1(M_\Gamma)} < \int_{\Gamma_R} \left|P_t^\Gamma f(x) - f(x)\right| M_\Gamma(x)\,\mathrm{d}x + \varepsilon. \tag{4.3}$$

Considering the integrand in (4.3), for all $x \in \Gamma_R$ we have

$$\left|P_t^\Gamma f(x) - f(x)\right| \leqslant \int_\Gamma p_t^\Gamma(x,y)|f(y) - f(x)|\,\mathrm{d}y + |f(x)|\mathbb{P}_x(\tau_\Gamma \leqslant t). \tag{4.4}$$

Since $P_t f \to f$ uniformly as $t \to 0^+$, for $t > 0$ small enough we get

$$\int_\Gamma p_t^\Gamma(x,y)|f(y) - f(x)|\,\mathrm{d}y \leqslant \int_{\mathbb{R}^d} p_t(x,y)|f(y) - f(x)|\,\mathrm{d}y < \varepsilon.$$

On the other hand,

$$|f(x)|\mathbb{P}_x(\tau_\Gamma \leqslant t) \leqslant \|f\|_\infty \sup_{x\in K} \mathbb{P}_x(\tau_\Gamma \leqslant t),$$

where $K := \operatorname{supp} f$. We have $r := \operatorname{dist}(K, \Gamma^c) > 0$, so

$$\mathbb{P}_x(\tau_\Gamma \leqslant t) \leqslant \mathbb{P}_x(\tau_{B(x,r)} \leqslant t) = \mathbb{P}_0(\tau_{B_r} \leqslant t) \leqslant ctr^{-\alpha} < \varepsilon,$$

for t small enough, see, e.g., [30]. By (4.3) and (4.4) we get, as required,

$$\|P_t^\Gamma f - f\|_{L^1(M_\Gamma)} < \varepsilon + (\varepsilon + \|f\|_\infty \varepsilon)|\Gamma_R| \sup_{\Gamma_R} M_\Gamma. \qquad \square$$

Recall that

$$\|f\|_{q,M_\Gamma} := \|f/M_\Gamma\|_{L^q(M_\Gamma^2)} = \left(\int_\Gamma |f(x)|^q M_\Gamma^{2-q}(x)\,\mathrm{d}x\right)^{\frac{1}{q}} = \|f\|_{L^q(M_\Gamma^{2-q})},$$

if $1 \leqslant q < \infty$, and

$$\|f\|_{\infty,M_\Gamma} := \operatorname*{ess\,sup}_{x\in\Gamma} |f(x)|/M_\Gamma(x).$$

The following characterization of *hypercontractivity* of P_t^Γ is crucial for the proof of (1.3).

Lemma 4.2. *Let $q \in [1,\infty)$. We have*

$$\|P_t^\Gamma f\|_{q,M_\Gamma} \leqslant Ct^{-\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|f\|_{1,M_\Gamma} \tag{4.5}$$

for all $t > 0$ and all nonnegative functions f on $\mathbb{R}^d$ if and only if

$$\sup_{y\in\Gamma} \int_\Gamma \rho_1(x,y)^q M_\Gamma^2(x)\,dx < \infty. \tag{4.6}$$

Proof. Assume (4.6). Let $f \geqslant 0$. With the notation $F := f/M_\Gamma$ we get

$$\|P_1^\Gamma f\|_{q,M_\Gamma} = \left(\int_\Gamma \left(\int_\Gamma \rho_1(x,y)F(y)M_\Gamma^2(y)\,\mathrm{d}y\right)^q M_\Gamma^2(x)\,\mathrm{d}x\right)^{1/q}.$$

Let c be the supremum in (4.6). By Minkowski integral inequality,

$$\begin{aligned}
&\left(\int_\Gamma \left(\int_\Gamma \rho_1(x,y)F(y)M_\Gamma^2(y)\,\mathrm{d}y\right)^q M_\Gamma^2(x)\,\mathrm{d}x\right)^{1/q}\\
\leqslant &\int_\Gamma \left(\int_\Gamma \rho_1(x,y)^q M_\Gamma^2(x)\,\mathrm{d}x\right)^{1/q} F(y)M_\Gamma^2(y)\,\mathrm{d}y\\
\leqslant &c^{1/q} \int_\Gamma F(y)M_\Gamma^2(y)\,\mathrm{d}y\\
= &c^{1/q}\|f\|_{1,M_\Gamma}.
\end{aligned}$$

For $t > 0$, by scaling we get (4.5) as follows:

$$\begin{aligned}\|P_t^\Gamma f\|_{q,M_\Gamma} &= t^{\frac{d+\beta(2-q)}{\alpha q}}\|P_1^\Gamma f(t^{1/\alpha}\,\cdot)\|_{q,M_\Gamma} \leqslant ct^{\frac{d+\beta(2-q)}{\alpha q}}\|f(t^{1/\alpha}\,\cdot)\|_{1,M_\Gamma}\\ &= ct^{\frac{d+\beta(2-q)}{\alpha q}}t^{-\frac{d+\beta}{\alpha}}\|f\|_{1,M_\Gamma} = ct^{-\frac{d+2\beta}{\alpha}\frac{q-1}{q}}\|f\|_{1,M_\Gamma}.\end{aligned}$$

Conversely, assume (4.5). Let $y \in \Gamma$. Let $g_n \geqslant 0$, $n \in \mathbb{N}$, be functions in $C_c^\infty(\Gamma)$ approximating δ_y, the Dirac measure at y, as follows:

$$\int_\Gamma g_n(x)\,\mathrm{d}x = 1, \qquad \text{and} \qquad \lim_{n\to\infty}\int_\Gamma h(x)g_n(x)\,\mathrm{d}x = h(y),$$

for every function h continuous near y. For $f_n := g_n/M_\Gamma$, $\|f_n\|_{1,M_\Gamma} = \|g_n\|_1 = 1$ and

$$P_1^\Gamma f_n(x) = \int_\Gamma p_1^\Gamma(x,z)\frac{g_n(z)}{M_\Gamma(z)}\,\mathrm{d}z \to \frac{p_1^\Gamma(x,y)}{M_\Gamma(y)},$$

as $n \to \infty$. By (4.5) and Fatou's lemma,

$$\begin{aligned}C^q &\geqslant \liminf_{n\to\infty}\|P_1^\Gamma f_n\|_{q,M_\Gamma}^q\\ &= \liminf_{n\to\infty}\int_\Gamma\left|\int_\Gamma\frac{p_1^\Gamma(x,z)}{M_\Gamma(z)}g_n(z)\,\mathrm{d}z\right|^q M_\Gamma^{2-q}(x)\,\mathrm{d}x\\ &\geqslant \int_\Gamma \rho_1(x,y)^q M_\Gamma^2(x)\,\mathrm{d}x.\end{aligned}$$

Since $y \in \Gamma$ was arbitrary, we obtain (4.6). □

Remark 4.3. Of course, (4.5) extends to arbitrary $f \in L^1(M_\Gamma)$.

Example 4.4. As in Example 3.14, we assume that $\beta = 0$. In fact, to simplify notation, let $\Gamma = \mathbb{R}^d$. Then (4.6) is trivially satisfied for every $q \in [1,\infty)$, because $\rho_1(x,y) = p_1(x,y)$ is a bounded density. Therefore, by (4.5), for each $f \in L^1$,

$$\|P_t f\|_q \leqslant Ct^{-\frac{d}{\alpha}\frac{q-1}{q}}\|f\|_1.$$

This agrees with [35], see also [13].

Here is a refinement of Lemma 4.2.

Lemma 4.5. *Let $q \in [1,\infty)$, assume (4.6) and suppose Γ is fat. If $f \in L^1(M_\Gamma)$, $\int_\Gamma f(x)M_\Gamma(x)\,dx = 0$ then*

$$\lim_{t\to\infty} t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}}\|P_t^\Gamma f\|_{q,M_\Gamma} = 0. \tag{4.7}$$

If, additionally, f has compact support, then (4.7) is true for $q = \infty$, too.

Proof. Let $\omega > 0$. First, we prove prove (4.7) for a compactly supported function $f \in L^1(M_\Gamma)$ satisfying

$$\int_\Gamma f(x)M_\Gamma(x)\,\mathrm{d}x = 0. \tag{4.8}$$

Step 1. Case $q = \infty$.
For $t > 0$ we let

$$I(t) := t^{\frac{d+2\beta}{\alpha}}\|P_t^\Gamma f\|_{\infty,M_\Gamma} = t^{\frac{d+2\beta}{\alpha}}\sup_{x\in\Gamma}\left|\int_\Gamma \rho_t(x,y)M_\Gamma(y)f(y)\,\mathrm{d}y\right|.$$

By (4.8),

$$I(t) = t^{\frac{d+2\beta}{\alpha}}\sup_{x\in\Gamma}\left|\int_\Gamma\left(\rho_t(x,y)-\rho_t(x,0)\right)M_\Gamma(y)f(y)\,\mathrm{d}y\right|.$$

Since f has compact support, for sufficiently large $t > 0$ we have

$$
\begin{aligned}
I(t) &= t^{\frac{d+2\beta}{\alpha}} \sup_{x\in\Gamma} \left| \int_{|y|\leqslant t^{\frac{1}{\alpha}}\omega} \left(\rho_t(x,y) - \rho_t(x,0)\right) M_\Gamma(y) f(y)\,\mathrm{d}y \right| \\
&\leqslant t^{\frac{d+2\beta}{\alpha}} \sup_{\substack{x\in\Gamma \\ |y|\leqslant t^{\frac{1}{\alpha}}\omega}} \left|\rho_t(x,y) - \rho_t(x,0)\right| \int_{|y|\leqslant t^{\frac{1}{\alpha}}\omega} M_\Gamma(y)|f(y)|\,\mathrm{d}y \\
&= \sup_{\substack{x\in\Gamma \\ |y|\leqslant\omega}} \left|\rho_1(x,y) - \rho_1(x,0)\right| \int_\Gamma M_\Gamma(y)|f(y)|\,\mathrm{d}y,
\end{aligned}
$$

where in the last line we used scaling (3.6) of ρ. By Theorem 3.8, we can make it arbitrary small by choosing small ω, and (4.7) follows in this case.
Step 2. Case $q = 1$.
For $t > 0$, we let

$$
\begin{aligned}
J(t) := \|P_t^\Gamma f\|_{1,M_\Gamma} &= \int_\Gamma \left| \int_\Gamma p_t^\Gamma(x,y) M_\Gamma(x) f(y)\,\mathrm{d}y \right| \mathrm{d}x \\
&= \int_\Gamma \left| \int_\Gamma \rho_t(x,y) M_\Gamma^2(x) f(y) M_\Gamma(y)\,\mathrm{d}y \right| \mathrm{d}x.
\end{aligned}
$$

Applying (4.8), we get

$$
J(t) \leqslant \int_\Gamma \int_\Gamma \left|\rho_t(x,y) - \rho_t(x,0)\right| M_\Gamma^2(x) |f(y)| M_\Gamma(y)\,\mathrm{d}y\,\mathrm{d}x.
$$

Since f has compact support,

$$
\begin{aligned}
J(t) &\leqslant \int_\Gamma \int_{|y|\leqslant t^{\frac{1}{\alpha}}\omega} \left|\rho_t(x,y) - \rho_t(x,0)\right| M_\Gamma^2(x)|f(y)| M_\Gamma(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant \sup_{|y|\leqslant t^{\frac{1}{\alpha}}\omega} \|\rho_t(\cdot,y) - \rho_t(\cdot,0)\|_{L^1(M_\Gamma^2)} \int_\Gamma M_\Gamma(y)|f(y)|\,\mathrm{d}y,
\end{aligned}
$$

for sufficiently large t. In view of (2.10) and (3.6), by changing variables $t^{-1/\alpha}x \to x$ and $t^{-1/\alpha}y \to y$ we obtain

$$
\begin{aligned}
&\sup_{|y|\leqslant t^{\frac{1}{\alpha}}\omega} \|\rho_t(\cdot,y) - \rho_t(\cdot,0)\|_{L^1(M_\Gamma^2)} \\
&= t^{-\frac{d+2\beta}{\alpha}} \sup_{|y|\leqslant t^{\frac{1}{\alpha}}\omega} \int \left| \rho_1\left(t^{-1/\alpha}x, t^{-1/\alpha}y\right) - \rho_1\left(t^{-1/\alpha}x, 0\right) \right| M_\Gamma^2(x)\,\mathrm{d}x \\
&= \sup_{|y|\leqslant\omega} \|\rho_1(\cdot,y) - \rho_1(\cdot,0)\|_{L^1(M_\Gamma^2)}.
\end{aligned}
$$

By Corollary 3.5, we can make it arbitrary small by choosing small ω, so (4.7) is true.
Step 3. Case $q \in (1,\infty)$.
By Hölder inequality we get that, as $t \to \infty$,

$$
\begin{aligned}
t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f\|_{q,M_\Gamma} &= t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \left(\int_\Gamma \left|P_t^\Gamma f(x)/M_\Gamma(x)\right|^{q-1} \left|P_t^\Gamma f(x) M_\Gamma(x)\right| \mathrm{d}x \right)^{\frac{1}{q}} \\
&\leqslant \left(t^{\frac{d+2\beta}{\alpha}} \|P_t^\Gamma f\|_{\infty,M_\Gamma} \right)^{\frac{q-1}{q}} \|P_t^\Gamma f\|_{1,M_\Gamma}^{\frac{1}{q}} \to 0,
\end{aligned}
$$

since both factors converge to zero as $t \to \infty$ by *Step 1.* and *Step 2.*

Finally, consider arbitrary $f \in L^1(M_\Gamma)$ with $\int_\Gamma f(x)M_\Gamma(x)\,\mathrm{d}x = 0$. Let $R > 0$ and $f_R(x) = (f(x) - c_R)\mathbf{1}_{|x|\leqslant R}$, where $c_R = \int_{|x|\leqslant R} f(x)M_\Gamma(x)\,\mathrm{d}x / \int_{|x|\leqslant R} M_\Gamma(x)\,\mathrm{d}x$. Of course,

$$\int_\Gamma M_\Gamma(x) f_R(x)\,\mathrm{d}x = 0, \tag{4.9}$$

and f_R is compactly supported. Furthermore, due to our assumptions,

$$\begin{aligned}\|f - f_R\|_{L^1(M_\Gamma)} &= |c_R| \int_{|x|\leqslant R} M_\Gamma(x)\,\mathrm{d}x + \int_{|x|>R} M_\Gamma(x)|f(x)|\,\mathrm{d}x \\ &= \Big|\int_{|x|\leqslant R} M_\Gamma(x) f(x)\,\mathrm{d}x\Big| + \int_{|x|>R} M_\Gamma(x)|f(x)|\,\mathrm{d}x \to 0\end{aligned}$$

as $R \to \infty$. Let $\varepsilon > 0$ and choose $R > 0$ so large that

$$\|f - f_R\|_{1,M_\Gamma} < \varepsilon.$$

For $q = 1$, by using the triangle inequality and Lemma 4.1, we get

$$\begin{aligned}\|P_t^\Gamma f\|_{1,M_\Gamma} &\leqslant \|P_t^\Gamma f_R\|_{1,M_\Gamma} + \|P_t^\Gamma (f - f_R)\|_{1,M_\Gamma} \\ &\leqslant \|P_t^\Gamma f_R\|_{1,M_\Gamma} + \|f - f_R\|_{1,M_\Gamma},\end{aligned}$$

and *Step 2.* yields

$$\limsup_{t\to\infty} \|P_t^\Gamma f\|_{1,M_\Gamma} \leqslant \varepsilon,$$

which proves (4.7) in this case.

If $1 < q < \infty$, then using the triangle inequality and Lemma 4.2, we obtain

$$\begin{aligned}t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f\|_{q,M_\Gamma} &\leqslant t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f_R\|_{q,M_\Gamma} + t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma (f - f_R)\|_{q,M_\Gamma} \\ &\leqslant t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f_R\|_{q,M_\Gamma} + C\|f - f_R\|_{1,M_\Gamma}.\end{aligned}$$

By (4.9) and *Step 3.*,

$$\limsup_{t\to\infty} t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f\|_{q,M_\Gamma} \leqslant 2C\varepsilon.$$

This completes the proof of (4.7) for $q \in (1, \infty)$. ☐

Theorem 4.6. *Let $q \in [1, \infty)$, assume (4.6) and suppose Γ is fat. Then for $f \in L^1(M_\Gamma)$ and $A = \int_\Gamma f(x)M_\Gamma(x)\,dx$,*

$$\lim_{t\to\infty} t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f - A\Psi_t\|_{q,M_\Gamma} = 0.$$

Remark 4.7. In view of (3.34) and Lemma 4.1, the constant A in Theorem 4.6 satisfies

$$\int_\Gamma \big(P_t^\Gamma f(x) - A\Psi_s(x)\big) M_\Gamma(x)\,\mathrm{d}x = 0, \quad s, t > 0.$$

Proof of Theorem 4.6. By (2.5), (1.2), Remark 4.7 and Lemma 4.5,

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma f - A\Psi_t\|_{q,M_\Gamma} &= \lim_{t\to\infty} t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_{t+1}^\Gamma f - A\Psi_{t+1}\|_{q,M_\Gamma} \\ &= \lim_{t\to\infty} t^{\frac{d+2\beta}{\alpha}\frac{q-1}{q}} \|P_t^\Gamma \big(P_1^\Gamma f - A\Psi_1\big)\|_{q,M_\Gamma} = 0.\end{aligned}$$ ☐

4.1 Applications

We conclude the article by providing several applications and examples which apply to our results. In particular, we draw the reader's attention to Lemma 4.9, which provides sharp distinction between cones bigger and smaller than the half-space $\mathbb{R}^d_+ := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$. The same behavior is displayed by the bigger class of smooth cones, as we assert in Corollary 4.10. First, we note a simple observation.

Example 4.8. Let $q = 1$. By (3.4), the condition (4.6) holds for every fat cone Γ.

Lemma 4.9. *Let $q \in (1, \infty)$ and suppose Γ is a right-circular cone. Then (4.6) holds if $\beta \geqslant \alpha/2$. Conversely, if $d \geqslant 2$ and $\beta < \alpha/2$, then (4.6) does not hold.*

Proof. Recall that by [10, Theorem 2 and Eq. (3)],

$$p_1^\Gamma(x,y) \approx p_1(x,y) \frac{\big(1 \wedge \delta_\Gamma(x)\big)^{\alpha/2} \big(1 \wedge \delta_\Gamma(y)\big)^{\alpha/2}}{\big(1 \wedge |x|\big)^{\alpha/2-\beta} \big(1 \wedge |y|\big)^{\alpha/2-\beta}}, \quad x, y \in \Gamma. \tag{4.10}$$

Moreover, [28, Lemma 3.3] entails that

$$M_\Gamma(x) \approx \delta_\Gamma(x)^{\alpha/2} |x|^{\beta-\alpha/2}, \quad x \in \mathbb{R}^d.$$

Using this together with (4.10) and (2.4), we infer that, for $x, y \in \Gamma$,

$$\begin{aligned} \rho_1(x,y) &\approx p_1(x,y) \frac{\big(1 \wedge \delta_\Gamma(x)\big)^{\alpha/2} \big(1 \wedge \delta_\Gamma(y)\big)^{\alpha/2}}{\big(1 \wedge |x|\big)^{\alpha/2-\beta} \big(1 \wedge |y|\big)^{\alpha/2-\beta} \delta_\Gamma(x)^{\alpha/2} |x|^{\beta-\alpha/2} \delta_\Gamma(y)^{\alpha/2} |y|^{\beta-\alpha/2}} \\ &\approx \big(1 + |x-y|\big)^{-d-\alpha} \frac{\big(1 + \delta_\Gamma(x)\big)^{-\alpha/2} \big(1 + \delta_\Gamma(y)\big)^{-\alpha/2}}{(1 + |x|)^{\beta-\alpha/2} (1 + |y|)^{\beta-\alpha/2}}. \end{aligned} \tag{4.11}$$

Let $q \in (1, \infty)$ and assume $\beta \geqslant \alpha/2$. Then it follows from (4.11) that $\rho_1(x,y) \lesssim 1$ for $x, y \in \Gamma$, and (3.4) entails that

$$\begin{aligned} \int_\Gamma \rho_1(x,y)^q M_\Gamma^2(x) \, \mathrm{d}x &\leqslant \|\rho_1(x, \cdot\,)\|_\infty^{q-1} \int_\Gamma \rho_1(x,y) M_\Gamma^2(y) \, \mathrm{d}y \\ &= \|\rho_1(x, \cdot\,)\|_\infty^{q-1} \lesssim 1. \end{aligned} \tag{4.12}$$

Thus, we get (4.6) as claimed.

Now assume that $d \geqslant 2$ and $\beta < \alpha/2$. Let $y \in \Gamma$ be such that $\delta_\Gamma(y) = 2$, so that $A := B(y, 1) \subseteq \Gamma$. Then for $x \in A$ one clearly has that $1 + |x - y| \approx 1$ and $\delta_\Gamma(x) \approx 1$. Then it follows from (4.11) that

$$\begin{aligned} \int_\Gamma \rho_1(x,y)^q M_\Gamma^2(x) \, \mathrm{d}x &\geqslant \int_A \rho_1(x,y)^q M_\Gamma^2(x) \, \mathrm{d}x \\ &\approx \int_A (1 + |x|)^{q(\alpha/2-\beta)} (1 + |y|)^{q(\alpha/2-\beta)} \delta_\Gamma(x)^\alpha |x|^{2\beta-\alpha} \, \mathrm{d}x \\ &\approx |y|^{(q-1)(\alpha-2\beta)}. \end{aligned}$$

Since $\alpha - 2\beta > 0$ and $q > 1$, by taking $|y| \to \infty$ we see that (4.6) cannot hold in this case. □

Corollary 4.10. *For $d \geqslant 2$ and smooth cone Γ, (4.6) holds if and only if $\beta \geqslant \alpha/2$.*

Proof. Recall that Γ is open and $C^{1,1}$ outside of the origin. From the harmonicity and homogeneity of M_Γ, by the boundary Harnack principle we get, as in [28, Lemma 3.3], that

$$M_\Gamma(x) \approx \delta_\Gamma(x)^{\alpha/2} |x|^{\beta-\alpha/2}, \quad x \in \Gamma.$$

Moreover, since a smooth cone is fat, its Dirichlet heat kernel satisfies (4.10). Thus, one can directly repeat the proof of Lemma 4.9 to conclude the claim. □

Example 4.11. For $d = 1$, either $\Gamma = (0, \infty)$ or $\Gamma = \mathbb{R} \setminus \{0\}$ and both cases are (trivially) smooth cones, with $\delta_\Gamma(y) = |y|$ for $y \in \Gamma$.

When $\Gamma = (0, \infty)$, then (4.11) yields the boundedness of ρ_1 and (4.6) holds through (4.12), since this Γ is right-circular. Recall that here one has $\beta = \alpha/2$ and $M_\Gamma(x) = (0 \vee x)^{\alpha/2}$ by [3, Example 3.2], and we refer to Example 1.3 for the rest of the summary of this case.

If $\alpha \in (1, 2)$ and $\Gamma = \mathbb{R} \setminus \{0\}$, then $\Gamma^c = \{0\}$ is a nonpolar set, $\beta = \alpha - 1$, and $M_\Gamma(x) = |x|^{\alpha-1}$; see [3, Example 3.3]. Note that in this example, Γ is not right-circular anymore. Nevertheless, by [10, Example 8], the survival probability is $\mathbb{P}_x(\tau_\Gamma > t) \approx (1 \wedge t^{-1/\alpha}|x|)^{\alpha-1}$ and

$$p_1^\Gamma(x, y) \approx (1 + |x - y|)^{-1-\alpha}(1 \wedge |x|)^{\alpha-1}(1 \wedge |y|)^{\alpha-1}, \quad x, y \in \Gamma.$$

Thus,

$$\rho_1(x, y) \approx (1 + |x - y|)^{-1-\alpha}(1 + |x|)^{1-\alpha}(1 + |y|)^{1-\alpha} \lesssim 1, \quad x, y \in \Gamma,$$

and one may apply (4.12) to get (4.6), too. It then follows from Theorem 4.6 with $q = 2$ that $\lim_{t\to\infty} t^{(2\alpha-1)/(2\alpha)} \|P_t^\Gamma f\|_2 = 0$ if $\int_\mathbb{R} f(x)|x|^{\alpha-1}\,\mathrm{d}x = 0$. Accordingly, by Lemma 3.6, the stationary density φ of Theorem 3.4 satisfies $\varphi(x) \approx (1 + |x|)^{-2\alpha}$. Here and below, $x \in \Gamma$ and $t > 0$. Then, $\Psi_t(x) \approx t^{-1}(1 + t^{-1/\alpha}|x|)^{-1-\alpha}(1 \wedge t^{-1/\alpha}|x|)^{\alpha-1}$ and the density of the Yaglom limit is comparable with $\varphi(x)|x|^{\alpha-1} \approx (1 + |x|)^{-2\alpha}|x|^{\alpha-1} \approx \Psi_1(x)$.

Example 4.12. Let $d \geqslant 2$ and Γ be a right-circular cone which is a subset of the half-space $\mathbb{R}^d_+$. Then by [3, Example 3.2 and Lemma 3.3], we have $\beta \geqslant \alpha/2$ and Lemma 4.9 gives (4.6). On the other hand, if Γ is such that $\mathbb{R}^d_+ \subsetneq \Gamma$, then $\beta < \alpha/2$ by [3, Lemma 3.3], and Lemma 4.9 asserts that (4.6) does not hold.

The following extends Example 1.3. Let $\Gamma = \mathbb{R}^d_+$, so that $M_\Gamma(x) = (0 \vee x_d)^{\alpha/2}$. By Theorem 4.6, $\lim_{t\to\infty} t^{(d+\alpha)/(2\alpha)} \|P_t^\Gamma f\|_2 = 0$ if $\int_{\mathbb{R}^d_+} f(x)x_d^{\alpha/2}\,\mathrm{d}x = 0$. Furthermore, the survival probability is $\mathbb{P}_x(\tau_\Gamma > t) \approx 1 \wedge t^{-1/2}(0 \vee x_d)^{\alpha/2}$ and

$$p_1^\Gamma(x, y) \approx (1 + |x - y|)^{-d-\alpha}(1 \wedge x_d)^{\alpha/2}(1 \wedge y_d)^{\alpha/2},$$

see [10, Example 2]. Here and below, $x, y \in \Gamma$. So, $\Psi_t(x) \approx (t^{1/\alpha} \vee |x|)^{-d-\alpha}(t^{1/\alpha} \wedge x_d)^{\alpha/2}$, $t > 0$. The stationary density φ is comparable to $(1+|x|)^{-d-\alpha}(1+x_d)^{-\alpha/2}$ and the density of the Yaglom distribution is $\varphi(x)x_d^{\alpha/2} / \int_{\mathbb{R}^d_+} \varphi(y)y_d^{\alpha/2}\,\mathrm{d}y \approx (1+|x|)^{-d-\alpha}(1 \wedge x_d)^{\alpha/2} \approx \Psi_1(x)$.

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