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# Phase transitions for a unidirectional elephant random walk with a power law memory*

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**Abstract**

For the standard elephant random walk, Laulin (2022) studied the case when the increment of the random walk is not uniformly distributed over the past history instead has a power law distribution. We study such a problem for the unidirectional elephant random walk introduced by Harbola, Kumar and Lindenberg (2014). Depending on the memory parameter $p$ and the power law exponent $\beta$, we obtain three distinct phases in one such phase the elephant travels only a finite distance almost surely, in the other phase there is a positive probability that the elephant travels an infinite distance and in the third phase the elephant travels an infinite distance with probability 1. For the critical case of the transition from the first phase to the second phase, the proof of our result requires coupling with a multi-type branching process.

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## 1 Introduction

The elephant random walk (ERW), introduced by Schütz and Trimper [12], is a one-dimensional discrete time random walk whose incremental steps are $\pm 1$. Unlike the simple random walk, the ERW keeps track of each of the incremental steps taken throughout its history. Formally, $S_0 \equiv 0$ and, for $s \in [0, 1]$,

$$S_1 = X_1 := \begin{cases} +1 & \text{with probability } s \\ -1 & \text{with probability } 1 - s. \end{cases} \tag{1.1}$$

Subsequently, for $n \geq 1$, the location of the ERW at time $(n+1)$ is $S_{n+1} = S_n + X_{n+1}$ where, for $p \in (0, 1)$ and $U_n$, a uniform random variable on $\{1, \ldots, n\}$,

$$X_{n+1} = \begin{cases} +X_{U_n} & \text{with probability } p \\ -X_{U_n} & \text{with probability } 1 - p. \end{cases}$$

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Here we assume that $\{U_k : k \in \mathbb{N}\}$ is a collection of independent random variables.

Depending on the *memory parameter* $p$, we observe three distinct phases, when $0 < p < 3/4$ we have a diffusive phase, when $p = 3/4$ a critical phase, and when $3/4 < p < 1$ a superdiffusive phase (see Baur and Bertoin [1], Coletti *et al.* [5], Bercu [2], and Kubota and Takei [7]). Other properties of the ERW as well as its multidimensional version have been studied, see the excellent thesis of Laulin [8] and the references therein.

Laulin [9] introduced smooth amnesia to the memory of the ERW in the following manner. Let $\beta > 0$ and $\{\beta_{n+1} : n \in \mathbb{N}\}$ be independent random variables with

$$P(\beta_{n+1} = k) = \begin{cases} \dfrac{\beta+1}{n} \cdot \dfrac{\mu_k}{\mu_{n+1}} & \text{for } 1 \le k \le n \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

where

$$\mu_n = \frac{\Gamma(n+\beta)}{\Gamma(n)\Gamma(\beta+1)} \sim \frac{n^\beta}{\Gamma(\beta+1)} \quad \text{as } n \to \infty. \tag{1.3}$$

Here and elsewhere in this article we use the notation $x_n \sim y_n$ to mean that $x_n/y_n \to 1$ as $n \to \infty$. The increments are given by (1.1) and, for $n \ge 1$,

$$X_{n+1} = \begin{cases} +X_{\beta_{n+1}} & \text{with probability } p \\ -X_{\beta_{n+1}} & \text{with probability } 1-p. \end{cases}$$

Laulin [9] showed that the ERW thus obtained exhibits a phase transition in the sense that, it is diffusive for $p < (4\beta+3)/(4\beta+4)$ and superdiffusive for $p > (4\beta+3)/(4\beta+4)$. See Chen and Laulin [3] for higher-dimensional analogues.

Harbola *et al.* [6] introduced a unidirectional ERW where the first step is

$$S_1 = X_1 := \begin{cases} +1 & \text{with probability } s \\ 0 & \text{with probability } 1-s, \end{cases}$$

and subsequently, with the collection $\{U_k : k \in \mathbb{N}\}$ as earlier,

$$\text{if } X_{U_n} = 1 \text{ then } X_{n+1} := \begin{cases} +1 & \text{with probability } p \\ 0 & \text{with probability } 1-p, \end{cases}$$
$$\text{if } X_{U_n} = 0 \text{ then } X_{n+1} := \begin{cases} +1 & \text{with probability } q \\ 0 & \text{with probability } 1-q, \end{cases}$$

here $s \in [0,1]$, $p \in (0,1)$ and $q \in [0,1]$ are the parameters of the process. Harbola *et al.* [6] showed that, for the random walk $S_n := \sum_{k=1}^n X_k$, with $\{X_k : k \in \mathbb{N}\}$ as above,

$$E[S_n] \sim \begin{cases} \dfrac{qn}{1-q} & \text{if } q > 0 \\ \dfrac{sn^p}{\Gamma(1+p)} & \text{if } q = 0, \end{cases}$$

and for $q > 0$, there are three distinct phases depending on $p - q$ (Coletti *et al.* [4]).

When $q = 0$ and $s = 1$ the walk is the 'laziest elephant random walk (LERW)':

$$S_0 = 0 \text{ and } S_n := \sum_{k=1}^n X_k \text{ with } X_1 \equiv 1,\ X_{n+1} = \begin{cases} X_{U_n} & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases} \tag{1.4}$$

Miyazaki and Takei [10] studied this model and showed that $\frac{S_n}{n^p} \xrightarrow{\mathrm{d}} \mathscr{W}$, where $\mathscr{W}$ has a Mittag–Leffler distribution with parameter $p$, and they proved that $\frac{S_n - \mathscr{W} n^p}{\sqrt{\mathscr{W} n^p}} \xrightarrow{\mathrm{d}} N(0,1)$.

Our study here is similar to Laulin [9], however for the LERW given by (1.4) and having a memory distribution (1.2) with $\beta > -1$. In particular, our model is given by $S_0 = 0$, and with $\{\beta_{n+1} : n \in \mathbb{N}\}$ as in (1.2) with $\beta > -1$,

$$S_n := \sum_{k=1}^{n} X_k \text{ with } X_1 \equiv 1,\ X_{n+1} = \begin{cases} X_{\beta_{n+1}} & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases} \tag{1.5}$$

For this model we obtain three distinct rates of growth of $S_n$ depending on the parameter $\beta$. We employ a martingale method and a coupling with a multi-type branching process. The latter is of intrinsic interest because it may be applied to other ERW models. In the next section we state our results and in subsequent sections we present the proofs of our results. We obtain different behaviours of $S_n$ in three different regimes separated by the critical values $\beta = 0$ and $\beta = p/(1-p)$.

Throughout the rest of the article we restrict ourselves to the model given by (1.5).

## 2 Results

Let $\mathscr{F}_0$ be the trivial $\sigma$-field, and $\mathscr{F}_n := \sigma(X_1, \ldots, X_n)$. From (1.2), we have

$$\begin{aligned} E[X_{n+1} \mid \mathscr{F}_n] &= p \cdot E[X_{\beta_{n+1}} \mid \mathscr{F}_n] = p \cdot \sum_{k=1}^{n} X_k P(\beta_{n+1} = k) \\ &= \frac{p(\beta+1)}{n\mu_{n+1}} \sum_{k=1}^{n} X_k \mu_k = \frac{p(\beta+1)}{n\mu_{n+1}} \cdot \Sigma_n, \end{aligned} \tag{2.1}$$

where $\Sigma_n := \sum_{k=1}^{n} X_k \mu_k$ for $n \in \mathbb{N}$. Noting that $\Sigma_1 = X_1 \mu_1 = 1$ a.s. we have

$$E[\Sigma_{n+1} \mid \mathscr{F}_n] = \Sigma_n + E[X_{n+1}\mu_{n+1} \mid \mathscr{F}_n] = \left(1 + \frac{p(\beta+1)}{n}\right) \Sigma_n.$$

For $\gamma > -1$, let

$$c_n(\gamma) := \frac{\Gamma(n+\gamma)}{\Gamma(n)\Gamma(\gamma+1)} \sim \frac{n^\gamma}{\Gamma(\gamma+1)} \quad \text{as } n \to \infty. \tag{2.2}$$

Note that $\mu_n = c_n(\beta)$. Put

$$M_n := \frac{\Sigma_n}{c_n(p(\beta+1))}. \tag{2.3}$$

Since $\{M_n\}$ is a non-negative martingale, there exists a non-negative random variable $M_\infty$ such that $\lim_{n\to\infty} M_n = M_\infty$ almost surely. As a consequence,

$$E[\Sigma_n] = c_n(p(\beta+1)) \cdot E[\Sigma_1] = c_n(p(\beta+1)) \sim \frac{n^{p(\beta+1)}}{\Gamma(p(\beta+1)+1)} \quad \text{as } n \to \infty. \tag{2.4}$$

Our main results are the following. For the ERW as defined in (1.5) we have:

**Theorem 2.1.** *For $\beta > -1$,*

$$E[S_n] = \begin{cases} \dfrac{p(\beta+1)}{p(\beta+1)-\beta} \cdot \dfrac{c_n(p(\beta+1))}{c_n(\beta)} + \dfrac{\beta}{\beta - p(\beta+1)} & \text{if } \beta \neq \dfrac{p}{1-p} \\ \displaystyle\sum_{k=0}^{n-1} \frac{\beta}{k+\beta} & \text{if } \beta = \dfrac{p}{1-p}. \end{cases}$$

Table 1: Summary of the results.

| Regime | Asymptotic behaviour |
|---|---|
| $-1 < \beta < 0$ | $P(S_n \geq Cn^{-\beta}$ for all $n) = 1$,<br>$P\left(M_\infty > 0,\, S_n \sim C(p,\beta)M_\infty n^{p(\beta+1)-\beta}\text{ as } n \to \infty\right) > 0$. |
| $\beta = 0$ | $P\left(M_\infty > 0,\, S_n \sim C(p,0)M_\infty n^{p}\text{ as } n \to \infty\right) = 1$. |
| $0 < \beta < \dfrac{p}{1-p}$ | $P(S_\infty < +\infty) > 0$,<br>$P\left(M_\infty > 0,\, S_n \sim C(p,\beta)M_\infty n^{p(\beta+1)-\beta}\text{ as } n \to \infty\right) > 0$. |
| $\beta = \dfrac{p}{1-p}$ | $E[S_n] \sim \beta \log n$, but $P(S_\infty < +\infty) = 1$. |
| $\beta > \dfrac{p}{1-p}$ | $E[S_\infty] < +\infty$, so $P(S_\infty < +\infty) = 1$. |

Let $C(p,\beta) := \dfrac{1}{p(\beta+1)-\beta} \cdot \dfrac{\Gamma(\beta+1)}{\Gamma(p(\beta+1))}$. As a corollary, from (2.2) we have

**Corollary 2.2.** *(i) If* $-1 < \beta < p/(1-p)$ *then* $\lim_{n\to\infty} \dfrac{E[S_n]}{n^{p(\beta+1)-\beta}} = C(p,\beta)$.

*(ii) If* $\beta = p/(1-p)$ *then* $\lim_{n\to\infty} \dfrac{E[S_n]}{\log n} = \beta$.

*(iii) If* $\beta > p/(1-p)$ *then* $\lim_{n\to\infty} E[S_n] = \dfrac{\beta}{\beta - p(\beta+1)}$*, so* $S_\infty := \lim_{n\to\infty} S_n < +\infty$ *a.s.*

**Theorem 2.3.** *If* $-1 < \beta < p/(1-p)$ *then* $P(M_\infty > 0) > 0$*, and*

$$S_n \sim C(p,\beta)M_\infty n^{p(\beta+1)-\beta} \quad \textit{a.s. on } \{M_\infty > 0\}.$$

**Theorem 2.4.** *(i) If* $\beta > 0$ *then* $P(S_\infty = k) > 0$ *for any* $k \in \mathbb{N}$.
*(ii) If* $\beta < 0$ *then* $P(S_\infty = \infty) = 1$.
*(iii) [Miyazaki and Takei [10]] If* $\beta = 0$ *then* $P(S_\infty = \infty) = 1$.

From Corollary 2.2 (ii) we have $E[S_\infty] = +\infty$ for $\beta = p/(1-p)$, however

**Theorem 2.5.** *If* $\beta = p/(1-p)$ *then* $P(S_\infty < +\infty) = 1$.

The proof of Theorem 2.5 needs coupling with an auxiliary multi-type branching process, which is presented in the next section, while the coupling is discussed in Section 4.4.

In the course of the proofs we also obtain asymptotic behaviour of the growth of $S_n$ and a summary of our results is presented in Table 1.

## 3 A multi-type branching process

Before we describe the branching process we need some analytic lemmas. With $\mu_n$ as defined in (1.3), let $\{q(x,y) : x, y \in \mathbb{N}^2 \cap \{x < y\}\}$, be defined as

$$q(x,y) = p \cdot P(\beta_y = x) = \frac{p(\beta+1)}{y-1} \cdot \frac{\mu_x}{\mu_y}. \tag{3.1}$$

**Lemma 3.1.** *Assume that* $\beta > 0$*. For each* $k \in \mathbb{N}$, $\displaystyle\sum_{x=k+1}^{\infty} q(k,x) = \frac{p(\beta+1)}{\beta}$.

*Proof.* For $K > k$, we have $\displaystyle\sum_{x=k+1}^{K} q(k,x) = p(\beta+1) \cdot \frac{\Gamma(k+\beta)}{\Gamma(k)} \sum_{x=k+1}^{K} \frac{\Gamma(x-1)}{\Gamma(x+\beta)}$. Now note that, for $\ell \geq 2$, $a \geq -1$, $b \geq 0$ and $b \neq a+1$,

$$\frac{\Gamma(\ell+a)}{\Gamma(\ell+b)} = \frac{1}{a-b+1}\left(\frac{\Gamma(\ell+1+a)}{\Gamma(\ell+b)} - \frac{\Gamma(\ell+a)}{\Gamma(\ell-1+b)}\right). \tag{3.2}$$

Continuing from above we have $\displaystyle\sum_{x=k+1}^{K} q(k,x) = \frac{p(\beta+1)}{\beta} \cdot \left(1 - \frac{\Gamma(k+\beta)}{\Gamma(k)} \cdot \frac{\Gamma(K)}{\Gamma(K+\beta)}\right)$. Noting that $\Gamma(K)/\Gamma(K+\beta) \sim K^{-\beta}$ as $K \to \infty$, we obtain the desired conclusion. □

Now we construct the branching process. Let $\{Z_1, Z_2, \ldots\}$ be a process where $Z_k$ denotes particles of the $k$-th generation and $\mathbb{N}$ be the space of types of particles.

(1) The first generation $Z_1$ consists of only one particle of type $y^{(1)} \equiv 1$.
(2) The particle of type $y^{(1)}$ in $Z_1$ gives birth to second generation particles each of whose type is $y^{(1)} + 1$ or more, *with the caveat that all its children are of distinct types*. The probability that the particle in $Z_1$ has a child of type $k > 1$ is $q(y^{(1)}, k)$. We also assume that the event that the particle has one child of type $k$ and another of type $\ell$ ($k \neq \ell$) are independent of each other.
(3) In case there is no particle in the second generation, we stop. Otherwise a particle of the second generation with type $y^{(2)}$ gives birth to a third generation particle of type $y^{(2)} + 1$ or more *with the caveat that all its children are of distinct types*. We assume that (i) the number and types of children of two distinct particles are independent of each other, (ii) the events that a particle has one child of type $k$ and another of type $\ell$ ($k \neq \ell$) are independent of each other. The probability that this particle has a child of type $k > y^{(2)}$ is $q(y^{(2)}, k)$.
(4) In case there is no particle in the $n$-th generation, we stop. Otherwise, first we note here that, for $n \geq 3$, there may be two particles in $Z_n$ of the same type born of two distinct parents. We assume that (i) the number and types of children of two distinct particles are independent of each other, (ii) the events that a particle has one child of type $k$ and another of type $\ell$ ($k \neq \ell$) are independent of each other. A particle of type $y^{(n)}$ in the $n$-th generation gives birth independently to a $(n+1)$-th generation particle of type $k > y^{(n)}$ with probability $q(y^{(n)}, k)$.

From Lemma 3.1 we see that the expected number of children of a particle of any type is $p(\beta+1)/\beta$, which is smaller or equal to one if and only if $\beta \geq p/(1-p)$. However, we still need to prove that the branching process dies out in this case, because the progeny distribution is different and depends on the type.

Let $C = C(p) := \{-\log(1-p)\}/p$. It is straightforward to see that $1 - t \geq e^{-Ct}$ for $t \in [0, p]$. Since $0 \leq q(k,x) \leq p$ for any $k \in \mathbb{N}$ and $x \geq k+1$, we have

$$p_0^{(k)} \geq \exp\left\{-C \sum_{x=k+1}^{\infty} q(k,x)\right\} = \exp\left\{-\frac{Cp(\beta+1)}{\beta}\right\} > 0 \quad \text{for all } k \in \mathbb{N}. \tag{3.3}$$

Let $N_n$ be the number of particles in the $n$-th generation of the branching process. Assume that $\beta > 0$, and so $m := p(\beta+1)/\beta < \infty$. We have $E[N_n] = m^{n-1}$ for $n \in \mathbb{N}$.

**Lemma 3.2.** *If* $\beta \geq p/(1-p)$ *then* $\lim_{n\to\infty} P(N_n > 0) = 0$.

*Proof.* If $\beta > p/(1-p)$ then $m < 1$. So, from (3.3) and Markov's inequality,

$$P(N_n > 0) \leq E[N_n] = m^{n-1} \to 0 \quad \text{as } n \to \infty.$$

Hereafter we assume that $\beta = p/(1-p)$, i.e. $m = 1$. For $K \in \mathbb{N}$, by Markov's inequality,

$$P(N_n > 0) = P(0 < N_n \leq K) + P(N_n > K) \leq P(0 < N_n \leq K) + \frac{1}{K+1}.$$

To obtain the desired conclusion it suffices to show that $\lim_{n\to\infty} P(0 < N_n \leq K) = 0$ for any $K \in \mathbb{N}$. Assume that $\limsup_{n\to\infty} P(0 < N_n \leq K) = \delta > 0$. We can find a subsequence

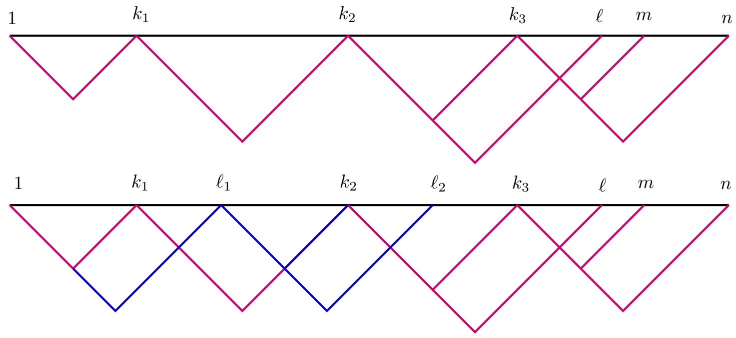


Figure 1: The figure on top represents the chain of memory involved in a positive increment at the $n$th step. The figure below represents the many ways at individual of type $n$ may appear in the branching process.

$\{n_j\}$ with $1 \leq n_1 < n_2 < \cdots$ and $P(0 < N_{n_j} \leq K) \geq \delta/2$ for $j \in \mathbb{N}$. For $j \in \mathbb{N}$, define $E_j := \{0 < N_{n_j} \leq K,\, N_{n_j+1} = 0\}$. Noting that $\{E_j : j \in \mathbb{N}\}$ are mutually disjoint,

$$1 \geq P\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} P(E_j) \geq \sum_{j=1}^{\infty} \frac{\delta}{2} \cdot \left(\inf_{k \in \mathbb{N}} p_0^{(k)}\right)^K = +\infty.$$

This is a contradiction. □

Thus, from Lemma 3.2, we have that the multi-type branching process dies out if and only if $m \leq 1$, i.e. $\beta \geq p/(1-p)$.

We end this section by explaining the connection between the ERW and this branching process with the help of Figure 1. The figure represents the event that the increment of the ERW at the $n$th step is $1$. This occurs because (i) at the $n$th step the elephant recalled the $k_3$th step *correctly*, (ii) at the $k_3$th step the elephant recalled the $k_2$th step *correctly*, (iii) at the $k_2$th step the elephant recalled the $k_1$th step *correctly* and (iv) at the $k_1$th step the elephant recalled the first step *correctly*. This occurs with probability $q(k_3, n)q(k_2, k_3)q(k_1, k_2)q(1, k_1)$. The figure at the bottom represents the event that an individual of type $n$ is born. In the figure there are two ways in which this could happen (i) the individual in generation $1$ gave birth to a child of type $k_1$, who gave birth to a child of type $k_2$, who gave birth to a child of type $k_3$ and who gave birth to a child of type $n$; (ii) alternately, the individual in generation $1$ gave birth to a child of type $l_1$, who gave birth to a child of type $k_2$, who gave birth to a child of type $k_3$ and who gave birth to a child of type $n$. The probability that the chain given in (i) occurred is $q(k_3, n)q(k_2, k_3)q(k_1, k_2)q(1, k_1)$, while the probability that the chain given in (ii) occurred is $q(k_3, n)q(k_2, k_3)q(l_1, k_2)q(1, l_1)$. A similar analysis may be done for the increments at the $m$th and the $\ell$th steps.

From the above it may be seen that, constructing both the ERW and the branching process on the same probability space, a chain of remembrances leading to an increment of the ERW at the $n$th step is also a chain of ancestry of an individual of the $n$th type.

# 4 Proofs

## 4.1 Proof of Theorem 2.1

Noting that by (2.1) and (2.4), $E[S_{n+1}] - E[S_n] = E[X_{n+1}] = \dfrac{p(\beta+1) \cdot c_n(p(\beta+1))}{n \cdot c_{n+1}(\beta)}$, we have

$$E[S_n] = E[S_1] + \sum_{k=1}^{n-1} \frac{p(\beta+1)\cdot c_k(p(\beta+1))}{k\cdot c_{k+1}(\beta)} = 1 + \frac{\Gamma(\beta+1)}{\Gamma(p(\beta+1)+1)} \sum_{k=1}^{n-1} \frac{\Gamma(k+p(\beta+1))}{\Gamma(k+\beta+1)}.$$

This together with (3.2) implies Theorem 2.1. □

### 4.2 Proof of Theorem 2.3

Regarding the non-negative martingale $M_n = \Sigma_n / c_n(p(\beta+1))$ we have

**Proposition 4.1.** *$\{M_n : n \in \mathbb{N}\}$ is an $L^2$-bounded martingale if and only if $\beta < p/(1-p)$. In particular, if $\beta < p/(1-p)$ then $P(M_\infty > 0) > 0$.*

*Proof.* Using (2.1) we have $E[\Sigma_{n+1}^2 \mid \mathscr{F}_n] = \left(1 + \dfrac{2p(\beta+1)}{n}\right)\cdot \Sigma_n^2 + \mu_{n+1}\cdot \dfrac{p(\beta+1)}{n}\cdot \Sigma_n$. Setting $L_n := \Sigma_n^2 / c_n(2p(\beta+1))$, we have, for some $C(\beta) > 0$,

$$E[L_{n+1}] - E[L_n] = c_{n+1}(\beta)\cdot \frac{p(\beta+1)\cdot E[\Sigma_n]}{n\cdot c_{n+1}(2p(\beta+1))} \sim C(\beta)\cdot n^{\beta - p(\beta+1)-1} \quad \text{as } n\to\infty. \tag{4.1}$$

Note that $\dfrac{E[M_n^2]}{E[L_n]} = \dfrac{c_n(2p(\beta+1))}{c_n(p(\beta+1))^2} \sim \dfrac{\Gamma(p(\beta+1)+1)^2}{\Gamma(2p(\beta+1)+1)}$ as $n\to\infty$. Using (4.1), we see that $\sup_{n\ge 1} E[M_n^2] < +\infty$ if and only if $\beta - p(\beta+1) < 0$. □

Theorem 2.3 follows from Proposition 4.1 and the following lemma.

**Lemma 4.2.** *If $-1 < \beta < p/(1-p)$ then $S_n \sim C(p,\beta) M_\infty n^{p(\beta+1)-\beta}$ a.s. on $\{M_\infty > 0\}$.*

*Proof.* Recalling (2.1), almost surely on $\{M_\infty > 0\}$, we have

$$E[X_{n+1} \mid \mathscr{F}_n] \sim \frac{p(\beta+1)}{n}\cdot \frac{\Gamma(\beta+1)}{n^\beta}\cdot \frac{M_\infty \cdot n^{p(\beta+1)}}{\Gamma(p(\beta+1)+1)} = \frac{\Gamma(\beta+1) M_\infty}{\Gamma(p(\beta+1))}\cdot n^{p(\beta+1)-\beta-1}$$

as $n\to\infty$. Define $A_n := \sum_{k=1}^n E[X_k \mid \mathscr{F}_{k-1}]$. From the conditional Borel–Cantelli lemma (see e.g. Williams [11], p.124),

$$S_\infty < +\infty \text{ a.s. on } \{A_\infty < +\infty\}, \text{ while } S_n \sim A_n \text{ as } n\to\infty, \text{ a.s. on } \{A_\infty = +\infty\}.$$

If $-1 < \beta < p/(1-p)$ then $A_n \sim C(p,\beta) M_\infty n^{p(\beta+1)-\beta}$ a.s. on $\{M_\infty > 0\}$, which completes the proof. □

**Remark 4.3.** If $\beta > p/(1-p)$, i.e. $p(\beta+1)-\beta-1 < -1$, then $A_\infty < +\infty$ and $S_\infty < +\infty$ a.s. (This is another proof of Corollary 2.2 (iii).) When $\beta = p/(1-p)$ we have $S_n \sim \beta M_\infty \log n$ a.s. on $\{M_\infty > 0\}$, but as we will see later $P(M_\infty > 0) = 0$ in this case.

### 4.3 Proof of Theorem 2.4

We assume that $\beta > 0$. Fix $k \in \mathbb{N}$. First note that $P(X_1 = 1) = 1$, and

$$P(X_1 = \cdots = X_k = 1) \ge \prod_{j=2}^k p\cdot P(\beta_j = 1) > 0 \quad \text{for } k = 2,3,\ldots.$$

On the other hand, $P(X_{k+1} = 0 \mid X_1 = \cdots = X_k = 1) = 1-p$, and

$$P(X_n = 0 \mid X_1 = \cdots = X_k = 1,\, X_{k+1} = \cdots = X_{n-1} = 0) \ge 1 - p\cdot P(\beta_n \le k)$$

for $n \ge k+2$. Since $P(\beta_n \le k) = \dfrac{\beta+1}{n-1}\cdot \dfrac{\sum_{j=1}^k \mu_j}{\mu_n} \sim \dfrac{\Gamma(\beta+2)}{n^{1+\beta}}\cdot \displaystyle\sum_{j=1}^k \mu_j$ as $n\to\infty$, we have $P(S_\infty = k) \ge P(X_1 = \cdots = X_k = 1)\cdot \prod_{n=k}^\infty \{1 - p\cdot P(\beta_n \le k)\} > 0$. This proves part (i) of the theorem.

To prove part (ii) we need the following

**Lemma 4.4.** *If $-1<\beta<0$, then there exists a positive constant $C=C(\beta,p)$ such that*

$$P\left(A_n \geq Cn^{-\beta} \text{ for all } n, \text{ and } S_n \sim A_n \text{ as } n\to\infty\right)=1.$$

*Proof.* Recall that $P(X_1=1)=1$ and $E[X_1]=1$. For each $k=2,3,\ldots$,

$$E[X_k \mid \mathscr{F}_{k-1}] \geq p\cdot P(\beta_k=1)=\frac{p(\beta+1)}{k\mu_{k+1}}.$$

The conclusion follows from $\mu_{k+1}\sim k^{\beta}/\Gamma(\beta+1)$ as $k\to\infty$. □

As stated after (1.4), part (iii) of the theorem is proved in Miyazaki and Takei [10]. □

### 4.4 Proof of Theorem 2.5

Throughout this subsection we assume that $\beta>0$.

To exhibit the coupling of $\{S_n : n\in\mathbb{N}\}$ and the branching process $\{Z_n : n\in\mathbb{N}\}$ we introduce a modified model. Let $\{\beta(i,j) : 1\leq i<j<\infty\}$ be a collection of $\{0,1\}$-valued independent random variables such that $P(\beta(i,j)=1)=P(\beta_j=i)$. We also assume that the collection $\{\beta(i,j) : 1\leq i<j<\infty\}$ is independent of all random processes considered so far. Let $Y_1=1$ and $Y_{n+1}=W_{n+1}\cdot\max_{1\leq i\leq n}\beta(i,n+1)Y_i$ for $n\in\mathbb{N}$, where $\{W_n : n\geq 1\}$ is an i.i.d. collection of Bernoulli($p$) random variables.

We put $\eta_n := \{k\in\mathbb{N} : k\leq n, Y_k=1\}$, $T_n := \#\eta_n=\sum_{k=1}^n Y_k$ and note that, $\{\eta_n : n\geq 1\}$ is a Markov process with

$$P(\eta_{n+1}=\zeta\cup\{n+1\}\mid \eta_n=\zeta)=P(W_{n+1}\cdot\beta(i,n+1)=1 \text{ for some } i\in\zeta) \quad \text{for } \zeta\subseteq\mathbb{N}.$$

Using the Skorokhod representation theorem and defining the processes $\eta_n$ and the first $n$ generations of the branching process on the same probability space, we have, from the discussion at the end of Section 3,

$$T_n \leq \#\{k\in\mathbb{N} : \text{there is a particle of type } k \text{ in the first } n \text{ generations}\} \quad \text{a.s.}$$

Thus the extinction of the branching process implies $T_\infty := \lim_{n\to\infty} T_n<\infty$ a.s.

Next, to study the relation between $\{S_n\}$ and $\{T_n\}$ we need to study the relation between $\xi_n$ and $\eta_n$, where $\xi_n := \{k\in\mathbb{N} : k\leq n, X_k=1\}$. Note that $\{\xi_n : n\geq 1\}$ is a Markov process with $P(\xi_{n+1}=\zeta\cup\{n+1\}\mid\xi_n=\zeta)=P(W_{n+1}=1,\,\beta_{n+1}\in\zeta)$ for $\zeta\subseteq\mathbb{N}$.

**Coupling of $\xi_n$ and $\eta_n$.** Noting that each of $\xi_n$ and $\eta_n$ are Markovian, the coupling is given as follows: For each $n\geq 1$, if $\xi_n\neq\eta_n$ then $\beta_{n+1}$ and $\{\beta(i,n+1) : 1\leq i\leq n\}$ are independent, while if $\xi_n=\eta_n$ then we ensure that

$$\{\beta_{n+1}=i\}\supseteq\{\beta(i,n+1)=1 \text{ and } \beta(j,n+1)=0 \text{ for } j\in\xi_n\setminus\{i\}\}.$$

For fixed $n$ and $\zeta\subset\{1,\ldots,n\}$, we fix some notation: Let $B_i := \{\beta(i,n+1)=1\}$, $C_i := \{\beta(i,n+1)=1 \text{ and } \beta(j,n+1)=0 \text{ for } j\in\zeta\setminus\{i\}\}$, and $D_i := \{\beta_{n+1}=i\}$.

**Lemma 4.5.** *For $\zeta\subset\{1,\ldots,n\}$, we have*

$$P(X_{n+1}\neq Y_{n+1}\mid\xi_n=\eta_n=\zeta)\leq 3p\cdot(\#\zeta)^2\cdot\frac{(\beta+1)^2}{n^2}. \tag{4.2}$$

*Proof.* We first note that, for any $n\in\mathbb{N}$ and $\zeta\subset\{1,\ldots,n\}$,

$$\begin{aligned}&P(X_{n+1}=1,\,Y_{n+1}=0\mid\xi_n=\eta_n=\zeta)=P\left(\{W_{n+1}=1\}\cap(\cup_{i\in\zeta}D_i\setminus\cup_{i\in\zeta}B_i)\right)\\&\leq p\cdot\sum_{i\in\zeta}P(D_i\setminus C_i)=p\cdot\sum_{i\in\zeta}\{P(D_i)-P(C_i)\},\end{aligned}$$

where the last line follows from the coupling in the case $\xi_n = \eta_n$, and in the inequality we used $D_i \supseteq C_i$ for $i \in \zeta$. Note that $\{C_i\}$ and $\{D_i\}$ are disjoint, and $P(D_i) = P(B_i)$. The inclusion-exclusion formula gives

$$\sum_{i\in\zeta} P(D_i) - P\Big(\bigcup_{i\in\zeta} B_i\Big) = \sum_{i\in\zeta} P(B_i) - P\Big(\bigcup_{i\in\zeta} B_i\Big) \le \sum_{i,j\in\zeta : i\neq j} P(B_i \cap B_j).$$

Because $\bigcup_{i\in\zeta} B_i \setminus \bigcup_{i\in\zeta} C_i = \bigcup_{i,j\in\zeta : i\neq j} B_i \cap B_j$,

$$P\Big(\bigcup_{i\in\zeta} B_i\Big) - \sum_{i\in\zeta} P(C_i) = P\Big(\bigcup_{i\in\zeta} B_i\Big) - P\Big(\bigcup_{i\in\zeta} C_i\Big) \le \sum_{i,j\in\zeta : i\neq j} P(B_i \cap B_j).$$

Next

$$\begin{aligned} &P(X_{n+1} = 0, Y_{n+1} = 1 \mid \xi_n = \eta_n = \zeta) = P\left(\{W_{n+1} = 1\} \cap (\cup_{i\in\zeta} B_i \setminus \cup_{i\in\zeta} D_i)\right) \\ &\le p \cdot P(\cup_{i\in\zeta} B_i \setminus \cup_{i\in\zeta} C_i) \le p \cdot \sum_{i,j\in\zeta : i\neq j} P(B_i \cap B_j). \end{aligned}$$

Noting that $B_i$ and $B_j$ are independent for $i \neq j$, and $\{\mu_n\}$ is increasing when $\beta > 0$, we obtain

$$\sum_{i,j\in\zeta : i\neq j} P(B_i \cap B_j) \le (\#\zeta)^2 \cdot \left(\frac{\beta+1}{n\mu_{n+1}} \max_{1\le i\le n} \mu_i\right)^2 \le (\#\zeta)^2 \cdot \frac{(\beta+1)^2}{n^2}.$$

Combining the above, we have (4.2). □

Let $\Lambda_n := \{\xi_n = \eta_n,\, \#\xi_n \le n^{(1-\delta)/2}\}$. Note that, by successive conditioning

$$P(\xi_\ell = \eta_\ell \text{ for all } \ell > n \mid \Lambda_n) \ge \prod_{\ell=n}^{\infty} P(\Lambda_{\ell+1} \mid \Lambda_\ell). \tag{4.3}$$

Because $P(S_{\ell+1} \le (\ell+1)^{(1-\delta)/2} \mid \Lambda_\ell) = P(\#\xi_{\ell+1} \le (\ell+1)^{(1-\delta)/2} \mid \#\xi_\ell \le \ell^{(1-\delta)/2}) \ge P(\#\xi_{\ell+1} \le (\ell+1)^{(1-\delta)/2}) = P(S_{\ell+1} \le (\ell+1)^{(1-\delta)/2})$ by the FKG inequality, we have

$$\begin{aligned} P(\Lambda_{\ell+1} \mid \Lambda_\ell) &\ge P(\xi_{\ell+1} = \eta_{\ell+1} \mid \Lambda_\ell) - P(S_{\ell+1} > (\ell+1)^{(1-\delta)/2} \mid \Lambda_\ell) \\ &\ge 1 - P(X_{\ell+1} \neq Y_{\ell+1} \mid \Lambda_\ell) - P(S_{\ell+1} > (\ell+1)^{(1-\delta)/2}). \end{aligned} \tag{4.4}$$

Suppose that

$$\sum_{n=1}^{\infty} P(S_n > n^{(1-\delta)/2}) < \infty. \tag{4.5}$$

Choose $N$ such that $\sum_{n=N}^{\infty} \frac{3(\beta+1)^2}{n^{1+\delta}} + \sum_{n=N}^{\infty} P(S_n > n^{(1-\delta)/2}) < \varepsilon$. From (4.2), (4.3), and (4.4), we have $P(\xi_\ell = \eta_\ell \text{ for all } \ell > N \mid \Lambda_N) \ge 1 - \varepsilon$. From Lemma 3.2, we have that the multi-type branching process dies out if and only if $m \le 1$, i.e. $\beta \ge p/(1-p)$. Hence, $P\left(T_\infty < \infty\right) = 1$ whenever $\beta \ge p/(1-p)$. For any $n \ge 1$, we have that $\#\eta_n \le n$ and so $P\left(T_\infty < \infty \mid \eta_n\right) = 1$ almost surely. Hence,

$$\begin{aligned} &P(S_\infty < \infty \mid S_N \le N^{(1-\delta)/2}) = P(S_\infty < \infty \mid \Lambda_N) \\ &\ge P(S_\infty < \infty, \xi_\ell = \eta_\ell \text{ for all } \ell > N \mid \Lambda_N) \\ &\ge P(T_\infty < \infty \mid T_N \le N^{(1-\delta)/2}) - P(\xi_\ell \neq \eta_\ell \text{ for some } \ell > N \mid \Lambda_N) \ge 1 - \varepsilon. \end{aligned}$$

Since $P(S_N \le N^{(1-\delta)/2}) \ge 1-\varepsilon$ from the property of $N$, we have

$$P(S_\infty < \infty) = P(S_\infty < \infty \mid S_N \le N^{(1-\delta)/2}) \cdot P(S_N \le N^{(1-\delta)/2}) \ge (1-\varepsilon)^2.$$

As $\varepsilon > 0$ is arbitrary, to complete the proof of $P(S_\infty < \infty) = 1$ for $\beta = p/(1-p)$, we need to establish (4.5) whose proof is obtained by using the Markov inequality and Proposition 4.6 for $k = 3$ and $\ell = 0$ stated below. □

**Proposition 4.6.** *Assume that $p \in (0,1)$ and $\beta = p/(1-p)$. For $k \in \{1,2,3\}$ and $\ell \in \{0,1,\dots,k\}$, we have that*

$$E\left[(S_n)^{k-\ell}(\Sigma_n)^{\ell}\right] \sim C_{k,\ell} \cdot n^{\ell\beta}(\log n)^{2k-1-\ell} \quad \textit{as } n \to \infty, \tag{4.6}$$

*where $C_{k,\ell}$ is a positive constant.*

To prove Proposition 4.6, we need the following lemma.

**Lemma 4.7.** *Let $\gamma \ge 0$. Assume that a sequence $\{x_n\}$ satisfies*

$$x_1 = c, \quad x_{n+1} = \left(1 + \frac{\gamma}{n}\right) x_n + f_n \quad \textit{for } n \in \mathbb{N}, \tag{4.7}$$

*where*

$$\frac{f_n}{c_{n+1}(\gamma)} \sim \frac{C(\log n)^m}{n} \quad \textit{as } n \to \infty \tag{4.8}$$

*for some $C > 0$ and $m \in \mathbb{Z}_+$. Then there exists a constant $K = K(\gamma, m) > 0$ such that $x_n \sim K n^\gamma (\log n)^{m+1}$ as $n \to \infty$.*

*Proof.* Putting $y_n := x_n / c_{n+1}(\gamma)$, we have that

$$y_n = y_1 + \sum_{k=1}^{n-1}(y_{k+1} - y_k) = \frac{c}{c_{n+1}(\gamma)} + \sum_{k=1}^{n-1} \frac{f_k}{c_{k+1}(\gamma)}.$$

From (4.8), $y_n \sim \{C(\log n)^{m+1}\}/(m+1)$ as $n \to \infty$. Since $c_{n+1}(\gamma) \sim n^\gamma/\Gamma(\gamma+1)$ as $n \to \infty$, we obtain the desired conclusion. □

*Proof of Proposition 4.6.* Assume that $\beta = p/(1-p)$, and so $p(\beta+1) = \beta$. For the case $k = 1$, Eq. (2.4) and Corollary 2.2 (ii) implies (4.6) with $C_{1,0} = \beta$ and $C_{1,1} = 1/\Gamma(\beta)$.

We turn to the case $k = 2$. Since

$$E[(\Sigma_{n+1})^2 \mid \mathscr{F}_n] = \left(1 + \frac{2\beta}{n}\right) \cdot (\Sigma_n)^2 + \frac{\beta\mu_{n+1}}{n} \cdot \Sigma_n,$$

we have that $x_n = E[(\Sigma_n)^2]$ satisfies (4.7) with $c = 1$, $\gamma = 2\beta$ and $f_n = \beta\mu_{n+1}E[\Sigma_n]/n$. From (2.4), we can see that (4.8) holds with $m = 0$. Lemma 4.7 implies that

$$E[(\Sigma_n)^2] \sim C_{2,2} n^{2\beta} \log n \quad \text{as } n \to \infty. \tag{4.9}$$

To obtain the asymptotics of $E[S_n \Sigma_n]$, note that

$$E[S_{n+1}\Sigma_{n+1} \mid \mathscr{F}_n] = \left(1 + \frac{\beta}{n}\right) \cdot S_n \Sigma_n + \frac{\beta}{n\mu_{n+1}} \cdot (\Sigma_n)^2 + \frac{\beta}{n} \cdot \Sigma_n.$$

We see that $x_n = E[S_n\Sigma_n]$ satisfies (4.7) with $c = 1$, $\gamma = \beta$ and $f_n = \dfrac{\beta E[(\Sigma_n)^2]}{n\mu_{n+1}} + \dfrac{\beta E[\Sigma_n]}{n}$. From (2.4) and (4.9), we have that (4.8) holds with $m = 1$. Lemma 4.7 implies that

$$E[S_n\Sigma_n] \sim C_{2,1} n^\beta (\log n)^2 \quad \text{as } n \to \infty. \tag{4.10}$$

As for $E[(S_n)^2]$, we have $E[(S_{n+1})^2 \mid \mathscr{F}_n] = (S_n)^2 + \dfrac{2\beta}{n\mu_{n+1}} \cdot S_n\Sigma_n + \dfrac{\beta}{n\mu_{n+1}} \cdot \Sigma_n$. Since $x_n = E[(S_n)^2]$ satisfies $c = 1$, $\gamma = 0$ and $f_n = \dfrac{2\beta E[S_n\Sigma_n]}{n\mu_{n+1}} + \dfrac{\beta E[\Sigma_n]}{n\mu_{n+1}}$. From (2.4) and (4.10), we can see that (4.8) holds with $m = 2$. Lemma 4.7 implies that $E[(S_n)^2] \sim C_{2,0}(\log n)^3$ as $n \to \infty$.

The case $k = 3$ can be handled in a similar manner. We can obtain the following equations:

$$\begin{aligned}
&E[(\Sigma_{n+1})^3 \mid \mathscr{F}_n] = \left(1 + \frac{3\beta}{n}\right) \cdot (\Sigma_n)^3 + \frac{3\beta\mu_{n+1}}{n} \cdot (\Sigma_n)^2 + \frac{3\beta\mu_{n+1}^2}{n} \cdot \Sigma_n, \\
&E[S_{n+1}(\Sigma_{n+1})^2 \mid \mathscr{F}_n] \\
&= \left(1 + \frac{2\beta}{n}\right) \cdot S_n(\Sigma_n)^2 + \frac{\beta\mu_{n+1}}{n} \cdot S_n\Sigma_n + \frac{\beta}{n\mu_{n+1}} \cdot (\Sigma_n)^3 + \frac{2\beta}{n} \cdot (\Sigma_n)^2 + \cdot\frac{\beta\mu_{n+1}}{n} \cdot \Sigma_n, \\
&E[(S_{n+1})^2\Sigma_{n+1} \mid \mathscr{F}_n] \\
&= \left(1 + \frac{\beta}{n}\right) \cdot (S_n)^2\Sigma_n + \frac{2\beta}{n\mu_{n+1}} \cdot S_n(\Sigma_n)^2 + \frac{2\beta}{n} \cdot S_n\Sigma_n + \frac{\beta}{n\mu_{n+1}} \cdot (\Sigma_n)^2 + \frac{\beta}{n} \cdot \Sigma_n, \\
&E[(S_{n+1})^3 \mid \mathscr{F}_n] = (S_n)^3 + \frac{3\beta}{n\mu_{n+1}} \cdot (S_n)^2\Sigma_n + \frac{3\beta}{n\mu_{n+1}} \cdot S_n\Sigma_n + \frac{\beta}{n\mu_{n+1}} \cdot \Sigma_n.
\end{aligned}$$

Using Lemma 4.7, $E\left[(S_n)^{3-\ell}(\Sigma_n)^{\ell}\right] \sim C_{3,\ell} \cdot n^{\ell\beta}(\log n)^{5-\ell}$ for $\ell = 3, 2, 1, 0$ can be successively proved. □

## 5 Concluding remarks

In Section 4.4 we introduced a comparison with the multi-type branching process. Here we introduce another comparison with the LERW, and obtain some different bounds.

Assume that $-1 < \beta < 0$. By (2.1),

$$P(X_{n+1} = 1 \mid \mathscr{F}_n) = \frac{p(\beta+1)}{n} \cdot \frac{n}{n+\beta} \cdot S_n \geq p(\beta+1) \cdot \frac{S_n}{n}. \tag{5.1}$$

Consider the LERW $\{S'_n = \sum_{k=1}^n X'_k\}$ defined by

$$X'_1 \equiv 1, \quad X'_{n+1} = \begin{cases} X'_{U_n} & \text{with probability } p(\beta+1) \\ 0 & \text{with probability } 1 - p(\beta+1). \end{cases}$$

We have

$$P(X'_{n+1} = 1 \mid \mathscr{F}'_n) = p(\beta+1) \cdot \frac{S'_n}{n} \quad \text{for } n \in \mathbb{N}, \tag{5.2}$$

where $\mathscr{F}'_n$ is the $\sigma$-algebra generated by $X'_1, \dots, X'_n$. By (5.1) and (5.2), we can construct a coupling such that with probability one,

$$S_n \geq S'_n \quad \text{for all } n \in \mathbb{N}, \tag{5.3}$$

and so with probability one, there exists a positive constant $C$ such that

$$S_n \geq Cn^{p(1+\beta)} \quad \text{for all } n \in \mathbb{N}.$$

Note that $-1 < \beta < 0$ and $p(1+\beta) > -\beta$ if and only if $-p/(1+p) < \beta < 0$. The above argument gives a better lower bound than Lemma 4.4.

For $\beta > 0$, using a similar argument leading to (5.3) we can show that, with probability one, $S_n \le S'_n$ for all $n \in \mathbb{N}$, and there exists a positive constant $C$ such that $S_n \le Cn^{p(1+\beta)}$ for all $n \in \mathbb{N}$. Note that $p(1+\beta) < 1/2$ if $p < 1/2$ and $0 < \beta < (1-2p)/2p$. As $(1-2p)/2p \ge p/(1-p)$ is equivalent to $p \le 1/3$,

$$\text{if } p \le \frac{1}{3} \text{ then } p(1+\beta) < \frac{1}{2} \text{ for all } \beta \in \left(0, \frac{p}{1-p}\right).$$

In view of (4.5), we hope this is useful for comparison with the branching process.

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