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Large deviation principle for complex solution to squared Bessel SDE*

Arnab Chowdhury† Atul Shekhar‡

Abstract

Complex solutions to squared Bessel SDEs appear naturally in relation to Schramm-Loewner evolutions. We prove the large deviation principle for such solutions as the dimension parameter tends to $-\infty$.

Keywords: large deviation principle; Bessel processes; Schramm-Loewner-evolutions.
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1 Introduction

1.1 Context

In this article we prove the Large deviation principle (LDP) for the complex solution to squared Bessel SDE. For a precise definition of a LDP and usual notions related to it, we refer to [1, 2]. For $\delta \geq 0$, the classical δ-dimensional squared Bessel process is the non-negative solution to the squared Bessel SDE

$$dX_t = 2\sqrt{X_t}dB_t + \delta dt, \quad X_0 = x \geq 0, \tag{1.1}$$

where B is a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see [[4]-ChapterXI]. In relation to Schramm-Loewner-Evolutions (SLEs), it is natural to consider a variant of (1.1) for $\delta < 0$ and with complex valued solutions. More precisely, for $\eta > 0$ (we write $\eta = -\delta$), we consider the SDE

$$dY_t = 2A_t dB_t - \eta dt, \quad Y_0 = 0, \tag{1.2}$$

where Y_t, A_t are complex valued adapted processes (w.r.t. the filtration of B) such that $A_t^2 = Y_t$ and $Im(A_t) \geq 0$. Note that for upper half plane $\mathbb{H} := \{x + iy | y > 0\}$, the square root function $\sqrt{z} : \mathbb{C} \setminus [0, \infty) \to \mathbb{H}$ is a conformal bijection. As such, if $Y_t \in \mathbb{C} \setminus [0, \infty)$ for some t, then $A_t = \sqrt{Y_t}$. Otherwise, if $Y_t \in [0, \infty)$, then A_t makes a choice from $\pm\sqrt{|Y_t|}$ in an adapted way[1]. In other words, A is an adapted branch chosen from all possible

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†Tata Institute of Fundamental Research-CAM, Bangalore, India. E-mail: arnab2020@tifrbng.res.in
‡Tata Institute of Fundamental Research-CAM, Bangalore, India. E-mail: atul@tifrbng.res.in

[1] If $z \in \mathbb{C} \setminus [0, \infty)$, we write $\sqrt{z}$ for its complex square root so that $\sqrt{z} \in \mathbb{H}$. If $z \in [0, \infty)$, we write $\sqrt{z}$ to mean the standard non-negative square root of z. Note that for $x > 0$, $\pm\sqrt{x}$ are two possible limit points of $\sqrt{z}$ as $z \to x$ in $\mathbb{C} \setminus [0, \infty)$.

square roots of Y. We thus refer to A as a branch square root of Y. It is proven in [5] that if Y is any solution to (1.2), then almost surely $Y_t \in \mathbb{C} \setminus [0,\infty)$ for all $t > 0$. As such, $A_t = \sqrt{Y_t}$ for all $t > 0$ and (1.2) is equivalent to

$$dY_t = 2\sqrt{Y_t}dB_t - \eta dt, \;\; Y_0 = 0. \tag{1.3}$$

The existence and uniqueness of strong solution to (1.3) is a consequence of the Rohde-Schramm estimate [6], see [[5]-Theorem 1.5]. In this article we prove the LDP for solutions Y^η as $\eta \to \infty$. The corresponding LDP result for X^δ as $\delta \to \infty$ was proven in [7].

1.2 Main result

Similarly as for X^δ in [7], we translate the LDP for Y^η into a small noise LDP as follows. Set $\varepsilon = 1/\sqrt{\eta}$ and $Z^\varepsilon_t = Y^\eta_t/\eta$. Then, Z^ε solves

$$dZ^\varepsilon_t = -dt + 2\varepsilon\sqrt{Z^\varepsilon_t}dB_t, \;\; Z^\varepsilon_0 = 0. \tag{1.4}$$

A LDP for the process Z^ε as $\varepsilon \to 0+$ falls in the framework of Freidlin-Wentzell theory which has been extensively studied in the literature, see [23, 25, 26, 24, 30, 16, 21, 22, 27, 29] and references therein for several related works. However, since our setup is rather specific, the existing literature does not give an out-of-the-box statement to imply our main result (at least to best of our knowledge). Note that for the process Z^ε, the diffusive vector field which is the complex square root is not even continuous on $\mathbb{C}$. One can alternatively view Z^ε as a $\mathbb{C} \setminus (0,\infty)$ valued process, but $\mathbb{C} \setminus (0,\infty)$ is not a complete space and $\sqrt{\cdot}$ is not Lipschitz on $\mathbb{C} \setminus (0,\infty)$. Another key distinguishing feature of (1.4) is that even though it is a two dimensional real valued system of equations, the noise term B is only one dimensional.

We now state our main result. Let $C_0([0,T],\mathbb{C}) = \{\varphi : [0,T] \to \mathbb{C} \mid \varphi \text{ is continuous and } \varphi_0 = 0\}$ be equipped with the uniform metric. We view Z^ε as a $C_0([0,T],\mathbb{C})$ valued random variable and denote the law of Z^ε by μ^ε. Let us first describe the LDP rate function I for $\{\mu^\varepsilon\}_{\varepsilon>0}$. The rate function I is finite for functions φ which satisfy condition: (H1) $\varphi \in C_0([0,T],\mathbb{C})$ such that $\varphi_t \in \mathbb{C} \setminus [0,\infty)$ for all $t > 0$, (H2) φ_t is absolutely continuous, i.e. both $Re(\varphi_t)$ and $Im(\varphi_t)$ are absolutely continuous, and (H3) $(\dot\varphi_t+1)/(2\sqrt{\varphi_t})$ is real valued with $(\dot\varphi_t+1)/(2\sqrt{\varphi_t}) \in L^2([0,T],\mathbb{R})$. Let $\mathcal{D}([0,T],\mathbb{C}) := \{\varphi \mid \varphi \text{ satisfies H1, H2, H3}\}$.

Theorem 1.1. *The family $\{\mu^\varepsilon\}_{\varepsilon>0}$ satisfies the LDP with speed ε^2 and a good rate function $I(\cdot)$ defined by*

$$I(\varphi) = \begin{cases} \int_0^T \frac{(\dot\varphi_t+1)^2}{8\varphi_t}dt \text{ if } \varphi \in \mathcal{D}([0,T],\mathbb{C}), \\ +\infty \text{ otherwise.} \end{cases} \tag{1.5}$$

Remark 1.2. The process Z^ε is scale invariant, i.e. for any $\lambda > 0$, $\{\lambda Z^\varepsilon_{\lambda^{-1}t}\}_{t\geq 0}$ has the same law as $\{Z^\varepsilon_t\}_{t\geq 0}$. As a consequence, the rate function $I(\cdot)$ should also invariant under the transformation $\{\varphi_t\}_{t\geq 0} \mapsto \{\lambda\varphi_{\lambda^{-1}t}\}_{t\geq 0}$. It can be easily verified from the explicit form of the rate function given in (1.5) that this is indeed the case.

1.3 Motivation

The process Y^η is related to SLE$_\kappa$ as follows: for $\kappa < 4$ and $\eta = 4/\kappa - 1$,

$$\{\sqrt{\kappa Y^\eta(T-t,T,0)}\}_{t\in[0,T]} \overset{d}{=} \{\gamma_t\}_{t\in[0,T]}, \tag{1.6}$$

where γ_t is the SLE$_\kappa$ curve and $\{Y^\eta(s,t,z)\}_{0\leq s\leq t, z\in\overline{\mathbb{H}}}$ is the flow of solutions obtained by solving (1.3) with the initial condition $Y_s = z \in \overline{\mathbb{H}}$, see [[5]-Corollary 1.7] for details. Our

motivation to prove a LDP for Y^η comes from the work of Y. Wang [8, 9, 10] on the LDP for SLE_κ as $\kappa \to 0+$. A LDP result for SLE_κ with respect to Hausdorff metric was proven in [3]. To establish a LDP for SLE_κ in the (stronger) uniform metric, one can utilise (1.6) and reduce this problem to proving a LDP for the stochastic flow of (1.3), see [13, 18] for some results in that direction. A natural first step in this approach is to prove a LDP for the solution Y^η itself which is addressed in this paper (note that $\eta \to \infty$ as $\kappa \to 0+$). We note that a LDP result for SLE_κ with respect to uniform metric (but in an incomplete space (S, τ)) has been established by V. Guskov [12]. However, the LDP for Y^η does not follow from results of [12].
As a corollary of Theorem 1.1, one can obtain a large deviation estimate for the tip of SLE_κ. Using (1.6), the tip γ_T^κ of SLE_κ is given by $\sqrt{\kappa Y_T^\eta}$, where $\eta = 4/\kappa - 1$. Theorem 1.1 can hence be applied to obtain a LDP for γ_T^κ. This can be compared to results of [19] which describes the exact law of the tip γ_T^κ. The corresponding rate function in the LDP for γ_T^κ is given by $I(z) = \inf\{I(\varphi) \mid \varphi \text{ joins } 0 \text{ to } z^2\}$, where $z \in \mathbb{H}$ and $I(\varphi)$ is given by (1.5). In the language of [8], this is the minimum Loewner energy required for a curve to pass through z. This can be explicitly computed and it turns out to be $-8\log(\sin(\arg(z)))$. This was already computed in [8] using probabilistic methods. A more direct deterministic proof has been given by T. Mesikepp [20]. A yet another proof of this fact can also be obtained by using Euler-Lagrange equation to directly compute the minimum value $I(z) = \inf\{I(\varphi) \mid \varphi \text{ joins } 0 \text{ to } z^2\}$ from (1.5). Since this computation is long and not the main point of this paper, we do not present the details here.
Theorem 1.1 is also a natural variant of LDP for squared Bessel processes X^δ as proven in [7]. It follows from central limit theorem and the additive property of squared Bessel processes (cf. [[7], Equation (1.3)]) that as $\delta \to \infty$,

$$\left\{\sqrt{\delta}\left(\frac{X_t^\delta}{\delta} - t\right)\right\}_{0\le t\le T} \xrightarrow{d} \left\{\sqrt{2}B_{t^2}\right\}_{0\le t\le T}. \tag{1.7}$$

In our setting, the process Y^η does not satisfy the additive property. Nevertheless, we can write

$$\sqrt{\eta}\left(\frac{Y_t^\eta}{\eta} + t\right) = \frac{1}{\varepsilon}\left(Z_t^\varepsilon + t\right) = 2\int_0^t \sqrt{Z_r^\varepsilon}\, dB_r.$$

It can be easily verified that $\{Z_t^\varepsilon\}_{t\in[0,T]} \to \{-t\}_{t\in[0,T]}$ in $L^2(\mathbb{P})$ as $\varepsilon \to 0$. Hence, it follows that as $\eta \to \infty$,

$$\left\{\sqrt{\eta}\left(\frac{Y_t^\eta}{\eta} + t\right)\right\}_{0\le t\le T} \xrightarrow{d} \left\{2i\int_0^t \sqrt{r}\, dB_r\right\}_{0\le t\le T} \stackrel{d}{=} \left\{\sqrt{2}iB_{t^2}\right\}_{0\le t\le T}. \tag{1.8}$$

Hence, even though Y^η does not satisfy the additivity property, we do have the above variant of CLT for Y^η. Obtaining a LDP for Y^η is a natural next step.

1.4 Idea of the proof of Theorem 1.1

To prove Theorem 1.1, we use the standard argument based on exponential martingales. We first show that the family $\{\mu^\varepsilon\}_{\varepsilon>0}$ is exponentially tight which is an easy consequence of estimates in [17] (Lemma 2.1 below). Then, we prove the weak upper and the weak lower bound (Proposition 3.2 and Proposition 3.6 below). The weak upper bound is obtained by weighting $\mathbb{P}$ with the exponential martingale $M_{f,g}^\varepsilon(Z^\varepsilon)$ (see (3.5) below). The choice of this appropriate martingale $M_{f,g}^\varepsilon(Z^\varepsilon)$ is a key observation of this paper. The weak upper bound is completed by obtaining a variational description of the rate function I which in itself is a two dimensional functional optimisation problem, see Proposition 3.3. For the weak lower bound, we use the classical change of measure appearing in Cameron-Martin theorem, see (3.20). The weak lower bound (3.18) boils down to Proposition 3.7 which is another key input of this paper, see Remark 3.8. The proof of Proposition 3.7 relies crucially on results of [11], particularly on the uniqueness of solution to (2.2), see Lemma 2.2 below. The paper [11] is a foundation for this paper and its results are used repeatedly in several instances.

Remark 1.3. As shown in [7], the LDP for X^δ can be obtained via two other methods besides the approach using exponential martingales: (1) by using an infinite dimensional Cramer's theorem approach which is based on additivity property of X^δ, and (2) by using contraction principle applied to Bessel processes $\sqrt{X^\delta}$, which in turn is based on the work of McKean [14, 15] giving the continuity of the associated Itô map. However, these two approaches fail to apply to Y^η. The process Y^η does not satisfy the additivity property. Also, the technique of [14, 15] does not apply to $\sqrt{Y^\eta}$ and the associated Itô map is not well defined.

Organization of the paper

In section 2 we recall some known results which will be useful in the proof of Theorem 1.1. Section 3 contains the proof of Theorem 1.1.

2 Preliminaries

In this section we recall some results which will be used in the proof of Theorem 1.1.

2.1 Cameron-Martin perturbations

Let $H_0^1([0,T],\mathbb{R}) = \{h : [0,T] \to \mathbb{R} \mid h_0 = 0, \dot{h} \in L^2([0,T],\mathbb{R})\}$ be the Cameron-Martin space equipped with the norm

$$||h||_{H_0^1} = \sqrt{\int_0^T \dot{h}_r^2 dr}.$$

Also define the Hölder (semi)norms for $\alpha \in (0,1]$,

$$||h||_\alpha := \sup_{0\le s<t\le T} \frac{|h_t - h_s|}{|t-s|^\alpha}.$$

For $h \in H_0^1([0,T],\mathbb{R})$, we will need to consider $Z_t^{\varepsilon,h}$ which are solutions to

$$dZ_t^{\varepsilon,h} = -dt + 2\sqrt{Z_t^{\varepsilon,h}}(\varepsilon dB_t + dh_t), \quad Z_0^{\varepsilon,h} = 0. \tag{2.1}$$

Using Girsanov theorem, $Z_t^{\varepsilon,h}$ has the same almost sure properties as Z_t^ε. The existence and uniqueness of strong solution $Z_t^{\varepsilon,h}$ to (2.1) follow similarly as for (1.4). We will write $\sqrt{Z_t^{\varepsilon,h}} = U_t^{\varepsilon,h} + iV_t^{\varepsilon,h}$ and $\sqrt{Z_t^\varepsilon} = U_t^\varepsilon + iV_t^\varepsilon$.

Lemma 2.1 (Lemma 2.1 in [17]). *For $U_t^{\varepsilon,h}, V_t^{\varepsilon,h}$ as above, we have*

$$|U_t^{\varepsilon,h}| \le 2 \sup_{s\in[0,t]} (\varepsilon|B_s| + |h_s|),$$

and

$$V_t^{\varepsilon,h} \le \sqrt{(\varepsilon^2+1)t}.$$

We will also need to consider solutions φ^h which solves

$$d\varphi_t^h = -dt + 2A_t dh_t, \quad \varphi_0^h = 0, \tag{2.2}$$

where A_t is $\overline{\mathbb{H}}$-valued measurable function such that $A_t^2 = \varphi_t^h$, i.e. $A_t = A_t(\varphi^h)$ is a branch square root of φ^h similarly as described in Section 1. Following results from [11] are crucial inputs in the proof of Theorem 1.1.

Lemma 2.2 (Proposition 2.6 in [11]). *Let $h \in H_0^1([0,T],\mathbb{R})$. For any solution $(\varphi^h, A(\varphi^h))$ of* (2.2)*, $A_t \in \mathbb{C}\setminus[0,\infty)$ for all $t>0$. Hence, $A_t = \sqrt{\varphi_t^h}$ and* (2.2) *is equivalent to*

$$d\varphi_t^h = -dt + 2\sqrt{\varphi_t^h}dh_t, \quad \varphi_0^h = 0. \tag{2.3}$$

Furthermore,

$$\liminf_{t\to 0+} \frac{Im(\sqrt{\varphi_t^h})}{\sqrt{t}} > 0,$$

and φ^h is the unique solution to (2.3).

Remark 2.3. The above result is in fact true under the assumption that h is bounded variation and it satisfies a certain slowpoint condition, see [11]. It can be easily checked that Cameron-Martin functions h satisfy this slowpoint condition.

Lemma 2.4 (Proposition 3.1 in [11]). *Let $h_n, h \in H_0^1([0,T],\mathbb{R})$ such that $h_n \to h$ uniformly as $n\to\infty$. Further assume that $\sup_n ||h_n||_{H_0^1} < \infty$. Then, φ^{h_n} converges to φ^h uniformly.*

Lemma 2.5 (Lemma 2.4 in [11]). *Let $\varphi^n, \varphi \in C_0([0,T],\mathbb{C})$ such that $\varphi^n \to \varphi$ uniformly. Suppose for all n and $t>0$, $\varphi_t^n \in \mathbb{C}\setminus[0,\infty)$. Then, there exists a subsequence φ^{n_k} and a branch square root $A = A(\varphi)$ of φ such that $\sqrt{\varphi^{n_k}}$ converges uniformly to A.*

3 Proof of Theorem 1.1

3.1 Goodness of rate function I

Recall the rate function $I(\varphi)$ from (1.5). Note that $I(\varphi) < \infty$ if and only if $\varphi \in \mathcal{D}([0,T],\mathbb{C})$. Hence, for Lebesgue almost every t,

$$\frac{\dot{\varphi}_t + 1}{2\sqrt{\varphi_t}} = \dot{h}_t$$

for some $h \in H_0^1([0,T],\mathbb{R})$. In other words, φ solves (2.3). Using Lemma 2.2, it follows that $\varphi = \varphi^h$. Hence, $I(\varphi) < \infty$ if and only if $\varphi = \varphi^h$ for some $h \in H_0^1([0,T],\mathbb{R})$. In that case, we have

$$I(\varphi) = I(\varphi^h) = \frac{1}{2}\int_0^T \dot{h}_r^2 dr.$$

To show that I is a good rate function, we check that level sets $\{\varphi \big| I(\varphi) \le L\}$ is sequentially compact for all $L \ge 0$. Let $\varphi_n \in$ be a sequence such that $I(\varphi_n) \le L$. Then, $\varphi_n = \varphi^{h_n}$ for some $h_n \in H_0^1([0,T],\mathbb{R})$ with $||h_n||_{H_0^1} \le \sqrt{2L}$. Since $(H_0^1, ||\cdot||_{H_0^1})$ is a Hilbert space, its closed balls are weakly compact. Hence, there exists a subsequence h_{n_k} converging weakly in H_0^1 to some $h_\infty \in H_0^1([0,T],\mathbb{R})$ with $||h_\infty||_{H_0^1} \le \sqrt{2L}$. Also, since $||h_n||_{H_0^1} \le \sqrt{2L}$, it follows that $\sup_n ||h_n||_{1/2} < \infty$. By Arzela-Ascoli theorem, possibly along a further subsequence, h_{n_k} converges uniformly to h_∞. Using Lemma 2.4, we obtain that $\varphi^{h_{n_k}}$ converges uniformly to $\varphi^{h_\infty} \in \{\varphi \big| I(\varphi) \le L\}$. This implies that $\{\varphi \big| I(\varphi) \le L\}$ is compact.

3.2 Exponential tightness

We prove that the family $\{\mu^\varepsilon\}$ is tight, and it is exponentially tight as well. More precisely:

Proposition 3.1. *For any $\alpha \in (0,1/2)$,*

$$\lim_{R\to+\infty} \sup_{\varepsilon\in(0,1)} \varepsilon^2 \log \mathbb{P}(||Z^\varepsilon||_\alpha \ge R) = -\infty.$$

Also, for any $h \in H_0^1([0,T],\mathbb{R})$,

$$\lim_{R\to\infty} \sup_{\varepsilon\in(0,1)} \mathbb{P}(||Z^{\varepsilon,h}||_\alpha \ge R) = 0.$$

Proof. Fix $\alpha \in (0,1/2)$. We write $Z_t^\varepsilon = -t + 2(M_t^\varepsilon + iN_t^\varepsilon)$, where

$$M_t^\varepsilon = \varepsilon \int_0^t \mathfrak{Re}(\sqrt{Z_r^\varepsilon})dB_r = \varepsilon \int_0^t U_r^\varepsilon dB_r, \text{ and } N_t^\varepsilon = \varepsilon \int_0^t \mathfrak{Im}(\sqrt{Z_r^\varepsilon})dB_r = \varepsilon \int_0^t V_r^\varepsilon dB_r$$

are local martingales. Clearly, it suffices to prove that for $f^\varepsilon = M^\varepsilon, N^\varepsilon$

$$\lim_{R\to+\infty} \sup_{\varepsilon\in(0,1)} \varepsilon^2 \log \mathbb{P}(||f^\varepsilon||_\alpha \geq R) = -\infty. \tag{3.1}$$

Using the Garsia-Rumsey-Rodemich (GRR) inequality, ref. [Lemma 1.1, [28]], with $\Psi(x) = e^{c\varepsilon^{-2}x} - 1$ and $p(x) = x^{\frac{1}{2}}$, where $0 < c < 1/2$ is properly chosen constant, we obtain for $f^\varepsilon = M^\varepsilon$ and $f^\varepsilon = N^\varepsilon$

$$|f_t^\varepsilon - f_s^\varepsilon| \leq \frac{8\varepsilon^2}{c} \int_0^{|t-s|} \log\left(1 + \frac{4K^\varepsilon}{u^2}\right) d\sqrt{u} \leq \frac{8\varepsilon^2}{c}(t-s)^{\frac{1}{2}} \left[\log(T^2 + 4K^\varepsilon) + 4\log\frac{e}{\sqrt{t-s}}\right],$$

where

$$K^\varepsilon := \int_0^T \int_0^T \left\{\exp\left(c\varepsilon^{-2}\frac{|f_t^\varepsilon - f_s^\varepsilon|}{|t-s|^{1/2}}\right) - 1\right\} dsdt.$$

It follows that

$$||f^\varepsilon||_\alpha \lesssim_T \varepsilon^2(\log(K^\varepsilon + 1) + 1). \tag{3.2}$$

Using Markov inequality, this implies that

$$\begin{aligned}
\sup_{\varepsilon\in(0,1)} \varepsilon^2 \log \mathbb{P}(||f^\varepsilon||_\alpha \geq R) &\leq \sup_{\varepsilon\in(0,1)} \varepsilon^2 \log \mathbb{P}(K^\varepsilon \geq e^{R/\varepsilon^2 - 1} - 1) \\
&\leq \sup_{\varepsilon\in(0,1)} \{\varepsilon^2 \log \mathbb{E}(K^\varepsilon) - \varepsilon^2 \log(e^{R/\varepsilon^2-1} - 1)\} \\
&\leq \sup_{\varepsilon\in(0,1)} \varepsilon^2 \log \mathbb{E}(K^\varepsilon) - \inf_{\varepsilon\in(0,1)} \varepsilon^2 \log(e^{R/\varepsilon^2-1} - 1).
\end{aligned}$$

Note that $\inf_{\varepsilon\in(0,1)} \varepsilon^2 \log(e^{R/\varepsilon^2-1} - 1) \to \infty$ as $R \to \infty$. Hence, to obtain (3.1), it suffices to verify that

$$\sup_{\varepsilon\in(0,1)} \varepsilon^2 \log \int_0^T \int_0^T \mathbb{E}\left[\exp\left(c\varepsilon^{-2}\frac{|f_t^\varepsilon - f_s^\varepsilon|}{|t-s|^{1/2}}\right)\right] dsdt < \infty. \tag{3.3}$$

We now use an exponential martingale inequality: for any continuous local martingale Y with $Y_0 = 0$, $\mathbb{E}(e^{\lambda|Y_t|}) \leq 2[\mathbb{E}(e^{2\lambda^2[Y]_t})]^{1/2}$. Therefore, using Lemma 2.1, we have for $f_t^\varepsilon = M_t^\varepsilon$

$$\begin{aligned}
\mathbb{E}\left[\exp\left(c\varepsilon^{-2}\frac{|M_t^\varepsilon - M_s^\varepsilon|}{|t-s|^{1/2}}\right)\right] &\leq 2\left[\mathbb{E}\left(\exp\left(\frac{2c^2\varepsilon^{-2}}{(t-s)}\int_s^t (U_r^\varepsilon)^2 dr\right)\right)\right]^{\frac{1}{2}} \\
&\leq 2\left[\mathbb{E}\left(\exp\left(8c^2 \sup_{r\in[0,T]} B_r^2\right)\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

The c is chosen small enough so that the right hand side above is finite using Fernique theorem. This implies (3.3) for $f_t^\varepsilon = M_t^\varepsilon$. For $f_t^\varepsilon = N_t^\varepsilon$, again using Lemma 2.1, we similarly have

$$\mathbb{E}\left[\exp\left(c\varepsilon^{-2}\frac{|N_t^\varepsilon - N_s^\varepsilon|}{|t-s|^{1/2}}\right)\right] \leq 2\left[\mathbb{E}\left(\exp\left(\frac{2c^2\varepsilon^{-2}}{(t-s)}\int_s^t (V_r^\varepsilon)^2 dr\right)\right)\right]^{\frac{1}{2}} \leq 2\exp\left(c^2\varepsilon^{-2}(\varepsilon^2+1)T\right)$$

which implies (3.3) for $f_t^\varepsilon = N_t^\varepsilon$.
Also, it easily follows from (3.2) and estimates above that $\sup_{\varepsilon\in(0,1)} \mathbb{E}(||Z^\varepsilon||_\alpha) < \infty$, which implies the tightness of $\{Z^\varepsilon\}_{\varepsilon\in(0,1)}$. The tightness of $\{Z^{\varepsilon,h}\}_{\varepsilon\in(0,1)}$ follows similarly. □

3.3 Upper bound

We now prove the LDP upper bound in Theorem 1.1. Since Z^ε is exponentially tight, it suffices to prove:

Proposition 3.2. *For $\varphi \in C_0([0,T],\mathbb{C})$, let $\mathcal{B}_r(\varphi)$ be the closed ball of radius r around φ. Then,*

$$\lim_{r\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(Z^\varepsilon\in\mathcal{B}_r(\varphi))\le -I(\varphi). \tag{3.4}$$

For proving the above claim, we will weight probabilities by exponential martingale $M^\varepsilon_{f,g}$ defined by

$$\begin{aligned}M^\varepsilon_{f,g}(Z^\varepsilon) &= \mathcal{E}\bigg(\frac{1}{\varepsilon}\int_0^T (f_rU^\varepsilon_r + g_rV^\varepsilon_r)dB_r\bigg)\\ &= \exp\bigg(\frac{1}{\varepsilon^2}\bigg(\int_0^T f_r\varepsilon U^\varepsilon_r dB_r + \int_0^T g_r\varepsilon V^\varepsilon_r dB_r\bigg)\\ &\qquad - \frac{1}{2\varepsilon^2}\int_0^T (f_r^2(U^\varepsilon_r)^2 + g_r^2(V^\varepsilon_r)^2 + 2f_rg_rU^\varepsilon_rV^\varepsilon_r)dr\bigg),\end{aligned} \tag{3.5}$$

where $f,g\in C^1([0,T],\mathbb{R})$. Note that we will need to have martingale $M^\varepsilon_{f,g}(Z^\varepsilon)$ to be parametrised by two functions f,g. This is owing to the fact that even though B is real valued, Z^ε is complex valued. Since Z^ε solves (1.4), we have

$$d(\mathfrak{Re}(Z^\varepsilon_t)+t) = d((U^\varepsilon_t)^2-(V^\varepsilon_t)^2+t) = 2\varepsilon U^\varepsilon_t dB_t,\quad d(\mathfrak{Im}(Z^\varepsilon_t))/2 = d(U^\varepsilon_tV^\varepsilon_t) = \varepsilon V^\varepsilon_t dB_t. \tag{3.6}$$

Therefore,

$$\begin{aligned}M^\varepsilon_{f,g}(Z^\varepsilon) = \exp\bigg(&\frac{1}{2\varepsilon^2}\bigg(\int_0^T f_rd(\mathfrak{Re}(Z^\varepsilon_r)+r) + \int_0^T g_rd(\mathfrak{Im}(Z^\varepsilon_r))\bigg)\\ &- \frac{1}{2\varepsilon^2}\int_0^T (f_r^2\frac{|Z^\varepsilon_r|+\mathfrak{Re}(Z^\varepsilon_r)}{2} + g_r^2\frac{|Z^\varepsilon_r|-\mathfrak{Re}(Z^\varepsilon_r)}{2} + f_rg_r\mathfrak{Im}(Z^\varepsilon_r))dr\bigg).\end{aligned}$$

Correspondingly, for any $\xi\in C_0([0,T],\mathbb{C})$, we define

$$M^\varepsilon_{f,g}(\xi) := \exp\bigg(\frac{1}{\varepsilon^2}J_{f,g}(\xi)\bigg),$$

where

$$\begin{aligned}J_{f,g}(\xi) :=& \frac{1}{2}\bigg(\int_0^T f_rd(\mathfrak{Re}(\xi_r)+r) + \int_0^T g_rd(\mathfrak{Im}(\xi_r))\bigg)\\ &- \frac{1}{2}\int_0^T (f_r^2\frac{|\xi_r|+\mathfrak{Re}(\xi_r)}{2} + g_r^2\frac{|\xi_r|-\mathfrak{Re}(\xi_r)}{2} + f_rg_r\mathfrak{Im}(\xi_r))dr.\end{aligned} \tag{3.7}$$

Note that, since $f,g\in C^1([0,T],\mathbb{R})$, the first two integrals appearing above is well defined for any continuous ξ as a Riemann-Stieltjes integral[2]. Furthermore, using integration by parts formula,

$$\int_0^T f_rd(\mathfrak{Re}(\xi_r)+r) = f_T(\mathfrak{Re}(\xi_T)+T) - \int_0^T (\mathfrak{Re}(\xi_r)+r)df_r,$$

and

$$\int_0^T g_rd(\mathfrak{Im}(\xi_r)) = g_T\mathfrak{Im}(\xi_T) - \int_0^T \mathfrak{Im}(\xi_r)dg_r.$$

Therefore, for each fixed $f,g\in C^1([0,T],\mathbb{R})$, the function $\xi\mapsto J_{f,g}(\xi)$ is continuous on $C_0([0,T],\mathbb{C})$. We further claim that:

[2] For continuous functions X,Y, using integration by parts, the Riemann Stieltjes integral $\int X_rdY_r$ is well defined if either of X or Y is of bounded variation.

Proposition 3.3. *For each $\varphi \in C_0([0,T],\mathbb{C})$ and the functional J defined as in (3.7),*

$$\sup_{f,g \in C^1([0,T],\mathbb{R})} J_{f,g}(\varphi) = I(\varphi). \tag{3.8}$$

The proof of Proposition 3.3 is postponed till section 3.5. As a result of this, we have:

Proof of Proposition 3.2. Since $M^\varepsilon_{f,g}(Z^\varepsilon)$ is a positive local martingale, it is a supermartingale. Hence, $\mathbb{E}(M^\varepsilon_{f,g}(Z^\varepsilon)) \le 1$. This implies that

$$\begin{aligned}\mathbb{P}(Z^\varepsilon \in \mathcal{B}_r(\varphi)) &= \mathbb{E}\left(1_{\mathcal{B}_r(\varphi)}(Z^\varepsilon)\frac{M^\varepsilon_{f,g}(Z^\varepsilon)}{M^\varepsilon_{f,g}(Z^\varepsilon)}\right)\\ &\le \sup_{\xi \in \mathcal{B}_r(\varphi)} (M^\varepsilon_{f,g}(\xi))^{-1}\mathbb{E}(M^\varepsilon_{f,g}(Z^\varepsilon))\\ &\le \sup_{\xi \in \mathcal{B}_r(\varphi)} (M^\varepsilon_{f,g}(\xi))^{-1}.\end{aligned}$$

This implies, using the continuity of $\xi \mapsto M^\varepsilon_{f,g}(\xi)$,

$$\lim_{r\to 0}\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(Z^\varepsilon \in \mathcal{B}_r(\varphi)) \le -J_{f,g}(\varphi). \tag{3.9}$$

Minimizing the right hand side over f, g and using Proposition 3.3 completes the proof. □

3.4 Some analytical lemmas

The proof of Proposition 3.3 requires following optimisation results. The following lemma is well known and it is a consequence of Riesz theorem, see [[7], Proposition 3.2] for details.

Lemma 3.4. *Let $\alpha, \beta \in C_0([0,T],\mathbb{R})$ such that β is non-negative. Assume that*

$$\sup_{f \in C^1([0,T],\mathbb{R})} \left\{\int_0^T f_r d\alpha_r - \frac{1}{2}\int_0^T f_r^2 \beta_r dr\right\} < \infty. \tag{3.10}$$

Then α is a absolutely continuous function and there exists a measurable function $k : [0,T] \to \mathbb{R}$ such that $\int_0^T k_r^2 \beta_r dr < \infty$ and $\dot{\alpha}_t = k_t \beta_t$ Lebesgue almost everywhere.

Besides the above one dimensional optimisation in f, we also need a two dimensional optimisation over functions f, g:

Lemma 3.5. *Let $u, v : [0,T] \to \mathbb{R}$ are bounded measurable functions and $p, q \in L^2([0,T], \mathbb{R})$. Then,*

$$\sup_{f,g \in C^1([0,T],\mathbb{R})} \int_0^T \{f_r u_r p_r + g_r v_r q_r - (f_r u_r + g_r v_r)^2\} dr < \infty \tag{3.11}$$

if and only if $p = q$ a.e. on the set $\{uv \neq 0\}$.

Proof. If $p = q$ a.e. on the set $\{uv \neq 0\}$, then for almost every r,

$$\begin{aligned}&f_r u_r p_r + g_r v_r q_r - (f_r u_r + g_r v_r)^2\\ &= (f_r u_r p_r + g_r v_r q_r - (f_r u_r + g_r v_r)^2)1_{u_r v_r \neq 0} + (f_r u_r p_r + g_r v_r q_r - (f_r u_r + g_r v_r)^2)1_{u_r v_r = 0}\\ &= (p_r(f_r u_r + g_r v_r) - (f_r u_r + g_r v_r)^2)1_{u_r v_r \neq 0} + (f_r u_r p_r + g_r v_r q_r - f_r^2 u_r^2 - g_r^2 v_r^2)1_{u_r v_r = 0}\\ &\le \frac{1}{4}(p_r^2 1_{u_r v_r \neq 0} + (p_r^2 + q_r^2)1_{u_r v_r = 0}),\end{aligned}$$

which implies (3.11).
Conversely, let us now assume (3.11) holds.

For constants $L, \varepsilon > 0$, consider functions

$$x_r := \frac{L(p_r - q_r) + p_r + q_r}{2u_r} 1_{|u_r| \wedge |v_r| \geq \varepsilon} + 1_{|u_r| \wedge |v_r| \leq \varepsilon},$$

and

$$y_r := \frac{p_r + q_r - L(p_r - q_r)}{2v_r} 1_{|u_r| \wedge |v_r| \geq \varepsilon} + 1_{|u_r| \wedge |v_r| \leq \varepsilon}.$$

Clearly, $x, y \in L^2([0,T], \mathbb{R})$. Since $C^1([0,T], \mathbb{R})$ is dense in $L^2([0,T], \mathbb{R})$, we can pick sequences $f^n, g^n \in C^1([0,T], \mathbb{R})$ such that $f^n \to x$ and $g^n \to y$ in $L^2([0,T], \mathbb{R})$. Since u, v are bounded, it follows that

$$f^n u + g^n v \to (p+q) 1_{|u| \wedge |v| \geq \varepsilon} + (u+v) 1_{|u| \wedge |v| \leq \varepsilon},$$

and

$$f^n u - g^n v \to L(p-q) 1_{|u| \wedge |v| \geq \varepsilon} + (u-v) 1_{|u| \wedge |v| \leq \varepsilon}$$

in $L^2([0,T], \mathbb{R})$. This in turn implies that as $n \to \infty$

$$\begin{aligned} &\frac{1}{2} \int_0^T (f_r^n u_r - g_r^n v_r)(p_r - q_r) dr \\ &\qquad \to \frac{L}{2} \int_0^T (p_r - q_r)^2 1_{|u_r| \wedge |v_r| \geq \varepsilon} dr + \frac{1}{2} \int_0^T (u_r - v_r)(p_r - q_r) 1_{|u_r| \wedge |v_r| \leq \varepsilon} dr, \end{aligned}$$

and

$$\frac{1}{2} \int_0^T (f_r^n u_r + g_r^n v_r)(p_r + q_r) dr - \int_0^T (f_r^n u_r + g_r^n v_r)^2 dr \to c$$

where c is independent of L. Note that sum of left hand sides of above two equations equals the integral appearing in (3.11), which is bounded in f, g. This implies that $L \int_0^T (p_r - q_r)^2 1_{|u_r| \wedge |v_r| \geq \varepsilon} dr$ is bounded. Since L is arbitrary, this implies that

$$\int_0^T (p_r - q_r)^2 1_{|u_r| \wedge |v_r| \geq \varepsilon} dr = 0.$$

By letting $\varepsilon \to 0+$, it follows using dominated convergence theorem that

$$\int_0^T (p_r - q_r)^2 1_{|u_r| \wedge |v_r| > 0} dr = 0,$$

which concludes the proof. □

3.5 Proof of Proposition 3.3

Let us first assume $I(\varphi) < \infty$. Then, $\varphi = \varphi^h$ for some $h \in H_0^1([0,T], \mathbb{R})$. Let $\sqrt{\varphi_t} = U_t + iV_t$. Since φ solves (2.3), we have

$$d(U_t^2 - V_t^2 + t) = 2U_t dh_t, \tag{3.12}$$

and

$$d(U_t V_t) = V_t dh_t. \tag{3.13}$$

Following a simple rewriting, this implies that

$$\begin{aligned} J_{f,g}(\varphi) &= \int_0^T (f_r U_r + g_r V_r) dh_r - \frac{1}{2} \int_0^T (f_r U_r + g_r V_r)^2 dr \\ &= \int_0^T \left\{ (f_r U_r + g_r V_r) \dot{h}_r - \frac{1}{2} (f_r U_r + g_r V_r)^2 \right\} dr \\ &\leq \frac{1}{2} \int_0^T \dot{h}_r^2 dr = I(\varphi), \end{aligned}$$

which implies $\sup_{f,g\in C^1([0,T],\mathbb{R})} J_{f,g}(\varphi) \le I(\varphi) < \infty$. Also, note that

$$J_{0,g}(\varphi) = \int_0^T g_r V_r \dot{h}_r dr - \frac{1}{2}\int_0^T g_r^2 V_r^2 dr = -\frac{1}{2}\int_0^T (\dot{h}_r - g_r V_r)^2 dr + I(\varphi).$$

Since $C^1([0,T],\mathbb{R})$ is dense in $L^2([0,T],\mathbb{R})$,

$$\inf_{g\in C^1([0,T],\mathbb{R})} \int_0^T (\dot{h}_r - g_r V_r)^2 dr = \inf_{g\in L^2([0,T],\mathbb{R})} \int_0^T (\dot{h}_r - g_r V_r)^2 dr.$$

Also, since $\dot{h} \in L^2([0,T],\mathbb{R})$ and V is a strictly increasing positive function,

$$\inf_{g\in L^2([0,T],\mathbb{R})} \int_0^T (\dot{h}_r - g_r V_r)^2 dr = 0.$$

Hence, $\sup_{f,g\in C^1([0,T],\mathbb{R})} J_{f,g}(\varphi) = I(\varphi)$.

Conversely, now assume that $\sup_{f,g\in C^1([0,T],\mathbb{R})} J_{f,g}(\varphi) < \infty$. This in particular implies that both $\sup_{f\in C^1([0,T],\mathbb{R})} J_{f,0}(\varphi) < \infty$ and $\sup_{g\in C^1([0,T],\mathbb{R})} J_{0,g}(\varphi) < \infty$. Since $J_{f,g}(\varphi)$ is given by (3.7), we have

$$J_{f,0}(\varphi) = \frac{1}{2}\int_0^T f_r d(\mathfrak{Re}(\varphi_r) + r) - \frac{1}{2}\int_0^T f_r^2 \frac{|\varphi_r| + \mathfrak{Re}(\varphi_r)}{2} dr.$$

$$J_{0,g}(\varphi) = \frac{1}{2}\int_0^T g_r d(\mathfrak{Im}(\varphi_r)) - \frac{1}{2}\int_0^T g_r^2 \frac{|\varphi_r| - \mathfrak{Re}(\varphi_r)}{2} dr$$

Applying Lemma 3.4 to above two equations, this implies that $Re(\varphi)$ and $Im(\varphi)$ are absolutely continuous functions. Furthermore, for some measurable functions $k, l : [0,T] \to \mathbb{R}$ such that

$$\int_0^T k_r^2 (|\varphi_r| + \mathfrak{Re}(\varphi_r)) dr + \int_0^T l_r^2 (|\varphi_r| - \mathfrak{Re}(\varphi_r)) dr < \infty, \tag{3.14}$$

we have

$$\mathfrak{Re}(\dot{\varphi}_t) + 1 = \frac{1}{2} k_t (|\varphi_t| + \mathfrak{Re}(\varphi_t)) \text{ a.e.}, \tag{3.15}$$

and

$$\mathfrak{Im}(\dot{\varphi}_t) = \frac{1}{2} l_t (|\varphi_t| - \mathfrak{Re}(\varphi_t)) \text{ a.e.}. \tag{3.16}$$

Next, let $u_t + iv_t = \sqrt{\varphi_t} 1_{\varphi_t \in \mathbb{C}\setminus[0,\infty)} + \sqrt{|\varphi_t|} 1_{\varphi_t \in [0,\infty)}$. Note that $u_t + iv_t$ is a branch square root of φ. It follows that $|\varphi_t| = u_t^2 + v_t^2, \mathfrak{Re}(\varphi_t) = u_t^2 - v_t^2$, and $\mathfrak{Im}(\varphi_t) = 2u_t v_t$. Hence, $J_{f,g}(\varphi)$ can be written as

$$J_{f,g}(\varphi) = \frac{1}{2}\int_0^T \{ f_r k_r u_r^2 + g_r l_r v_r^2 - (f_r u_r + g_r v_r)^2 \} dr. \tag{3.17}$$

Using Lemma 3.5, we obtain that $k_r u_r = l_r v_r$ a.e. on the set $\{uv \neq 0\}$. Now, (3.15), (3.16), $\mathfrak{Re}(\dot{\varphi}_t) + 1 = k_t u_t^2$, $\mathfrak{Im}(\dot{\varphi}_t) = l_t v_t^2$, which implies that

$$\begin{aligned}\varphi_t &= -t + \int_0^t (k_r u_r^2 + i l_r v_r^2) dr \\ &= -t + \int_0^t (u_r + iv_r)(k_r u_r 1_{u_r v_r \neq 0} + l_r v_r 1_{u_r = 0, v_r \neq 0} + k_r u_r 1_{u_r \neq 0, v_r = 0}) dr.\end{aligned}$$

Therefore, φ solves (2.2) with $A_t = u_t + iv_t$ and

$$h_t = \frac{1}{2}\int_0^t (k_r u_r 1_{u_r v_r \neq 0} + l_r v_r 1_{u_r = 0, v_r \neq 0} + k_r u_r 1_{u_r \neq 0, v_r = 0}) dr.$$

Note that by (3.14), $h \in H_0^1([0,T],\mathbb{R})$. Hence, using Lemma 2.2, $\varphi_t \in \mathbb{C} \setminus [0,\infty)$ for all $t > 0$ and $\varphi = \varphi^h$. Hence $I(\varphi) < \infty$ and (3.3) follows from the previous case.

3.6 Lower bound

We now prove the LDP lower bound in Theorem 1.1. Let $C_0^2([0,T],\mathbb{R})$ be the space of continuously twice differentiable $h:[0,T]\to\mathbb{R}$ with $h_0=0$ and $Y:=\{\varphi^h \mid h\in C_0^2([0,T],\mathbb{R})\}$. It follows using density of $C_0^2([0,T],\mathbb{R})$ in $H_0^1([0,T],\mathbb{R})$ and Lemma 2.4 that for each φ with $I(\varphi)<\infty$, there exists a sequence $\varphi_n\in Y$ such that $\varphi_n\to\varphi$ uniformly and $I(\varphi_n)\to I(\varphi)$. Thus, it suffices to prove the following to obtain the LDP lower bound for Z^ε.

Proposition 3.6. *For any* $\varphi\in Y=\{\varphi^h \mid h\in C_0^2([0,T],\mathbb{R})\}$,

$$\lim_{r\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(Z^\varepsilon\in\mathcal{B}_r(\varphi))\geq -I(\varphi). \tag{3.18}$$

The key ingredient in the proof of above claim is the following observation:

Proposition 3.7. *Let* $h\in H_0^1([0,T],\mathbb{R})$ *and* $Z^{\varepsilon,h},\varphi^h$ *be as described in Section 2. Then, as* $\varepsilon\to 0+$,

$$Z^{\varepsilon,h}\xrightarrow{\mathbb{P}}\varphi^h. \tag{3.19}$$

The proof of Proposition 3.7 is postponed till next section. As a result of this, we have:

Proof of Proposition 3.6. Let $\varphi=\varphi^h$ for some $h\in C^2([0,T],\mathbb{R})$. We introduce a change of measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}}=N^\varepsilon$$

where

$$N^\varepsilon=\exp\left(\frac{1}{\varepsilon}\int_0^T\dot{h}_r dB_r-\frac{1}{2\varepsilon^2}\int_0^T\dot{h}_r^2 dr\right). \tag{3.20}$$

By Girsanov theorem, B_t-h_t/ε is a standard Brownian motion under $\mathbb{Q}$. Also, using integration by parts,

$$\int_0^T\dot{h}_r dB_r=\dot{h}_T B_T-\int_0^T B_r\ddot{h}dr\leq C||B||_\infty$$

for some constant C depending only on h. Therefore,

$$\begin{aligned}
\mathbb{P}(Z^\varepsilon\in\mathcal{B}_r(\varphi)) &= \mathbb{E}\left(1_{\mathcal{B}_r(\varphi)}(Z^\varepsilon)\frac{N^\varepsilon}{N^\varepsilon}\right)\\
&= \mathbb{E}^{\mathbb{Q}}\left(1_{\mathcal{B}_r(\varphi)}(Z^\varepsilon)\exp\left(-\frac{1}{\varepsilon}\int_0^T\dot{h}_r dB_r+\frac{1}{2\varepsilon^2}\int_0^T\dot{h}_r^2 dr\right)\right)\\
&= \mathbb{E}\left(1_{\mathcal{B}_r(\varphi)}(Z^{\varepsilon,h})\exp\left(-\frac{1}{\varepsilon}\int_0^T\dot{h}_r dB_r-\frac{1}{2\varepsilon^2}\int_0^T\dot{h}_r^2 dr\right)\right)\\
&\geq \mathbb{P}\left(Z^{\varepsilon,h}\in\mathcal{B}_r(\varphi),||B||_\infty\leq 1\right)e^{-\frac{C}{\varepsilon}}\exp\left(-\frac{1}{2\varepsilon^2}\int_0^T\dot{h}_r^2 dr\right).
\end{aligned}$$

Using Proposition 3.7, as $\varepsilon\to 0$, $\mathbb{P}\left(Z^{\varepsilon,h}\in\mathcal{B}_r(\varphi),||B||_\infty\leq 1\right)\to\mathbb{P}\left(||B||_\infty\leq 1\right)>0$. Therefore,

$$\lim_{r\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(Z^\varepsilon\in\mathcal{B}_r(\varphi))\geq -I(\varphi). \tag{3.21}$$

□

3.7 Proof of Proposition 3.7

Using Lemma 2.1, it can be easily seen that as $\varepsilon \to 0+$

$$\varepsilon \int_0^{\cdot} \sqrt{Z_r^{\varepsilon,h}} dB_r \xrightarrow{\mathbb{P}} 0.$$

Since $Z^{\varepsilon,h}$ solve (2.1), we get that

$$Z_t^{\varepsilon,h} + t + 2\int_0^t \sqrt{Z_r^{\varepsilon,h}} dh_r \xrightarrow{\mathbb{P}} 0. \tag{3.22}$$

Now, let $\varepsilon_n \to 0+$ be any sequence. Let us write $Z_t^n = Z^{\varepsilon_n,h}$. Using the tightness of $Z^{\varepsilon,h}$ (Proposition 3.1), we get that along a subsequence ε_{n_k}, $Z^{n_k} \xrightarrow{d} \varphi$, where φ is some $C_0([0,T],\mathbb{C})$-valued random variable. Using Skorokhod's representation theorem, there exists $C_0([0,T],\mathbb{C})$-valued random variables Y^k and Ψ such that $Y^k \stackrel{d}{=} Z^{n_k}$, $\Psi \stackrel{d}{=} \varphi$, and $Y^k \to \Psi$ almost surely. Clearly, (3.22) implies that

$$Y_t^k + t + 2\int_0^t \sqrt{Y_r^k} dh_r \xrightarrow{\mathbb{P}} 0. \tag{3.23}$$

Next, using Lemma 2.5, possibly along a subsequence, $\sqrt{Y^k}$ converges uniformly to a branch square root $A_t = A_t(\Psi)$. Therefore, it follows by taking $k \to \infty$ in the above that

$$\Psi_t + t + 2\int_0^t A_r dh_r = 0 \text{ a.s..}$$

Using Lemma 2.2, this implies that $\Psi = \varphi^h$ a.s.. Hence, $\varphi = \varphi^h$ a.s.. Since φ^h is deterministic, it follows that $Z^{n_k} \xrightarrow{\mathbb{P}} \varphi^h$. Since the limiting object φ^h is the same for any sequence $\varepsilon_n \to 0+$, the (3.19) follows.

Remark 3.8. The Proposition 3.7 is similar in spirit to continuity of Loewner traces with respect to perturbations in the driving function. This in general is a delicate and difficult problem. However, since we only need convergence in probability in (3.19), we get around this difficulty by relying on the uniqueness of solution to (2.2).

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