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The density of imaginary multiplicative chaos is positive*

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Abstract

Consider a log-correlated Gaussian field Γ and its associated imaginary multiplicative chaos $:e^{i\beta\Gamma}:$ where β is a real parameter. In [3], we showed that for any nonzero test function f, the law of $\int f :e^{i\beta\Gamma}:$ possesses a smooth density with respect to Lebesgue measure on $\mathbb{C}$. In this note, we show that this density is strictly positive everywhere on $\mathbb{C}$. Our simple and direct strategy could be useful for studying other functionals on Gaussian spaces.

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1 Introduction

Let Γ be a logarithmically correlated Gaussian field on some domain $U \subset \mathbb{R}^d$ whose covariance kernel C (formally $C(x,y) = \mathbb{E}[\Gamma(x)\Gamma(y)]$, $x,y \in U$) can be written as

$$C(x,y) = \log \frac{1}{|x-y|} + g(x,y), \quad x,y \in U, \tag{1.1}$$

where $g \in H^{d+\varepsilon}_{\mathrm{loc}}(U \times U) \cap L^2(U \times U)$ for some $\varepsilon > 0$, is symmetric ($g(x,y) = g(y,x)$) and bounded from above. Throughout this article and as in [3], we will make the assumption that

$$\star \ \Gamma \text{ is nondegenerate in the sense that } C \text{ is an injective operator on } L^2(U). \tag{1.2}$$

Let us now fix $\beta \in (0, \sqrt{d})$. For any $f \in L^\infty(U, \mathbb{C})$ we may define the imaginary chaos μ tested against f via the regularisation and renormalisation procedure

$$\mu(f) \coloneqq \lim_{\varepsilon \to 0} \int_U f(x) e^{i\beta\Gamma_\varepsilon(x) + \frac{\beta^2}{2}\mathbb{E}\Gamma_\varepsilon(x)^2}\, dx,$$

where $\Gamma_\varepsilon = \Gamma * \phi_\varepsilon$ is a convolution approximation of Γ against some smooth mollifier $\phi_\varepsilon = \varepsilon^{-d}\phi(\cdot/\varepsilon)$. The above limit takes place in L^2 and the resulting limiting random

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variable does not depend on the specific choice of mollifier [7, 8]. We will sometimes denote this random variable by $\int_U f : e^{i\beta\Gamma}:$, where $: e^{i\beta\Gamma}:$ stands for the Wick exponential of $i\beta\Gamma$.

In [3] and under the above assumptions, we showed that for any nonzero $f \in C_c(U, \mathbb{C})$, the law of $\mu(f)$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{C}$ and the density is a Schwartz function[1]. The main result of the current paper shows that this density is everywhere positive:

Theorem 1.1. *Consider a nonzero test function $f \in C_c(U, \mathbb{C})$. Then for any $z_0 \in \mathbb{C}$, the limit*

$$\lim_{r \to 0^+} r^{-2} \mathbb{P}\left(|\mu(f) - z_0| < r\right) \tag{1.3}$$

is strictly positive. In particular, the density of $\mu(f)$ is strictly positive everywhere.

Note that the existence of the limit (1.3) follows from the existence of a density for $\mu(f)$. The contribution of the current paper is to show that it does not vanish. As a direct consequence of the positivity of the density at the origin, we now have a complete understanding of the blow-up of the moments of $|\mu(f)|$:

$$\forall p \in (-2, +\infty), \quad \mathbb{E}[|\mu(f)|^p] < +\infty \qquad \text{and} \qquad \forall p \in (-\infty, -2], \quad \mathbb{E}[|\mu(f)|^p] = +\infty.$$

A particular case of interest is the imaginary chaos corresponding to the Gaussian field Γ on the circle $\mathbb{S}^1$ whose covariance is given by

$$\mathbb{E}[\Gamma(e^{i\theta})\Gamma(e^{i\theta'})] = -\log|e^{i\theta} - e^{i\theta'}|, \quad \theta, \theta' \in [0, 2\pi],$$

together with the test function $f \equiv 1$, i.e. the total mass of the corresponding imaginary chaos. This case is not covered by the theorem above, however we explain in Section 2.4 how one can modify our arguments to treat this case too. This log-correlated Gaussian field gives rise to an exactly solvable real chaos [6, 11]: the Fyodorov–Bouchaud (FB) formula is an explicit expression for all the moments of the total mass of the real chaos and in fact determines its law. Moreover, the analytic continuation of the moment-formula to $\gamma = i\beta$ yields finite negative moments up to $-2/\beta^2$. Now, in [3], we showed that the analytic continuation of this formula from the real case to the imaginary chaos cannot in general correspond to the -1th moment of the imaginary chaos. In that argument we made use of the negative moments of the absolute value of the total mass. Here, our result implies that, for the absolute value, moments of order $p \leq -2$ blow up. Still, given the FB formula, one may wonder whether thanks to some cancellations it may or may not be possible to make sense of the negative moments $\mathbb{E}[\mu(f)^p]$ without the absolute values for some values of $p \leq -2$.

Understanding the density of imaginary chaos is of importance in studying the properties of imaginary chaos itself [3, 2] but, as just explained, also has implications for related objects like real multiplicative chaos [3] and possibly also continuum limits of spin models [7]. More widely, the problem of proving existence and positivity of densities of functionals on Gaussian spaces can be put in the wider context of Malliavin calculus [10]. In particular, there are known conditions for obtaining positivity for Wiener functionals using Mallivan calculus, e.g. [9, 4, 1]. None of these, nor small modifications thereof seem to apply in our concrete setting, hence we propose a new, simple and direct general strategy that could potentially apply in other contexts too; see Section 1.1 below.

As a possibly interesting side-result and recalling that one can make sense of $: e^{i\beta\Gamma}:$ as a random element of $H^{-d/2-\varepsilon}(\mathbb{R}^d)$ for any $\varepsilon > 0$ [7, Theorem 1.1], we prove that morally the support of imaginary chaos is all of $H^{-d/2-\varepsilon}$:

[1] This result was stated for real-valued test functions in [3] but extends readily to complex-valued test functions. See [2, Proof of Theorem 3.2] for further explanations.

Proposition 1.2. *Let $\varepsilon > 0$, $f \in L^\infty(U,\mathbb{C})$ not identically zero and $K \subseteq U$ a compact subset of U. For any $\eta > 0$, the probability*

$$\mathbb{P}\left(\|1_K(f : e^{i\beta\Gamma} : -1)\|_{H^{-d/2-\varepsilon}(\mathbb{R}^d)} \le \eta\right)$$

is strictly positive.

This note is structured as follows. Our general strategy is explained in the following subsection, and we deal with the rigorous set-up in Section 1.2. Section 2 contains the proofs of our main results. We will start in Section 2.1 by proving a deterministic result which guarantees the existence, for any given function f, of some smooth oscillating function $e^{i\beta a(\cdot)}$ such that the integral $\int f e^{i\beta a}$ takes the desired value. We will then move to the proof of Theorem 1.1 in Section 2.2, assuming Proposition 1.2. We will finally prove Proposition 1.2 in Section 2.3.

1.1 High level strategy

Let $f \in C_c(U,\mathbb{C})$ be a nonzero test function and $z_0 \in \mathbb{C}$. To bound from below the probability that $|\mu(f) - z_0| < r$, we will use the following high level strategy. This strategy could be proved useful in other contexts, especially in settings that can be studied with Malliavin calculus.

- We find an orthonormal basis $(h_n)_{n\ge1}$ of the Cameron–Martin space H_Γ of Γ (see Section 1.2 for the definition of H_Γ), which may depend on z_0 and f, and such that the following holds. Decomposing $\Gamma = \sum_{n\ge1} A_n h_n$ where A_n, $n \ge 1$, are i.i.d. standard Gaussian random variables, we can view the random variable $\mu(f)$ as a function of $(A_n)_{n\ge1}$:
$$\mu(f) = \psi(A_n, n \ge 1), \qquad \text{with} \qquad \psi : \mathbb{R}^{\mathbb{N}} \longmapsto \mathbb{C}.$$
- We find $n_0 \ge 2$ and $a_1, \dots, a_{n_0} \in \mathbb{R}$ such that the map
$$\varphi_0 : (u_1,u_2) \in \mathbb{R}^2 \longmapsto \mathbb{E}[\psi(a_1+u_1, a_2+u_2, a_3, \dots, a_{n_0}, A_{n_0+1}, A_{n_0+2}, \dots)]$$
satisfies $\varphi_0(0,0) = z_0$ and $\varphi_0 : B \to \varphi_0(B)$ is a diffeomorphism for some neighbourhood B of $(0,0)$.
- We show that the above properties are stable in the following sense. There exists an event $E \in \sigma(A_3, A_4, \dots)$ with positive probability which informally requires A_n to be close to a_n for $n = 3, \dots, n_0$, and A_n to have a typical behaviour for $n \ge n_0 + 1$ and such that the following holds. On the event E, the map
$$\Phi : (u_1,u_2) \in \mathbb{R}^2 \longmapsto \psi(a_1+u_1, a_2+u_2, A_3, A_4, \dots)$$
satisfies $\Phi(\mathbf{u}) = z_0$ for some (random) $\mathbf{u} \in B/2$ and $\Phi : B \to \Phi(B)$ is a diffeomorphism. Moreover, on the event E, the determinant of the derivative map $D\Phi$ is uniformly bounded from above and below on the set B by positive deterministic constants.
- We conclude by noticing that, on the event E, we have
$$\mathbb{P}\left(|\mu(f) - z_0| < r | A_n, n \ge 3\right) = \mathbb{P}\left((A_1 - a_1, A_2 - a_2) \in \Phi^{-1}(B(z_0,r)) | A_n, n \ge 3\right) \ge cr^2.$$

For technical reasons, we actually use a slight variant of the above strategy but, roughly speaking, the above steps correspond to the following intermediate results. Finding an orthonormal basis and real numbers $a_1, \dots, a_{n_0} \in \mathbb{R}$ such that $\varphi_0(0,0) = z_0$ is the content of the deterministic Lemma 2.1. The fact that φ_0 is a diffeomorphism on some ball centred at $(0,0)$ is proved in the proof of Lemma 2.3. The stability step is contained in (the proof of) Proposition 1.2.

1.2 Setup

We recall some basic facts concerning the log-correlated Gaussian field Γ. Note that its covariance operator C defines a Hilbert–Schmidt operator on $L^2(U)$, and hence C is self-adjoint and compact. Since C is positive definite, by the spectral theorem there exists a nonincreasing sequence of strictly positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ and corresponding orthogonal eigenfunctions $(f_k)_{k\geq 1}$ spanning the subspace $L \coloneqq (\operatorname{Ker} C)^\perp$ in $L^2(U)$ (which agrees with $L^2(U)$ under our assumption (1.2)). We may now construct the log-correlated field Γ via its Karhunen–Loève expansion

$$\Gamma = \sum_{k\geq 1} A_k C^{1/2} f_k = \sum_{k\geq 1} A_k \sqrt{\lambda_k} f_k, \tag{1.4}$$

where $(A_k)_{k\geq 1}$ is an i.i.d. sequence of standard normal random variables. It has been shown in [7, Proposition 2.3] that the above series converges in $H^{-\varepsilon}(\mathbb{R}^d)$ for any fixed $\varepsilon > 0$ (extending the relevant functions/field by 0 outside of U).

From the KL-expansion one can see that heuristically Γ is a standard Gaussian on the space $H \coloneqq C^{1/2}L$. The space H is called the *Cameron–Martin space* of Γ, and it becomes a Hilbert space by endowing it with the inner product $\langle f, g\rangle_H = \langle C^{-1/2} f, C^{-1/2} g\rangle_{L^2}$, where $C^{-1/2} f, C^{-1/2} g \in L$. This definition makes sense since $C^{1/2}$ is an injection on L. Alternatively, the space H is the space of distributions f such that $\Gamma + f$ is absolutely continuous with respect to Γ; see for instance [5, Section 1.9] in the case of the 2D GFF.

We now record a lemma concerning Cameron–Martin spaces for ease of future reference.

Lemma 1.3. *Let $K \subset U$ be any compact subset of U. There exists a Gaussian field Γ' defined on $\mathbb{R}^d$ such that $\Gamma \overset{(d)}{=} \Gamma'$ in K and such that the Cameron–Martin space of Γ' contains $C_c^\infty(\mathbb{R}^d)$.*

We would like to point out that it can be the case that two fields Γ and Γ' have the same law when restricted to K, but the subsets $\{f \in H_\Gamma : \operatorname{supp} f \subset K\}$ and $\{f \in H_{\Gamma'} : \operatorname{supp} f \subset K\}$ of the Cameron–Martin spaces of Γ and Γ' do not agree. For an example consider two independent standard Gaussians X and Y, the fields (X, Y) and (X, X) and as K the first coordinate. In the case of (X, Y), the whole Cameron–Martin space is spanned by $(1, 0)$ and $(0, 1)$, so its subset of "functions" supported by the first coordinate is simply $\mathbb{R}$. In the second case, the Cameron–Martin space is spanned by $(1, 1)$ so the only "function" with support in the first coordinate is 0. Lemma 1.3 was stated in this way because of this counter-intuitive property.

Proof. By [3, Theorem 4.5] and because Γ is nondegenerate (see (1.2)), we may decompose in K, $\Gamma = L + R$ as the sum of a Hölder continuous field R and an independent almost $\star$-scale invariant field L whose covariance equals

$$\mathbb{E}[L(x)L(y)] = \int_0^\infty k(e^u(x-y))(1-e^{-\delta u})\, du.$$

Here, $\delta > 0$ is a parameter and $k : \mathbb{R}^d \to [0,\infty)$ is a rotationally symmetric seed covariance with $k(0) = 1$, $\operatorname{supp} k \subset B(0,1)$ and such that

$$\exists s > \frac{d+1}{2}, \quad \forall \xi \in \mathbb{R}^d, \quad 0 \leq \hat{k}(\xi) \leq (1+|\xi|^2)^{-s}. \tag{1.5}$$

By [3, Lemma 4.8], the Cameron–Martin space of L contains $C_c^\infty(\mathbb{R}^d)$. By [3, Lemma 4.1], this implies that the Cameron–Martin space of $L + R$ also contains $C_c^\infty(\mathbb{R}^d)$ concluding the proof. □

2 Proofs

2.1 A deterministic result

Lemma 2.1. *Let $f \in L^1(\mathbb{R}^d, \mathbb{C})$ not identically zero. For all $z_0 \in \mathbb{C}$ with $|z_0| < \|f\|_{L^1}$, there exists a function $a \in C^\infty(\mathbb{R}^d, \mathbb{R})$ such that*

$$\int f(x) e^{i\beta a(x)}\, dx = z_0 \tag{2.1}$$

and such that $\beta a + \arg(f)$ is not constant in the support of f.

We emphasise that the function a is required to be smooth. Of course, if f has compact support within some open domain U, then one can also require a to have compact support in U.

In general, this result cannot be extended to $|z_0| \geq \|f\|_{L^1}$. Indeed, by the triangle inequality, one cannot find such a function if $|z_0| > \|f\|_{L^1}$. If $|z_0| = \|f\|_{L^1}$ and if (2.1) holds for some function a, then $fe^{i\beta a}$ agrees with $|f|z_0/|z_0|$ Lebesgue-almost everywhere. In general this equality cannot be achieved by a continuous function a.

Proof. Let us first assume the existence of such a function a_0 when $z_0 = 0$ (this is actually the difficult part of the proof). Let $z_0 \in \mathbb{C}$ with $0 < |z_0| < \|f\|_{L^1}$. Let $v = -\arg(f)/\beta$ where we arbitrarily define $\arg(f) = 0$ when $f = 0$. By construction, $fe^{i\beta v} = |f|$. Let $\rho_\varepsilon = \varepsilon^{-d}\rho(\cdot/\varepsilon)$, $\varepsilon \geq 0$, be a sequence of smooth mollifiers. Define $b_\varepsilon = (1-\varepsilon) v * \rho_\varepsilon + \varepsilon a_0$. By dominated convergence theorem, the map

$$\varepsilon \in [0,1] \longmapsto \left| \int f(x) e^{i\beta b_\varepsilon(x)}\, dx \right|$$

is continuous, equal to $\|f\|_{L^1}$ when $\varepsilon = 0$ and equal to 0 when $\varepsilon = 1$. Therefore, there exists some $\varepsilon \in (0,1)$ such that the integral of $fe^{i\beta b_\varepsilon}$ has the same modulus as z_0. Setting $a \in C^\infty(\mathbb{R}^d, \mathbb{R})$ to be equal to $b_\varepsilon(\cdot) + \theta$ on the support of f, where $\theta \in \mathbb{R}$ appropriately changes the phase, we obtain that $\int fe^{i\beta a} = z_0$.

We now treat the delicate case of $z_0 = 0$. We will first find a continuous function $\tilde{a}$ such that $\int |f| e^{i\beta\tilde{a}} = 0$. The function $\hat{a} := \arg(f)/\beta + \tilde{a}$ will thus satisfy $\int fe^{i\beta\hat{a}} = 0$ but will not have the desired regularity. We will then show that there is a smooth perturbation a of $\hat{a}$ which has the same property. We define

$$\tilde{a}(x_1, \dots, x_d) = b(x_d) = \frac{2\pi}{\beta \|f\|_{L^1}} \int_{-\infty}^{x_d} \int_{\mathbb{R}^{d-1}} |f(u_1, \dots, u_{d-1}, u_d)|\, du_1 \dots du_d.$$

We have

$$\lim_{x_d \to +\infty} b(x_d) = 2\pi/\beta \quad \text{and} \quad \lim_{x_d \to -\infty} b(x_d) = 0.$$

Hence,

$$\int_{\mathbb{R}^d} |f(x)| e^{i\beta\tilde{a}(x)}\, dx = \frac{\beta \|f\|_{L^1}}{2\pi} \int_{-\infty}^{\infty} b'(x_d) e^{i\beta b(x_d)}\, dx_d = \frac{\beta \|f\|_{L^1}}{2\pi} \cdot \frac{e^{2\pi i} - 1}{i\beta} = 0$$

as desired.

To define the perturbation a of $\hat{a}$, notice first that, since the phase of $fe^{i\beta\hat{a}}$ is not constant, there exist smooth functions $g_1, g_2 \in C^\infty(\mathbb{R}^d, \mathbb{R})$ such that

$$\int f(x) g_j(x) e^{i\beta\hat{a}(x)} = \theta_j, \quad j = 1, 2,$$

where θ_1, θ_2 are two complex numbers with modulus one which are not linearly correlated. We will make an ansatz for a of the form

$$a(x) = (\rho_\varepsilon * \hat{a})(x) + s_1 g_1(x) + s_2 g_2(x),$$

where $\rho_\varepsilon = \varepsilon^{-d}\rho(\cdot/\varepsilon)$ is a sequence of standard smooth mollifiers and $s_j \in \mathbb{R}$, $j = 1,2$, are some parameters. Let us now look at the family of smooth maps $(\eta_\varepsilon)_{\varepsilon\geq 0}$ given by

$$\eta_\varepsilon : \mathbf{s} = (s_1, s_2) \in \mathbb{R}^2 \longmapsto \int f(x) e^{i\beta(\rho_\varepsilon * \hat{a})(x) + s_1 g_1(x) + s_2 g_2(x)}\, dx \in \mathbb{C},$$

where we used the convention that $\rho_0 * \hat{a} = \hat{a}$. Note that $\eta_\varepsilon \to \eta_0$ uniformly on $\mathbb{R}^2$ by elementary inequalities and the convergence $\rho_\varepsilon * \hat{a}\mathbb{1}_{f\neq 0} \to \hat{a}\mathbb{1}_{f\neq 0}$ in $L^1(\mathbb{R}^d)$. To conclude the proof it is enough to find some small $\varepsilon > 0$ and $\mathbf{s} \in \mathbb{R}^2$ such that $\eta_\varepsilon(\mathbf{s}) = 0$. Note that η_0 is a map with $\eta_0(0,0) = 0, \partial_1\eta_0(0,0) = \theta_1, \partial_2\eta_0(0,0) = \theta_2$. In particular, $D\eta_0(0,0)$ is invertible and, by the inverse function theorem, there exists a neighbourhood $B(0,\tilde{R})$ of $(0,0)$ such that η_0 is invertible with inverse $h\colon \eta_0(B(0,\tilde{R})) \to \mathbb{R}^2$. Let us fix $R > 0$ such that $B(0,R) \subset \eta_0(B(0,\tilde{R}))$ and consider $r > 0$ for which $\eta_0(\overline{B(0,r)}) \subset B(0,R/2)$. We next note the following simple corollary of Brouwer fixed-point theorem:

Lemma 2.2. *Let $F : \overline{B(0,r)} \to \mathbb{R}^2$ be a continuous function such that $|F(\mathbf{s}) - \mathbf{s}| \leq r$ for all $\mathbf{s} \in \overline{B(0,r)}$. Then $F(s) = 0$ for some $\mathbf{s} \in \overline{B(0,r)}$.*

Proof of Lemma 2.2. Even though it is likely that this is a classical result, we provide a proof for completeness. Consider the map $G : \mathbf{s} \in \overline{B(0,r)} \mapsto -F(\mathbf{s}) + \mathbf{s}$. By assumption, $G(\overline{B(0,r)}) \subset \overline{B(0,r)}$ and G is continuous. So Brouwer fixed-point theorem applies giving the existence of a fixed point for G, or equivalently a point $\mathbf{s}$ such that $F(\mathbf{s}) = 0$. □

To apply the lemma, we consider the functions $h \circ \eta_\varepsilon$ on $\overline{B(0,r)}$. For ε small enough we see that they are well-defined since then $|\eta_\varepsilon(\mathbf{s})| \leq 3R/4$ for all $\mathbf{s} \in B(0,r)$. Moreover since h is Lipschitz in $B(3R/4)$, we have that $|h(\eta_\varepsilon(\mathbf{s})) - \mathbf{s}| \leq L|\eta_\varepsilon(\mathbf{s}) - \eta_0(\mathbf{s})|$ for some constant $L > 0$. Thus for small enough ε the right hand side is less than r and the lemma above applies and we have that $h(\eta_\varepsilon(\mathbf{s})) = 0$ for some $\mathbf{s} \in B(0,r)$, implying that $\eta_\varepsilon(\mathbf{s}) = 0$ and in particular yielding the required a. This concludes the proof of Lemma 2.1. □

2.2 Proof of Theorem 1.1 assuming Proposition 1.2

We start with a key intermediate result.

Lemma 2.3. *Let $f \in C_c(U,\mathbb{C})$ not identically zero and $z_0 \in \mathbb{C}$. By changing Γ by a field which agrees with Γ on the support of f if necessary, there exist an orthonormal basis $(h_n)_{n=1}^\infty$ of the Cameron–Martin space of Γ, a constant $C > 1$, $a_1, a_2 \in \mathbb{R}$ and a ball $B = B(0,\delta) \subset \mathbb{R}^2$ such that the following holds. Let $A_n = \langle \Gamma, h_n \rangle_H, n \geq 1$. On an event $E \in \sigma(A_n, n \geq 3)$ with positive probability, the random smooth map*

$$\Phi : (u_1,u_2) \longmapsto \int f(x) e^{i\beta((a_1+u_1)h_1(x) + (a_2+u_2)h_2(x)) + \frac{\beta^2}{2}(h_1(x)^2 + h_2(x)^2)} : e^{i\beta\sum_{n\geq 3} A_n h_n(x)} : dx \tag{2.2}$$

is a diffeomorphism $B \to \Phi(B)$. Moreover, on the event E, $\Phi(\mathbf{u}) = z_0$ for some random $\mathbf{u} \in B(0,\delta/2)$, and for all $\mathbf{u} \in B$ the derivative map $D\varphi\colon T_\mathbf{u}\mathbb{R}^2 \to T_{\Phi(\mathbf{u})}\mathbb{C}$ is bounded in norm by C and $1/C$ from above and away from 0 respectively.

Proof of Lemma 2.3, assuming Proposition 1.2. By Lemma 2.1, there exist $t \in \mathbb{R}$ and $g_3 \in C_c^\infty(U,\mathbb{R})$ such that $\arg f + \beta g_3$ is not constant in the support of f and such that

$$\int f(x) e^{i\beta g_3(x) + t}\, dx = z_0. \tag{2.3}$$

Since the phase of $fe^{i\beta g_3}$ is not constant, we can pick two functions g_1 and $g_2 \in C_c^\infty(U,\mathbb{R})$ such that

$$\int f(x) g_j(x) e^{i\beta g_3(x) + t} dx = \theta_j, \quad j = 1,2, \tag{2.4}$$

where θ_1 and θ_2 are two complex numbers with modulus one which are not linearly correlated. Let $V \subset U$ be an open subset of U containing the support of f. We can modify the definitions of the functions g_1, g_2 and g_3 inside $U \setminus V$ without changing the value of the integrals (2.3) and (2.4). We will in particular assume the existence of three disjoint open sets $W_1, W_2, W_3 \subset U \setminus V$ such that for all $i \neq j$,

$$g_j \mathbf{1}_{W_j} \in C_c^\infty(W_j, \mathbb{R}), \quad \left\| g_j \mathbf{1}_{W_j} \right\|_H > 0 \quad \text{and} \quad g_j \mathbf{1}_{W_i} = 0. \tag{2.5}$$

These conditions in particular imply that g_1, g_2 and g_3 are linearly independent, but will be also useful at the end of the current proof for our "restriction trick".

Let us now fix an ON-basis $(h_n)_{n=1}^\infty$ of H with the first two elements h_1, h_2 spanning $\operatorname{span}\{g_1, g_2\}$ and the first three elements h_1, h_2, h_3 spanning $\operatorname{span}\{g_1, g_2, g_3\}$. This can be done thanks to Lemma 1.3 (without loss of generality, we may consider another field Γ' that agree with Γ on the support of f and whose Cameron–Martin space contains $C_c^\infty(\mathbb{R}^d)$). Because $g_3 \in \operatorname{span}\{h_1, h_2, h_3\}$, there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that $g_3 = a_1 h_1 + a_2 h_2 + a_3 h_3$. The map

$$\varphi_0 : (u_1, u_2) \in \mathbb{R}^2 \longmapsto \int f(x) e^{i\beta((a_1+u_1)h_1(x) + (a_2+u_2)h_2(x) + a_3 h_3(x)) + t} \, dx \in \mathbb{C}.$$

satisfies

$$\varphi_0(0,0) = z_0 \quad \text{and} \quad \partial_j \varphi_0(0,0) = i\beta \int f(x) h_j(x) e^{i\beta g_3(x) + t} \, dx, \quad j = 1, 2.$$

Because θ_1 and θ_2 are not linearly correlated (see (2.4)), $D\varphi_0(0,0)$ is invertible. By the inverse function theorem, there exists a ball $B = B(0, 2\delta)$ such that $\varphi_0 : B \to \varphi_0(B)$ is a diffeomorphism. Our main task now is to show that the same is true when we perturb φ_0 and consider Φ instead. It is enough to show that for all $\eta > 0$ arbitrarily small, the probability

$$\mathbb{P}(\|\Phi - \varphi_0\|_{L^\infty(B)} < \eta, \|\partial_j \Phi - \partial_j \varphi_0\|_{L^\infty(B)} < \eta, j = 1, 2) > 0. \tag{2.6}$$

We can bound $|\Phi(u_1, u_2) - \varphi_0(u_1, u_2)|$ by

$$\begin{aligned}
&\left| \int f e^{i\beta((a_1+u_1)h_1 + (a_2+u_2)h_2) + t} (e^{i\beta A_3 h_3} - e^{i\beta a_3 h_3}) \right| \\
&+ \left| \int f e^{i\beta((a_1+u_1)h_1 + (a_2+u_2)h_2 + A_3 h_3)} \left(e^{\frac{\beta^2}{2}(h_1^2 + h_2^2 + h_3^2)} : e^{i\beta \sum_{n \geq 4} A_n h_n} : - e^t \right) \right| \\
&\leq \|f\|_{L^1(U)} e^t \left\| e^{i\beta A_3 h} - e^{i\beta a_3 h} \right\|_{L^\infty(U)} + \left\| f e^{i\beta((a_1+u_1)h_1 + (a_2+u_2)h_2 + A_3 h_3)} \right\|_{H^d(\mathbb{R}^d)} \times \\
&\qquad \times \left\| \left(e^{\frac{\beta^2}{2}(h_1^2 + h_2^2 + h_3^2)} : e^{i\beta \sum_{n \geq 4} A_n h_n} : - e^t \right) \mathbf{1}_V \right\|_{H^{-d}(\mathbb{R}^d)}.
\end{aligned}$$

Similarly, for $j = 1, 2$, $|\partial_j \Phi(u_1, u_2) - \partial_j \varphi_0(u_1, u_2)|$ is also bounded by the above sum of two terms. The first term can be made arbitrarily small with positive probability by making A_3 close to a_3. Assuming for a moment that we can apply Proposition 1.2 to the field $\sum_{n \geq 4} A_n h_n$ in V (i.e. that it satisfies the assumption (1.2)), we can make the second term arbitrarily small with positive probability conditionally on A_3. This would then show (2.6) and conclude the proof of Lemma 2.3.

Restriction trick. It remains to prove that the field $\sum_{n \geq 4} A_n h_n$ restricted to V satisfies the assumption (1.2) (notice that it would not be the case without the restriction to a subset V). Let

$$\widetilde{C} : (x, y) \in V \times V \mapsto \sum_{n \geq 4} h_n(x) h_n(y).$$

We want to show that $\widetilde{C}$ is injective on $L^2(V)$. Let $\tilde{f} \in L^2(V)$ be such that $\widetilde{C}\tilde{f} = 0$. We want to show that $\tilde{f} = 0$. Recall the existence of the functions g_1, g_2, g_3 and the subsets $W, W_1, W_2 \subset U \setminus V$ introduced in (2.5) and above. Let $j \in \{1,2,3\}$. Because $\{h_n, n \geq 4\}$ does not span $L^2(W_j)$, there exists $\tilde{f}_j \in L^2(U)$ vanishing outside of W_j such that $\int \tilde{f}_j h_n = 0$ for $n \geq 4$ and $\int \tilde{f}_j g_j \neq 0$. Let $f \in L^2(U)$ be defined as

$$f = \tilde{f}\mathbf{1}_V + \sum_{j=1}^{3} \lambda_j \tilde{f}_j, \quad \text{where} \quad \lambda_j = -\int \tilde{f} g_j \Big/ \int \tilde{f}_j g_j.$$

Since $\widetilde{C}\tilde{f} = 0$, we have

$$Cf = \sum_{n=1}^{3} \Big(\int \tilde{f} h_n \Big) h_n + \sum_{n=1}^{3} \sum_{j=1}^{3} \lambda_j \Big(\int \tilde{f}_j h_n \Big) h_n.$$

By definition of λ_j and because g_n vanishes on the support of $\tilde{f}_j$ for $n \in \{1,2,3\} \setminus \{j\}$, we have for all $n = 1,2,3$,

$$\sum_{j=1}^{3} \lambda_j \Big(\int \tilde{f}_j g_n \Big) = -\int \tilde{f} g_n.$$

Since h_1, h_2 and h_3 are linear combinations of g_1, g_2 and g_3, we deduce that for all $n = 1,2,3$,

$$\sum_{j=1}^{3} \lambda_j \Big(\int \tilde{f}_j h_n \Big) = -\int \tilde{f} h_n,$$

implying that $Cf = 0$. By assumption (1.2), C is injective on $L^2(U)$ and thus $f = 0$. We deduce that $\tilde{f} = f_{|V} = 0$ as desired. This concludes the proof. □

We can now prove:

Proof of Theorem 1.1, assuming Proposition 1.2. Let $z_0 \in \mathbb{C}$. By definition (2.2) of the map Φ, $\mu(f) = \Phi(A_1 - a_1, A_2 - a_2)$ a.s. By Lemma 2.3, on the event E, if $r > 0$ is small enough,

$$\mathbb{P}(\mu(f) \in B(z_0, r) | A_n, n \geq 3) = \mathbb{P}((A_1 - a_1, A_2 - a_2) \in \Phi^{-1}(B(z_0, r)) | A_n, n \geq 3).$$

By Lemma 2.3 and on the event E, the Lebesgue measure of $\Phi^{-1}(B(z_0, r))$ is bounded from below by cr^2 for some deterministic constant $c > 0$. It follows that for $r > 0$ small enough and on the event E,

$$\mathbb{P}(\mu(f) \in B(z_0, r) | A_n, n \geq 3) \geq cr^2. \tag{2.7}$$

We deduce that $\mathbb{P}(\mu(f) \in B(z_0, r)) \geq c\mathbb{P}(E) r^2$ proving that the limit in (1.3) is positive. □

2.3 Proof of Proposition 1.2

We now turn to the proof of Proposition 1.2.

Proof of Proposition 1.2. Let $f : U \to \mathbb{C}$ and $K \Subset U$ be as in the statement of the proposition. By [3, Theorem 4.5] and because Γ is nondegenerate (see (1.2)), we may decompose in the compact K, $\Gamma = L + R$ as the sum of a Hölder continuous field R and an independent almost $\star$-scale invariant field L whose covariance equals

$$\mathbb{E}[L(x)L(y)] = \int_0^\infty k(e^u(x-y))(1 - e^{-\delta u})\, du.$$

Here, $\delta > 0$ is a parameter and $k : \mathbb{R}^d \to \mathbb{R}$ is a seed covariance satisfying the same assumptions as above (1.5). The process L is the limit of the smooth processes L_t as $t \to \infty$ where L_t has the covariance structure

$$\mathbb{E}[L_t(x)L_t(y)] = \int_0^t k(e^u(x-y))(1-e^{-\delta u})\,du.$$

We now define the approximations, for $T > 0$,

$$\Gamma_T = L_T + R, \quad \text{and} \quad \Gamma_{T,\infty} = \Gamma - \Gamma_T$$

for the field Γ and its tail.

We may bound

$$\|1_K(f : e^{i\beta\Gamma} : -1)\|_{H^{-d/2-\varepsilon}(\mathbb{R}^d)} \le X_T + Y_T \tag{2.8}$$

where

$$X_T = \|1_K(f : e^{i\beta\Gamma_T(x)} : -1) : e^{i\beta\Gamma_{T,\infty}} :\|_{H^{-d/2-\varepsilon}(\mathbb{R}^d)}$$

and

$$Y_T = \|1_K(: e^{i\beta\Gamma_{T,\infty}(x)} : -1)\|_{H^{-d/2-\varepsilon}(\mathbb{R}^d)}.$$

We now deal with X_T and Y_T separately. We first claim that $\mathbb{E}[Y_T^2] \to 0$ as $T \to \infty$. To show this, we compute

$$\begin{aligned}
&\mathbb{E}[Y_T^2] = \mathbb{E}[\|\mathbf{1}_K(: e^{i\beta\Gamma_{T,\infty}(\cdot)} : -1)\|^2_{H^{-d/2-\varepsilon}(\mathbb{R}^d)}] \\
&= \mathbb{E}\int_{\mathbb{R}^d} d\xi(1+|\xi|^2)^{-d/2-\varepsilon}\int_{K\times K} dx\,dy(: e^{i\beta\Gamma_{T,\infty}(x)} : -1)(: e^{-i\beta\Gamma_{T,\infty}(y)} : -1)e^{-2\pi i\xi\cdot(x-y)} \\
&= \int_{\mathbb{R}^d} d\xi(1+|\xi|^2)^{-d/2-\varepsilon}\int_{K\times K} dx\,dy(e^{\beta^2\int_T^\infty k(e^s(x-y))(1-e^{-\delta s})\,ds} - 1)e^{-2\pi i\xi\cdot(x-y)}.
\end{aligned}$$

Denote by $u_T(\xi)$ the above integral over $K \times K$. By dominated convergence theorem, for all $\xi \in \mathbb{R}^d$, $u_T(\xi) \to 0$ as $T \to \infty$. Since

$$\sup_{T\ge 0}|u_T(\xi)| \le \int_{K\times K} dx\,dy(e^{\beta^2\int_0^\infty k(e^s(x-y))(1-e^{-\delta s})\,ds} - 1) < \infty,$$

we can conclude by dominated convergence theorem that $\mathbb{E}[Y_T^2] \to 0$ as $T \to \infty$ as claimed. In the rest of the proof, we will pick $T > 0$ large enough so that $\mathbb{E}[Y_T] \le \eta/4$.

It remains to deal with X_T. By Lemma 2.1, there exists $t_0 \in \mathbb{R}$ such that for all $t \ge t_0$, there exists $a_t \in C_c^\infty(U, \mathbb{R})$, such that

$$\int f e^{\beta^2\mathbb{E}[R^2]/2} e^{i\beta a_t + t} = 1.$$

$\mathbb{E}[L_T(x)^2]$ does not depend on x and goes to infinity as $T \to \infty$. We thus also pick T large enough so that $\mathbb{E}[L_T(x)^2]$ exceeds the above value of t_0. We can now find $a \in C_c^\infty(U, \mathbb{R})$ such that

$$\int f e^{i\beta a + \beta^2\mathbb{E}[\Gamma_T^2]/2} = 1.$$

Since the Cameron–Martin space of Γ_T contains $C_c^\infty(U)$ (see the proof of [3, Lemma 4.8] for details), Γ_T can be made arbitrarily close to a with positive probability in say $H^d(\mathbb{R}^d)$-norm and $f : e^{i\beta\Gamma_T} : -1$ can be made arbitrarily close to 0. Moreover, a computation similar to the computation of $\mathbb{E}[Y_T^2]$ shows that $\mathbb{E}[X_T^2|\Gamma_T]$ is controlled by $\|f : e^{i\beta\Gamma_T} : -1\|_{H^d(\mathbb{R}^d)}$. Altogether, this shows that the probability of the event

$$E := \{\mathbb{E}[X_T|\Gamma_T] \le \eta/4\}$$

is positive. Wrapping up and by Markov's inequality, we have

$$\begin{aligned}\mathbb{P}\left(X_T+Y_T\leq\eta\right)&\geq\mathbb{P}\left(X_T+Y_T\leq\eta,E\right)=\mathbb{P}\left(E\right)-\mathbb{P}\left(X_T+Y_T>\eta,E\right)\\&\geq\mathbb{P}\left(E\right)-\mathbb{E}[\mathbb{E}[X_T+Y_T|\Gamma_T]\mathbf{1}_E]/\eta.\end{aligned}$$

Since $\mathbb{E}[Y_T]\leq\eta/4$ and, on the event E, $\mathbb{E}[X_T|\Gamma_T]\leq\eta/4$, we have shown that

$$\mathbb{P}\left(X_T+Y_T\leq\eta\right)\geq\mathbb{P}(E)/2>0.$$

Together with (2.8), this concludes the proof of Proposition 1.2. □

2.4 GFF on the circle

In this section we briefly explain how one can modify some of the arguments in the proof of Theorem 1.1 in order to be able to treat the case of the total mass of the multiplicative chaos associated to the GFF on the circle. This field can be explicitly decomposed as

$$\Gamma(e^{i\theta})=\sum_{k\geq1}A_k\frac{\sin(k\theta)}{\sqrt{k}}+B_k\frac{\cos(k\theta)}{\sqrt{k}},$$

where $A_k,B_k,k\geq1$, are i.i.d. standard normal Gaussians. Because the average of the GFF on the circle vanishes, the resulting field is not nondegenerate in the sense of (1.2). However, since the underlying structure is explicit, we can make the appropriate changes. The following result is the main deterministic result we use instead of Lemma 2.1.

Lemma 2.4. *There exists two open sets $O_1\subset\mathbb{R}^2,O_2\subset\mathbb{C}$ with O_2 containing the origin such that the map*

$$F:(s_1,s_2)\in O_1\longmapsto\int_0^{2\pi}e^{i(s_1\sin(\theta)+s_2\cos(2\theta))}\,\mathrm{d}\theta$$

is a diffeomorphism from O_1 onto O_2.

Proof. For $n\geq0$, we will denote by J_n the n-th Bessel function of the first kind. When $s_2=0$, $F(s_1,s_2)$ is explicit and is equal to $2\pi J_0(|s|)$. Let $j_0>0$ be the smallest positive root of J_0. One can show that, when $(s_1,s_2)=(j_0,0)$,

$$\frac{\partial F(s_1,s_2)}{\partial s_1}=2i\pi J_1(j_0)\quad\text{and}\quad\frac{\partial F(s_1,s_2)}{\partial s_2}=2\pi J_2(j_0).$$

Since $J_1(j_0)$ and $J_2(j_0)$ do not vanish (this is a general fact concerning Bessel functions: the zeros of J_n and J_m are distinct when $n\neq m$), this shows that the determinant of $DF(j_0,0)$ does not vanish. We then conclude by the inverse function theorem. □

We now follow our strategy described in Section 1.1. We fix $z_0\in\mathbb{C}$. Let $(h_n)_{n\geq1}$ be the orthonormal basis of the Cameron–Martin space composed of the functions $\theta\mapsto k^{-1/2}\sin(k\theta)$, $\theta\mapsto k^{-1/2}\cos(k\theta)$, $k\geq1$. We order these functions so that $h_1=\sin(\cdot)$ and $h_2=2^{-1/2}\cos(2\cdot)$. Decomposing the GFF on the circle as $\sum A_nh_n$, where A_n, $n\geq1$, are i.i.d. standard Gaussian random variables, we view the total mass of the imaginary chaos as a function of $(A_n)_{n\geq1}$:

$$\int_{\mathbb{S}^1}:e^{i\beta\Gamma}:\quad=\quad\psi(A_1,A_2,\dots).$$

Let $n_0\geq1$ be large. For $n=3,\dots,n_0$, let $a_n=0$ and let φ_0 be the map

$$(s_1,s_2)\in\mathbb{R}^2\longmapsto\mathbb{E}[\psi(s_1,s_2,a_3,\dots,a_{n_0},A_{n_0+1},A_{n_0+2},\dots)]=\int_{\mathbb{S}^1}e^{\frac{\beta^2}{2}\sum_1^{n_0}h_n^2}e^{i\beta(s_1h_1+s_2h_2)}.$$

Let $K_{n_0}=\frac{1}{2\pi}\int_{\mathbb{S}^1}e^{\frac{\beta^2}{2}\sum_1^{n_0}h_n^2}$. We take n_0 large enough to ensure that:

- $K_{n_0}O_2$ contains the ball $B(0,2|z_0|)$ where $O_2 \subset \mathbb{C}$ is the open set from Lemma 2.4;
- $\|e^{\frac{\beta^2}{2}\sum_1^{n_0} h_n^2} - K_{n_0}\|_\infty$ is as small as desired, exploiting that $\sum_1^{n_0} h_n(x)^2$ is asymptotically independent of x (the field Γ is rotationally invariant);
- the chaos $: e^{i\beta(\sum_{n\geq n_0+1} A_n h_n)}:$ coming from the tail field is close to its expectation in $H^{-1/2-\varepsilon}(\mathbb{S}^1)$-norm, with positive probability.

We can then conclude as before. More precisely, thanks to the first two properties, φ_0 is a small perturbation of the map $K_{n_0}F$ from Lemma 2.4. Thus, there exist $(a_1,a_2) \in \mathbb{R}^2$ and a neighbourhood B of (a_1,a_2) such that $\varphi_0 : B \to \varphi(B)$ is a diffeomorphism and $\varphi_0(B)$ is a neighbourhood of z_0. Using the third property, we can then conclude that this property is stable in the following sense. Let $E \in \sigma(A_n, n\geq 3)$ be the event that A_n, $n = 3,\dots,n_0$, stays close to 0 and that the $H^{-1/2-\varepsilon}(\mathbb{S}^1)$-norm of $: e^{i\beta(\sum_{n\geq n_0+1} A_n h_n)}:$ is close to its expectation. The event E occurs with positive probability and, on this event, the map

$$\Phi : (s_1,s_2) \in \mathbb{R}^2 \longmapsto \psi(s_1,s_2,A_3,A_4,\dots)$$

is also a diffeomorphism $\tilde{B} \to \Phi(\tilde{B})$ where $\Phi(\tilde{B})$ contains a neighbourhood of z_0. Altogether, this allows us to conclude as before that:

$$\lim_{r\to 0^+} r^{-2}\mathbb{P}\Big(\Big|\int_{\mathbb{S}^1} : e^{i\beta\Gamma} : -z_0\Big| < r\Big) > 0. \tag{2.9}$$

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