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Matrix models for cyclic monotone and monotone independences*

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Abstract

Cyclic monotone independence is an algebraic notion of non-commutative independence, introduced in the study of multi-matrix random matrix models with small rank. Its algebraic form turns out to be surprisingly close to monotone independence, which is why it was named cyclic monotone independence. This paper conceptualizes this notion by showing that the same random matrix model is also a model for the monotone independence with an appropriately chosen state. This observation provides a unified nonrandom matrix model for both types of monotone independences.

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1 Introduction

Monotone independence was introduced by Muraki [13], and Lu [12] in the context of non-commutative probability theory. Later on, Muraki [13, 14, 15, 16], Hasebe [6, 7] and Hasebe and Saigo [10] developed monotone probability theory, which is a non-commutative probability theory with monotone independence, inspired by Voiculesu's free probability theory and Speicher's universal products [20]. The construction of non-commutative probability spaces which realize monotone independence was achieved with the help of Fock spaces and universal products. This theory triggered substantial interest because monotone independence connects different subjects. For example, Accardi, Ghorbal, and Obata [1] realized monotone independence via the spectral analysis of the comb graph. Schleissinger [18] found a relation between monotone independence and SLE theory. The relation between Löewner chains and monotone probability theory is developing rapidly [8, 5]. On the other hand, there was no matrix model for cyclic monotone independence, and only random models in specific cases for

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monotone independence – see Theorem 7.1 in [11]. The goal of this paper is to provide such a model.

Recently, motivated by the study of outliers in random matrix theory, Collins, Hasebe and Sakuma found in [3] cyclic monotone independence. One cannot observe outliers from the empirical eigenvalue distributions of random matrices, but can from their operator norm. To overcome this problem from the point of view of eigenvalue distributions, they proposed considering non-commutative probability spaces with a weight. The weight corresponds to the non-normalized trace. Computations of moments evidenced the notion of cyclic monotone independence – a rule to compute joint moments that is quite similar to monotone independence, with the additional property that it conserves traciality. This similarity was left as a curiosity to explore. However, it raised the natural question of the relation between both notions and the existence of a unified model for both independences.

This paper is organized as follows. In Section 2, we recall the notions of monotone and cyclic monotone independences, and then state and prove a theoretical result about the structure of the free product algebra quotiented by the monotone (resp. cyclic monotone) free product state (Theorem 2.7). In Section 3, we apply the result and provide a unified matrix model for monotone (resp. cyclic monotone) variables. After that, we discuss random matrix models for monotone and cyclic monotone independences.

2 Notation and abstract result

Let us first review basic notations for monotone and cyclic monotone independences. For details, see [3]. A *non-commutative measure space* is a pair $(\mathcal{A},\omega)$, where $\mathcal{A}$ is a (unital or non-unital) $*$-algebra over $\mathbb{C}$ endowed with a weight ω, i.e.

- ω is defined on a (possibly non-unital) $*$-subalgebra $\mathrm{D}(\omega)$ of $\mathcal{A}$ and $\omega : \mathrm{D}(\omega) \to \mathbb{C}$ is linear,
- ω is positive, i.e. $\omega(a^*a) \geq 0$ for every $a \in \mathrm{D}(\omega)$,
- ω is respects the involution, i.e. $\omega(a^*) = \overline{\omega(a)}$ for all $a \in \mathrm{D}(\omega)$.

Additionally, if ω is tracial, i.e. $\omega(ab) = \omega(ba)$ for all $a, b \in \mathrm{D}(\omega)$, we call ω a tracial weight. If $\mathcal{A}$ is unital, $\mathrm{D}(\omega) = \mathcal{A}$ and $\omega(1_{\mathcal{A}}) = 1$ then we call $(\mathcal{A},\omega)$ a *non-commutative probability space* and ω is called a state. Moreover if $(\mathcal{A},\omega)$ is non-commutative measure space and in addition $(\mathcal{A},\tau)$ is a non-commutative probability space, we call the triple $(\mathcal{A},\omega,\tau)$ a *non-commutative probability space with a weight ω and a state τ.*

Let $(\mathcal{A},\omega)$ be a non-commutative measure space and let $a_1,\dots,a_k \in \mathrm{D}(\omega)$. The *non-commutative distribution* of $(a_1,\dots,a_k)$ is the family of *(mixed) moments*

$$\{\omega(a_{i_1}^{\varepsilon_1}\dots a_{i_p}^{\varepsilon_p}) : p \geq 1, 1 \leq i_1,\dots,i_p \leq k, (\varepsilon_1,\dots,\varepsilon_p) \in \{1,*\}^p\}.$$

Given non-commutative measure spaces $(\mathcal{A},\omega), (\mathcal{B},\xi)$ and elements $a_1,\dots,a_k \in \mathrm{D}(\omega)$, $b_1,\dots,b_k \in \mathrm{D}(\xi)$, we say that $(a_1,\dots,a_k)$ has *the same distribution* as $(b_1,\dots,b_k)$ if

$$\omega(a_{i_1}^{\varepsilon_1}\cdots a_{i_p}^{\varepsilon_p}) = \xi(b_{i_1}^{\varepsilon_1}\cdots b_{i_p}^{\varepsilon_p}) \tag{2.1}$$

for any choice of $p \in \mathbb{N}$, $1 \leq i_1,\dots,i_p \leq k$ and $(\varepsilon_1,\dots,\varepsilon_p) \in \{1,*\}^p$.

Let $(\mathcal{C},\omega,\tau)$ be a non-commutative probability space with a weight ω (or $\tilde{\omega}$). Let $\mathcal{A}$, $\mathcal{B}$ be $*$-subalgebras of $\mathcal{C}$ such that $1_{\mathcal{C}} \in \mathcal{B}$. Let $\mathrm{Ideal}_{\mathcal{B}}(\mathcal{A})$ be the ideal generated by $\mathcal{A}$ over $\mathcal{B}$. More precisely,

$$\mathrm{Ideal}_{\mathcal{B}}(\mathcal{A}) := \mathrm{span}\{b_0a_1b_1\cdots a_nb_n : n \in \mathbb{N}, a_1,\dots,a_n \in \mathcal{A}, b_0,\dots,b_n \in \mathcal{B}\},$$

which is a $*$-subalgebra of $\mathcal{C}$ containing $\mathcal{A}$. Note $\mathrm{Ideal}_{\mathcal{B}}(\mathcal{A})$ is not necessarily an ideal in $\mathcal{C}$ but in $\mathcal{A} * \mathcal{B}$.

We start with the definition of monotone independence.

Definition 2.1. *Let $(\mathcal{C}, \tilde{\omega}, \tau)$ non-commutative probability space with a weight $\tilde{\omega}$ and $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{C}$ be $*$-subalgebras.*

(1) *We say that the pair $(\mathcal{A}, \mathcal{B})$ is* monotonically independent *or simply* monotone *with respect to $(\tilde{\omega}, \tau)$ if*

- $\mathrm{Ideal}_{\mathcal{B}}(\mathcal{A}) \subset \mathrm{D}(\tilde{\omega})$*;*
- *for any $n \in \mathbb{N}, a_1, \ldots, a_n \in \mathcal{A}$ and any $b_0, \ldots, b_n \in \mathcal{B}$, we have that*

$$\tilde{\omega}(b_0 a_1 b_1 a_2 b_2 \cdots a_n b_n) = \tilde{\omega}(a_1 a_2 \cdots a_n)\tau(b_0)\tau(b_1)\tau(b_2)\cdots\tau(b_n),$$

(2) *Given $a_1, \ldots, a_k \in \mathrm{D}(\tilde{\omega})$ and $b_1, \ldots, b_\ell \in \mathcal{C}$, the pair $(\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_\ell\})$ is monotone if $(\mathrm{alg}\{a_1, \ldots, a_k\}, \mathrm{alg}\{1_{\mathcal{C}}, b_1, \ldots, b_\ell\})$ is monotone. Note that we do not assume that $\mathrm{alg}\{a_1, \ldots, a_k\}$ contains the unit of $\mathcal{C}$.*

Remark 2.2. Note that for the monotone case ω is not tracial. In fact, monotone product of tracial states is not tracial in general.

Next, we recall the definition of cyclic monotone independence.

Definition 2.3. *Let $(\mathcal{C}, \omega, \tau)$ be a non-commutative probability space with a tracial weight ω and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ be $*$-subalgebras.*

(1) *We say that the pair $(\mathcal{A}, \mathcal{B})$ is* cyclic monotonically independent *or simply* cyclic monotone *with respect to (ω, τ) if*

- $\mathrm{Ideal}_{\mathcal{B}}(\mathcal{A}) \subset \mathrm{D}(\omega)$*;*
- *for any $n \in \mathbb{N}, a_1, \ldots, a_n \in \mathcal{A}$ and any $b_0, \ldots, b_n \in \mathcal{B}$, we have that*

$$\omega(b_0 a_1 b_1 a_2 b_2 \cdots a_n b_n) = \omega(a_1 a_2 \cdots a_n)\tau(b_1)\tau(b_2)\cdots\tau(b_n b_0).$$

(2) *Given $a_1, \ldots, a_k \in \mathrm{D}(\omega)$ and $b_1, \ldots, b_\ell \in \mathcal{C}$, the pair $(\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_\ell\})$ is cyclic monotone if $(\mathrm{alg}\{a_1, \ldots, a_k\}, \mathrm{alg}\{1_{\mathcal{C}}, b_1, \ldots, b_\ell\})$ is cyclic monotone. As in the previous definition, we do not assume that $\mathrm{alg}\{a_1, \ldots, a_k\}$ contains the unit of $\mathcal{C}$.*

If $(\mathcal{A}, \omega_0)$ be a (non-unital) measure space with a tracial weight ω_0 and $(\mathcal{B}, \tau)$ a non-commutative probability space with a tracial state τ then there is a natural way to construct a non-commutative probability space with a tracial weight such that $(\mathcal{A}, \mathcal{B})$ is cyclic monotone.

Definition 2.4. *Let $(\mathcal{A}, \omega_0)$ be a (non-unital) measure space with a tracial weight ω_0 and $(\mathcal{B}, \tau)$ a non-commutative probability space with a tracial state τ. We consider the algebraic free product $\mathcal{A} * \mathcal{B}$ and the unit $1_{\mathcal{B}}$ identified with $\mathbb{C}1_{\mathcal{A}*\mathcal{B}}$. More precisely if $\mathcal{B} = \mathbb{C}1_{\mathcal{B}} \oplus \mathring{\mathcal{B}}$ is a direct sum decomposition then*

$$\mathcal{A} * \mathcal{B} = \mathbb{C}1_{\mathcal{A}*\mathcal{B}} \oplus \mathring{\mathcal{B}} \oplus \mathcal{A} \oplus (\mathcal{A} \otimes \mathring{\mathcal{B}}) \oplus (\mathring{\mathcal{B}} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathring{\mathcal{B}} \otimes \mathcal{A}) \ldots$$

Then the cyclic monotone product $\omega_0 \trianglerighteq \tau$ of ω and τ is defined by

$$\mathrm{D}(\omega_0 \trianglerighteq \tau) := \mathrm{D}(\omega_0) \oplus (\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}) \oplus (\mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0)) \oplus (\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0)) \ldots,$$
$$\omega_0 \trianglerighteq \tau(b_0 a_1 b_1 \ldots a_n b_n) := \omega_0(a_1 \ldots a_n)\tau(b_1) \ldots \tau(b_{n-1})\tau(b_n b_0).$$

It is positive and tracial, and then $(\mathcal{A} * \mathcal{B}, \omega_0 \trianglerighteq \tau)$ is a non-commutative measure space such that $(\mathrm{D}(\omega_0), \mathcal{B})$ are cyclic monotone in $(\mathcal{A} * \mathcal{B}, \omega_0 \trianglerighteq \tau, \tilde{\tau})$, where $\tilde{\tau}$ is the free product of the zero map on $\mathcal{A}$ and tracial state τ on $\mathcal{B}$. So $\tilde{\tau}$ is a tracial state on $\mathcal{A} * \mathcal{B}$.

Remark 2.5. 1. The construction of $\mathcal{A} * \mathcal{B}$ is similar to the free product construction of non-commutative probability spaces (see [17]), though here one needs to take into account that $\mathcal{A}$ is not unital and ω_0 may not be defined on all of $\mathcal{A}$.

2. There is analogous construction for monotone independence. It is denoted by $\triangleright$ instead of $\trianglerighteq$, see Definition 2.3 on page 341 in [16] for details. The details of the cyclic monotone case can be found in page 1122 in [3].

Let us motivate our main result of the section by the following calculation.

Let $(\mathcal{A}, \omega_0)$ be a (non-unital) measure space with a tracial weight ω_0, $(\mathcal{B}, \tau)$ a non-commutative probability space with a tracial state τ and consider the monotone and cyclic monotone product constructions of in Definition 2.4 and Remark 2.5, where we take $\mathring{\mathcal{B}} := \ker \tau$. Thus we have the decomposition

$$\mathcal{B} = \mathbb{C}1_{\mathcal{B}} \oplus \mathring{\mathcal{B}} = \mathbb{C}1_{\mathcal{B}} \oplus \ker(\tau).$$

i.e. for any given $b \in \mathcal{B}$ there exists $\mathring{b} \in \mathring{\mathcal{B}}$ such that $b = \mathring{b} + \tau(b)1_{\mathcal{B}}$. Following Definition 2.4 we define

$$\omega := \omega_0 \trianglerighteq \tau \colon \mathrm{D}(\omega) \to \mathbb{C}, \quad \tilde{\omega} := \omega_0 \triangleright \tau \colon \mathrm{D}(\tilde{\omega}) \to \mathbb{C},$$

i.e., the cyclic monotone product and the monotone product of ω_0 and τ. Note that $\mathrm{D}(\omega)$ does not have any elements in $\mathcal{B}$. Then any element of $\mathrm{D}(\omega)$ can be written as

$$b_0 a_1 b_1 \dots a_n b_n \in \mathrm{D}(\omega),$$

where $n \geq 1$, $a_j \in \mathrm{D}(\omega_0)$ and $b_j \in \mathring{\mathcal{B}}$ with the exception of b_0, b_n which may be equal to $1_{\mathcal{B}}$ if the word begins resp. ends with an element of $\mathrm{D}(\omega)$. Let us explore how this decomposition interacts with the tracial weight ω and the weight $\tilde{\omega}$, and compute some easy examples. Let $0 \neq a = a^* \in \mathrm{D}(\omega_0)$ and $0 \neq b = b^* \in \mathcal{B}$, then we have

$$\omega(ab) = \omega(ba) = \omega(a\tau(b)1 + a\mathring{b}) = \tau(b)\omega(a).$$

For words of length 3 we have

$$\begin{aligned} \omega(aba) &= \omega(a^2)\tau(b) + \omega(a^2)\tau(\mathring{b}) = \omega(a^2)\tau(b) \\ \omega(bab) &= \omega(a)\tau(b^2). \end{aligned}$$

Moreover consider

$$\begin{aligned} \omega(abab) &= \omega(a(\mathring{b} + \tau(b)1)a(\mathring{b} + \tau(b)1)) \\ &= \omega_0(a^2)\tau(\mathring{b})^2 + 2\tau(b)\omega_0(a^2)\tau(\mathring{b}) + \tau(b)^2\omega_0(a^2) = \tau(b)^2\omega_0(a^2). \end{aligned}$$

Now if $b \in \mathring{\mathcal{B}}$, then all of these expressions vanish besides the word bab. This illustrates that the only expressions that do not vanish are words that do not have an element $b \in \mathring{\mathcal{B}}$ *enclosed* by elements of $\mathrm{D}(\omega_0)$. We collect the results of the computations in the following table, where $*$ denotes values which may differ from 0. On the other hand, we

Table 1: Values of monomials evaluated in $\omega/\tilde{\omega}$

m	a_1	$a_1\mathring{b}_1$	$\mathring{b}_1 a_1$	$\mathring{b}_1 a_1 \mathring{b}_2$	others monomials
$\omega(m)$	$*$	0	0	$*$	0
$\tilde{\omega}(m)$	$*$	0	0	0	0

can compute

$$\omega(baab) = \omega(abba) = \omega(a(\mathring{b} + \tau(b)1)(\mathring{b} + \tau(b)1)a)$$

$$= \omega_0(a^2)\tau(\mathring{b}^2) + \tau(b)^2\omega_0(a^2)$$

and if $\tau(b) = 0$ i.e. $b = \mathring{b}$ then $\omega(abba) = \omega_0(a^2)\tau(\mathring{b}^2) \geq 0$. The latter is an example on how such a non vanishing term may be produced by a product of monomials, here $ab \cdot ba$. There are only a few of these instances where a product of two monomials may not vanish, we capture the whole picture in the following tables. The first column and row consist of monomials m_1, m_2 alternating in elements $b_j \in \mathring{B}, a_j \in \mathrm{D}(\omega_0)$ whereas the 0 and $*$ denote whether $\omega(m_1 \cdot m_2)$ (respectively $\tilde{\omega}(m_1 \cdot m_2)$) is zero or might be different from 0 (again denoted by an $*$).

Table 2: Values of ω of monomial products, i.e. $\omega(m_1 m_2)$

$m_1 \backslash m_2$	a_1	$a_1\mathring{b}_1$	$\mathring{b}_1 a_1$	$\mathring{b}_2 a_1 \mathring{b}_2$	others
a_2	$*$	0	0	0	0
$a_2\mathring{b}_3$	0	0	$*$	0	0
$\mathring{b}_3 a_2$	0	$*$	0	0	0
$\mathring{b}_3 a_2 \mathring{b}_4$	0	0	0	$*$	0
others	0	0	0	0	0

Table 3: Values of $\tilde{\omega}$ of monomial products, i.e. $\tilde{\omega}(m_1 m_2)$

$m_1 \backslash m_2$	a_1	$a_1\mathring{b}_1$	$\mathring{b}_1 a_1$	$\mathring{b}_2 a_1 \mathring{b}_2$	others
a_2	$*$	0	0	0	0
$a_2\mathring{b}_3$	0	0	$*$	0	0
$\mathring{b}_3 a_2$	0	0	0	0	0
$\mathring{b}_3 a_2 \mathring{b}_4$	0	0	0	0	0
others	0	0	0	0	0

This shows that only a certain part of $\mathrm{D}(\omega)$ is not in the kernel of ω. The main theorem of this section describes this observation. We introduce more notation and then state the theorem. First, we define I and J through the following equations:

$$I := \mathrm{D}(\omega_0) \oplus (\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}) \oplus (\mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0)) \oplus (\mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}) \oplus \underbrace{(\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0)) \oplus \cdots}_{J}$$

$$J := (\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0)) \oplus \cdots$$

Namely, J is the sum of all tensor products with at least three legs and at least two $D(\omega_0)$'s, so that, in particular:

$$I = \mathrm{D}(\omega_0) \oplus (\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}) \oplus (\mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0)) \oplus (\mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}) \oplus J.$$

Then we have the following

Lemma 2.6. *J is an ideal in I.*

Proof. The fact that J is closed under linear operations is clear. The elements of J are linear combinations of words in alternating elements of $\mathrm{D}(\omega_0)$ and $\mathring{\mathcal{B}}$ of length bigger than 2 and having at least two a's. Thus let $n > 1$, $x := b_0 a_1 b_1 a_2 \dots a_n b_n \in J$, where b_0, b_n may be $1_{\mathcal{B}}$ in case the word begins or ends on an a. and similarly let $m \geq 1$ such that $y := \tilde{b}_0\tilde{a}_1\tilde{b}_1 \dots \tilde{a}_m\tilde{b}_m \in I$ where we allow $\tilde{b}_0, \tilde{b}_n = 1_{\mathcal{B}}$. We write $b = b_n\tilde{b}_0$, then we can decompose $b = \tau(b)1 + \mathring{b} \in 1 \oplus \mathring{B}$ and hence

$$\begin{aligned} xy &= b_0a_1b_1a_2 \dots a_n \underbrace{(b_n\tilde{b}_0)}_{b} \tilde{a}_1\tilde{b}_1 \dots \tilde{a}_m\tilde{b}_m \\ &= \tau(b)b_0a_1b_1a_2 \dots a_n\tilde{a}_1\tilde{b}_1 \dots \tilde{a}_m + b_0a_1b_1a_2 \dots a_n\mathring{b}\tilde{a}_1\tilde{b}_1 \dots \tilde{a}_m\tilde{b}_m \end{aligned}$$

Both summands are alternating products of length bigger than 2 i.e. elements of J. Note this is due to the fact that $\mathcal{A}$ is non-unital, i.e. does not contain any invertible elements and $\mathcal{A}$ and $\mathcal{B}$ have no algebraic relations i.e. the products cannot be reduced. □

Let us introduce more notation: Let H be arbitrary vector space, we denote by $\mathrm{End}_{\mathrm{fin}}(H)$ the collection of finite rank endomorphisms on H. As a vector space, it is canonically isomorphic to $H^* \otimes H$. We are particularly interested in the case of $H = \mathcal{B}$. In this case, $H^* \otimes H$ becomes $\mathrm{End}_{\mathrm{fin}}(\mathcal{B})$. Note we also have the embedding

$$\mathcal{B} \otimes \mathcal{B} \to \mathrm{End}_{\mathrm{fin}}(\mathcal{B}), \quad b_1 \otimes b_2 \mapsto (x \mapsto b_1\tau(b_2x))$$

We define ψ_τ on $\mathrm{End}_{\mathrm{fin}}(\mathcal{B})$ as the linear extension of

$$\psi_\tau(h_1^* \otimes h_2) = h_1^*(1)\tau(h_2).$$

Intuitively, it is the upper left coefficient of the matrix of the endomorphism. The map ψ_τ is defined on $\mathrm{End}_{\mathrm{fin}}(\mathcal{B})$ and via the embedding of $\mathcal{B} \otimes \mathcal{B} \to \mathrm{End}_{\mathrm{fin}}(\mathcal{B})$ we can evaluate it in elements of $\mathcal{B} \otimes \mathcal{B}$. Note that ψ_τ depends on the state τ; however, we will omit this dependence in the notation and write ψ. Finally we can state the theorem.

Theorem 2.7. *The weights ω and $\tilde{\omega}$ vanish on the ideal J, and we have a canonical map*

$$\chi : I \to I/J \cong \mathcal{B} \otimes \mathcal{B} \otimes \mathrm{D}(\omega_0),$$

which satisfies the following two properties:

$$\omega = \omega_0 \otimes \mathsf{Tr} \circ \chi,$$

where Tr is given by $\mathsf{Tr}(b_1 \otimes b_2) = \tau(b_1b_2)$ and

$$\tilde{\omega} = \omega_0 \otimes \psi \circ \chi.$$

Proof. First note that $\mathrm{D}(\omega) = \mathrm{D}(\tilde{\omega}) = I$. We need to show that J annihilates ω and $\tilde{\omega}$. Take a monomial $b_0a_1b_1 \dots a_nb_n = x \in J$, where $n > 1$, then as we have seen in the discussion above an *enclosed* element $b \in \mathring{\mathcal{B}}$ will annihilate ω,

$$\omega(x) = \omega(b_0a_1b_1 \dots a_nb_n) = \omega(a_1 \dots a_n)\tau(b_0)\underbrace{\tau(b_1)}_{=0} \dots \tau(b_n) = 0,$$

thus J is an ideal contained in the kernel of ω and we have

$$I/J = \mathrm{D}(\omega_0) \oplus (\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}) \oplus (\mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}) \oplus (\mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}),$$

the isomorphisms in the theorem are given by the identification

$$\mathrm{D}(\omega_0) \cong 1_B \otimes \mathrm{D}(\omega_0) \otimes 1_B, \mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}} \cong 1_B \otimes \mathrm{D}(\omega_0) \otimes \mathring{\mathcal{B}}, \mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0) \cong \mathring{\mathcal{B}} \otimes \mathrm{D}(\omega_0) \otimes 1_B.$$

Hence ω factorized over I/J and by definition $\omega_0 \otimes \mathsf{Tr}(y) = \omega(y)$ for every element in $I/J \cong \mathcal{B} \otimes \mathcal{B} \otimes \mathrm{D}(\omega_0)$, i.e. it must be the unique map that satisfies $\omega_0 \otimes \mathsf{Tr} \circ \chi$. A similar calculation shows the result for $\tilde{\omega}$. □

Remark 2.8. This result is the abstract version of the matrix model in the next section. Note that there we have a large but finite n model such that we actually have an isomorphism of $B \cong B^*$ which means that $B \otimes B \otimes \mathrm{D}(\omega_0)$ from the main result is isomorphic to $\mathrm{End}_{\mathrm{fin}}(\mathcal{B}) \otimes \mathrm{D}(\omega_0)$. And indeed the elements in section 3 will be endomorphisms in $M_n(\mathbb{C}) \otimes M_{2^q}(\mathbb{C})$.

3 Matrix model

This section is an application of the result of the previous section: it exhibits a matrix model for monotone independence and cyclic monotone independence. Basically, this is a concrete version of the above abstract result.

3.1 Setup

Let us denote by $M_\infty(\mathbb{C})$ the inductive limit given by the non-unital embeddings $M_n(\mathbb{C})$ into $M_m(\mathbb{C})$ for $n < m$

$$f_{n,m}\colon M_n(\mathbb{C}) \to M_m(\mathbb{C}), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

i.e. we are plugging $a \in M_n(\mathbb{C})$ into the left upper corner and padding it by zeros. Note that these embeddings are compatible with the non-normalized traces on Tr_n on $M_n(\mathbb{C})$, i.e. $\mathsf{Tr}_m \circ f_{n,m} = \mathsf{Tr}_n$. This induces the trace Tr on $M_\infty(\mathbb{C})$. Given any collection of elements $a_1, \dots, a_p \in M_\infty(\mathbb{C})$, we may always choose $n \in \mathbb{N}$ big enough such that $a_i \in M_n(\mathbb{C})$, i.e. in an upper corner of size n of $M_\infty(\mathbb{C})$.

From now on let $(\mathcal{C}, \omega, \tau)$ be a non-commutative probability space with a tracial weight ω and a tracial state τ, where $\mathcal{A}$, $\mathcal{B}$ are $*$-subalgebras of $\mathcal{C}$ and $\mathcal{C} = \mathcal{A} * \mathcal{B}$. We also assume that $\mathcal{A} = M_\infty(\mathbb{C})$. Moreover we consider the following setup:

- $a_1, \dots, a_p \in \mathcal{A}$ are self-adjoint;
- $\tau(a) = 0$ for all $a \in \mathcal{A}$;
- $b_0 = 1$;
- $b_1, \dots, b_q \in \mathcal{B}$ are self-adjoint, $\tau(b_i) = 0$ and $\tau(b_i^2) = 1$ for any $i = 1, 2, \dots q$, and $b_i \perp b_j$, that is, $\tau(b_i b_j) = 0$ if $i \neq j$.

Roughly speaking τ measures elements in $\mathcal{B}$, which we consider as large elements and $\tau(a)$ is 0 for all $a \in \mathcal{A}$ which we consider as small elements.

Remark 3.1. There is no loss in making these assumptions on b_i because we can simultaneously take their real and imaginary parts if they are not self-adjoint. As for orthogonality, we can subsequently make a Gram-Schmidt orthogonalization to ensure that this property is satisfied too.

We consider a linear combination of words in $a_1, \dots a_p, b_1, \dots b_q$ with the property that each word non-trivially involved in the linear combination has a non-zero valuation in at least one of the a_i's. Such a linear combination can be written as

$$P := P(a_1, \dots, a_p, b_0, \dots, b_q) = \sum_{i_1, i_2 = 0}^{q} \sum_{j_1 = 1}^{p} \lambda_{i_1, i_2, j_1} b_{i_1} a_{j_1} b_{i_2} + \cdots , \tag{3.2}$$

where P can be viewed as a non-commuting polynomial in $p + q$ variables and "$\cdots$" stands for a sum of alternating monomials in a_i, b_j with at least two a's.

In the following we model the polynomial, we denote by $I_n \in M_n(\mathbb{C})$ the identity matrix. Then we consider the operator $\tilde{b}_{j,n} \in M_n(\mathbb{C}) \otimes M_{2^q}(\mathbb{C})$, $j = 0, 1, \dots q$, defined by

$$\begin{aligned}
\tilde{b}_{0,n} &= I_n \otimes I_{2^q} =: I_n \otimes B_0 \\
\tilde{b}_{1,n} &= I_n \otimes \underbrace{J \otimes I_2 \otimes \cdots \otimes I_2}_{=:B_1,\, q \text{ terms}} \\
\tilde{b}_{2,n} &= I_n \otimes \underbrace{I_2 \otimes J \otimes \cdots \otimes I_2}_{=:B_2,\, q \text{ terms}} \\
&\vdots \\
\tilde{b}_{q,n} &= I_n \otimes \underbrace{I_2 \otimes \cdots \otimes I_2 \otimes J}_{=:B_q,\, q \text{ terms}},
\end{aligned} \tag{3.3}$$

where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The tensor model for P in $M_n(\mathbb{C}) \otimes M_{2^q}(\mathbb{C})$ is the matrix

$$\tilde{P} := \tilde{P}(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots \tilde{b}_{q,n}) = \sum_{i_1, i_2, j_1} \lambda_{i_1, i_2, j_1} \tilde{b}_{i_1,n} \cdot \phi(a_{j_1}) \cdot \tilde{b}_{i_2,n},$$

where

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \phi(a) := a \otimes E_{11}^{\otimes q} = \begin{pmatrix} a & 0 & \dots 0 \\ 0 & 0 & \dots 0 \\ 0 & 0 & \dots 0 \end{pmatrix} \in M_n(\mathbb{C}) \otimes M_{2^q}(\mathbb{C})$$

and we assumed $n \in \mathbb{N}$ large enough to fit $a_i \in M_n(\mathbb{C})$ as described earlier. Here for simplicity, we do not write $\tilde{P}(\phi(a_1), \dots, \phi(a_p), \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})$ but $\tilde{P}(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots \tilde{b}_{q,n})$.

We aim to prove that noncommutative distribution of $(a_1, \dots, a_p, b_0, \dots, b_q)$ under the tracial weight ω is same as noncommutative distribution of $(\phi(a_1), \dots, \phi(a_p), \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})$. To do it, we shall see that the moments of P equal with the moments of P'.

3.2 A tensor model for cyclic monotone independence

Now, we assume that the pair $(\mathcal{A}, \mathcal{B})$ is cyclic monotone with respect to (ω, τ). Then the moments of P under ω can be obtained from the tensor model $\tilde{P}$ in the following sense.

Theorem 3.2. *We have*

$$\omega(P(a_1, \dots, a_p, b_0, \dots, b_q)^k) = \mathrm{Tr}_n \otimes \mathrm{Tr}_2^{\otimes q}(\tilde{P}(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})^k)$$

for all $n \in \mathbb{N}$.

Proof. Let us start by computing the k-moment of P. Recall that by Theorem 2.7, only the terms containing one a contribute to the calculation. We can ignore higher moments. So its computation will be quite simple.

$$\begin{aligned}
\omega(P^k) &= \omega\left(\left(\sum_{i_1, i_2=0}^{q} \sum_{j_1=1}^{p} \lambda_{i_1, i_2, j_1} b_{i_1} a_{j_1} b_{i_2} + \cdots\right)^k\right) \\
&= \sum_{i_1, i_2, \dots, i_{2k}=0}^{q} \sum_{j_1, \dots, j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1}, i_{2r}, j_r}\right) \omega(b_{i_1} a_{j_1} b_{i_2} \dots b_{i_{2k-1}} a_{j_k} b_{i_{2k}}) \\
&= \sum_{i_1, i_2, \dots, i_{2k}=0}^{q} \sum_{j_1, \dots, j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1}, i_{2r}, j_r}\right) \omega(a_{j_1} \dots a_{j_k}) \tau(b_{i_{2k}} b_{i_1}) \dots \tau(b_{i_{2k-2}} b_{i_{2k-1}})
\end{aligned}$$

$$= \sum_{i_1,i_2,\dots,i_{2k}=0}^{q} \sum_{j_1,\dots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r},j_r}\right) \omega(a_{j_1}\dots a_{j_k}) \delta_{i_{2k},i_1}\delta_{i_2,i_3}\dots\delta_{i_{2k-2},i_{2k-1}}$$
$$= \sum_{i_1,i_3,\dots,i_{2k-1}=0}^{q} \sum_{j_1,\dots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r+1},j_r}\right) \omega(a_{j_1}\dots a_{j_k}).$$

We show that it is equal to $\mathrm{Tr}_n \otimes \mathrm{Tr}_2^{\otimes q}(\tilde{P}(a_1,\dots,a_p,\tilde{b}_{0,n},\dots,\tilde{b}_{q,n})^k)$. We have

$$\mathrm{Tr}_2^{\otimes q}(B_{i_1}E_{11}^{\otimes q}B_{i_2}\dots B_{i_{2k-1}}E_{11}^{\otimes q}B_{i_{2k}}) = \begin{cases} 1 & i_{2k}=i_1,\dots,i_{2k-2}=i_{2k-1} \\ 0 & \text{otherwise} \end{cases}$$

from $JE_{11}J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $E_{11}JE_{11} = 0$. Here are examples in the $q=2$ case:

$$\mathrm{Tr}_2^{\otimes 2}(B_1E_{11}^{\otimes 2}B_2B_2E_{11}^{\otimes 2}B_1) = 1,$$
$$\mathrm{Tr}_2^{\otimes 2}(B_1E_{11}^{\otimes 2}B_2B_1E_{11}^{\otimes 2}B_2) = 0.$$

Thus, its computation will be quite straightforward again and we obtain

$$\mathrm{Tr}_n \otimes \mathrm{Tr}_2^{\otimes q}(\tilde{P}(a_1,\dots,a_p,\tilde{b}_{0,n},\dots,\tilde{b}_{q,n})^k)$$
$$= \sum_{i_1,\dots,i_{2k}=0}^{q} \sum_{j_1,\dots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r},j_r}\right) \mathrm{Tr}_n(a_{j_1}\dots a_{j_k})\mathrm{Tr}_2^{\otimes q}(B_{i_1}E_{11}^{\otimes q}B_{i_2}\dots B_{i_{2k-1}}E_{11}^{\otimes q}B_{i_{2k}})$$
$$= \sum_{i_1,\dots,i_{2k}=0}^{q} \sum_{j_1,\dots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r},j_r}\right) \mathrm{Tr}_n(a_{j_1}\dots a_{j_k})\delta_{i_{2k},i_1}\delta_{i_2,i_3}\dots\delta_{i_{2k-2},i_{2k-1}}$$
$$= \sum_{i_1,i_3,\dots,i_{2k-1}=0}^{q} \sum_{j_1,\dots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r+1},j_r}\right) \mathrm{Tr}_n(a_{j_1}\dots a_{j_k})$$
$$= \sum_{i_1,i_3,\dots,i_{2k-1}=0}^{q} \sum_{j_1,\dots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r+1},j_r}\right) \omega(a_{j_1}\dots a_{j_k}).$$

Recall that we have $\mathrm{Tr}_n(a_{j_1}\dots a_{j_k}) = \omega(a_{j_1}\dots a_{j_k})$, which concludes the proof. □

3.3 A tensor model for monotone independence

As it follows from the main theorem, we can also treat monotone independence with the very same model, provided that we modify the weight. Let us assume that the pair $(\mathcal{A},\mathcal{B})$ is monotone independent with respect to $(\tilde{\omega},\tau)$.

Theorem 3.3. *Let* $\eta(B) = B_{11}$ *for* $B = (B_{ij})_{ij} \in M_2(\mathbb{C})$. *Then we have*

$$\tilde{\omega}(P(a_1,\dots,a_p,b_0,\dots,b_q)^k) = \mathrm{Tr}_n \otimes \eta^{\otimes q}(\tilde{P}(a_1,\dots,a_p,\tilde{b}_{0,n},\dots,\tilde{b}_{q,n})^k).$$

for all $n \in \mathbb{N}$.

Note that $\mathrm{Tr}_n \otimes \eta^{\otimes q}$ is a non-tracial weight.

Proof. First, note that similar to the cyclic monotone case, we may omit all terms containing at least two a's. By monotone independence, we have for the remaining terms:

$$\tilde{\omega}(P(a_1,\dots,a_p,b_0,\dots,b_q)^k))$$

$$= \sum_{i_1,\ldots,i_{2k}=0}^{q} \sum_{j_1,\ldots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r},j_r}\right) \tilde{\omega}(a_{j_1}\ldots a_{j_k})\tau(b_{i_{2k}})\tau(b_{i_1})\tau(b_{i_2}b_{i_3})\ldots\tau(b_{i_{2k-2}}b_{i_{2k-1}})$$
$$= \sum_{i_1,\ldots,i_{2k}=0}^{q} \sum_{j_1,\ldots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r},j_r}\right) \tilde{\omega}(a_{j_1}\ldots a_{j_k})\delta_{i_{2k},0}\delta_{i_1,0}\delta_{i_2,i_3}\ldots\delta_{i_{2k-2},i_{2k-1}}$$
$$= \sum_{i_2,i_4,\ldots,i_{2k-2}=0}^{q} \sum_{j_1,\ldots,j_k=1}^{p} \lambda_{0,i_2,j_1}\left(\prod_{r=2}^{k-1} \lambda_{i_{2r-2},i_{2r},j_r}\right)\lambda_{i_{2k-2},0,j_k}\tilde{\omega}(a_{j_1}\ldots a_{j_k}).$$

On the other hand, since $\eta(JE_{11}J)=0$, we obtain

$$\begin{aligned}
&\mathrm{Tr}_n\otimes\eta^{\otimes q}(\tilde{P}^k)\\
&= \sum_{i_1,\ldots,i_{2k}=0}^{q} \sum_{j_1,\ldots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r},j_r}\right) \mathrm{Tr}_n(a_{j_1}\ldots a_{j_k})\eta^{\otimes q}(B_{i_1}E_{11}^{\otimes q}B_{i_2}\cdots B_{i_{2k-1}}E_{11}^{\otimes q}B_{i_{2k}})\\
&= \sum_{i_1,\ldots,i_{2k}=0}^{q} \sum_{j_1,\ldots,j_k=1}^{p} \left(\prod_{r=1}^{k} \lambda_{i_{2r-1},i_{2r},j_r}\right) \mathrm{Tr}_n(a_{j_1}\ldots a_{j_k})\delta_{i_1 0}\delta_{i_2 i_3}\ldots\delta_{i_{2k-2}i_{2k-1}}\delta_{i_{2k}0}\\
&= \sum_{i_2,i_4,\ldots,i_{2k-2}=0}^{q} \sum_{j_1,\ldots,j_k=1}^{p} \lambda_{0,i_2,j_1}\left(\prod_{r=2}^{k-1} \lambda_{i_{2r-2},i_{2r},j_r}\right)\lambda_{i_{2k-2},0,j_k}\tilde{\omega}(a_{j_1}\ldots a_{j_k}).
\end{aligned}$$

□

3.4 Replacing tensors by limit swaps

This subsection is a simple observation: the previous proofs rely on the same model that relies on tensors and considers two different states – one for monotone cyclic independence, and one for monotone independence. Here, we show that in a context of a *sequence* of matrix models, we can avoid resorting to tensors.

Let us first recall that we are interested in polynomials $\tilde{P}_n \in M_n(\mathbb{C})\otimes M_2(\mathbb{C})^{\otimes q}$. In addition, $\tilde{P}_n$ depends tacitly on n, q and is well defined for any q, n large enough via the embedding described in Equation (3.1). Likewise, with the same embedding, we will freely view $\tilde{P}_n$ (as a double sequence in n, q) as elements of $M_\infty(\mathbb{C})$. In the sequel of this paper, we denote by η_l the function

$$\mathcal{A} = M_\infty(\mathbb{C}) \to \mathbb{C}, \quad x \mapsto \sum_{k=0}^{l} x_{kk}.$$

This is sometimes called a partial trace, e.g. in the context of Horn inequalities (although it is not the partial trace of quantum information theory). We have

Theorem 3.4. *The following holds true*

$$\omega(P^k) = \lim_{n\to\infty}\lim_{l\to\infty}\eta_l(\tilde{P}_n^k),$$

i.e. convergence to the cyclic monotone independent moments and

$$\tilde{\omega}(P^k) = \lim_{l\to\infty}\lim_{n\to\infty}\eta_l(\tilde{P}_n^k),$$

i.e. convergence to the monotone independent moments.

Proof. First, recall that we have

$$\omega(P(a_1,\ldots,a_p,b_0,\ldots,b_q)^k) = \mathrm{Tr}(\tilde{P}_n(a_1,\ldots,a_p,\tilde{b}_{0,n},\ldots,\tilde{b}_{q,n})^k)$$

by Theorem 3.2. Note that the right hand side has a dependency on the size n that can be easily removed by letting $n \to \infty$ (taking n large enough is sufficient in the proof), and we get

$$\omega(P(a_1, \dots, a_p, b_0, \dots, b_q)^k) = \lim_{n\to\infty} \mathrm{Tr}(\tilde{P}_n(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})^k)$$

which proves the first claim.

On the other hand we note $\mathrm{Tr} = \lim_{l\to\infty} \eta_l$, and therefore we get

$$\omega(P(a_1, \dots, a_p, b_0, \dots, b_q)^k) = \lim_{n\to\infty} \lim_{l\to\infty} \eta_l(\tilde{P}_n(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})^k)$$

Likewise, Theorem 3.3 gives

$$\tilde{\omega}(P(a_1, \dots, a_p, b_0, \dots, b_q)^k) = \lim_{n\to\infty} \mathrm{Tr}(f_{n,2^q n}(I_n)\tilde{P}_n(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})^k).$$

Rewriting it as

$$\tilde{\omega}(P(a_1, \dots, a_p, b_0, \dots, b_q)^k) = \mathrm{Tr}(\lim_{n\to\infty} f_{n,2^q n}(I_n)\tilde{P}_n(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})^k),$$

we get

$$\tilde{\omega}(P(a_1, \dots, a_p, b_0, \dots, b_q)^k) = \lim_{l\to\infty} \eta_l(\lim_{n\to\infty} f_{n,2^q n}(I_n)\tilde{P}_n(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})^k),$$

but clearly, for $n \geq l$, we have $\eta_l(f_{n,2^q n}(I_n)P) = \eta_l(P)$ and therefore

$$\tilde{\omega}(P(a_1, \dots, a_p, b_0, \dots, b_q)^k) = \lim_{l\to\infty} \eta_l(\lim_{n\to\infty} \tilde{P}_n(a_1, \dots, a_p, \tilde{b}_{0,n}, \dots, \tilde{b}_{q,n})^k),$$

which concludes the proof. □

3.5 Example

Let us illustrate our result with an example. We consider a non-commutative probability space $(\mathcal{C}, \omega, \tau)$ with a tracial weight ω, finite rank operator $a \in \mathrm{D}(\omega)$ with the eigenvalues $(2^{-1}, 2^{-2}, 2^{-3})$ and a operator $b \in \mathcal{B}$ with $\tau(b) = 0$ and $\tau(b^2) = 1$. We, in addition, assume that the operators a and b are cyclic monotone independent.

From our result we can give the following matrix model A and B for a and b:

$$A = \mathrm{diag}(2^{-1}, 2^{-2}, 2^{-3}) \otimes E_{11} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = I_3 \otimes J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We consider $X = A + BAB$ and $Y = AB + BA$:

$$X = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \end{pmatrix}$$

The lists of eigenvalues X and Y are

$$\{2^{-1}, 2^{-1}, 2^{-2}, 2^{-2}, 2^{-3}, 2^{-3}\}$$

and

$$\{2^{-1}, -2^{-1}, 2^{-2}, -2^{-2}, 2^{-3}, -2^{-3}\},$$

respectively. They correspond with the eigenvalues of $a + bab$ and $ab + ba$, respectively.

3.6 Random matrix models and concluding remarks

This paper presents a matrix model for monotone and cyclic monotone independences, which is not random. In free probability, there exist random matrix models which have few symmetries [19], but symmetric random matrix models are much more common. Therefore, it is natural to wonder whether there is a random model in the case of monotone and cyclic monotone independences. This turns out to be the case, and we can easily show that the model introduced by Collins, Hasebe and Sakuma in [3] is also a model for monotone independence, provided that we consider $\lim_l \lim_n \eta_l$ as our limiting weight.

Theorem 3.5. *Let $A_1^{(n)}, \dots, A_p^{(n)}, B_1^{(n)}, \dots, B_q^{(n)} \in M_n(\mathbb{C})$ be matrices such that there is $C > 0$ such that for any $m \in \mathbb{N}$, $i_1, \dots, i_m \in \{1, \dots, p\}$ and $j_0, j_1, \dots, j_m \in \{1, \dots, q\}$ we have*

$$|\mathsf{Tr}(A_{i_1}^{(n)} \dots A_{i_m}^{(n)})| \le C$$

and

$$|\mathsf{tr}(B_{j_0}^{(n)} \dots B_{j_m}^{(n)})| \le C.$$

Moreover let $U = U(n)$ be a Haar unitary random matrix. Then, if

$$\lim_{l\to\infty} \lim_{n\to\infty} \eta_l(A_{i_1}^{(n)} \dots A_{i_m}^{(n)}) \mathsf{tr}(B_{j_0}^{(n)}) \dots \mathsf{tr}(B_{j_m}^{(n)})$$

exists, it implies that

$$\lim_{l\to\infty} \lim_{n\to\infty} \eta_l(UB_{j_0}^{(n)}U^* A_{i_1}^{(n)} UB_{j_1}^{(n)} U^* \dots A_{i_m}^{(n)} UB_{j_m}^{(n)} U^*))$$

exists too, and that they are both equal.

Proof. It follows from a standard computation using Weingarten calculus (see for details Section 4 in [4]) that

$$\begin{aligned} &\mathbb{E}_U(\mathsf{tr}(UB_{j_0}^{(n)}U^* A_{i_1}^{(n)} UB_{j_1}^{(n)} U^* \dots A_{i_m}^{(n)} UB_{j_m}^{(n)} U^*)) \\ &\qquad = \sum_{\pi \in NC(m)} K_\pi(A_{i_1}^{(n)}, \dots, A_{i_m}^{(n)}) K_{\pi^c}(B_{i_0}^{(n)}, \dots, B_{i_m}^{(n)}) + O(n^{-2}), \end{aligned}$$

where $NC(m)$ are the non-crossing partitions on m elements, K_π are Speicher's free cumulants, and π^c is Kreweras's complement (specifically, the largest partition on $\{0, \dots, m\}$ such that its concatenation with π respecting the alternating order remains non-crossing. See Section 9, 11 and 14 in [17] for definitions and details of these. Dividing both sides by n and observing that
$K_\pi(A_{i_1}^{(n)}, \dots, A_{i_m}^{(n)}) = n^{-1}(\delta_{1,\#block(\pi)} \mathsf{Tr}(A_{i_1}^{(n)} \dots A_{i_m}^{(n)})) + O(n^{-1}))$, we get

$$\begin{aligned} &\mathbb{E}_U |\mathsf{Tr}(UB_{j_0}^{(n)}U^* A_{i_1}^{(n)} UB_{j_1}^{(n)} U^* \dots A_{i_m}^{(n)} UB_{j_m}^{(n)} U^*) \\ &\qquad - \mathsf{Tr}(A_{i_1}^{(n)} \dots A_{i_1}^{(n)}) \mathsf{tr}(B_{j_0}^{(n)}) \dots \mathsf{tr}(B_{j_m}^{(n)})| = O(n^{-1}). \end{aligned}$$

It is similar with (4.15) in [3].

For $l \leq n$, recall $f_{l,n}(I_l)$ is the matrix whose first l diagonal entries are 1 and all other entries in $M_n(\mathbb{C})$ are zero, the above formula implies

$$\mathbb{E}_U|\mathsf{Tr}(f_{l,n}(I_l)UB_{j_0}^{(n)}U^*A_{i_1}^{(n)}UB_{j_1}^{(n)}U^* \dots A_{i_m}^{(n)}UB_{j_m}^{(n)}U^*) - \mathsf{Tr}(f_{l,n}(I_l)A_{i_1}^{(n)} \dots A_{i_1}^{(n)})\mathsf{tr}(B_{j_0}^{(n)}) \dots \mathsf{tr}(B_{j_m}^{(n)})| = O(n^{-1}).$$

Noting that

$$\mathsf{Tr}(f_{l,n}(I_l)A_{i_1}^{(n)} \dots A_{i_m}^{(n)}) = \eta_l(A_{i_1}^{(n)} \dots A_{i_m}^{(n)}),$$

we obtain our desired model for monotone convergence by letting $n \to \infty$ followed by $l \to \infty$. □

In the space of compact operators of l^2, this is compared to the result of Collins, Hasebe and Sakuma for the very same model, where we first take $l \to \infty$ (to get the non-normalized trace), followed by $n \to \infty$ (to get monotone convergence).

Let us discuss the relation between this result and previous results on this model. In the paper [3], the same model was in consideration, with the additional assumption of moment convergence for the sequences $A_1^{(n)}, \dots, A_p^{(n)}, B_1^{(n)}, \dots$ and $B_q^{(n)} \in M_n(\mathbb{C})$ respectively. Namely, in addition to assuming $|\mathsf{Tr}(A_{i_1}^{(n)} \dots A_{i_m}^{(n)})| \leq C$ and $|\mathsf{tr}(B_{j_1}^{(n)} \dots B_{j_m}^{(n)})| \leq C$, it was assumed that

$$\lim_n \mathsf{Tr}(A_{i_1}^{(n)} \dots A_{i_m}^{(n)}) = f(i_1, \dots, i_m) \text{ and } \lim_n \mathsf{tr}(B_{j_1}^{(n)} \dots B_{j_m}^{(n)}) = g(j_1, \dots, j_m).$$

There, the main result was to prove that

$$\lim_n \mathsf{Tr}(A_{i_1}^{(n)}UB_{i_1}^{(n)}U^* \dots A_{i_m}^{(n)}UB_{i_m}^{(n)}U^*)$$

converges almost surely to

$$\lim_n \mathsf{Tr}(A_{i_1}^{(n)} \dots A_{i_m}^{(n)})\mathsf{tr}(B_{i_1}^{(n)}) \dots \mathsf{tr}(B_{i_m}^{(n)}),$$

which defines the cyclic monotone independence. Given that $\mathsf{Tr} = \lim_l \eta_l$, Theorem 3.5 implies that the monotone state can be obtained from the same model provided that we swap the limits n and l. In this respect, we completely generalize and conceptualize the results of [3].

We conclude by remarking that in a work [2] by Arizmendi, Hasebe and Lehner, they obtain models for cyclic monotone independence and monotone independence which are different from ours.

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