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Upper deviation probabilities for the range of a supercritical super-Brownian motion*

Shuxiong Zhang†

Abstract

Let $\{X_t\}_{t\geq 0}$ be a d-dimensional supercritical super-Brownian motion started from the origin with branching mechanism ψ. Denote by $R_t := \inf\{r > 0 : X_s(\{x \in \mathbb{R}^d : |x| \geq r\}) = 0,\ \forall\ 0 \leq s \leq t\}$ the radius of the minimal ball (centered at the origin) containing the range of $\{X_s\}_{s\geq 0}$ up to time t. In [8], Pinsky proved that condition on non-extinction, $\lim_{t\to\infty} R_t/t = \sqrt{2\beta}$ in probability, where $\beta := -\psi'(0)$. Afterwards, Engländer [1] studied the lower deviation probabilities of R_t. For the upper deviation probabilities, Engländer [1, Conjecture 8] conjectured that for $\rho > \sqrt{2\beta}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(R_t \geq \rho t | X_s(\mathbb{R}^d) > 0, \forall s > 0) = -\left(\frac{\rho^2}{2} - \beta\right).$$

In this note, we confirmed this conjecture.

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1 Introduction

In this paper, we consider a d-dimensional measure-valued Markov process $\{X_t\}_{t\geq 0}$, called super-Brownian motion (henceforth SBM). For convenience of the reader, we give a brief introduction to the SBM and some pertinent results needed in this article.

Let $M_F(\mathbb{R}^d)$ be the space of finite measure on $\mathbb{R}^d$ and ψ be a function on $[0,\infty)$ of the form:

$$\psi(u) := -\beta u + \alpha u^2 + \int_0^\infty \left(e^{-uy} - 1 + uy\right) n(dy),\ u \geq 0, \tag{1.1}$$

where $\beta \in \mathbb{R},\ \alpha \geq 0$ and n is a σ-finite measure on $(0,\infty)$ satisfying

$$\int_0^\infty (y^2 \wedge y) n(dy) < \infty.$$

The function ψ is called a branching mechanism. The SBM with initial value $\mu \in M_F(\mathbb{R}^d)$ and branching mechanism ψ is an $M_F(\mathbb{R}^d)$-valued process, whose transition probabilities are characterized through their Laplace transforms. Write $\nu(f) := \int_{\mathbb{R}^d} f(x)\nu(dx)$ for a

*Supported by the Institute of Mathematical Statistics (IMS) and the Bernoulli Society.
†School of Mathematics and Statistics, Anhui Normal University, China. E-mail: shuxiong.zhang@mail.bnu.edu.cn

(random) measure ν and positive function f. For all $\mu \in M_F(\mathbb{R}^d)$ and positive bounded function g on $\mathbb{R}^d$,

$$\mathbb{E}_\mu\left[e^{-X_t(g)}\right] = e^{-\mu(v(t,x))}, \tag{1.2}$$

where $v(t,x),\ t \geq 0,\ x \in \mathbb{R}^d$, is the unique locally bounded positive solution of

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v - \psi(v), \\ v(0,x) = g(x). \end{cases} \tag{1.3}$$

$\{X_t\}_{t\geq 0}$ is called a supercritical (critical, subcritical) super-Brownian motion if $\beta > 0$ $(= 0,\ < 0)$. We refer the reader to [2, 7] for an explicit definition and more elaborate discussions on the super-Brownian motion. In this paper, we only deal with the supercritical case, that is $\beta > 0$. Moreover, to simplify the statement, we always assume $X_0 = \delta_0$ in the rest of the article.

In this work, we are interested in the radius of the range of $\{X_s\}_{s\geq 0}$ up to time t, which is defined as

$$R_t := \inf\{r > 0 : X_s(B^c(r)) = 0,\ \forall\ 0 \leq s \leq t\},$$

where $B^c(r)$ stands for the complementary set of the ball $B(r) := \{x \in \mathbb{R}^d : |x| < r\}$. Denote by $S := \{\forall t \geq 0, X_t(\mathbb{R}^d) > 0\}$ the survival set of the SBM. It is well-known that if ψ satisfies the Grey's condition [7, Theorem 3.8] (i.e., there exists some $\theta > 0$ such that and $\int_\theta^\infty \psi(z)^{-1}dz < \infty$ and $\psi(z) > 0$ for $z \geq \theta$), then $\mathbb{P}(S) = 1 - e^{-\lambda^*}$, where $\lambda^* \in (0,\infty)$ is the largest root of the equation $\psi(u) = 0$.

In [1, Theorems 1], Engländer proved that if $\alpha > 0$, n is the zero measure and $\rho \in (0, \sqrt{2\beta})$, then

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(R_t \leq \rho t | S) = -\beta + \sqrt{\beta/2}\rho, \tag{1.4}$$

which is the so-called lower deviation probabilities. Engländer also [1] tried to consider relevant upper deviation probabilities. He found that the upper deviation probabilities of the branching Brownian motion can be deduced from Freidlin [3, 4]. For the the upper deviation probabilities of the super-Brownian motion, he conjectured that it has the same asymptotics as the branching Brownian motion. But different ideas should be used.

In the next, we first consider the upper deviation probabilities under the probability measure $\mathbb{P}$. Then, the deviation probabilities under $\mathbb{P}(\ |S)$ are given in Remark 1.2, which confirms the conjecture of Engländer [1, Conjecture 8].

Theorem 1.1. *Suppose that there exists some constant $\gamma > 0$ such that*

$$\int_1^\infty y(\log y)^{2+\gamma} n(dy) < \infty. \tag{1.5}$$

Assume $\alpha > 0$ and $\rho \in (\sqrt{2\beta}, \infty)$. Then,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(R_t \geq \rho t) = -\left(\frac{\rho^2}{2} - \beta\right). \tag{1.6}$$

Remark 1.2. [1, Conjecture 8] states as follows. Assume $\psi(u) = -\beta u + \alpha u^2$ with $\beta,\ \alpha > 0$. If $\rho \in (\sqrt{2\beta}, \infty)$, then

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(R_t \geq \rho t | S) = -\left(\frac{\rho^2}{2} - \beta\right). \tag{1.7}$$

Now, let's prove (1.7). In fact, the upper bound of (1.7) follows from Theorem 1.1 and the fact that

$$\mathbb{P}(R_t \geq \rho t | S) \leq \frac{\mathbb{P}(R_t \geq \rho t)}{\mathbb{P}(S)}.$$

The lower bound of (1.7) can be obtained by using [1, Propostion 4] and [10, Theorem 1.4].

Remark 1.3. Condition (1.5) comes from Ren et al. [9, Theorem 1.2]. The reason is that, in the proof of Theorem 1.1 (see (2.8) below), we use the maximum of the one-dimensional super-Brownian motion to derive the lower bound. The assumption $\alpha > 0$ is because in the proof of upper bound, we use the lower deviation probabilities of R_t. Nevertheless, the assumptions on the branching mechanism is weaker than Pinsky [8] and Engländer [1], who assumed $\psi(u) = -\beta u + \alpha u^2$ with $\beta,\ \alpha > 0$.

Remark 1.4. In fact, besides using [9, Theorem 1.2], one can also obtain the precise lower bound of Theorem 1.1 easily through the comparison lemma; see [1, Proposition 4]. So, the difficulty lies in the upper bound. To overcome it, we use the the weighted occupation time of the super-Brownian motion to characterize R_t, and then the precise upper bound is obtained through the Feynman-Kac formula and lower deviation probabilities of R_t.

2 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. We first give a brief discussion about the strategy of proofs. The proof of the theorem is divided into two parts–lower bound and upper bound. The lower bound is easy by using the maximum to control the radius of the range. To establish the upper bound, we first prove a Feynman-Kac formula for the solution of a certain PDE, and then combine a stopping time trick with the Feynman-Kac formula to show that the solution of the PDE can be bounded above by the deviation probabilities of Brownian motion and lower deviation probabilities of R_t

To establish the Feynman-Kac formula, we should first give a formulation of the weighted occupation time $\int_0^t X_s(\phi)ds$, which was first introduced by Iscoe [5] for the super-stable process. He then used it to study the supporting properties of super-Brownian motions [6]. Li [7, Section 5.4] gives a detailed introduction of the weighted occupation time of a general superprocess.

Let $\mathbb{P}_{\delta_x}$ ($\mathbb{P}_x$) be the probability measure under which the SBM $\{X_t\}_{t\geq 0}$ (the d-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$) starts from the Dirac measure (the point) at x. Moreover, we use $\mathbb{P}$ to stand for the probability measure under which $\{X_t\}_{t\geq 0}$ starts from δ_0 and $\{B_t\}_{t\geq 0}$ starts from 0. Assume $\phi(x)$ and $g(x)$ are non-negative, bounded and continuous function on $\mathbb{R}^d$. From [7, Corollary 5.17, Theorem 7.12], we have

$$\mathbb{E}_{\delta_x}\left[e^{-X_t(g)-\int_0^t X_s(\phi)ds}\right] = e^{-v(t,x)}, \tag{2.1}$$

where $v(t,x),\ t\geq 0,\ x\in\mathbb{R}^d$, is the unique locally bounded positive solution of

$$\begin{cases}\partial_t v = \frac{1}{2}\Delta v - \psi(v) + \phi, \\ v(0,x) = g(x).\end{cases} \tag{2.2}$$

Now, we are ready to present the Feynman-Kac formula to the above partial differential equation. Set

$$k(v) := \frac{\psi(v)}{v},$$

where we make the convention that $k(0) = -\beta$ if $v = 0$.

Lemma 2.1 (Feynman-Kac formula). *Let $v(t,x), t \geq 0, x \in \mathbb{R}^d$, be the unique locally bounded positive solution of (2.2). Then, we have*

$$v(t,x) = \mathbb{E}_x\left[g(B_t)e^{-\int_0^t k(v(t-r,B_r))dr} + \int_0^t \phi(B_s)e^{-\int_0^s k(v(t-r,B_r))dr}ds\right]. \tag{2.3}$$

Proof. We first cast (2.2) into the following mild form (see [7, Corollary 5.17]):

$$v(t,x) = \mathbb{E}_x\left[g(B_t)\right] + \int_0^t \mathbb{E}_x[\phi(B_s)]ds - \mathbb{E}_x\left[\int_0^t \psi(v(t-s,B_s))ds\right], \tag{2.4}$$

which implies

$$v(t,x) \leq \|g\|_\infty + \|\phi\|_\infty t - \left(\min_{u\geq 0}\psi(u)\right)t. \tag{2.5}$$

Let ∇_x be the gradient operator. By It ô's formula,

$$\begin{aligned}
&d\left[v(t-s,B_s)e^{-\int_0^s k(v(t-r,B_r))dr}\right] \\
&= e^{-\int_0^s k(v(t-r,B_r))dr}dv(t-s,B_s) + v(t-s,B_s)de^{-\int_0^s k(v(t-r,B_r))dr} \\
&= e^{-\int_0^s k(v(t-r,B_r))dr}\left[-\partial_t v(t-s,B_s)ds + \nabla_x v(t-s,B_s)dB_s + \frac{1}{2}\Delta v(t-s,B_s)ds\right] \\
&\quad - e^{-\int_0^s k(v(t-r,B_r))dr}\psi(v(t-s,B_s))ds \\
&= e^{-\int_0^s k(v(t-r,B_r))dr}\nabla_x v(t-s,B_s)dB_s - e^{-\int_0^s k(v(t-r,B_r))dr}\phi(B_s).
\end{aligned}$$

Integrating on both sides gives that

$$\begin{aligned}
&v(t-s,B_s)e^{-\int_0^s k(v(t-r,B_r))dr} - v(t,x) \\
&= \int_0^s e^{-\int_0^{s'} k(v(t-r,B_r))dr}\partial_x v(t-s',B_s')dB_{s'} - \int_0^s e^{-\int_0^{s'} k(v(t-r,B_r))dr}\phi(B_{s'})ds'.
\end{aligned} \tag{2.6}$$

For $s \in [0,t]$, set

$$\begin{aligned}
M_s &:= \int_0^s e^{-\int_0^{s'} k(v(t-r,B_r))dr}\partial_x v(t-s',B_s')dB_{s'} \\
&= v(t-s,B_s)e^{-\int_0^s k(v(t-r,B_r))dr} - v(t,x) + \int_0^s e^{-\int_0^{s'} k(v(t-r,B_r))dr}\phi(B_{s'})ds'.
\end{aligned} \tag{2.7}$$

This, combined with (2.5), implies that $M_s,\ s \in [0,t]$ is a bounded martingale. Therefore,

$$\begin{aligned}
&\mathbb{E}_x\left[g(B_t)e^{-\int_0^t k(v(t-r,B_r))dr} - v(t,x) + \int_0^t e^{-\int_0^s k(v(t-r,B_r))dr}\phi(B_s)ds\right] \\
&= \mathbb{E}_x[M_t] = \mathbb{E}_x[M_0] = 0,
\end{aligned}$$

which concludes the lemma. □

At this moment, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. **Lower bound.** Let H_t be the right-most position of the one-dimensional super-Brownian motion $\{X'_s\}_{s\geq 0}$ at time t. Namely,

$$H_t := \sup\{x \in \mathbb{R} : X'_t((x,\infty)) > 0\},$$

with the convention that $\sup\emptyset = -\infty$. It is simple to see that

$$\mathbb{P}(R_t \geq \rho t) \geq \mathbb{P}(H_t \geq \rho t). \tag{2.8}$$

Then, the desired lower bound follows by using [9, (1.23)]. In fact, [9, (1.23)] also needs the assumption that there exists some $\theta \in (0,1]$ and $a,\ b>0$ such that

$$\psi(u) \geq -au + bu^{1+\theta} \text{ for } u \geq 0.$$

But this condition is satisfied, since we have assumed $\alpha > 0$.

Upper bound. Let N be a d-dimensional standard normal random vector. Then, by the polar coordinates transformation, we have for $z > 0$,

$$\begin{aligned}\mathbb{P}(|N| \geq z) &= \int_{|y|\geq z} \frac{1}{(2\pi)^{d/2}} e^{-\frac{|y|^2}{2}} dy \\ &= \int_z^\infty \frac{dw_d}{(2\pi)^{d/2}} e^{-\frac{r^2}{2}} r^{d-1} dr, \end{aligned} \tag{2.9}$$

where w_d is the volume of a d-dimensional unit ball. Thus, by L'H ô pital's rule, we have

$$\lim_{z\to\infty} \frac{\mathbb{P}(|N| \geq z)}{e^{-\frac{z^2}{2}} z^{d-2}} = 1.$$

Thus, there exists a positive constant C_d depending only on d such that for any $z > 1$,

$$\mathbb{P}(|N| \geq z) \leq C_d e^{-\frac{z^2}{2}} z^{d-2}. \tag{2.10}$$

For $M > 0$, let $f(x)$ be a smooth and non-negative function on $\mathbb{R}^d$ such that $\{x \in \mathbb{R}^d : f(x) > 0\} = B^c(M)$. Put $\phi_\lambda(x) := \lambda f(x)$ for $\lambda > 0$. Denote by $v_\lambda(t,x)$ the solution of (2.2) with $\phi = \phi_\lambda$, $g = 0$. Since $X_s(\phi_\lambda)$ is right continuous with respect to s, we have

$$\begin{aligned}\mathbb{P}(R_t \geq M) &= \mathbb{P}(\exists s \in [0,t], X_s(B^c(M)) > 0) \\ &= \mathbb{P}(\exists s \in [0,t], X_s(\phi_\lambda) > 0) \\ &= \mathbb{P}\left(\int_0^t X_s(\phi_\lambda) ds > 0\right) \\ &= 1 - \lim_{\lambda\to\infty} \mathbb{E}\left[e^{-\int_0^t X_s(\phi_\lambda)ds}\right] \\ &= 1 - \lim_{\lambda\to\infty} e^{-v_\lambda(t,0)} \\ &\leq \lim_{\lambda\to\infty} v_\lambda(t,0). \end{aligned} \tag{2.11}$$

Let $u_\lambda(t,x)$ be the unique locally bounded positive solution of

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u - (-\beta u + \alpha u^2) + \phi_\lambda, \\ u(0,x) = 0. \end{cases} \tag{2.12}$$

Since

$$\begin{aligned}\partial_t u - \frac{1}{2}\Delta u + \psi(u) &= \psi(u) - [-\beta u + \alpha u^2] + \phi_\lambda \\ &\geq \phi_\lambda \\ &= \partial_t v - \frac{1}{2}\Delta v + \psi(v), \end{aligned} \tag{2.13}$$

by the maximum priciple (see [9, Lemma 2.2]), we have for any $t,\ \lambda \geq 0$ and $x \in \mathbb{R}^d$,

$$v_\lambda(t,x) \leq u_\lambda(t,x).$$

This means, in the rest of this proof, we can assume n is a zero measure in (1.1). Namely, without loss of generality, we assume

$$k(v) = -\beta + \alpha v.$$

By Lemma 2.1,

$$v_\lambda(t,x) = \int_0^t e^{\beta s}\mathbb{E}_x\left[\phi_\lambda(B_s)e^{-\int_0^s \alpha v_\lambda(t-r,B_r)dr}\right]ds. \tag{2.14}$$

In the following proofs, we sometimes write $v(t,x) := v_\lambda(t,x)$ for short. For $\varepsilon \in (0, 1 \wedge (\sqrt{\beta}/\rho))$, let T be the first time of $\{B_s\}_{s\geq 0}$ to hit the boundary of $B((1-\varepsilon)M)$, i.e.,

$$T := \inf\{s \geq 0 : |B_s| = (1-\varepsilon)M\}.$$

Let $\{B'_s\}_{s\geq 0}$ be an independent copy of $\{B_s\}_{s\geq 0}$. Note that if $s \leq T$, then $\phi_\lambda(B_s) = 0$. This, together with (2.14) and Fubini's theorem, yields that

$$\begin{aligned}
v(t,0) &= \int_0^t e^{\beta s}\mathbb{E}_0\left[\phi_\lambda(B_s)e^{-\int_0^s \alpha v(t-r,B_r)dr}\right]ds \\
&= \int_0^t e^{\beta s}\mathbb{E}_0\left[\mathbf{1}_{\{T<s\}}\phi_\lambda(B_s)e^{-\int_0^s \alpha v(t-r,B_r)dr}\right]ds \\
&= \int_0^t e^{\beta s}\mathbb{E}_0\left[\mathbf{1}_{\{T<s\}}e^{-\int_0^T \alpha v(t-r,B_r)dr}\mathbb{E}_{B_T}\left[\phi_\lambda(B'_{s-T})e^{-\int_T^s \alpha v(t-r,B'_{r-T})dr}\right]\right]ds \\
&\leq \mathbb{E}_0\left[\int_0^t e^{\beta s}\mathbf{1}_{\{T<s\}}\mathbb{E}_{B_T}\left[\phi_\lambda(B'_{s-T})e^{-\int_0^{s-T} \alpha v(t-T-r,B'_r)dr}\right]ds\right],
\end{aligned} \tag{2.15}$$

where third equality follows from the strong Markov property of Brownian motion. Again, using (2.14), (2.15) yields that

$$\begin{aligned}
v(t,0) &\leq \mathbb{E}_0\left[\mathbf{1}_{\{T\leq t\}}\int_T^t e^{\beta s}\mathbb{E}_{B_T}\left[\phi_\lambda(B'_{s-T})e^{-\int_0^{s-T} \alpha v(t-T-r,B'_r)dr}\right]ds\right] \\
&= \mathbb{E}_0\left[\mathbf{1}_{\{T\leq t\}}\int_0^{t-T} e^{\beta(T+\eta)}\mathbb{E}_{B_T}\left[\phi_\lambda(B'_\eta)e^{-\int_0^{\eta} \alpha v(t-T-r,B'_r)dr}\right]d\eta\right] \\
&= \mathbb{E}_0\left[\mathbf{1}_{\{T\leq t\}}e^{\beta T}v_\lambda(t-T,B_T)\right].
\end{aligned} \tag{2.16}$$

Since $g = 0$, from (2.1), it is simple to see that $v_\lambda(t,x)$ is increasing with respect to λ and t. Thus, if we let $M = \rho t$, then there exists a positive constant $C_{\varepsilon,\rho,\beta}$ depending only on ε, ρ and β such that for t large enough, one the event $\{T \leq t\}$, we have

$$\begin{aligned}
v_\lambda(t-T,B_T) &\leq v_\lambda(t,B_T) \\
&\leq \lim_{\lambda\to\infty} v_\lambda(t,B_T) \\
&= -\log \mathbb{P}_{\delta_{B_T}}(\forall s \in [0,t], X_s(B^c(M)) = 0) \\
&\leq -\log \mathbb{P}_{\delta_0}(R_t \leq \varepsilon\rho t) \\
&\leq -\log\left[\mathbb{P}_{\delta_0}(R_t \leq \varepsilon\rho t|S)\mathbb{P}(S)\right] \\
&\leq C_{\varepsilon,\rho,\beta}t,
\end{aligned} \tag{2.17}$$

where the first equality follows from similar arguments to obtain the last equality of (2.11) and the last inequality comes from (1.4). For $1 \leq i \leq d$, let $B_s^{(i)}$ be the i-th component of the vector B_s. It is easy to see that

$$\begin{aligned}\max_{s\in[0,t]}|B_s| &= \max_{s\in[0,t]}\left[\sum_{i=1}^{d}(B_s^{(i)})^2\right]^{1/2}\\ &\leq \left[\sum_{i=1}^{d}(\max_{s\in[0,t]}|B_s^{(i)}|)^2\right]^{1/2}\\ &= \left[\sum_{i=1}^{d}\left[(\max_{s\in[0,t]}B_s^{(i)})\vee(-\min_{s\in[0,t]}B_s^{(i)})\right]^2\right]^{1/2}. \qquad (2.18)\end{aligned}$$

By the reflection principle, $\max_{s\in[0,t]}B_s^{(i)}$ and $-\min_{s\in[0,t]}B_s^{(i)}$ have the same distribution as $|B_t^i|$. Hence, it follows as above that if $(1-\varepsilon)\rho\sqrt{t}>1$, then

$$\begin{aligned}&\mathbb{P}(T\leq t)\\ &=\mathbb{P}\left(\max_{s\in[0,t]}|B_s|\geq(1-\varepsilon)M\right)\\ &\leq 2^d\mathbb{P}\left(\left[\sum_{i=1}^{d}\left(\max_{s\in[0,t]}B_s^{(i)}\right)^2\right]^{1/2}\geq(1-\varepsilon)M\right)\\ &=2^d\mathbb{P}\left(\left[\sum_{i=1}^{d}\left(B_t^{(i)}\right)^2\right]^{1/2}\geq(1-\varepsilon)M\right)\\ &=2^d\mathbb{P}\left(|B_t|\geq(1-\varepsilon)M\right)\\ &=2^d\mathbb{P}\left(|B_t|/\sqrt{t}\geq(1-\varepsilon)\rho\sqrt{t}\right)\\ &\leq 2^dC_de^{-\frac{[(1-\varepsilon)\rho\sqrt{t}]^2}{2}}[(1-\varepsilon)\rho\sqrt{t}]^{d-2}, \qquad (2.19)\end{aligned}$$

where the last inequality comes from (2.10). Plugging (2.17) and (2.19) into (2.16) yields that for t large enough and any $\lambda>0$,

$$\begin{aligned}v(t,0)&\leq e^{\beta t}\mathbb{P}\left(T\leq t\right)C_{\varepsilon,\rho,\beta}t\\ &\leq e^{\beta t}2^dC_de^{-\frac{[(1-\varepsilon)\rho\sqrt{t}]^2}{2}}\left[(1-\varepsilon)\rho\sqrt{t}\right]^{d-2}C_{\varepsilon,\rho,\beta}t. \qquad (2.20)\end{aligned}$$

Plugging (2.20) into (2.11) and then taking limits yields that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(R_t\geq\rho t)\leq-\left(\frac{(1-\varepsilon)^2\rho^2}{2}-\beta\right).$$

The desired upper bound follows by letting $\varepsilon\to 0$. □

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