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From quenched invariance principle to semigroup convergence with applications to exclusion processes

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Abstract

Consider a random walk on $\mathbb{Z}^d$ in a translation-invariant and ergodic random environment and starting from the origin. In this short note, assuming that a quenched invariance principle for the opportunely-rescaled walks holds, we show how to derive an L^1-convergence of the corresponding semigroups. We then apply this result to obtain a quenched pathwise hydrodynamic limit for the simple symmetric exclusion process on $\mathbb{Z}^d$, $d \geq 2$, with i.i.d. symmetric nearest-neighbors conductances $\omega_{xy} \in [0, \infty)$ only satisfying

$$\mathbb{Q}(\omega_{xy} > 0) > p_c,$$

where p_c is the critical value for bond percolation.

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1 Introduction

Random walks in random environment received a lot of attention in the recent years, and studying their scaling limits is one of the main challenges (see, e.g., the surveys [9, 28]). Often the limit theorem is stated in the form of a *quenched invariance principle* (QIP) for the random walk *starting from the origin*. Namely, calling $\mathbb{Q}$ the law of the environment, under some natural assumptions of stationarity and ergodicity of $\mathbb{Q}$, one has that, for $\mathbb{Q}$-a.e. realization of the environment, the random walk $(X_t)_{t\geq 0}$ with $X_0 = 0$ rescales to a process $(\mathbb{X}_t)_{t\geq 0}$ whose law does not depend on the specific realization of the environment.

Examples that we have in mind are random walks on $\mathbb{Z}^d$ with generator of the form

$$Af(x) := \frac{1}{\nu_x} \sum_{y\in\mathbb{Z}^d} \omega_{xy} \left(f(y) - f(x)\right), \qquad x \in \mathbb{Z}^d, \tag{1.1}$$

with random bond- and site-weights ω_{xy} and ν_x, respectively. In particular, if $\omega_{xy} = \omega_{yx}$, the walk is referred to as *Random Conductance Model* (see, e.g., [9]), with ν_x playing the role of the invariant measure for the walk: for $\nu_x \equiv 1$ one obtains the so-called variable

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speed random walk, while for $\nu_x = \sum_{y\in\mathbb{Z}^d} \omega_{xy}$ the constant-speed random walk. Another relevant example is the *Bouchaud Trap Model* (see, e.g., [7]): given a collection of i.i.d. $\mathbb{N}$-valued random variables $\alpha = (\alpha_x)_{x\in\mathbb{Z}^d}$ satisfying, for some $\beta \in (0,1)$,

$$\mathbb{Q}\big(\alpha_0 \geq u\big) = u^{-\beta}\,(1+o(1))\,,\quad \text{as } u\to\infty,$$

such a model is the continuous-time random walk with infinitesimal generator

$$Af(x) := \sum_{y\sim x} \alpha_x^{a-1}\alpha_y^{a}\,(f(y)-f(x))\,,\qquad x\in\mathbb{Z}^d. \tag{1.2}$$

In this formula, the summation runs over nearest-neighbor sites of x in $\mathbb{Z}^d$, while $a\in[0,1]$ is an additional parameter tuning the asymmetry of the model.

For all these models, QIP has been established under various conditions (see, e.g., [1–10, 12, 14, 16]). In many cases, the limiting process is a Brownian motion with a diffusion matrix which does not depend on the realization of the environment. In other cases, e.g., for the Bouchaud trap model and the constant speed random walk with i.i.d. and heavy-tailed ω_{xy}^{-1} for $d\geq 2$, the scaling limit is a semi-Markov process known as *Fractional Kinetics Process* [4].

Such scaling limits of random walks in random environment can be useful when studying hydrodynamic limits of interacting particle systems (IPS) in random environment. Indeed, as shown in, e.g., [15, 18, 30], for IPS in random environment which satisfy self-duality, the quenched convergence of the empirical density fields of the IPS (in the sense of finite-dimensional distribution convergence) can be obtained from the quenched convergence of the semigroup of the one-particle system.

In most papers on random walks in random environment, only the convergence of the walk starting from the origin is addressed. However for applications in hydrodynamic limits of IPS in random environment, one typically needs a suitable convergence of the random walk's semigroups, which is implied by the strenghtening of the QIP from the origin in the form of an *arbitrary-starting-point* QIP. It is worth mentioning that the problem of deriving an arbitrary-starting-point QIP was posed in [32], and only recently established in [14] (see also [24, 31]) for a class of random environments for which a finer analysis on Green functions and heat kernels is available.

In this paper, we show how to obtain from a QIP from the origin a suitable form of L^1-convergence of the corresponding semigroups (or pseudo-semigroups if the limiting process is not Markovian), under the only additional assumption of translation-invariance and ergodicity of the law of the underlying environment (plus some mild assumptions, cf. (2.12) and (2.13)). We then apply our result to derive a quenched functional hydrodynamic limit for the simple symmetric exclusion process on $\mathbb{Z}^d$, $d\geq 2$, with i.i.d. conductances in $[0,\infty)$ only satisfying

$$\mathbb{Q}(\omega_{xy}>0) > p_c,$$

so that bonds with strictly positive conductance percolate. The hydrodynamic equation is the heat equation with a diffusivity matrix of the form $\sigma^2\boldsymbol{I}$, with $\sigma>0$ depending on d and $\mathbb{Q}$, but not on the specific realization of the environment. The simple symmetric exclusion process is one of the most studied IPS and consists of many (possibly infinite) symmetric nearest-neighbor random walks, subject to the exclusion rule: only one particle per site is allowed. The hydrodynamic limit of the simple symmetric exclusion process with random conductances has been widely studied (see, e.g., [18, 20–22, 24–26, 31]); however, the assumptions on the conductances that we assume here have not been considered before.

The rest of the paper is organized as follows. In Section 2, we introduce the general setting and we state the first main results of the paper, Theorem 2.3 and Proposition 2.4, on the convergence of semigroups. In that same section, we list some relevant examples to which our result applies. In Section 3.1, we introduce the symmetric exclusion process on the infinite cluster with i.i.d. unbounded conductances and we derive its hydrodynamic limit in Section 3.2. In Section 3.3, we discuss some possible further applications of Theorem 2.3 to other kinds of exclusion processes.

2 From QIP from the origin to semigroup L^1-convergence

Consider a random environment $\omega \in \Omega$, in the following form:

$$\omega = ((\omega_{xy})_{x,y\in\mathbb{Z}^d}, (\nu_x)_{x\in\mathbb{Z}^d}), \tag{2.1}$$

with $\omega_{xy} \in [0,\infty)$ and $\nu_x \in (0,\infty)$, and equip Ω with a σ-field $\mathcal{F}$. The environment is sampled according to a probability measure $\mathbb{Q}$, which we assume to be invariant and ergodic under translations in $\mathbb{Z}^d$ (Assumption 2.1), and for which a QIP from the origin holds (possibly, for a proper subset of environments, see Assumption 2.2). There are several examples of random environments satisfying such assumptions; for the reader's convenience, some of them are discussed in Section 2.2 below.

In what follows, for all $z \in \mathbb{Z}^d$, $\tau_z : \Omega \to \Omega$ denotes the translation map given by

$$\omega = ((\omega_{xy})_{x,y\in\mathbb{Z}^d}, (\nu_x)_{x\in\mathbb{Z}^d}) \longmapsto \tau_z\omega := ((\omega_{x+z,y+z})_{x,y\in\mathbb{Z}^d}, (\nu_{x+z})_{x\in\mathbb{Z}^d}).$$

Assumption 2.1 (Ergodicity of the environment)**.** *All $(\tau_z)_{z\in\mathbb{Z}^d}$ are $\mathcal{F}$-measurable. Moreover, $\mathbb{Q}$ is invariant and ergodic under translations in $\mathbb{Z}^d$, i.e., $\mathbb{Q}(A) = \mathbb{Q}(\tau_z(A))$ for all $A \in \mathcal{F}$ and $z \in \mathbb{Z}^d$, and $\mathbb{Q}(A) \in \{0,1\}$ for all $A \in \mathcal{F}$ such that $\tau_z(A) = A$ for all $z \in \mathbb{Z}^d$.*

Given a realization of the environment ω sampled according to $\mathbb{Q}$ satisfying Assumption 2.1, we consider the random walk X, with $X = (X_t)_{t\geq 0}$, on $\mathbb{Z}^d$, $d \geq 1$, with infinitesimal generator $A = A^\omega$ given in (1.1) and assumed to be non-explosive (cf. Assumption 2.2 below). Furthermore, $\mathbf{P}_z^\omega$ and $\mathbf{E}_z^\omega$ denote the law and corresponding expectation, respectively, of the random walk X with $X_0 = z \in \mathbb{Z}^d$. Moreover, $\mathcal{D}([0,\infty);\mathbb{R}^d)$ stands for the Polish space of $\mathbb{R}^d$-valued *càdlàg* paths equipped with the J_1-Skorokhod topology (see, e.g., [11, §16]).

Assumption 2.2 (QIP from the origin)**.** *There exist:*

1. *a sequence $(\theta_n)_{n\in\mathbb{N}} \subset (0,\infty)$;*
2. *a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{Q}(\Omega_0) > 0$, and such that, for all $\omega \in \Omega_0$ and $z \in \mathbb{Z}^d$, $X = (X_t)_{t\geq 0}$ is a.s. non-explosive under $\mathbf{P}_z^\omega$;*
3. *a process $\mathbb{X} = (\mathbb{X}_t)_{t\geq 0}$ with paths in $\mathcal{D}([0,\infty);\mathbb{R}^d)$ and translation-invariant law $(\mathbf{P}_x^{\mathbb{X}})_{x\in\mathbb{R}^d}$, that is, for all $x \in \mathbb{R}^d$, $\mathbb{X}$ under $\mathbf{P}_x^{\mathbb{X}}$ has the same law as $x + \mathbb{X}$ under $\mathbf{P}_0^{\mathbb{X}}$;*

such that, for $\mathbb{Q}(\,\cdot \mid \Omega_0)$-a.e. ω, $X^n = (X_{t\theta_n}/n)_{t\geq 0}$ under $\mathbf{P}_0^\omega$ converges in law to $\mathbb{X}$ under $\mathbf{P}_0^{\mathbb{X}}$ in $\mathcal{D}([0,\infty);\mathbb{R}^d)$ as $n \to \infty$.

While for many examples the QIP from the origin holds for $\mathbb{Q}$-a.e. realization of the environment, the role of the set $\Omega_0 \in \mathcal{F}$ in Assumption 2.2 will become clear when looking at random walks on supercritical percolation clusters, which is the example treated in Section 3.1 below: in that case, Ω_0 coincides with the set of environments for which the origin belongs to the infinite cluster.

2.1 Main result

We are now ready to present our main result. In what follows, $\mathbf{E}_x^{\mathbb{X}}$ denotes the expectation of $\mathbb{X}$ when $\mathbb{X}_0 = x \in \mathbb{R}^d$. Finally, recall $X^n = (X_{t\theta_n}/n)_{t\geq 0}$ from Assumption 2.2.

Theorem 2.3. *Under Assumptions 2.1 and 2.2, we have, for $\mathbb{Q}$-a.e. ω, for all compact sets $\mathcal{A} \subset \mathbb{R}^d$, and for all uniformly continuous bounded functions $G : \mathcal{D}([0,\infty);\mathbb{R}^d) \to \mathbb{R}$,*

$$\frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\mathcal{A}}(x/n) \mathbb{1}_{\Omega_0}(\tau_x\omega) \left| \mathbf{E}^\omega_x[G(X^n)] - \mathbf{E}^{\mathbb{X}}_{x/n}[G(\mathbb{X})] \right| \xrightarrow{n\to\infty} 0. \tag{2.2}$$

Proof. Fix $L > 0$ and $M > 0$. Let $G : \mathcal{D}([0,\infty);\mathbb{R}^d) \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_{\mathcal{D}}(G) \le L$, and bounded by M: for all $z, w \in \mathcal{D}([0,\infty);\mathbb{R}^d)$,

$$|G(z) - G(w)| \le L\, \mathrm{d}_{\mathcal{D}}(z,w), \qquad \|G\|_{\mathcal{D},\infty} := \sup_{z\in\mathcal{D}([0,\infty);\mathbb{R}^d)} |G(z)| \le M, \tag{2.3}$$

where $\mathrm{d}_{\mathcal{D}}(\cdot,\cdot)$ denotes the metric inducing the J_1-Skorokhod topology on $\mathcal{D}([0,\infty);\mathbb{R}^d)$.

Fix $K > 0$. Define, for all $k, n \in \mathbb{N}$, $x \in \mathbb{Z}^d$, and $\omega \in \Omega$,

$$\begin{aligned} f_{n,x}(\omega) &:= \left| \mathbf{E}^\omega_0 \left[G(X^n + x/n)\right] - \mathbf{E}^{\mathbb{X}}_{x/n} \left[G(\mathbb{X})\right] \right| \\ g_n(\omega) &:= \sup\left\{ f_{m,y}(\omega) : m \ge n,\ |y| \ge n \right\}. \end{aligned} \tag{2.4}$$

Let $\mathcal{A} = \mathcal{A}^K := \{y \in \mathbb{R}^d : |y| \le K\}$ and $\mathcal{A}_n = \mathcal{A}^K_n := \{y \in \mathbb{Z}^d : |y/n| \le K\}$; hence, $n\mathcal{A} \cap \mathbb{Z}^d = \mathcal{A}_n$, and $\frac{1}{n^d}\sum_{x\in\mathcal{A}_n} \mathbb{1}_{\Omega_0}(\tau_x\omega)\, f_{n,x}(\tau_x\omega)$ coincides with the left-hand side of (2.2).

For all $n, k \in \mathbb{N}$ with $k \le n$, by estimating $f_{n,x} \le g_k$ for $x \in \mathcal{A}_n \setminus \mathcal{A}_k$ and $f_{n,x} \le 2\|G\|_{\mathcal{D},\infty} \le 2M$ for $x \in \mathcal{A}_k$, we get

$$\frac{1}{n^d} \sum_{x\in\mathcal{A}_n} \mathbb{1}_{\Omega_0}(\tau_x\omega) f_{n,x}(\tau_x\omega) \le \frac{1}{n^d} \sum_{x\in\mathcal{A}_n} \mathbb{1}_{\Omega_0}(\tau_x\omega)\, g_k(\tau_x\omega) + \frac{2}{n^d} \sum_{x\in\mathcal{A}_k} M. \tag{2.5}$$

The second term on the right-hand side of (2.5) vanishes, for every fixed $k \in \mathbb{N}$, as $n \to \infty$.

We now estimate the first term. By translation-invariance of $\mathbb{X}$ (see item (3) in Assumption 2.2), we have

$$f_{n,x}(\omega) = \left| \mathbf{E}^\omega_0 \left[G(X^n + x/n)\right] - \mathbf{E}^{\mathbb{X}}_0 \left[G(\mathbb{X} + x/n)\right] \right|.$$

Moreover, by the convergence in Assumption 2.2 and Skorokhod's representation theorem (see, e.g., [11, Theorem 6.7 and Theorem 16.3]), for $\mathbb{Q}(\,\cdot \mid \Omega_0)$-a.e. ω, there exists a coupling of $(\mathbb{X}, X^1, X^2, \ldots)$, say

$$(\bar{\mathbb{X}}, \bar{X}^1, \bar{X}^2, \ldots), \qquad \text{with law } \bar{\mathbf{P}}^\omega_0 \text{ and expectation } \bar{\mathbf{E}}^\omega_0, \tag{2.6}$$

for which the convergence in $\mathcal{D}([0,\infty);\mathbb{R}^d)$ occurs $\bar{\mathbf{P}}^\omega_0$-a.s.:

$$\bar{\mathbf{P}}^\omega_0\text{-a.s.}, \qquad \mathrm{d}_{\mathcal{D}}(\bar{X}^n, \bar{\mathbb{X}}) \xrightarrow{n\to\infty} 0. \tag{2.7}$$

By combining these facts, we get, for all $L, M > 0$, and $\mathbb{Q}$-a.s. for all functions G as in (2.3),

$$\begin{aligned} \mathbb{1}_{\Omega_0}(\omega)\, g_k(\omega) &:= \mathbb{1}_{\Omega_0}(\omega) \sup\left\{ f_{n,y}(\omega) : n \ge k, |y| \ge k \right\} \\ &= \mathbb{1}_{\Omega_0}(\omega)\, \sup\left\{ \left| \bar{\mathbf{E}}^\omega_0 \left[G(\bar{X}^n + y/n) - G(\bar{\mathbb{X}} + y/n) \right] \right| : n \ge k, |y| \ge k \right\} \\ &\le \mathbb{1}_{\Omega_0}(\omega)\, \sup\left\{ \bar{\mathbf{E}}^\omega_0 \left[\left(L\, \mathrm{d}_{\mathcal{D}}(\bar{X}^n, \bar{\mathbb{X}}) \right) \wedge 2M \right] : n \ge k \right\} \\ &=: \mathbb{1}_{\Omega_0}(\omega)\, h_k(\omega) \xrightarrow{k\to\infty} 0, \end{aligned} \tag{2.8}$$

where the last inequality is a consequence of (2.3), while the last step follows by (2.7) and the dominated convergence theorem. In conclusion, the above upper bounds and the pointwise ergodic theorem applied to the bounded function $\mathbb{1}_{\Omega_0} h_k : \Omega \to \mathbb{R}$ ensure that, for all $k \in \mathbb{N}$, $K, L, M > 0$ and some $c = c(K,d) > 0$, $\mathbb{Q}$-a.s. for all functions G as in (2.3),

$$\limsup_{n\to\infty} \frac{1}{n^d} \sum_{x\in\mathcal{A}_n} \mathbb{1}_{\Omega_0}(\tau_x\omega)\, g_k(\tau_x\omega) \le \lim_{n\to\infty} \frac{1}{n^d} \sum_{x\in\mathcal{A}_n} \mathbb{1}_{\Omega_0}(\tau_x\omega)\, h_k(\tau_x\omega) = c\,\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\Omega_0} h_k\right]. \tag{2.9}$$

Going back to (2.5), we have, for all $k \in \mathbb{N}$, $K, L, M > 0$, $\mathbb{Q}$-a.s. for all functions G as in (2.3),

$$\limsup_{n\to\infty} \frac{1}{n^d} \sum_{x\in\mathcal{A}_n} \mathbb{1}_{\Omega_0}(\tau_x\omega) f_{n,x}(\tau_x\omega) \le c\,\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\Omega_0} h_k\right] \xrightarrow{k\to\infty} 0, \tag{2.10}$$

where for the last step we used (2.8) and the dominated convergence theorem. Thus, for all $K, L, M > 0$, $\mathbb{Q}$-a.s. for all functions G as in (2.3), the left-hand side of (2.5) vanishes as $n \to \infty$.

We thus have proved that, given $K, L, M > 0$, there exists $\Omega^{K,L,M} \in \mathcal{F}$ such that:

- $\mathbb{Q}(\Omega^{K,L,M}) = 1$;
- (2.2) holds for all $\omega \in \Omega^{K,L,M}$, for all compact sets $\mathcal{A} \subset \mathbb{R}^d$ contained in the centered Euclidean ball of radius $K > 0$, and for all Lipschitz continuous bounded functions G satisfying $\mathrm{Lip}_{\mathcal{D}}(G) \le L$ and $\|G\|_{\mathcal{D},\infty} \le M$.

By taking $\Omega' := \cap_{K,L,M\in\mathbb{N}} \Omega^{K,L,M}$, $\mathbb{Q}(\Omega') = 1$, and (2.2) holds for all $\omega \in \Omega'$, for all compact sets $\mathcal{A} \subset \mathbb{R}^d$, and for all Lipschitz continuous bounded functions on $\mathcal{D}([0,\infty), \mathbb{R}^d)$. Since this latter function space is dense in the space of uniformly continuous bounded functions with respect to the uniform norm $\|\cdot\|_{\mathcal{D},\infty}$, this concludes the proof. $\square$

As a direct consequence of the proof of Theorem 2.3, we obtain the following result, which will be used in the applications of Section 3. In what follows, $\mathcal{C}_c^+(\mathbb{R}^d)$ denotes the space of non-negative and compactly supported continuous functions on $\mathbb{R}^d$.

Proposition 2.4. *Under Assumptions 2.1 and 2.2, and further assuming that $\mathbb{X}$ has continuous sample paths, it holds that, for $\mathbb{Q}$-a.e. ω and for all $t > 0$, compact sets $\mathcal{A} \subset \mathbb{R}^d$, and uniformly continuous bounded functions $g : \mathbb{R}^d \to \mathbb{R}$,*

$$\frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\mathcal{A}}(x/n) \mathbb{1}_{\Omega_0}(\tau_x\omega) \left| \mathbf{E}_x^\omega[g(X_{t\theta_n}/n)] - \mathbf{E}_{x/n}^{\mathbb{X}}[g(\mathbb{X}_t)] \right| \xrightarrow{n\to\infty} 0. \tag{2.11}$$

Furthermore, assume that, for $\mathbb{Q}$-a.e. ω, the following two conditions hold true: for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $t > 0$,

$$\frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\Omega_0}(\tau_x\omega) \left(\mathbf{E}_x^\omega[f(X_{t\theta_n}/n)] - \mathbf{E}_{x/n}^{\mathbb{X}}[f(\mathbb{X}_t)] \right) \xrightarrow{n\to\infty} 0, \tag{2.12}$$

$$\lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{n^d} \sum_{\substack{x\in\mathbb{Z}^d \\ |x|>kn}} \mathbb{1}_{\Omega_0}(\tau_x\omega)\, \mathbf{E}_{x/n}^{\mathbb{X}}\left[f(\mathbb{X}_t)\right] = 0. \tag{2.13}$$

Then, for $\mathbb{Q}$-a.e. ω, for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $t > 0$,

$$\frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\Omega_0}(\tau_x\omega) \left| \mathbf{E}_x^\omega[f(X_{t\theta_n}/n)] - \mathbf{E}_{x/n}^{\mathbb{X}}[f(\mathbb{X}_t)] \right| \xrightarrow{n\to\infty} 0. \tag{2.14}$$

Proof. The first claim, namely (2.11), is proved as Theorem 2.3. We outline the main differences. First observe that the continuity of the paths of $\mathbb{X}$ allows us to improve the convergence in $\mathcal{D}([0,\infty); \mathbb{R}^d)$ by using the uniform metric (see, e.g., [11, Theorems 13.2 & 13.4] combined with Eqs. (12.7)–(12.9) therein). More precisely, for $\mathbb{Q}(\,\cdot \mid \Omega_0)$-a.e. ω, there exists a coupling of $(\mathbb{X}, X^1, X^2, \ldots)$ (with a slight abuse of notation, we adopt the same symbols as in (2.6) for this coupling) for which the following holds:

$$\bar{\mathbf{P}}_0^\omega\text{-a.s.}, \qquad \mathrm{d}_{\mathcal{C}}(\bar{X}^n, \bar{\mathbb{X}}) := \sum_{T=1}^{\infty} 2^{-T} \left(\sup_{t\in[0,T]} |\bar{X}^n_t - \bar{\mathbb{X}}_t| \wedge 1 \right) \xrightarrow{n\to\infty} 0. \tag{2.15}$$

The rest follows as in Theorem 2.3 by approximating uniformly continuous bounded functions $g : \mathbb{R}^d \to \mathbb{R}$ with Lipschitz bounded ones $g_{L,M} : \mathbb{R}^d \to \mathbb{R}$, $L, M > 0$, satisfying

$$|g_{L,M}(x) - g_{L,M}(y)| \le L\,|x-y|, \qquad \|g_{L,M}\|_\infty := \sup_{x\in\mathbb{R}^d} |g_{L,M}(x)| \le M, \tag{2.16}$$

and by using the following estimate (which holds $\mathbb{Q}$-a.s. for all $T \in \mathbb{N}$ and all $0 < t < T$)

$$\begin{aligned}\Big|\mathbf{E}_x^\omega[g_{L,M}(X_{t\theta_n}/n)] - \mathbf{E}^{\mathbb{X}}_{x/n}[g_{L,M}(\mathbb{X}_t)]\Big| &\le \bar{\mathbf{E}}_0^\omega\big[\big(L\,\big|\bar{X}^n_t - \bar{\mathbb{X}}_t\big|\big) \wedge 2M\big] \\ &\le \bar{\mathbf{E}}_0^\omega\big[2^T (L \vee 2M)\, \mathrm{d}_{\mathcal{C}}(\bar{X}^n, \bar{\mathbb{X}})\big]\end{aligned}$$

as a replacement of the second inequality in (2.8).

The second claim, namely (2.14), follows by combining (2.11) with the arguments in the proof of [31, Proposition 5.3], which we briefly recall here for the reader's convenience. Fix $f \in \mathcal{C}_c^+(\mathbb{R}^d)$. Using $|a| = a + 2\max\{-a, 0\}$, $a \in \mathbb{R}$, one splits, for all $k > 0$, the expression in (2.14) as follows:

$$\begin{aligned}&\frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\Omega_0}(\tau_x\omega) \left(\mathbf{E}_x^\omega[f(X_{t\theta_n}/n)] - \mathbf{E}^{\mathbb{X}}_{x/n}[f(\mathbb{X}_t)]\right) \\ &\quad + \frac{2}{n^d} \sum_{\substack{x\in\mathbb{Z}^d\\ |x|\le kn}} \mathbb{1}_{\Omega_0}(\tau_x\omega) \max\left\{\mathbf{E}^{\mathbb{X}}_{x/n}[f(\mathbb{X}_t)] - \mathbf{E}_x^\omega[f(X_{t\theta_n}/n)]\,,\, 0\right\} \\ &\quad + \frac{2}{n^d} \sum_{\substack{x\in\mathbb{Z}^d\\ |x|> kn}} \mathbb{1}_{\Omega_0}(\tau_x\omega) \max\left\{\mathbf{E}^{\mathbb{X}}_{x/n}[f(\mathbb{X}_t)] - \mathbf{E}_x^\omega[f(X_{t\theta_n}/n)]\,,\, 0\right\}.\end{aligned}$$

The first term vanishes as $n \to \infty$ by the assumption in (2.12); the second one vanishes, for every $k > 0$, as $n \to \infty$ by (2.11); the third one is bounded by twice the expression in (2.13), which vanishes taking first $n \to \infty$ and then $k \to \infty$. □

2.2 Examples

We list here six examples for which both Theorem 2.3 and Proposition 2.4 apply, the limiting process $\mathbb{X}$ having continuous sample paths. We divide these examples into two sub-classes, depending on the type of space-time scaling involved. In either case, we always consider environments as in (2.1) and satisfying Assumption 2.1, with $\mathcal{F}$ being the product Borel σ-field.

2.2.1 Diffusive scaling

Assumption 2.2 holds for $\theta_n = n^2$, $\Omega_0 = \Omega$, and $\mathbb{X}$ a d-dimensional Brownian motion with a non-degenerate $\mathbb{Q}$-a.s. constant covariance matrix, if the random environment ω fulfills either one of the following conditions:

(1) *moment conditions, $d \ge 1$:* ω_{xy} symmetric, nearest-neighbor, satisfying:

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\big[\omega_{xy}\big] < \infty \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\big[\omega_{xy}^{-1}\big] < \infty, \qquad d = 1, 2, \\ &\mathbb{E}_{\mathbb{Q}}\big[\omega_{xy}^{p}\big] < \infty \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\big[\omega_{xy}^{-q}\big] < \infty, \qquad d \ge 3\,,\end{aligned}$$

for $p,q\in(1,\infty]$, $\frac{1}{p}+\frac{1}{q}<\frac{2}{d-1}$, with either $\nu_x=1$ or $\nu_x=\sum_y \omega_{xy}$ for all $x\in\mathbb{Z}^d$; see, e.g., [9, §4.4] and references therein for $d=1,2$; see [6, Theorem 1 and Remark 1] for $d\geq 3$;

(2) *elliptic & i.i.d., $d\geq 2$:* ω_{xy} symmetric, nearest-neighbor, i.i.d., satisfying

$$\mathbb{Q}(\omega_{xy}\geq c)=1,\quad \text{for some } c>0,$$

with either $\nu_x=1$ for all $x\in\mathbb{Z}^d$ or, under the additional assumption that $\mathbb{E}_{\mathbb{Q}}\left[\omega_{xy}\right]<\infty$, $\nu_x=\sum_y\omega_{xy}$ for all $x\in\mathbb{Z}^d$; see [5, Theorem 1.1];

(3) *long-range, $d\geq 2$:* ω_{xy} symmetric, satisfying

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\sum_{x\in\mathbb{Z}^d}\omega_{0x}|x|^2\right)^p\right]<\infty\quad\text{and}\quad \mathbb{E}_{\mathbb{Q}}\left[\left(1/\omega_{0x}\right)^q\right]<\infty,\ |x|=1,$$

for $p,q\in(1,\infty)$, $\frac{1}{p}+\frac{1}{q}<\frac{2}{d}$; moreover, $\nu_x=\sum_y\omega_{xy}$ for all $x\in\mathbb{Z}^d$ and $\mathbb{E}_{\mathbb{Q}}[\omega_{00}]<\infty$; this follows from [10, Propositions 4.1 and 4.2] and a straightforward adaptation of standard random-time change arguments (see for example §6.2 in [1]);

(4) *balanced & i.i.d., $d\geq 1$:* $\omega_{xy}:=c_x(y-x)$, where $(c_x)_{x\in\mathbb{Z}^d}$ is an i.i.d. collection of probability measures c_x on $\mathbb{Z}^d$ satisfying

$$c_x(z)=c_x(-z)>0 \text{ if } |z|=1,\qquad \text{and}\quad c_x(z)=0 \text{ if } |z|\neq 1\ ;$$

further, $\nu_x=\sum_y\omega_{xy}$; this follows from [8, §6 Remark 1] and standard random-time change arguments.

2.2.2 Sub-diffusive scaling

Assumption 2.2 holds for

$$\theta_n=n^{2/\beta} \text{ if } d\geq 3,\qquad \text{or}\quad \theta_n=n^{2/\beta}\left(\log(n)\right)^{1-1/\beta} \text{ if } d=2, \tag{2.17}$$

$\Omega_0=\Omega$, and $\mathbb{X}:=$ a multiple of the Fractional Kinetics process if ω fulfills either one of the following conditions, for some $\beta\in(0,1)$ and $c_1,c_2>0$ (see [4, Theorems 1.2 and 1.3] for $d\geq 3$, [12, Theorems 1.1 and 1.2] for $d=2$):

(5) *heavy-tailed conductances:* ω_{xy} symmetric, nearest-neighbor, i.i.d., satisfying,

$$\mathbb{Q}(\omega_{xy}>u)=c_1u^{-\beta}(1+o(1)) \text{ as } u\to\infty,\quad\text{and}\quad \mathbb{Q}(\omega_{xy}\geq c_2)=1\ ; \tag{2.18}$$

here, $\nu_x=\sum_y\omega_{xy}$;

(6) *heavy-tailed site-weights:* $\omega_{xy}:=\alpha_x^a\alpha_y^a$, with $a\in[0,1]$, α_x i.i.d. satisfying

$$\mathbb{Q}(\alpha_x>u)=c_1u^{-\beta}(1+o(1)) \text{ as } u\to\infty,\quad\text{and}\quad \mathbb{Q}(\alpha_x\geq c_2)=1\ ; \tag{2.19}$$

and $\nu_x=\alpha_x$ for all $x\in\mathbb{Z}^d$.

The list above is far to be complete, and we refer the interested reader also to, e.g., [3, 16, 17], the variable-speed random walk on the supercritical percolation cluster discussed in Section 3.1 (cf. (3.7)), as well as to examples of time-dependent environments, e.g., [2].

3 Applications to exclusion processes

3.1 SSEP with i.i.d. conductances. Setting and hydrodynamic limit

Consider $\mathbb{Z}^d$, $d \geq 2$, and let E_d denote the set of unoriented nearest-neighbor bonds of $\mathbb{Z}^d$ (with a slight abuse of notation, we now identify $xy = yx$). Let $\omega = (\omega_e)_{e\in E_d}$ be i.i.d. random variables on $[0,\infty)$, for which we assume

$$\mathbb{Q}(\omega_e > 0) > p_c, \tag{3.1}$$

where $p_c = p_c(d) \in (0,1)$ denotes the critical probability for i.i.d. bond percolation on $\mathbb{Z}^d$. Define $\mathcal{O}(\omega) := \{e \in E_d : \omega_e > 0\}$, and, for $\mathbb{Q}$-a.e. ω, set $\mathcal{C}(\omega) \subset \mathbb{Z}^d$ to be the unique infinite connected supercritical percolation cluster of sites with at least one adjacent bond in $\mathcal{O}(\omega)$. Finally, define $\Omega_0 := \{\omega \in \Omega \;:\; 0 \in \mathcal{C}(\omega)\}$; then $\mathbb{Q}(\Omega_0) > 0$.

For any ω sampled according to $\mathbb{Q}$, we consider $(\eta_t)_{t\geq 0}$ as the simple symmetric exclusion process on $\mathcal{C}(\omega)$ with conductances $\omega = (\omega_e)_{e\in E_d}$. More precisely, let $\Xi = \Xi^\omega := \{0,1\}^{\mathcal{C}(\omega)}$ be the state space, while $\mathcal{L}^\omega$ is the Markov pre-generator, defined on the dense subspace $\mathfrak{D}_{\ell oc}(\Xi)$ of $(\mathcal{C}(\Xi), \|\cdot\|_\infty)$ of local functions as follows: for $\varphi \in \mathfrak{D}_{\ell oc}(\Xi)$,

$$\mathcal{L}^\omega \varphi(\eta) := \sum_{\substack{xy\in E_d\\ x,y\in\mathcal{C}(\omega)}} \omega_{xy} \left(\varphi(\eta^{xy}) - \varphi(\eta)\right). \tag{3.2}$$

Here, $\eta^{xy} \in \Xi$ denotes the configuration obtained from η by exchanging $\eta(x)$ with $\eta(y)$. Since the single-particle system (i.e., the random walk X_t defined in Section 3.2.1 below) is $\mathbb{Q}$-a.s. non-explosive [1] and $\omega_x := \sum_{y\in\mathcal{C}(\omega)} \mathbb{1}_{xy\in E_d}\, \omega_{xy} < \infty$ for all $x \in \mathcal{C}(\omega)$, by the results in [19, Section 3], the closure of $\mathcal{L}^\omega$ (which, with a slight abuse of notation, we still refer to as $\mathcal{L}^\omega$) is a Markov generator. Hence, $\mathcal{L}^\omega$ generates a Feller process on Ξ, referred to as $(\eta_t)_{t\geq 0}$, and we let $\mathbb{P}_\eta$ denote the law of η_t when $\eta_0 = \eta$, while, for a probability distribution μ on Ξ, we write $\mathbb{P}_\mu := \mu(\mathrm{d}\eta)\mathbb{P}_\eta$. We write $\mathbb{E}_\eta$ and $\mathbb{E}_\mu$ for the corresponding expectations.

It is well-known (see, e.g., [14, Proposition 3.14]) that $m_n := \frac{1}{n^d}\sum_{x\in\mathcal{C}(\omega)} \delta_{x/n}$ converges vaguely in a $\mathbb{Q}$-a.e. sense to a deterministic multiple of the Lebesgue measure of $\mathbb{R}^d$. More precisely, setting $q = q(\mathbb{Q}, d) := \mathbb{Q}(0 \in \mathcal{C}(\omega)) \in (0,1]$, $\mathbb{Q}$-a.s., the following convergence

$$\int_{\mathbb{R}^d} f \,\mathrm{d}m_n \underset{n\to\infty}{\longrightarrow} q \int_{\mathbb{R}^d} f \,\mathrm{d}x, \qquad f \in \mathcal{C}_c^+(\mathbb{R}^d), \tag{3.3}$$

holds. Our goal is to determine the scaling limit of the diffusively rescaled empirical density fields of the particle system on $\mathcal{C}(\omega)$, i.e.,

$$\mathcal{X}_t^n \;:=\; \frac{1}{n^d} \sum_{x\in\mathcal{C}(\omega)} \eta_{tn^2}(x)\,\delta_{x/n}, \qquad t \geq 0. \tag{3.4}$$

This is the content of Proposition 3.1 below. In what follows, we view $\mathcal{X}_t^n$ as a random measure in $\mathcal{M}_v(\mathbb{R}^d)$, i.e., the Polish space of locally-finite measures endowed with the vague topology (see, e.g., [27, Lemma A5.5]). Moreover, we write $\mathcal{D}([0,\infty);\mathcal{M}_v(\mathbb{R}^d))$ for the space of $\mathcal{M}_v(\mathbb{R}^d)$-valued *càdlàg* paths endowed with the J_1-Skorokhod topology (see, e.g., [11]).

Proposition 3.1 (SSEP with i.i.d. conductances: hydrodynamic limit)**.** *Let $d \geq 2$ and $\omega = (\omega_e)_{e\in E_d}$ be i.i.d. and fulfilling* (3.1)*. Assume that, $\mathbb{Q}$-a.s., a sequence of probability distributions $(\mu_n)_n$ on $\{0,1\}^{\mathcal{C}(\omega)}$ satisfies*

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}\left(\left| \int_{\mathbb{R}^d} f \,\mathrm{d}\mathcal{X}_0^n - q \int_{\mathbb{R}^d} f\,\gamma\,\mathrm{d}x \right| > \varepsilon \right) = 0, \qquad f \in \mathcal{C}_c^+(\mathbb{R}^d),\ \varepsilon > 0, \tag{3.5}$$

for some deterministic and continuous function $\gamma : \mathbb{R}^d \to [0,1]$. Then, there exists $\sigma = \sigma(\mathbb{Q}, d) > 0$ such that, $\mathbb{Q}$-a.s., the family $((\mathcal{X}_t^n)_{t\geq 0})_n$ is tight in $\mathcal{D}([0,\infty); \mathcal{M}_v(\mathbb{R}^d))$ and satisfies

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}\left(\left|\int_{\mathbb{R}^d} f\,\mathrm{d}\mathcal{X}_t^n - q\int_{\mathbb{R}^d} f\,\rho_t\,\mathrm{d}x\right| > \varepsilon\right) = 0, \qquad t>0,\ f\in\mathcal{C}_c^+(\mathbb{R}^d)\ ,\ \varepsilon>0, \tag{3.6}$$

where $(\rho_t)_{t\geq 0}$ is the unique bounded classical solution, on $\mathbb{R}^d$, to

$$\partial_t \rho_t = \sigma^2 \Delta \rho_t, \quad \text{with } \rho_0 = \gamma.$$

In other words, the above proposition states that, for $\mathbb{Q}$-a.e. ω, the law on $\mathcal{D}([0,\infty),\mathcal{M}_v(\mathbb{R}^d))$ of the random path $(\mathcal{X}_t^n)_{t\geq 0}$, given in (3.4) and with $\mathcal{X}_0^n$ sampled according μ_n, converges in law to the Dirac measure concentrated on the deterministic path $(\rho_t(x)\mathrm{d}x)_{t\geq 0}$, $\rho_0=\gamma$. We emphasize that it suffices to assume that the i.i.d. conductances percolate (in particular, without any moment assumptions) so to ensure that the quenched hydrodynamic limit is non-degenerate. Further, we note that hydrodynamic limits for SSEP have also been established for less regular initial limiting profiles γ; although possible, it is not our purpose in this work to relax this condition.

3.2 SSEP with i.i.d. conductances. Proof of Proposition 3.1

The hydrodynamic limit in Proposition 3.1 comes as a direct consequence of Proposition 2.4, self-duality of the symmetric exclusion process, and the strategy developed in [30] and further refined in, e.g., [18,21,31]. We discuss these steps separately, starting by deriving the required results on the single random walk and the aforementioned duality relation.

3.2.1 Single random walk

Fix a realization of the environment ω sampled according to $\mathbb{Q}$, and consider the variable-speed random walk $X = (X_t)_{t\geq 0}$ on $\mathcal{C}(\omega)\subset\mathbb{Z}^d$, having $A^\omega : \mathcal{D}\subset L^2(\mathcal{C}(\omega))\to L^2(\mathcal{C}(\omega))$,

$$A^\omega f(x) = \sum_{y\in\mathcal{C}(\omega)} \omega_{xy}\left(f(y)-f(x)\right), \qquad x\in\mathcal{C}(\omega). \tag{3.7}$$

as its infinitesimal generator. Recall that $\mathbf{P}_x^\omega$ and $\mathbf{E}_x^\omega$ denote the law and corresponding expectation of X_t when $X_0 = x\in\mathcal{C}(\omega)$. Moreover, let, for all $n\in\mathbb{N}$ and continuous bounded functions $g\in\mathcal{C}_b(\mathbb{R}^d)$,

$$P_t^n g(x/n) := \mathbf{E}_x^\omega\left[g(X_{tn^2}/n)\right], \qquad x\in\mathcal{C}(\omega),\ t\geq 0, \tag{3.8}$$

denote the semigroup of the diffusively rescaled random walk $X^n = (X_{n^2t}/n)_{t\geq 0}$.

We now recall the scaling limit of the random conductance model with i.i.d. unbounded conductances, first obtained in [5] under the assumption that $\mathbb{Q}(\omega_{xy}\geq 1)=1$, and further generalized in [1]. Here, $B^\sigma = (B_t^\sigma)_{t\geq 0}$ denotes a d-dimensional Brownian motion with diffusion matrix $\sigma^2\boldsymbol{I}$, while $(S_t^\sigma)_{t\geq 0}$ its semigroup.

Theorem 3.2 ([1, Theorem 1.1])**.** *Let $d\geq 2$ and $\omega = (\omega_e)_{e\in E_d}$ be i.i.d. and fulfilling (3.1). Then, there exists $\sigma>0$ such that for $\mathbb{Q}(\,\cdot\mid\Omega_0)$-a.e. ω, X^n under $\mathbf{P}_0^\omega$ converges in law to B^σ with $B_0^\sigma = 0$.*

As a consequence of the above result, Assumptions 2.1 and 2.2, and, thus, (2.11), hold true in this case. Furthermore:

- For all $t>0$, $S_t^\sigma:\mathcal{C}_c^+(\mathbb{R}^d)\to\mathcal{C}_b(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ because $|S_t^\sigma f(x)|\le c\exp(-C|x|^2)$ for some $c,C>0$ depending on $f\in\mathcal{C}_c^+(\mathbb{R}^d)$ and $t>0$; thus, for all $\omega\in\Omega$, for all $f\in\mathcal{C}_c^+(\mathbb{R}^d)$ and all $t>0$, we have
$$\lim_{k\to\infty}\limsup_{n\to\infty}\frac{1}{n^d}\sum_{\substack{x\in\mathbb{Z}^d\\|x|>kn}}\mathbb{1}_{\Omega_0}(\tau_x\omega)\,S_t^\sigma f(x/n)\le\lim_{k\to\infty}\limsup_{n\to\infty}\frac{1}{n^d}\sum_{\substack{x\in\mathbb{Z}^d\\|x|>kn}}S_t^\sigma f(x/n)=0,$$
and, in particular, condition (2.13) holds true.
- Fix $\omega\in\Omega$ for which (3.3) holds true. Then, due to symmetry of P_t^n with respect to the counting measure on $\mathcal{C}(\omega)$, we have, for all $f\in\mathcal{C}_c^+(\mathbb{R}^d)$, $t>0$, and $k\in\mathbb{N}$,
$$\begin{aligned}\frac{1}{n^d}\sum_{x\in\mathcal{C}(\omega)}\big(P_t^n f(x/n)-S_t^\sigma f(x/n)\big)&=\int_{\mathbb{R}^d}(f-S_t^\sigma f)\,\mathrm{d}m_n\\&=\int_{\mathbb{R}^d}\varphi_k\,(f-S_t^\sigma f)\,\mathrm{d}m_n+\int_{\mathbb{R}^d}(1-\varphi_k)\,(f-S_t^\sigma f)\,\mathrm{d}m_n,\end{aligned}\tag{3.9}$$
where we introduced a cutoff function $\varphi_k\in\mathcal{C}_c^+(\mathbb{R}^d)$, equal to one on $\{x\in\mathbb{R}^d:|x|\le k\}$, vanishing outside $\{x\in\mathbb{R}^d:|x|\le 2k\}$ and such that $0\le\varphi_k\le 1$. On the one hand, since $\varphi_k\,(f-S_t^\sigma f)\in\mathcal{C}_c^+(\mathbb{R}^d)$, (3.3) ensures that the first term on the right-hand side of (3.9) satisfies
$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi_k\,(f-S_t^\sigma f)\,\mathrm{d}m_n=q\int_{\mathbb{R}^d}\varphi_k\,(f-S_t^\sigma f)\,\mathrm{d}x\xrightarrow{k\to\infty}q\int_{\mathbb{R}^d}(f-S_t^\sigma f)\,\mathrm{d}x=0,$$
where the limit as $k\to\infty$ holds by integrability of $f-S_t^\sigma f$, while the last identity is a consequence of the symmetry of S_t^σ with respect to the Lebesgue measure on $\mathbb{R}^d$. On the other hand, the second term on the right-hand side of (3.9) vanishes as $n\to\infty$ and $k\to\infty$:
$$\lim_{k\to\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^d}(1-\varphi_k)\,(f-S_t^\sigma f)\,\mathrm{d}m_n\right|\le\lim_{k\to\infty}\limsup_{n\to\infty}\frac{1}{n^d}\sum_{\substack{x\in\mathbb{Z}^d\\|x|>kn}}S_t^\sigma f(x/n)=0.$$
All in all, owing to $\mathbb{1}_{\Omega_0}(\omega)=\mathbb{1}_{x\in\mathcal{C}(\omega)}$, this shows that, for all $\omega\in\Omega$ satisfying (3.3) (which form a subset of $\mathbb{Q}$-measure one), condition (2.14) holds for all $f\in\mathcal{C}_c^+(\mathbb{R}^d)$ and $t>0$.

Hence, (2.14) in Proposition 2.4 holds true and thus, for $\mathbb{Q}$-a.e. ω, $t\ge 0$, and $f\in\mathcal{C}_c^+(\mathbb{R}^d)$,
$$\lim_{n\to\infty}\frac{1}{n^d}\sum_{x\in\mathcal{C}(\omega)}|P_t^n f(x/n)-S_t^\sigma f(x/n)|=0.\tag{3.10}$$

3.2.2 Duality and convergence of finite-dimensional distributions

The claim in (3.10) above is the main step in the proof of Proposition 3.1. Given (3.10), the proof of convergence in probability of the one-time distributions of $\mathcal{X}_t^n$ goes as in, e.g., [18, 24, 30]. Since, for a random vector, convergence in probability of the marginals implies convergence in probability of the joint distribution, this yields convergence of all finite-dimensional distributions.

For employing (3.10), one crucially exploits the fact that, for any fixed ω, the symmetric exclusion process η_t with (pre-)generator given in (3.2) and X_t are in stochastic duality relation with duality function given by $D(x,\eta)=\eta(x)$. More precisely, recalling (3.7), one can check by a direct computation that, formally,
$$\mathcal{L}^\omega D(x,\cdot)(\eta)=A^\omega D(\cdot,\eta)(x),\tag{3.11}$$

for all $\eta \in \Xi$ and $x \in \mathcal{C}(\omega)$ (see also [19, Lemma 3.7]). Moreover, let $A^{\omega,(2)}$ be the L^2-generator that describes the infinitesimal evolution of the positions of the two-particle exclusion process as follows: for all $x, y \in \mathcal{C}(\omega)$ with $x \neq y$,

$$A^{\omega,(2)} f(x,y) = \sum_{z \in \mathcal{C}(\omega)} \omega_{xz} \left(1 - \mathbb{1}_z(y)\right) \left(f(z,y) - f(x,y)\right) + \sum_{z \in \mathcal{C}(\omega)} \omega_{yz} \left(1 - \mathbb{1}_z(x)\right) \left(f(x,z) - f(x,y)\right),$$

while $A^{\omega,(2)} f(x,x) := 0$. Then, letting $D^{(2)}((x,y),\eta) = \eta(x)\left(\eta(y) - \mathbb{1}_x(y)\right)$, it is not difficult to check that, at least formally,

$$\mathcal{L}^\omega D^{(2)}((x,y),\cdot)(\eta) = A^{\omega,(2)} D(\cdot,\eta)((x,y)) \tag{3.12}$$

holds true, for all $\eta \in \Xi$ and $x, y \in \mathcal{C}(\omega)$ with $x \neq y$.

Note that, for all $x, y \in \mathcal{C}(\omega)$ with $x \neq y$, both $\eta \mapsto D(x,\eta)$ and $\eta \mapsto D^{(2)}((x,y),\eta)$ are local functions in $\mathfrak{D}_{\ell oc}(\Xi)$ and, thus, belong to the domain of $\mathcal{L}^\omega$. On the contrary, for $\eta \in \Xi$ with $\sum_{x \in \mathcal{C}(\omega)} \eta(x) = \infty$, $x \mapsto D(x,\eta)$ and $(x,y) \mapsto D((x,y),\eta)$ are not compactly supported functions; hence, such functions do not necessarily belong to the domains of A^ω and $A^{\omega,(2)}$. Nevertheless, both relations (3.11) and (3.12) become rigorous when tested against compactly supported functions, as we now detail: for all $\eta \in \Xi$ and for all functions $f : \mathcal{C}(\omega) \to \mathbb{R}$ and $g : \mathcal{C}(\omega) \times \mathcal{C}(\omega) \to \mathbb{R}$ with finite support, we have

$$\mathcal{L}^\omega \Bigg(\sum_{x \in \mathcal{C}(\omega)} \eta(x)\, f(x) \Bigg) = \sum_{x \in \mathcal{C}(\omega)} \eta(x)\, A^\omega f(x),$$

and (note that summands equal zero if $x = y$)

$$\mathcal{L}^\omega \Bigg(\sum_{x,y \in \mathcal{C}(\omega)} \eta(x) \left(\eta(y) - \mathbb{1}_x(y)\right) g(x,y) \Bigg) = \sum_{x,y \in \mathcal{C}(\omega)} \eta(x) \left(\eta(y) - \mathbb{1}_x(y)\right) A^{\omega,(2)} g(x,y).$$

Analogous identities hold true at the semigroup level: letting P_t^ω and $P_t^{\omega,(2)}$ denote the L^2-semigroups associated to the generators A^ω and $A^{\omega,(2)}$, respectively, we have, for all $t \geq 0$ (keeping the same notation as above),

$$\mathbb{E}_\eta \Bigg[\sum_{x \in \mathcal{C}(\omega)} \eta_t(x)\, f(x) \Bigg] = \sum_{x \in \mathcal{C}(\omega)} \eta(x)\, P_t^\omega f(x), \tag{3.13}$$

and

$$\mathbb{E}_\eta \Bigg[\sum_{x,y \in \mathcal{C}(\omega)} \eta_t(x) \left(\eta_t(y) - \mathbb{1}_x(y)\right) g(x,y) \Bigg] = \sum_{x,y \in \mathcal{C}(\omega)} \eta(x) \left(\eta(y) - \mathbb{1}_x(y)\right) P_t^{\omega,(2)} g(x,y). \tag{3.14}$$

In view of these duality relations, we readily obtain the following decomposition of the empirical density fields $\mathcal{X}_t^n$ given in (3.4) in terms of a first expression (in parenthesis) with $\mathbb{E}_\eta$-expectation equal to zero and a second $\mathcal{X}_0^n$-measurable one: for $\mathbb{Q}$-a.e. $\omega \in \Omega$, for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$, and for all $t \geq 0$,

$$\int_{\mathbb{R}^d} f \, \mathrm{d}\mathcal{X}_t^n = \left(\int_{\mathbb{R}^d} f \, \mathrm{d}\mathcal{X}_t^n - \int_{\mathbb{R}^d} P_t^n f \, \mathrm{d}\mathcal{X}_0^n \right) + \int_{\mathbb{R}^d} P_t^n f \, \mathrm{d}\mathcal{X}_0^n. \tag{3.15}$$

Our first task is to show that the first term on the right-hand side vanishes in probability. For this purpose, we use both duality relations in (3.13) and (3.14), as well as the negative dependence of $\mathrm{SSEP}(\omega)$.

Lemma 3.3. *$\mathbb{Q}$-a.s., for all $n \in \mathbb{N}$, $\eta \in \Xi$, $t \geq 0$ and $f \in \mathcal{C}_c^+(\mathbb{R}^d)$,*

$$\mathbb{E}_\eta \left[\left(\int_{\mathbb{R}^d} f \, \mathrm{d}\mathcal{X}_t^n - \int_{\mathbb{R}^d} P_t^n f \, \mathrm{d}\mathcal{X}_0^n \right)^2 \right] \leq \frac{1}{n^d} \int_{\mathbb{R}^d} f^2 \, \mathrm{d}m_n. \tag{3.16}$$

Proof. Writing $\eta_t(x,y) := \eta_t(x)\,(\eta_t(y) - \mathbb{1}_x(y))$ and $\eta_0 = \eta$, the duality in (3.13) yields

$$\begin{aligned}
&\mathbb{E}_\eta \left[\left(\int_{\mathbb{R}^d} f \, \mathrm{d}\mathcal{X}_t^n - \int_{\mathbb{R}^d} P_t^n f \, \mathrm{d}\mathcal{X}_0^n \right)^2 \right] \\
&= \frac{1}{n^{2d}} \sum_{x,y \in \mathcal{C}(\omega)} \left(\mathbb{E}_\eta \left[\eta_{tn^2}(x,y) \right] - \mathbb{E}_\eta \left[\eta_{tn^2}(x) \right] \mathbb{E}_\eta \left[\eta_{tn^2}(y) \right] \right) f(x/n) f(y/n) \\
&\quad + \frac{1}{n^{2d}} \sum_{x \in \mathcal{C}(\omega)} \mathbb{E}_\eta \left[\eta_{tn^2}(x) \right] f(x/n)^2.
\end{aligned}$$

Since $\eta_t(x) \leq 1$, the second term on the r.h.s. above is smaller than the expression on the right-hand side of (3.16). After a simple manipulation based on (3.13) and (3.14), and adopting the notation "$\otimes$" for tensor product (e.g., $(f \otimes f)(a,b) := f(a)f(b)$), the first term on the right-hand side above reads as

$$\begin{aligned}
&\frac{1}{n^{2d}} \sum_{x,y \in \mathcal{C}(\omega)} \eta_0(x,y) \left\{ \left(P_t^{(2),n} - (P_t^n \otimes P_t^n) \right) (f \otimes f)\,(x/n, y/n) \right\} \\
&\qquad - \frac{1}{n^{2d}} \sum_{x \in \mathcal{C}(\omega)} \left(P_t^n f(x/n) \right)^2 \eta_0(x),
\end{aligned} \tag{3.17}$$

where $P_t^{(2),n} = P_t^{\omega,(2),n}$ is defined in terms of $P_t^{\omega,(2)}$, after a diffusive space-time rescaling, as similarly done in (3.8). The second term in (3.17) above is clearly non-positive; we conclude by showing that also the first term is non-positive. Since we want to verify such non-positivity for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $\eta \in \Xi$, it suffices to show this for $n = 1$ (this simplifies a bit the notation).

Let $A \oplus A$ denote the infinitesimal generator of $P_t \otimes P_t$, and recall that $A^{(2)}$ is the generator associated to $P_t^{(2)}$. Note that, since $f \otimes f$ is in the L^2-domains of both $A^{(2)}$ and $A \oplus A$ (indeed, $f \otimes f$ has finite support), then $P_t^{(2)}(f \otimes f)$ and $(P_t \otimes P_t)(f \otimes f)$ both belong to the aforementioned domains. Therefore, by arguing as in [29, Chapter VIII. Proposition 1.7] via the integration-by-parts formula, we get, for all $x, y \in \mathcal{C}(\omega)$ with $x \neq y$,

$$\begin{aligned}
&(P_t^{(2)} - (P_t \otimes P_t))(f \otimes f)(x,y) \\
&\quad = \int_0^t P_{t-s}^{(2)} (A^{(2)} - A \oplus A)(P_s \otimes P_s)(f \otimes f)(x,y) \, \mathrm{d}s \\
&\quad = \sum_{z,w \in \mathcal{C}(\omega)} \int_0^t P_{t-s}^{(2)} \mathbb{1}_{(z,w)}(x,y)\, (A^{(2)} - A \oplus A)(P_s f \otimes P_s f)(z,w) \\
&\quad = - \sum_{\substack{z,w \in \mathcal{C}(\omega) \\ |z-w|=1}} \int_0^t (P_{t-s}^{(2)} \mathbb{1}_{\{z,w\}}(x,y))\, (P_s f(z) - P_s f(w))^2 \, \mathrm{d}s \leq 0.
\end{aligned}$$

Since $\eta_0(x,y) \geq 0$ and $\eta_0(x,x) = 0$ for all $x, y \in \mathcal{C}(\omega)$, we get the desired claim. □

By (3.3), we have, $\mathbb{Q}$-a.s. and for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$, $\limsup_{n\to\infty} \int_{\mathbb{R}^d} f^2 \, \mathrm{d}m_n < \infty$; hence, the above lemma ensures that the first expression within parenthesis in (3.15) vanishes,

$\mathbb{Q}$-a.s., as $n \to \infty$ in probability. By the triangle inequality, convergence (in probability) of the finite-dimensional distributions follows from the convergence to $q\int_{\mathbb{R}^d} f\,\rho_t\,\mathrm{d}x$ of the second term in (3.15). After observing that $\int f\rho_t\,\mathrm{d}x = \int S_t^\sigma f\,\gamma\,\mathrm{d}x$, we get:

$$\int_{\mathbb{R}^d} P_t^n f\,\mathrm{d}\mathcal{X}_0^n - q\int_{\mathbb{R}^d} f\,\rho_t\,\mathrm{d}x = \left(\int_{\mathbb{R}^d} P_t^n f\,\mathrm{d}\mathcal{X}_0^n - \int_{\mathbb{R}^d} S_t^\sigma f\,\mathrm{d}\mathcal{X}_0^n\right) + \left(\int_{\mathbb{R}^d} S_t^\sigma f\,\mathrm{d}\mathcal{X}_0^n - q\int_{\mathbb{R}^d} S_t^\sigma f\,\gamma\,\mathrm{d}x\right).$$

Observe that, by $\mathcal{X}_0^n \le m_n$ and (3.10), $\mathbb{Q}$-a.s., the first parenthesis vanishes:

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}\left[\left|\int_{\mathbb{R}^d} P_t^n f\,\mathrm{d}\mathcal{X}_0^n - \int_{\mathbb{R}^d} S_t^\sigma f\,\mathrm{d}\mathcal{X}_0^n\right|\right] = 0. \tag{3.18}$$

As for the second one, (3.5), $\mathcal{X}_0^n \le m_n$, $S_t^\sigma f \in \mathcal{C}_b(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$, $S_t^\sigma f \ge 0$, and (3.3) yield, $\mathbb{Q}$-a.s.,

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}\left(\left|\int_{\mathbb{R}^d} S_t^\sigma f\,\mathrm{d}\mathcal{X}_0^n - q\int_{\mathbb{R}^d} S_t^\sigma f\,\gamma\,\mathrm{d}x\right| > \varepsilon\right) = 0, \qquad \varepsilon > 0. \tag{3.19}$$

The desired claim in (3.6) follows by the triangle inequality.

3.2.3 Tightness

Tightness is ensured, e.g., by the arguments in [22, Section 5.1] (see also [21, Section 8] for a refinement) or [31]. Here, we follow the same steps of the tightness proof in [31].

We need to show that, for some measurable $\tilde\Omega \subset \Omega$ satisfying $\mathbb{Q}(\tilde\Omega) = 1$, for all $\omega \in \tilde\Omega$, and for all $n \in \mathbb{N}$, $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $\varepsilon > 0$, there exists a *non-decreasing* function $\psi_n = \psi_{n,f,\varepsilon}^\omega : [0,T] \to [0,\infty)$ satisfying

$$\mathbb{P}_{\mu_n}\left(\left|\int_{\mathbb{R}^d} f\,\mathrm{d}\mathcal{X}_{t+h}^n - \int_{\mathbb{R}^d} f\,\mathrm{d}\mathcal{X}_t^n\right| > \varepsilon \,\middle|\, \mathcal{F}_t^n\right) \le \psi_n(h), \qquad t,h \ge 0, \tag{3.20}$$

where $\mathcal{F}_t^n := \sigma\left(\mathcal{X}_s^n : s \le t\right)$, and

$$\psi(h) \underset{h\to 0}{\longrightarrow} 0, \qquad \text{where } \ \psi(h) := \limsup_{n\to\infty} \psi_n(h). \tag{3.21}$$

Combining this with the tightness criterion in [31, Appendix B] would ensure $\mathbb{Q}$-a.s. tightness of all projections $(\int f\,\mathrm{d}\mathcal{X}_\cdot^n)_n$, $f \in \mathcal{C}_c^+(\mathbb{R}^d)$, in $\mathcal{D}([0,\infty);\mathbb{R})$; by, e.g., [27, Theorem 23.23], this implies tightness of $(\mathcal{X}_\cdot^n)_n$ in $\mathcal{D}([0,\infty);\mathcal{M}_v(\mathbb{R}^d))$.

Lemma 3.4. *Writing, for all $p \in [1,\infty)$ and $g \in \mathcal{C}_b(\mathbb{R}^d)$, $\|g\|_{p,n} := \left(\frac{1}{n^d}\sum_{x\in\mathcal{C}(\omega)} |g(x/n)|^p\right)^{1/p}$, $\mathbb{Q}$-a.s., for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $\varepsilon > 0$, there exist $C_1, C_2 > 0$ (depending only on $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $\varepsilon > 0$) for which the inequality in (3.20) holds true with*

$$\psi_n(h) := C_1\sqrt{\|f\|_{2,n}^2 - \|P_{h/2}^n f\|_{2,n}^2} + C_2\,\frac{1}{n^{d/2}}\|f\|_{2,n}, \qquad h \ge 0. \tag{3.22}$$

Furthermore, for all $n \in \mathbb{N}$, $\psi_n(h)$ given here is non-decreasing in $h \ge 0$.

Proof. By the triangle inequality and the Markov property, we have, $\mathbb{Q}$-a.s., for all $n \in \mathbb{N}$, $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{P}_{\mu_n}\left(\left|\int_{\mathbb{R}^d} f\,\mathrm{d}\mathcal{X}_{t+h}^n - \int_{\mathbb{R}^d} f\,\mathrm{d}\mathcal{X}_t^n\right| > \varepsilon \,\middle|\, \mathcal{F}_t^n\right)\\ &\quad \le \mathbb{P}_{\mu_n}\left(\left|\int_{\mathbb{R}^d} (f - P_h^n f)\,\mathrm{d}\mathcal{X}_t^n\right| > \frac{\varepsilon}{2} \,\middle|\, \mathcal{F}_t^n\right) + \frac{2}{\varepsilon}\frac{\|f\|_{2,n}}{n^{d/2}}.\end{aligned}$$

We obtained the second term on the right-hand side above by applying Markov and Cauchy-Schwarz inequalities, Lemma 3.3, and $\mathcal{X}_t^n \le m_n$. In particular, the second term on the right-hand side is independent of $h > 0$ and thus is non-decreasing. As for the first term, by $\mathcal{X}_t^n \le m_n$ (which holds for all $\omega \in \Omega$) and Markov inequality, we get

$$\mathbb{P}_{\mu_n}\left(\left|\int_{\mathbb{R}^d}(f - P_h^n f)\,\mathrm{d}\mathcal{X}_t^n\right| > \frac{\varepsilon}{2}\,\middle|\,\mathcal{F}_t^n\right) \le \frac{2}{\varepsilon}\left\|P_h^n f - f\right\|_{1,n}.$$

Using $|v| = v + 2\max\{0,-v\}$, $v \in \mathbb{R}$, and $\int P_h^n f\,\mathrm{d}m_n = \int f\,\mathrm{d}m_n$ (which holds $\mathbb{Q}$-a.s. for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $h \ge 0$), we get

$$\begin{aligned}\left\|P_h^n f - f\right\|_{1,n} &= \frac{2}{n^d}\sum_{x\in\mathcal{C}(\omega)}\max\{0, f(x/n) - P_h^n f(x/n)\}\\ &\le \frac{2}{n^d}\sum_{x\in\mathcal{C}(\omega)\cap n\mathcal{A}}|f(x/n) - P_h^n f(x/n)| + \frac{2}{n^d}\sum_{x\in\mathcal{C}(\omega)\cap(n\mathcal{A})^c} f(x/n)\\ &= 2\left\|\mathbb{1}_{\mathcal{A}}(f - P_h^n f)\right\|_{1,n},\end{aligned}$$

where the last step follows by choosing a compact $\mathcal{A} \supseteq \mathrm{supp}(f)$. In order to obtain a non-decreasing function, we use Cauchy-Schwarz inequality, $\|P_h^n f\|_{2,n} \le \|f\|_{2,n}$ and get

$$\begin{aligned}\left\|\mathbb{1}_{\mathcal{A}}(f - P_h^n f)\right\|_{1,n} &\le \left\|\mathbb{1}_{\mathcal{A}}\right\|_{2,n}\left\|f - P_h^n f\right\|_{2,n}\\ &\le \left\|\mathbb{1}_{\mathcal{A}}\right\|_{2,n}\sqrt{2\|f\|_{2,n}^2 - 2\|P_{h/2}^n f\|_{2,n}^2}.\end{aligned}$$

This finally defines a non-decreasing function in h. This concludes the proof of the lemma. □

We now argue that also (3.21) holds true for the function $\psi_n(h)$ given in (3.22). The second term on the right-hand side of (3.22) vanishes for all ω as $n \to \infty$. As for the first term, $\mathbb{Q}$-a.s. for all $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $h > 0$, we have, by $\|P_{h/2}^n f\|_{2,n}^2 = \frac{1}{n^d}\sum_{x\in\mathcal{C}(\omega)} f(x/n)\,P_h^n f(x/n)$, (3.10) and (3.3),

$$\lim_{n\to\infty}\left\|f\right\|_{2,n}^2 - \left\|P_{h/2}^n f\right\|_{2,n}^2 = q\int_{\mathbb{R}^d} f\,(f - S_h^\sigma f)\,\mathrm{d}x, \tag{3.23}$$

which vanishes as $h \to 0$ since $f \in \mathcal{C}_c^+(\mathbb{R}^d)$ and $\|f - S_h^\sigma f\|_\infty \to 0$ as $h \to 0$ by the strong continuity of S_h^σ in $\mathcal{C}_0(\mathbb{R}^d)$ (the Banach space of continuous functions vanishing at infinity). This establishes tightness, and, thus, concludes the proof of the proposition. □

3.3 Further discussion

We conclude by discussing some possible further applications of Theorem 2.3 and Proposition 2.4.

Remark 3.5 (SSEP on random conductance model)**.** The quenched hydrodynamic limit for the symmetric exclusion process with symmetric random conductances satisfying either the conditions in Examples (1) and (3) from Section 2.2 (the model is still self-dual in these cases, see, e.g., [23, §4.1]) can be obtained by following the same proof strategy adopted in the previous section: in both case the hydrodynamic equation is the heat equation with a non-degenerate constant diffusion matrix which does not depend on the realization of the environment.

Remark 3.6 (Interacting Bouchaud trap models)**.** Theorem 2.3 applies also to the context in which the limiting process is not Markovian, as arising from sub-diffusive walks discussed in Section 2.2.2. Recalling Example (6) from Section 2.2.2, one might consider the partial exclusion process (also known as $\mathrm{SEP}(\alpha)$, see, e.g., [24]) $(\eta_t)_{t\geq 0}$ that evolves on the state space $\Xi := \prod_{x\in\mathbb{Z}^d}\{0,1,\ldots,\alpha_x\}$, with integer α_x's, and jumps from η to $\eta^{x,y} := \eta - \delta_x + \delta_y$ with rate $\eta(x)\,(\alpha_y - \eta(y))$. Up to imposing the conditions (2.19) on the underlying environment $\alpha = (\alpha_x)_{x\in\mathbb{Z}^d}$ and checking the duality relations in [24], Theorem 2.3 should be helpful to prove that the following rescaled fields

$$t \in [0,\infty) \longmapsto \mathcal{Z}_t^n := \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d} \frac{\eta_{t\theta_n}(x)}{\alpha_x}\,\delta_{x/n} \in \mathcal{M}_v(\mathbb{R}^d), \tag{3.24}$$

where $\theta_n = \theta_{n,\beta,d}$ is given in (2.17), would admit as a quenched hydrodynamic limit the solution on $\mathbb{R}^d$, $d\geq 2$, to

$$\frac{\partial^\beta}{\partial t^\beta}\rho_t = \sigma^{2/\beta}\Delta\rho_t. \tag{3.25}$$

In this last formula, $\frac{\partial^\beta}{\partial t^\beta}$ stands for the Caputo derivative of order $\beta\in(0,1)$, i.e., for all $t\geq 0$ and $h\in\mathcal{C}^1(\mathbb{R})$, $\frac{\partial^\beta}{\partial t^\beta}h(t) := \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{1}{(t-s)^\beta}\,h'(s)\,\mathrm{d}s$. For background on the equation (3.25), we refer the interested reader to [13, 33] and references therein.

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