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Estimation and inference in sparse multivariate regression and conditional Gaussian graphical models under an unbalanced distributed setting

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Abstract: This paper proposes a distributed estimation and inferential framework for sparse multivariate regression and conditional Gaussian graphical models under the unbalanced splitting setting. This type of data splitting arises when the datasets from different sources cannot be aggregated on one single machine or when the available machines are of different powers. In this paper, the number of covariates, responses and machines grow with the sample size, while sparsity is imposed. Debiased estimators of the coefficient matrix and of the precision matrix are proposed on every single machine and theoretical guarantees are provided. Moreover, new aggregated estimators that pool information across the machines using a pseudo log-likelihood function are proposed. It is shown that they enjoy efficiency and asymptotic normality as the number of machines grows with the sample size. The performance of these estimators is investigated via a simulation study and a real data example. It is shown empirically that the performances of these estimators are close to those of the non-distributed estimators which use the entire dataset.

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1. Introduction

A natural way to investigate the relationship among gene expressions in a genetic study is via graphical models. The relationships between variables usually are mediated by external influences under the form of covariates effects. For instance, when looking at how genes are connected in a genome-wide expression quantitative trait loci (eQTL) analysis, the genetic variation can be viewed as external effects. It is crucial to account for these external factors to unravel the real connections in the gene network. For instance consider the connection between microRNA gene expressions in cancer research, in which several covariates such as the presence and composition of immune cells within the tumor microenvironment can affect the microRNA expressions and cancer progression. By considering these additional covariates in the analysis of microRNAs, one can construct more comprehensive models that better capture the complexity of cancer biology. This, in turn, can lead to more robust and clinically relevant findings that benefit cancer diagnosis and treatment. The reader can refer to [1, 23] and [27] for more details among many others. In Section 7 of the paper a cancer dataset is used to illustrate the benefits of accounting for covariates when describing interactions between various genes.

By adjusting the effect of covariates on the mean of the random variables in a Gaussian graphical model (GGM), one is able to estimate the structure of a conditional graphical model constructed using the elements of the precision matrix. Most studies focusing on the estimation of the precision matrix for GGMs assume that the random vector has zero or constant mean. For a treatment on the subject in the high-dimensional context for a mean zero GGM, we refer the reader to [2, 6, 7, 13, 24, 26, 32] and [39] among many others. However, in many real applications, adjusting for the effect of covariates on the mean of the random vector is important for understanding the underlying graph structure. This problem can be viewed as a multivariate linear regression problem, where multiple response variables (say p) are regressed on multiple predictors (say q),

and one is going to estimate the elements of the precision matrix related to the response vector. This model has many applications in the real world, especially in genomic data analysis, where one can model the dependence of RNA levels on DNA copy numbers through a multivariate regression model with RNA levels being responses and the DNA copy numbers being predictors as in [30]. Estimation of the coefficient matrix in multivariate regression models is also of interest and has many applications in the real world. For instance, consider the case when one wants to predict the expression levels of multiple microRNA mature stands based on a set of gene-level copy numbers in tumor samples. This is a common problem in cancer research, where the goal is to understand the regulatory mechanisms involving microRNAs and their relationship with gene copy number alterations. For the estimation of the coefficient matrix one requires to estimate pq coefficients, which becomes challenging with high-dimensional predictors and responses. To allow for consistent estimation in high-dimensional setting, a sparsity assumption is imposed on the model.

Several studies used regularization-based approaches to estimate the coefficient and precision matrices with adjusted covariates. The works of [29] and [37] proposed a joint regularization penalty to estimate iteratively both the multivariate regression coefficients corresponding to the covariates and the precision matrix of a GGM. In [33] and [36], the coefficient and precision matrices were estimated simultaneously via a joint penalized likelihood function. The works of [8] and [38] proposed a two-stage strategy which first estimates the regression coefficients and then using the residuals from the first stage, estimates the precision matrix. In [9], a tuning-free parameter estimator was proposed that is asymptotically normal and efficient for the estimation of every finite sub-graph for covariate adjusted GGMs. In these studies, due to the use of penalization, the estimators are biased. In this paper, by introducing debiased estimators, we are able to perform not only the estimation, but also inference and hypothesis testing.

The mentioned works investigated the covariate adjusted graphical models when the size of the dataset is not too large. However, with the development of technology, the size of the datasets grows at a high rate, such that in certain situations it is not possible to store all needed datasets in the memory of one single machine. Moreover, in recent frameworks, like federated learning [25], due to privacy concerns, it may be impossible to collect datasets from different resources on one single central machine. As such, the dataset is partitioned onto a cluster of machines. Distributed statistical approaches, also known as 'divide and conquer' approaches, have drawn a lot of attention in the last decade and have been developed for various statistical problems. The two most popular techniques in distributed statistical inference and estimation problems are 'averaging' estimators from local machines and the 'one-step' approach, which combines the simple averaging estimator with a classical Newton's method to generate a one-step estimator. In [19], the authors presented a 'Communication-efficient Surrogate Likelihood' framework for solving distributed statistical inference problems in low-dimensional, high-dimensional and Bayesian frameworks. In [3] and [22] the authors considered a high-dimensional sparse parameter vector estimation prob-

lem, where they adopted the penalized M-estimator setting. In [10] the authors proposed an aggregated estimator for the coefficients of a univariate generalized linear model, where the weights were determined by the majority voting method, and they showed the asymptotic normality for the estimators of the active set components. Recently, [12] proposed a weighted combination of ridge regression estimators in a univariate non-sparse linear regression problem via a balanced distributed setting. However, ridge regression is not an efficient procedure in high-dimensional setting. For a detailed review on aggregation methods for distributed estimators, the reader can refer to [16].

Nevertheless, most studies in the distributed setting have focused on the balanced sub-samples case, while in recent approaches like federated learning, some of the machines are more powerful than others and it is not efficient to distribute a dataset on different machines with equal sizes. In this situation, just taking a simple average is not an optimal approach for aggregating estimators. Recently, [28] proposed a new weighted, aggregated estimator for the elements of the precision matrix in a zero-mean GGM, where the weights are a function of sub-sample sizes and the variances estimated based on the sub-samples. However, as explained earlier, adjusting for the effect of covariates is a crucial issue in many fields, such as genomic data analysis. In this paper, we introduce new aggregated estimators for both the coefficient and precision matrices in covariate adjusted GGMs using a pseudo log-likelihood function which is constructed using the asymptotic distribution of the debiased estimators. It is shown empirically that these estimators perform better than the simple average in terms of accuracy and coverage probabilities. These estimators are constructed for the setting where the number of responses (p), covariates (q) and machines (K) grow with the sample size (n). As a consequence, sparsity assumptions are imposed on the true matrices as a function of p, q and n. Different upper bounds are derived on the number of machines to guarantee the consistency and asymptotic normality of the estimators.

The content of this paper is organized as follows. Notation and preliminaries are presented in Section 2. The debiased distributed estimators and their statistical properties are provided for the coefficient matrix and the precision matrix separately in Sections 3 and 4, respectively. The final aggregated estimators for both target matrices are introduced in Section 5. Theoretical properties of these estimators are also investigated in this section. In Section 6, the performance of the estimators is evaluated by means of a controlled simulation study and in Section 7, the performance on a real data set is illustrated. We close with a discussion on the method in Section 8. Proofs of the supporting lemmas and theorems and more simulation results can be found in the Appendix.

2. Notation and preliminaries

The multivariate regression model in this paper is defined as

$$\dot{\mathbf{Y}} = \mathbf{\Gamma}^{\top}\dot{\mathbf{X}} + \dot{\boldsymbol{\varepsilon}}, \tag{2.1}$$

where $\dot{\mathbf{Y}} = (Y^1, \ldots, Y^p)^\top \in \mathbb{R}^p$ and $\dot{\mathbf{X}} = (X^1, \ldots, X^q)^\top \in \mathbb{R}^q$ are the random response and covariate vectors, respectively. Moreover, the matrix $\mathbf{\Gamma}$ is a $q \times p$ regression coefficient matrix. Consider a random noise vector $\dot{\boldsymbol{\varepsilon}} = (\varepsilon^1, \ldots, \varepsilon^p)^\top \in \mathbb{R}^p$ independent of $\dot{\mathbf{X}}$, which follows a p-dimensional Gaussian distribution with mean zero, covariance matrix $\mathbf{\Sigma}$ and precision matrix $\mathbf{\Theta} = \mathbf{\Sigma}^{-1}$. The components of $\dot{\mathbf{Y}}$ are mapped to the node set $\mathcal{V} = \{1, \ldots, p\}$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ describes the set of edges between all pairs $(a, b) \in \mathcal{V} \times \mathcal{V}$, $a \neq b$. An undirected edge between nodes $a, b \in \mathcal{V}$ is drawn if $(a, b) \in \mathcal{E}$ and $(b, a) \in \mathcal{E}$. A pair (a, b) is included in the edge set $\mathcal{E}$ if and only if the variables Y^a and Y^b are conditionally dependent given $\dot{\mathbf{X}}$ and all remaining random variables in $\dot{\mathbf{Y}}$.

According to model (2.1), conditionally on $\dot{\mathbf{X}} = \dot{\mathbf{x}}$, $\dot{\mathbf{Y}}$ follows a p-dimensional Gaussian distribution with mean vector $\mathbf{\Gamma}^\top \dot{\mathbf{x}}$, covariance matrix $\mathbf{\Sigma}$ and precision matrix $\mathbf{\Theta}$. Every off-diagonal entry (a, b) of $\mathbf{\Theta}$ is proportional to the partial correlation between Y^a and Y^b given $\dot{\mathbf{X}}$ and all other variables in $\dot{\mathbf{Y}}$. As such, a pair of variables in $\dot{\mathbf{Y}}$ is conditionally independent given all remaining variables of $\dot{\mathbf{Y}}$ and $\dot{\mathbf{X}}$, if and only if the corresponding entry in the precision matrix $\mathbf{\Theta}$ is zero. Denote by $\mathbf{A}_{ab}$ the (a, b)-th element of an arbitrary matrix $\mathbf{A}$. The support of the precision matrix $\mathbf{\Theta}$ is defined as the index set of its non-zero off-diagonal elements

$$S_1 := \{(a, b) \in \mathcal{V} \times \mathcal{V}, a \neq b : \quad \mathbf{\Theta}_{ab} \neq 0\},$$

with cardinality $s_1 = \#S_1$ and the maximum node degree or row sparsity of $\mathbf{\Theta}$ is defined as

$$d_1 := \max_{a \in \mathcal{V}} \#\{b \in \mathcal{V}, b \neq a : \quad \mathbf{\Theta}_{ab} \neq 0\}.$$

Analogously, the support of the regression coefficient matrix $\mathbf{\Gamma}$ is considered as

$$S_2 := \big\{(a, b) \in \{1, \ldots, q\} \times \{1, \ldots, p\} : \quad \mathbf{\Gamma}_{ab} \neq 0\big\},$$

which is the index set of its non-zero elements, with cardinality $s_2 = \#S_2$. Moreover, the support of the b-th column of the regression coefficient matrix $\mathbf{\Gamma}$ is defined as $S_2(b) = \big\{a \in \{1, \ldots, q\} : \quad \mathbf{\Gamma}_{ab} \neq 0\big\}$, $b = 1, \ldots, p$, with cardinality $s_2(b) = \#S_2(b)$.

Given n independent observations from the pair $(\dot{\mathbf{Y}}^\top, \dot{\mathbf{X}}^\top)$, our goal is to introduce distributed, debiased estimators for $\mathbf{\Gamma}$ and $\mathbf{\Theta}$ from model (2.1). Suppose that the n samples are randomly divided into K non-overlapping unbalanced sub-samples with size n_k for the k-th sub-sample, $k = 1, \ldots, K$, and denote by $n_\dagger = \min_{1 \leqslant k \leqslant K} n_k$, while $n = \sum_{k=1}^K n_k$. Suppose that $n_k/n \to c_k \in (0, 1)$, as $n_k \to \infty$, such that $\lim_{K \to \infty} \sum_{k=1}^K c_k = 1$. In this paper, p and q can grow with n, such that they might be larger than n. However, we suppose that $\log(p) = o(n_\dagger)$ and $\log(q) = o(\sqrt{n_\dagger})$, where $o(\cdot)$ expresses the asymptotic behavior of a sequence and is defined later in this section. The usual assumption in the high-dimensional context is that the underlying matrices (coefficient and precision matrices in this paper) are sparse, which means that the number of non-zero elements in the matrices cannot grow too fast and a certain bound is imposed on it. This sparsity

reflects that many predictors in the regression model are redundant and that the graph related to the precision matrix has a rather low number of edges. To impose this sparsity condition on the estimation, one common approach is to add an ℓ_1 penalty to the function which is going to be optimized. This kind of regularization effectively forces some of the elements to zero, thus resulting in sparse solutions. There exists a wide variety of methods making use of ℓ_1 regularization, see for example [13, 32] and [34]. For convenience of notation, denote by $\mathbf{Y}_k = [\dot{\mathbf{Y}}_{1,k}, \ldots, \dot{\mathbf{Y}}_{n_k,k}]^\top$ and $\boldsymbol{\xi}_k = [\dot{\boldsymbol{\varepsilon}}_{1,k}, \ldots, \dot{\boldsymbol{\varepsilon}}_{n_k,k}]^\top$ the k-th sub-sample of the response vector $\dot{\mathbf{Y}}$ and the random noise $\dot{\boldsymbol{\varepsilon}}$, respectively, both arranged as matrices of dimension $n_k \times p$, with $\dot{\mathbf{Y}}_{l,k} \in \mathbb{R}^p$, $\dot{\boldsymbol{\varepsilon}}_{l,k} \in \mathbb{R}^p$ as the l-th row, $l = 1, \ldots, n_k$, and by $\mathbf{X}_k = [\dot{\mathbf{X}}_{1,k}, \ldots, \dot{\mathbf{X}}_{n_k,k}]^\top$ the k-th design matrix of dimension $n_k \times q$, with $\dot{\mathbf{X}}_{l,k} \in \mathbb{R}^q$ as the l-th row, $l = 1, \ldots, n_k$. With this notation, the sample version of the regression model (2.1) corresponds to

$$\mathbf{Y}_k = \mathbf{X}_k \boldsymbol{\Gamma} + \boldsymbol{\xi}_k, \tag{2.2}$$

where the rows of $\boldsymbol{\xi}_k$ are i.i.d. p-dimensional Gaussian vectors with mean zero, covariance matrix $\boldsymbol{\Sigma}$ and precision matrix $\boldsymbol{\Theta}$. In the next two sections, debiased estimators for $\boldsymbol{\Gamma}$ and $\boldsymbol{\Theta}$ based on the k-th sub-sample are derived.

Before starting the discussion, we introduce some order notation which is needed later in the paper. For two sequences $\{a_n; n \geqslant 1\}$ and $\{b_n; n \geqslant 1\}$, $b_n = O(a_n)$ if there exist positive numbers M_0 and N_0 such that $|b_n/a_n| \leqslant M_0$ for all $n \geqslant N_0$. Similarly, for a random sequence $\{X_n; n \geqslant 1\}$, we write $X_n = O_p(a_n)$ if for every $\epsilon > 0$, there exist finite numbers $M_0 > 0$ and $N_0 > 0$ such that $\mathbb{P}(|X_n/a_n| > M_0) < \epsilon$ for all $n \geqslant N_0$. We write $b_n \asymp a_n$ if both $b_n = O(a_n)$ and $a_n = O(b_n)$ hold. Moreover, $b_n = o(a_n)$ if $\lim_{n\to\infty} b_n/a_n = 0$. In the case of a random sequence $\{X_n; n \geqslant 1\}$, we write $X_n = o_p(a_n)$ if $X_n/a_n \xrightarrow{p} 0$, as $n \to \infty$, where the notation $\xrightarrow{p}$ denotes convergence in probability. For a matrix $\mathbf{A}$, we use the notation $|||\mathbf{A}|||_\infty = \max_a \sum_b |\mathbf{A}_{ab}|$ and $\|\mathbf{A}\|_\infty = \max_{a,b} |\mathbf{A}_{ab}|$ for the matrix and elementwise ℓ_∞ norms, respectively. The same symbol $\|\mathbf{x}\|_\infty = \max_b |\mathbf{x}_b|$ is used for the ℓ_∞ norm of a vector $\mathbf{x}$, where $\mathbf{x}_b$ is the b-th element of $\mathbf{x}$. Moreover, $\|\mathbf{A}\|_1 = \sum_{a,b} |\mathbf{A}_{ab}|$ and $\|\mathbf{x}\|_1 = \sum_b |\mathbf{x}_b|$ are used for the elementwise ℓ_1 norm of a matrix $\mathbf{A}$ and of a vector $\mathbf{x}$, respectively. We use $\|\mathbf{A}\|_F = \sqrt{\sum_{a,b} \mathbf{A}_{ab}^2} = \sqrt{\operatorname{trace}(\mathbf{A}^\top \mathbf{A})}$ for the Frobenius norm of a matrix and $\|\mathbf{x}\|_2 = \sqrt{\sum_b \mathbf{x}_b^2}$ for the ℓ_2 norm of a vector $\mathbf{x}$. Finally, by $\mathbf{A} \otimes \mathbf{B}$ we denote the Kronecker product of two arbitrary matrices $\mathbf{A}$ and $\mathbf{B}$ of dimension $m \times n$ and $p \times q$ as a $pm \times qn$ block matrix with $\mathbf{A}_{ab}\mathbf{B}$ for the block (a, b) where $\mathbf{A}_{ab}$ is the (a, b)-th element of matrix $\mathbf{A}$, $a = 1, \ldots, m$, and $b = 1, \ldots, n$.

3. Distributed estimator of the coefficient matrix using the k-th sub-sample

We make the following assumptions in this section.

(A1) The rows of the design matrix $\mathbf{X}_k$ are i.i.d. q-dimensional Gaussian vectors with mean vector zero and positive definite covariance matrix $\mathbf{Q}$, where $\max_a \mathbf{Q}_{aa} = O(1)$.

Assumption (A1) can be relaxed to a deterministic design matrix. However, in this case, one needs the mutual incoherence or irrepresentability condition on the design matrix to exhibit model selection consistency (see Chapter 6 of [5] for more details). Another equivalent condition is the restricted eigenvalue condition [4]. The work of [31] showed that this condition holds for the random Gaussian design matrices. As such, one does not need any mutual incoherence assumption in this setting.

(A2) The eigenvalues of $\mathbf{Q}$ are bounded from above and below, i.e., there exists a constant $\Lambda_1 \geqslant 1$ such that $1/\Lambda_1 \leqslant \Lambda_{\min}(\mathbf{Q}) \leqslant \Lambda_{\max}(\mathbf{Q}) \leqslant \Lambda_1$, where $\Lambda_{\min}(\mathbf{Q})$ and $\Lambda_{\max}(\mathbf{Q})$ are the minimum and maximum eigenvalues of $\mathbf{Q}$, respectively.

Denote the maximum row sparsity of the inverse covariance matrix $\mathbf{Q}^{-1}$ with $d_2 := \max_{a\in\{1,\ldots,q\}} \#\{a' \in \{1,\ldots,q\}, a' \neq a : \quad \mathbf{Q}^{-1}_{aa'} \neq 0\}$. To find an initial estimator for the regression coefficient matrix $\mathbf{\Gamma}$ in the k-th sub-sample, following [38], we minimize the following joint penalized residual sum of squares and denote by $\hat{\mathbf{\Gamma}}_k$,

$$\hat{\mathbf{\Gamma}}_k = \arg\min_{\mathbf{\Gamma}} \left\{ \frac{1}{2n_k} \operatorname{trace}\big\{(\mathbf{Y}_k - \mathbf{X}_k\mathbf{\Gamma})^\top(\mathbf{Y}_k - \mathbf{X}_k\mathbf{\Gamma})\big\} + \rho_k\|\mathbf{\Gamma}\|_1 \right\}, \tag{3.1}$$

where $\rho_k > 0$ is a regularization parameter that forces $\hat{\mathbf{\Gamma}}_k$ to be sparse. The optimization problem (3.1) contains p decoupled Lasso regressions with q coefficients. This equation ignores the correlation among response variables when estimating the multiple regression coefficients. The work of [33] showed empirically that only when the correlation of the errors is high, incorporation of such a dependency can lead to increased efficiency in estimating $\mathbf{\Gamma}$. Lemma 5 in Appendix B, provides a bound of the form $O_p\big(s_2\sqrt{\log(pq)/n_k}\big)$ on the ℓ_1 norm of the difference between $\hat{\mathbf{\Gamma}}_k$ and the true matrix $\mathbf{\Gamma}$ under additional regularity conditions.

Due to the ℓ_1 penalty which is added to this loss function, $\hat{\mathbf{\Gamma}}_k$ is a biased estimator. To obtain a debiased estimator of $\mathbf{\Gamma}$, we use the idea of [34] by inverting the Karush-Kuhn-Tucker (KKT) conditions. They showcased this method in the linear regression and generalized linear models framework. Using the KKT conditions in (3.1), we have

$$-\mathbf{X}_k^\top\mathbf{Y}_k/n_k + \mathbf{X}_k^\top\mathbf{X}_k\hat{\mathbf{\Gamma}}_k/n_k + \rho_k\mathbf{B}_k = \mathbf{0}, \tag{3.2}$$

where $\mathbf{B}_k$ belongs to the sub-differential of the ℓ_1 norm evaluated at $\hat{\mathbf{\Gamma}}_k$. By adding and subtracting $\mathbf{X}_k^\top\boldsymbol{\xi}_k/n_k$ to (3.2), performing some algebra calculations and finally vectorizing, we get

$$\sqrt{n_k}\operatorname{vec}\big(\hat{\mathbf{\Gamma}}_k^d - \mathbf{\Gamma}\big) = \mathbf{T}_k + \mathbf{R}_{k,\mathbf{\Gamma}}, \tag{3.3}$$

where vec(·) is an operator which converts the matrix into a column vector by stacking the columns of the matrix on top of one another. Moreover

$$\begin{aligned}\hat{\mathbf{\Gamma}}_k^d &= \hat{\mathbf{\Gamma}}_k + \mathbf{M}_k\mathbf{X}_k^\top(\mathbf{Y}_k - \mathbf{X}_k\hat{\mathbf{\Gamma}}_k)/n_k,\\ \mathbf{T}_k &= \mathrm{vec}\big(\mathbf{M}_k\mathbf{X}_k^\top\boldsymbol{\xi}_k\big)/\sqrt{n_k},\\ \mathbf{R}_{k,\mathbf{\Gamma}} &= \sqrt{n_k}\mathrm{vec}\big((\mathbf{I}_q - \mathbf{M}_k\mathbf{C}_k)(\hat{\mathbf{\Gamma}}_k - \mathbf{\Gamma})\big),\end{aligned}$$

where $\mathbf{I}_q$ is the identity matrix of dimension $q\times q$ and $\mathbf{M}_k$ is a reasonable approximation for the inverse of the sample covariance matrix $\mathbf{C}_k = \mathbf{X}_k^\top\mathbf{X}_k/n_k$. As q can be larger than n_k, the sample covariance matrix $\mathbf{C}_k$ is not always invertible. As such, we find $\mathbf{M}_k$ such that $\mathbf{M}_k\mathbf{C}_k \approx \mathbf{I}_q$. The estimator $\hat{\mathbf{\Gamma}}_k^d$, $k = 1,\dots,K$, is the multivariate version of the one introduced in [34]. Note that the quantities in (3.3) are indexed by k, and as such one obtains a collection of debiased estimators $\hat{\mathbf{\Gamma}}_1^d,\dots,\hat{\mathbf{\Gamma}}_K^d$, each using a particular sub-sample. In Section 5, we propose a novel, aggregated estimator based on the collection $\hat{\mathbf{\Gamma}}_1^d,\dots,\hat{\mathbf{\Gamma}}_K^d$. In order to construct the aggregated estimator one needs the asymptotic distribution of $\hat{\mathbf{\Gamma}}_k^d$, $k = 1,\dots,K$, which is investigated in the sequel.

Similarly to the case of a univariate response, by conditioning on $\mathbf{X}_k$, one can show the normality of $\mathbf{T}_k$ from (3.3). One just needs next to show that the remainder term $\mathbf{R}_{k,\mathbf{\Gamma}}$ vanishes with increasing n_k. To this end, we consider a suitable approximation for $\mathbf{M}_k$, such that with increasing n_k, the entries in $(\mathbf{I}_q - \mathbf{M}_k\mathbf{C}_k)$ get closer to zero. Several works (for instance, [18, 34, 40]) assumed that $\mathbf{Q}^{-1}$ is sparse and then using the method of nodewise Lasso, estimated $\mathbf{Q}^{-1}$ and have set $\mathbf{M}_k = \hat{\mathbf{Q}}^{-1}$, where $\hat{\mathbf{Q}}^{-1}$ is the nodewise Lasso estimator of $\mathbf{Q}^{-1}$. We follow the same procedure, and for the reader's convenience, a brief description of the method is provided in Appendix A.1. Furthermore, we need to control the randomness of the product between the noise matrix $\boldsymbol{\xi}_k$ and the design matrix $\mathbf{X}_k$ in the multivariate linear regression. Many studies, focusing on Lasso regression models, control the randomness of the noise by conditioning on an event of interest (see for instance, [5, 18]). Similarly, considering the threshold $\rho_{0,k}$, such that $2\rho_{0,k} \leqslant \rho_k$, recall that ρ_k is the regularization parameter in (3.1), we consider the event

$$\mathcal{F}_k(n_k,p,q) = \left\{\|\mathrm{vec}\big(\mathbf{X}_k^\top\boldsymbol{\xi}_k\big)\|_\infty/n_k \leqslant \rho_{0,k}\right\}.$$

Lemmas 2 and 3 in Appendix B show that for a suitable value of $\rho_{0,k}$, the event $\mathcal{F}_k(n_k,p,q)$ happens with a large probability for every fixed k and jointly in $k = 1,\dots,K$, respectively. By controlling the randomness of the product between $\mathbf{X}_k$ and $\boldsymbol{\xi}_k$, one can control the ℓ_1 error bound on the estimation of the coefficient matrix. The reader can refer to Theorem 2 and its proof for more details. To show the asymptotic normality of $\mathbf{T}_k$ from (3.3), we need to impose as well a sparsity condition on the inverse covariance matrix $\mathbf{Q}^{-1}$. The sparsity condition in [34] is considered as $d_2 = o(n/\log(q))$. In this paper, we need to restrict the elements of $\mathbf{Q}^{-1}$ to a sparser regime such that the maximum row sparsity grows at the rate $d_2 = o(\sqrt{n_\dagger}/\log(q))$. With this sparsity condition, we

show the asymptotic normality of the final aggregated estimator in Section 5 as this condition is needed in Lemma 8, where we show that under such an assumption two sequences are equivalent in probability. Moreover, due to the fact that both K and n_k, $k = 1, \dots, K$, grow in the distributed case, we impose the sparsity condition $s_2 = o\big(n_\dagger^{\pi_1}/(\log(q)\log(pq))\big)$, $0 < \pi_1 \leqslant 1/2$ on $\mathbf{\Gamma}$ to guarantee the theoretical properties of the aggregated estimator.

Theorem 1. *Consider the regression model* (2.2) *for the k-th sub-sample with zero-mean Gaussian noise matrix $\boldsymbol{\xi}_k$ having covariance matrix $\mathbf{\Sigma}$ and random design matrix $\mathbf{X}_k$, which satisfies assumptions* (A1) *and* (A2) *from Section 3 with sparsity condition $d_2 = o(\sqrt{n_\dagger}/\log(q))$. On the event $\mathcal{F}_k(n_k, p, q)$, with regularization parameter $\rho_k \asymp \sqrt{\log(pq)/n_k}$ in* (3.1) *and $\tilde{\rho}_{j,k} \asymp \sqrt{\log(q)/n_k}$, $j = 1, \dots, q$, from the nodewise Lasso procedure, defined in Appendix A.1, we have*

$$\mathbf{T}_k|\mathbf{X}_k \sim \mathcal{N}_{pq}\big(\mathbf{0}, \mathbf{\Sigma} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)\big), \quad \|\mathbf{R}_{k,\mathbf{\Gamma}}\|_\infty = O_p(s_2\sqrt{\log(q)\log(pq)/n_k}), \tag{3.4}$$

and under the additional sparsity condition $s_2 = o\big(n_\dagger^{\pi_1}/(\log(q)\log(pq))\big)$, $0 < \pi_1 \leqslant 1/2$, we have $\|\mathbf{R}_{k,\mathbf{\Gamma}}\|_\infty = o_p(1)$.

The proof is given in Appendix A.1.

One can use Theorem 1 to construct asymptotic inferential tools for $\mathbf{\Gamma}$ based on the k-th sub-sample. However, the covariance matrix $\mathbf{\Sigma}$ is in general unknown and it can be replaced in practice by a consistent estimator. Lemma 6 in Appendix B shows that if the eigenvalues of $\mathbf{\Sigma}$ are bounded from below and above, then under the random design setting, with sparsity condition $s_2 = o\big(n_\dagger^{\pi_1}/(\log(q)\log(pq))\big)$, $0 < \pi_1 \leqslant 1/2$, the estimator $\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} = (\mathbf{Y}_k - \mathbf{X}_k\hat{\mathbf{\Gamma}}_k)^\top \times (\mathbf{Y}_k - \mathbf{X}_k\hat{\mathbf{\Gamma}}_k)/n_k$ is a consistent estimator for the covariance matrix $\mathbf{\Sigma}$ with convergence rate in ℓ_∞ norm of order $O_p(\max\{\sqrt{\log(p)/n_k}, s_2\log(pq)/n_k\})$.

Using (3.3), one can also obtain the convergence rate of the debiased estimator $\hat{\mathbf{\Gamma}}_k^d$. This bound is investigated in Theorem 2.

Theorem 2. *Consider the regression model* (2.2) *for the k-th sub-sample with design matrix $\mathbf{X}_k$ which satisfies assumptions* (A1) *and* (A2) *from Section 3. On the event $\mathcal{F}_k(n_k, p, q)$, with regularization parameter $\rho_k \asymp \sqrt{\log(pq)/n_k}$, we have*

$$\|\hat{\mathbf{\Gamma}}_k^d - \mathbf{\Gamma}\|_\infty = O_p\big(\max\{\sqrt{d_2\log(pq)/n_k}, s_2\sqrt{\log(q)\log(pq)}/n_k\}\big), \tag{3.5}$$

where under the assumption $\log(p)/\log(q) = o(n_\dagger^{\pi_2})$, $0 < \pi_2 < 1/2$, and the sparsity conditions $d_2 = o(\sqrt{n_\dagger}/\log(q))$ and $s_2 = o\big(n_\dagger^{\pi_1}/(\log(q)\log(pq))\big)$, $0 < \pi_1 \leqslant 1/2$, we obtain $\|\hat{\mathbf{\Gamma}}_k^d - \mathbf{\Gamma}\|_\infty = o_p(1)$.

The proof is given in Appendix A.2. In the next section, the distributed estimation of $\mathbf{\Theta}$ based on the k-th sub-sample is provided.

4. Distributed estimator of the precision matrix using the k-th sub-sample

Given the design matrix $\mathbf{X}_k$, the following assumptions (similarly to [17] and [32]) are adapted to our context and are considered for providing theoretical guarantees in the estimation procedure of $\boldsymbol{\Theta}$.

(B1) The eigenvalues of the precision matrix $\boldsymbol{\Theta}$ are bounded from below and above, i.e., there exists a constant $\Lambda_2 \geqslant 1$ such that $1/\Lambda_2 \leqslant \Lambda_{\min}(\boldsymbol{\Theta}) \leqslant \Lambda_{\max}(\boldsymbol{\Theta}) \leqslant \Lambda_2$, where $\Lambda_{\min}(\boldsymbol{\Theta})$ and $\Lambda_{\max}(\boldsymbol{\Theta})$ are the minimum and maximum eigenvalues of $\boldsymbol{\Theta}$, respectively. Moreover, $\max_a \boldsymbol{\Theta}_{aa} = O(1)$, where $\boldsymbol{\Theta}_{aa}$ is the a-th diagonal element of $\boldsymbol{\Theta}$.

Recall that $\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} = (\mathbf{Y}_k - \mathbf{X}_k\hat{\boldsymbol{\Gamma}}_k)^\top(\mathbf{Y}_k - \mathbf{X}_k\hat{\boldsymbol{\Gamma}}_k)/n_k$ and consider $\mathbf{H}$ as the Hessian of the negative log-likelihood function proportional to $l(\boldsymbol{\Theta}) = \operatorname{trace}(\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\boldsymbol{\Theta}) - \log\det(\boldsymbol{\Theta})$. By definition, $\mathbf{H}$ is a $p^2 \times p^2$ matrix indexed by the pair of elements from the node set, such that $\mathbf{H} = [\mathbf{H}_{(a,b),(c,d)}]$, where $(a,b),(c,d) \in \mathcal{V}\times\mathcal{V}$. Let $\mathbb{S}_1 = \{S_1 \bigcup \{(1,1),(2,2),\ldots,(p,p)\}\}$ with cardinality ϖ, which is equal to $\varpi = s_1 + p$, and denote its complement set with $\mathbb{S}_1^c$.

(B2) The irrepresentability condition holds for the true precision matrix $\boldsymbol{\Theta}$, i.e., there exists $\alpha_1 \in (0,1]$ such that $\max_{e\in\mathbb{S}_1^c} \|\mathbf{H}_{e\mathbb{S}_1}\left(\mathbf{H}_{\mathbb{S}_1\mathbb{S}_1}\right)^{-1}\|_1 \leqslant 1-\alpha_1$, where $\mathbf{H}_{\mathbb{S}_1\mathbb{S}_1} \in \mathbb{R}^{\varpi\times\varpi}$ is a sub-matrix of $\mathbf{H}$ whose rows and columns are indexed by the elements of $\mathbb{S}_1$. Moreover, e is a pair $(a,b) \in \mathbb{S}_1^c$ such that $\mathbf{H}_{e\mathbb{S}_1}$ is an ϖ-dimensional column vector with elements $\mathbf{H}_{e,(c,d)}$, where $(c,d) \in \mathbb{S}_1$.

Given $\mathbf{X}_k$ and using $\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} = (\mathbf{Y}_k - \mathbf{X}_k\hat{\boldsymbol{\Gamma}}_k)^\top(\mathbf{Y}_k - \mathbf{X}_k\hat{\boldsymbol{\Gamma}}_k)/n_k$, one can construct an estimator for $\boldsymbol{\Theta}$ using the following graphical Lasso optimization problem

$$\hat{\boldsymbol{\Theta}}_k = \arg\min_{\boldsymbol{\Theta}\in\mathcal{S}_{++}^p} \left\{\operatorname{trace}(\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\boldsymbol{\Theta}) - \log\det(\boldsymbol{\Theta}) + \lambda_k\|\boldsymbol{\Theta}\|_{1,\text{off}}\right\}, \tag{4.1}$$

where $\|\cdot\|_{1,\text{off}}$ is the ℓ_1 off-diagonal norm of the matrix defined as $\|\boldsymbol{\Theta}\|_{1,\text{off}} = \sum_{a\neq b}|\boldsymbol{\Theta}_{ab}|$ and $\mathcal{S}_{++}^p$ is the space of positive definite matrices of dimension $p\times p$. To obtain a debiased estimator of $\boldsymbol{\Theta}$, we invert the KKT conditions from the optimization problem (4.1), which is of the form

$$\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} - \hat{\boldsymbol{\Theta}}_k^{-1} + \lambda_k\hat{\mathbf{D}}_k = \mathbf{0}, \tag{4.2}$$

where the matrix $\hat{\mathbf{D}}_k$ belongs to the sub-differential of the ℓ_1 off-diagonal norm evaluated at $\hat{\boldsymbol{\Theta}}_k$. The difference between (4.2) and the problem in [17] is that the previous work used the sample covariance matrix $\mathbf{Y}_k^\top\mathbf{Y}_k/n_k$, but here due to the non-zero mean of the model, we use $\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}$ which contains the plug-in estimation of the regression coefficient matrix $\boldsymbol{\Gamma}$ therein, thus making the analysis more complicated. To simplify notation, define

$$\mathbf{W}_k = \hat{\boldsymbol{\Sigma}}_{k,\boldsymbol{\Gamma}} - \boldsymbol{\Sigma}, \quad \mathbf{W}_k' = \hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} - \boldsymbol{\Sigma}, \quad \mathbf{W}_k'' = \hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} - \hat{\boldsymbol{\Sigma}}_{k,\boldsymbol{\Gamma}},$$

where

$$\hat{\mathbf{\Sigma}}_{k,\mathbf{\Gamma}} = (\mathbf{Y}_k - \mathbf{X}_k\mathbf{\Gamma})^\top(\mathbf{Y}_k - \mathbf{X}_k\mathbf{\Gamma})/n_k.$$

Using the KKT condition (4.2) and performing some algebra calculations, leads to

$$\sqrt{n_k}(\hat{\mathbf{\Theta}}_k^d - \mathbf{\Theta}) = -\sqrt{n_k}\mathbf{\Theta}\mathbf{W}_k\mathbf{\Theta} + \mathbf{R}_{k,\mathbf{\Theta}}, \tag{4.3}$$

where $\hat{\mathbf{\Theta}}_k^d = 2\hat{\mathbf{\Theta}}_k - \hat{\mathbf{\Theta}}_k\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\hat{\mathbf{\Theta}}_k$ is the k-th debiased estimator of $\mathbf{\Theta}$ and the term $\mathbf{R}_{k,\mathbf{\Theta}}$ is defined as

$$\mathbf{R}_{k,\mathbf{\Theta}} := -\sqrt{n_k}(\hat{\mathbf{\Theta}}_k\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} - \mathbf{I}_p)(\hat{\mathbf{\Theta}}_k - \mathbf{\Theta}) - \sqrt{n_k}(\hat{\mathbf{\Theta}}_k - \mathbf{\Theta})\mathbf{W}_k'\mathbf{\Theta} - \sqrt{n_k}\mathbf{\Theta}\mathbf{W}_k''\mathbf{\Theta}. \tag{4.4}$$

In the sequel, it is shown that under suitable conditions, $\|\mathbf{R}_{k,\mathbf{\Theta}}\|_\infty = o_p(1)$ and that the term $\sqrt{n_k}\mathbf{\Theta}\mathbf{W}_k\mathbf{\Theta}$ is elementwise asymptotically normal.

Relative to the work of [17], in addition to different covariance matrices used in (4.4), our remainder contains the extra term $\sqrt{n_k}\mathbf{\Theta}\mathbf{W}_k''\mathbf{\Theta}$. This extra term is bounded in elementwise ℓ_∞ norm at the rate of $O_p(d_1 s_2 \log(pq)/\sqrt{n_k})$, as it is shown in Lemma 1. In the estimation procedure, if one considers $\mathbf{\Gamma} = \mathbf{0}$ as a special case, then the KKT condition (4.2) will simplify to the one in [17], and as such the estimator we propose, $\hat{\mathbf{\Theta}}_k^d$, will simplify to the same debiased estimator with the same convergence rate from that work. We recall $\kappa_{\mathbf{\Sigma}} := |||\mathbf{\Sigma}|||_\infty$ as the matrix ℓ_∞ norm of $\mathbf{\Sigma}$ and $\kappa_{\mathbf{H}} = |||\left(\mathbf{H}_{\mathbb{S}_1,\mathbb{S}_1}\right)^{-1}|||_\infty$ as the matrix ℓ_∞ norm of the inverse of $\mathbf{H}_{\mathbb{S}_1,\mathbb{S}_1}$ from the irrepresentability condition (B2). In Lemma 1, negligibility of $\mathbf{R}_{k,\mathbf{\Theta}}$ is shown.

Lemma 1. Consider the multivariate regression model (2.2) with random Gaussian noise matrix $\boldsymbol{\xi}_k$ having zero-mean rows, covariance matrix $\mathbf{\Sigma}$ and precision matrix $\mathbf{\Theta}$. Let assumptions (B1)–(B2), and (C1)–(C3) from Appendix A.3 hold. Consider the debiased estimator in (4.3) with regularization parameter $\lambda_k \asymp \sqrt{\log(p)/n_k}$. On the event $\mathcal{F}_k(n_k, p, q)$ with $2\rho_{0,k} \leqslant \rho_k$, where $\rho_k \asymp \sqrt{\log(pq)/n_k}$, and under the assumptions $1/\alpha_1 = O(1)$, $\kappa_{\mathbf{\Sigma}} = O(1)$ and $\kappa_{\mathbf{H}} = O(1)$, we have

$$\|\mathbf{R}_{k,\mathbf{\Theta}}\|_\infty = O_p\Big(\max\big\{d_1^{3/2}\log(p)/\sqrt{n_k}, d_1^2(\log(p))^{3/2}/n_k, d_1 s_2 \log(pq)/\sqrt{n_k}\big\}\Big), \tag{4.5}$$

and under the sparsity conditions $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$ and $s_2 = o(n_\dagger^{\pi_3}/\log(pq))$, where $0 < \pi_3 \leqslant 1/6$, we get $\|\mathbf{R}_{k,\mathbf{\Theta}}\|_\infty = o_p(1)$.

The proof is given in Appendix A.3.

Remark 1. A similar convergence rate for the remainder term is obtained in [17] but for the zero-mean model. Their rate is of order $O_p\big(\max\big\{d_1^{3/2}\log(p)/\sqrt{n_k}, d_1^2(\log(p))^{3/2}/n_k\big\}\big)$. Comparing it with (4.5), it is observed that the convergence rate of the extra term $\sqrt{n_k}\mathbf{\Theta}\mathbf{W}_k''\mathbf{\Theta}$ is of order $O_p(d_1 s_2 \log(pq)/\sqrt{n_k})$ which also adds more constraints on the negligibility of $\mathbf{R}_{k,\mathbf{\Theta}}$.

Theorem 3 investigates the elementwise asymptotic normality of the debiased estimator $\hat{\mathbf{\Theta}}_k^d$, which will be leveraged further in Section 5 to construct an aggregated estimator using all K estimators $\hat{\mathbf{\Theta}}_1^d, \ldots, \hat{\mathbf{\Theta}}_K^d$.

Theorem 3. *Under the assumptions of* Lemma 1 *and the sparsity conditions* $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$ *and* $s_2 = o(n_\dagger^{\pi_3}/\log(pq))$, $0 < \pi_3 \leqslant 1/6$, *for all* $(a,b) \in \mathcal{V} \times \mathcal{V}, a \neq b$, *it holds that*

$$\sqrt{n_k}(\hat{\mathbf{\Theta}}_{ab,k}^d - \mathbf{\Theta}_{ab})/\sigma_{ab} = \mathbf{Z}_{ab,k} + o_p(1), \tag{4.6}$$

where $\mathbf{\Theta}_{ab}$ *and* $\hat{\mathbf{\Theta}}_{ab,k}^d$ *are the* (a,b)*-th element of* $\mathbf{\Theta}$ *and* $\hat{\mathbf{\Theta}}_k^d$, *respectively, and* $\sigma_{ab}^2 = \mathbf{\Theta}_{aa}\mathbf{\Theta}_{bb} + \mathbf{\Theta}_{ab}^2$. *Moreover,* $1/\sigma_{ab} = O(1)$ *and* $\mathbf{Z}_{ab,k}$ *converges weakly to* $\mathcal{N}(0,1)$ *as* n_k *grows.*

The proof is given in Appendix A.4.

Remark 2. One can use Theorem 3 to construct asymptotic confidence intervals and hypothesis testing procedures. However, to perform inference, one needs a consistent estimator for σ_{ab} as it is unknown. By similar arguments as in Lemma 2 of [17] and considering $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$, the estimator $\hat{\sigma}_{ab,k}^2 = \hat{\mathbf{\Theta}}_{aa,k}\hat{\mathbf{\Theta}}_{bb,k} + \hat{\mathbf{\Theta}}_{ab,k}^2$ is a consistent estimator for σ_{ab}^2 with convergence rate $O_p(\max\{\log(p)/n_k, \sqrt{d_1\log(p)/n_k}\})$.

Remark 3. To quantify the convergence rate of the debiased estimator $\hat{\mathbf{\Theta}}_k^d$, from (4.3), we have that

$$\|\hat{\mathbf{\Theta}}_k^d - \mathbf{\Theta}\|_\infty \leqslant \|\|\mathbf{\Theta}\|\|_\infty^2 \|\mathbf{W}_k\|_\infty + \|\mathbf{R}_{k,\mathbf{\Theta}}\|_\infty/\sqrt{n_k}.$$

Given $\mathbf{X}_k$ and under assumption (B1), by considering $\boldsymbol{\xi}_k = \mathbf{Y}_k - \mathbf{X}_k\mathbf{\Gamma}$ and setting $\mathbf{A} = \mathbf{B} = \mathbf{I}_p$ in Lemma 2 of [21], for which the conditions are fulfilled, one can write $\|\mathbf{W}_k\|_\infty = O_p(\sqrt{\log(p)/n_k})$. Combining this bound with (4.5) and (A.9) from Appendix A.3, we get

$$\|\hat{\mathbf{\Theta}}_k^d - \mathbf{\Theta}\|_\infty = O_p\Big(\max\big\{d_1\sqrt{\log(p)/n_k}, d_1^{3/2}\log(p)/n_k, \\ d_1^2(\log(p)/n_k)^{3/2}, d_1 s_2 \log(pq)/n_k\big\}\Big),$$

and under the sparsity conditions $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$, and $s_2 = o(n_\dagger^{\pi_3}/\log(pq))$, $0 < \pi_3 \leqslant 1/6$, the consistency of $\hat{\mathbf{\Theta}}_k^d$ follows.

5. The final aggregated estimators across the sub-samples

The results in the previous sections are provided at the level of the k-th sub-sample, $k = 1, \ldots, K$. To construct more reliable estimators, we aggregate estimators drawn from different distributed locations. There are multiple ways to aggregate these K estimators into a combined estimator. In this paper, due to

the asymptotic normal distribution of the local estimators, we derive the final aggregated estimator by maximizing a pseudo log-likelihood function, which is constructed using the asymptotic normal density of local estimators constructed based on the sub-samples. Consider Δ as a general parameter of interest, for example $\boldsymbol{\Theta}_{ab}$ or $\text{vec}(\boldsymbol{\Gamma})_a$ in this paper, where $\text{vec}(\boldsymbol{\Gamma})_a$ is the a-th element of the vectorized form of $\boldsymbol{\Gamma}$, and $\hat{\Delta}_k$ as the k-th estimator based on the k-th sub-sample with variance σ^2 and denote its consistent estimator by $\hat{\sigma}_k^2$ which is also derived based on the k-th sub-sample. Consider the asymptotic normal density of $\hat{\Delta}_k$ at point ι_k as $\hat{f}_k(\iota_k \mid \Delta, \hat{\sigma}_k)$, where the variance σ^2 is replaced by its consistent estimator $\hat{\sigma}_k^2$. By maximizing the pseudo log-likelihood function constructed as

$$l(\Delta) = \log\left(\prod_{k=1}^{K} \hat{f}_k(\iota_k \mid \Delta, \hat{\sigma}_k)\right) \propto \sum_{k=1}^{K} (-n_k/2)(\iota_k - \Delta)^2/\hat{\sigma}_k^2,$$

with respect to Δ, the final aggregated estimator is of the form

$$\tilde{\Delta}_{\text{owAvg}} = \frac{1}{\sum_{k=1}^{K} \frac{n_k}{\hat{\sigma}_k^2}} \times \sum_{k=1}^{K} \frac{n_k}{\hat{\sigma}_k^2} \hat{\Delta}_k, \tag{5.1}$$

where the subscript "owAvg" stands for the optimally weighted average. We call this estimator an "optimally weighted average", as it is obtained using a maximization problem. As it is shown in [11], when K is fixed, a convex combination of K estimators $\hat{\Delta}_1, \ldots, \hat{\Delta}_K$ is of the form $\tilde{\Delta}_c = \boldsymbol{w}^\top \hat{\boldsymbol{\Delta}}$, where $\hat{\boldsymbol{\Delta}} = (\hat{\Delta}_1, \ldots, \hat{\Delta}_K)^\top$ and $\boldsymbol{w} = (w_1, \ldots, w_K)^\top$ is the vector of weights satisfying the constraint $\sum_{k=1}^{K} w_k = 1$. When the covariance between every two estimators is zero, the optimal weight for the k-th estimator in this convex combination is of the form $w_k = \frac{1/\sigma_k^2}{\sum_{k=1}^{K} 1/\sigma_k^2}$, where σ_k^2 is the variance of the k-th estimator. This result can be also encountered in portfolio theory, where the same weights are used to find the global minimum variance portfolio. The reader can refer to [14] and [20] among many other references for more details. In the distributed setting, when the sub-samples are equal (balanced case), our proposed estimator simplifies to the optimal weights convex combination, where we substitute the variance σ_k^2 with its consistent estimator $\hat{\sigma}_k^2$, as it is unknown in practice. Substituting debiased estimators $\hat{\boldsymbol{\Gamma}}_k^d$ and $\hat{\boldsymbol{\Theta}}_k^d$ in (5.1), one can aggregate distributed estimators of the coefficient and precision matrices from the sub-samples into the final combined estimators $\tilde{\boldsymbol{\Gamma}}_{\text{owAvg}}$ and $\tilde{\boldsymbol{\Theta}}_{\text{owAvg}}$, respectively. Note that if the variance σ^2 is known, then replacing $\hat{\sigma}_k^2$ by σ^2 simplifies the aggregated estimator $\tilde{\Delta}_{\text{owAvg}}$ to two special cases. In the case of unbalanced sub-samples, $\tilde{\Delta}_{\text{owAvg}}$ is simplified to the sample size weighted average estimator

$$\tilde{\Delta}_{\text{wAvg}} = \sum_{k=1}^{K} \frac{n_k}{n} \hat{\Delta}_k, \tag{5.2}$$

where "wAvg" stands for the sample size weighted average proportional to the sub-sample sizes. Similarly to the optimally weighted average estimator, by substituting debiased estimators $\hat{\mathbf{\Gamma}}_k^d$ and $\hat{\mathbf{\Theta}}_k^d$ in (5.2), the weighted average aggregated estimators of $\mathbf{\Gamma}$ and $\mathbf{\Theta}$ can be constructed and denoted by $\tilde{\mathbf{\Gamma}}_{\text{wAvg}}$ and $\tilde{\mathbf{\Theta}}_{\text{wAvg}}$, respectively. Moreover, if the sub-samples are also balanced, $\tilde{\Delta}_{\text{owAvg}}$ further simplifies to the simple average estimator

$$\tilde{\Delta}_{\text{sAvg}} = \frac{1}{K}\sum_{k=1}^{K}\hat{\Delta}_k, \tag{5.3}$$

where "sAvg" stands for the simple average. By the same argument as for the optimally weighted and sample size weighted averages, the simple average estimators of $\mathbf{\Gamma}$ and $\mathbf{\Theta}$ can be constructed and denoted by $\tilde{\mathbf{\Gamma}}_{\text{sAvg}}$ and $\tilde{\mathbf{\Theta}}_{\text{sAvg}}$, respectively.

Using (3.3) and (5.1), by considering the consistent estimator $\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}$ for $\mathbf{\Sigma}$, the optimally weighted average estimator for the a-th element of $\text{vec}(\mathbf{\Gamma})$, $a = 1,\ldots,qp$, is of the form

$$\begin{aligned}\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a = &\left(\sum_{k=1}^{K}\frac{n_k}{\big[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)\big]_{aa}}\right)^{-1}\\ &\times\sum_{k=1}^{K}\frac{n_k}{\big[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)\big]_{aa}}\text{vec}\big(\hat{\mathbf{\Gamma}}_k^d\big)_a.\end{aligned} \tag{5.4}$$

It can be shown that

$$\sqrt{\sum_{k=1}^{K}n_k/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\big(\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a-\text{vec}(\mathbf{\Gamma})_a\big)=\mathbf{W}_{a,\mathbf{\Gamma}}+\mathbf{R}_{a,\mathbf{\Gamma}}, \tag{5.5}$$

where

$$\begin{aligned}\mathbf{R}_{a,\mathbf{\Gamma}} = &\sqrt{\frac{\sum_{k=1}^{K}n_k/[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^{K}n_k/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}\\ &\times\sum_{k=1}^{K}\frac{\sqrt{n_k[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}}{\sqrt{n}[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\mathbf{R}_{a,k,\mathbf{\Gamma}},\\ \mathbf{W}_{a,\mathbf{\Gamma}} = &\sqrt{\frac{\sum_{k=1}^{K}n_k/[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^{K}n_k/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}\\ &\times\sum_{k=1}^{K}\frac{\sqrt{n_k[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}}{\sqrt{n}[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\mathbf{T}_{a,k},\end{aligned}$$

and $\mathbf{R}_{a,k,\mathbf{\Gamma}}$ and $\mathbf{T}_{a,k}$ are the a-th element of $\mathbf{R}_{k,\mathbf{\Gamma}}$ and $\mathbf{T}_k$, respectively, defined in (3.3).

Similarly, using (4.3) and (5.1), by considering the consistent estimator $\hat{\sigma}^2_{ab,k}$ for σ^2_{ab} in (4.6), the aggregated estimator for the (a,b)-th element of $\mathbf{\Theta}$, $(a,b) \in \mathcal{V} \times \mathcal{V}, a \neq b$, is of the form

$$\tilde{\mathbf{\Theta}}_{ab,\mathrm{owAvg}} = \left(\sum_{k=1}^{K} \frac{n_k}{\hat{\sigma}^2_{ab,k}} \right)^{-1} \times \sum_{k=1}^{K} \frac{n_k}{\hat{\sigma}^2_{ab,k}} \hat{\mathbf{\Theta}}^d_{ab,k}. \tag{5.6}$$

Moreover,

$$\sqrt{\sum_{k=1}^{K} \frac{n_k}{\hat{\sigma}^2_{ab,k}}} \left(\tilde{\mathbf{\Theta}}_{ab,\mathrm{owAvg}} - \mathbf{\Theta}_{ab} \right) = \mathbf{W}_{ab,\mathbf{\Theta}} + \mathbf{R}_{ab,\mathbf{\Theta}}, \tag{5.7}$$

where

$$\begin{aligned}
\mathbf{R}_{ab,\mathbf{\Theta}} &= \sqrt{\frac{\sum_{k=1}^{K} n_k/\sigma^2_{ab}}{\sum_{k=1}^{K} n_k/\hat{\sigma}^2_{ab,k}}} \sum_{k=1}^{K} \frac{\sqrt{n_k}\sigma_{ab}}{\sqrt{n}\hat{\sigma}^2_{ab,k}} \mathbf{R}_{ab,k,\mathbf{\Theta}}, \\
\mathbf{W}_{ab,\mathbf{\Theta}} &= \sqrt{\frac{\sum_{k=1}^{K} n_k/\sigma^2_{ab}}{\sum_{k=1}^{K} n_k/\hat{\sigma}^2_{ab,k}}} \sum_{k=1}^{K} \frac{\sigma_{ab}}{\sqrt{n}\hat{\sigma}^2_{ab,k}} \sum_{l=1}^{n_k} \big(\mathbf{\Theta}_a^\top (\dot{\mathbf{Y}}_{l,k} - \mathbf{\Gamma}^\top \dot{\mathbf{X}}_{l,k}) \\
&\qquad \times (\dot{\mathbf{Y}}_{l,k} - \mathbf{\Gamma}^\top \dot{\mathbf{X}}_{l,k})^\top \mathbf{\Theta}_b - \mathbf{\Theta}_{ab} \big),
\end{aligned}$$

and $\mathbf{R}_{ab,k,\mathbf{\Theta}}$ is the (a,b)-th element of $\mathbf{R}_{k,\mathbf{\Theta}}$ from (4.3), $\dot{\mathbf{Y}}_{l,k} \in \mathbb{R}^p$ and $\dot{\mathbf{X}}_{l,k} \in \mathbb{R}^q$ are the l-th row of $\mathbf{Y}_k$ and $\mathbf{X}_k$, respectively, while $\mathbf{\Theta}_a$ and $\mathbf{\Theta}_b$ are p-dimensional vectors coming from the a-th row and the b-th row of $\mathbf{\Theta}$, respectively.

Theorem 4. *Consider the regression model (2.2), where $\mathbf{X}_k$ satisfies assumptions* (A1) *and* (A2) *from Section 3, and consider the maximum row sparsity of $\mathbf{Q}^{-1}$ as $d_2 = o(\sqrt{n_\dagger}/\log(q))$. Moreover, consider the coefficient matrix $\mathbf{\Gamma}$ with sparsity condition $s_2 = o\big(n_\dagger^{\pi_1}/(\log(pq)\log(q))\big)$, $0 < \pi_1 \leqslant 1/2$. Suppose that the event $\mathcal{F}_k(n_k,p,q)$ holds jointly in $k = 1,\dots,K$. Moreover, suppose that $\hat{\mathbf{\Gamma}}^d_k$, $k = 1,\dots,K$, is the k-th debiased estimator in (3.3) with tuning parameter $\rho_k \asymp \sqrt{\log(pq)/n_k}$ and let $n_k/n \to c_k \in (0,1)$ as n_k grows such that $\lim_{K\to\infty} \sum_{k=1}^{K} c_k = 1$.*

a) If K grows at the rate $K = O\big(n^{1/4}/(\sqrt{\log(pq)\log(q)}\max\{s_2, \sqrt{d_2}\})\big)$, and $\log(p)/\log(q) = o(n_\dagger^{\pi_2})$, $0 < \pi_2 < 1/2$, for the a-th element of the vectorized form of $\tilde{\mathbf{\Gamma}}_{\mathrm{owAvg}}$, $a = 1,\dots,qp$, in (5.5), we have

$$\mathbf{W}_{a,\mathbf{\Gamma}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{and} \quad |\mathbf{R}_{a,\mathbf{\Gamma}}| = o_p(1), \tag{5.8}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

b) If K grows at the rate $K = O\big(n^{1/3}/(\sqrt{\log(pq)\log(q)}\max\{s_2, d_2\})\big)$, in the spirit of the special case (5.2), for the a-th element of the vectorized form

of $\tilde{\mathbf{\Gamma}}_{\text{wAvg}}$, $a = 1, \ldots, qp$, *which is denoted by* $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{wAvg}})_a$, *we have*

$$\frac{\sqrt{nK}}{\sqrt{\sum_{k=1}^{K}[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}\big(\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{wAvg}})_a - \text{vec}(\mathbf{\Gamma})_a\big) = \mathbf{W}'_{a,\mathbf{\Gamma}} + \mathbf{R}'_{a,\mathbf{\Gamma}},$$

where $\mathbf{W}'_{a,\mathbf{\Gamma}}$ *converges weakly to* $\mathcal{N}(0,1)$ *and* $|\mathbf{R}'_{a,\mathbf{\Gamma}}| = o_p(1)$.

c) *If* K *grows at the rate* $\sqrt{K} = O\big(n_\dagger^{1/3}/(\sqrt{\log(pq)\log(q)}\max\{s_2, d_2\})\big)$, *in the spirit of the special case* (5.3), *for the* a*-th element of the vectorized form of* $\tilde{\mathbf{\Gamma}}_{\text{sAvg}}$, $a = 1, \ldots, qp$, *which is denoted by* $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{sAvg}})_a$, *we have*

$$\frac{K^{3/2}}{\sqrt{(\sum_{k=1}^{K}[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa})(\sum_{k=1}^{K} 1/n_k)}}\big(\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{sAvg}})_a - \text{vec}(\mathbf{\Gamma})_a\big) = \mathbf{W}''_{a,\mathbf{\Gamma}} + \mathbf{R}''_{a,\mathbf{\Gamma}},$$

where $\mathbf{W}''_{a,\mathbf{\Gamma}}$ *converges weakly to* $\mathcal{N}(0,1)$ *and* $|\mathbf{R}''_{a,\mathbf{\Gamma}}| = o_p(1)$.

The proof is given in Appendix A.5.

Remark 4. Note that Theorem 1 in Section 3 provides the asymptotic normality, conditionally on the design matrix $\mathbf{X}_k$, while the asymptotic normality result in Theorem 4 is more general as it is an unconditional result. By using a similar argument to the proof of Theorem 4, it can be shown that both $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a$ and $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{wAvg}})_a$ have an asymptotic variance equal to $[\mathbf{\Sigma} \otimes \mathbf{Q}^{-1}]_{aa}$ which implies that their asymptotic relative efficiency is equal to 1, and they are as efficient as the full estimator using the full sample data. This result is expected since (i) the definitions of $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a$ and $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{wAvg}})_a$ are closely related, and (ii) the estimated variances based on the sub-samples are consistent estimators for the true variance, i.e., $[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}/[\mathbf{\Sigma} \otimes \mathbf{Q}^{-1}]_{aa} \xrightarrow{p} 1$. However, it has been brought to our attention that the same final result can be obtained via the asymptotic normality result characterizing M estimators (see for instance, Theorem 5.21 of [35]) since the proposed estimators are asymptotically linear and hence asymptotically equivalent to the estimator obtained using the full sample data. A comparison of the owAvg and wAvg estimators from a finite sample perspective is also provided through simulation and real data examples in Sections 6 and 7, respectively.

In Theorem 5, the asymptotic normality of the estimator in (5.6) and its two special cases is investigated as $n_\dagger$ and K both grow.

Theorem 5. *Consider the regression model* (2.2) *and suppose that assumptions* (B1)–(B2), *and* (C1)–(C3) *from Appendix A.3 hold. Moreover, consider the coefficient matrix* $\mathbf{\Gamma}$ *with sparsity condition* $s_2 = o(n_\dagger^{\pi_3}/\log(pq))$, $0 < \pi_3 \leqslant 1/6$. *Suppose that the event* $\mathcal{F}_k(n_k, p, q)$ *holds jointly in* $k = 1, \ldots, K$. *Moreover, suppose that* $\hat{\mathbf{\Theta}}_k^d$, $k = 1, \ldots, K$, *is the* k*-th debiased estimator in* (4.3) *with tuning parameter* $\lambda_k \asymp \sqrt{\log(p)/n_k}$ *and let* $n_k/n \to c_k \in (0,1)$ *as* n_k *grows, such that* $\lim_{K\to\infty}\sum_{k=1}^{K} c_k = 1$.

a) If K and the maximum node degree of $\mathbf{\Theta}$ grow at the rates $K = O\big(n^{1/3}/(d_1 \log(p))\big)$, and $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$, respectively, then for every pair $(a,b) \in \mathcal{V} \times \mathcal{V}, a \neq b$, of the pooled estimator $\tilde{\mathbf{\Theta}}_{\text{owAvg}}$ in (5.7), we have

$$\mathbf{W}_{ab,\mathbf{\Theta}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{and} \quad |\mathbf{R}_{ab,\mathbf{\Theta}}| = o_p(1). \tag{5.9}$$

b) Under the same conditions as in part a), in the spirit of the special case (5.2), for every pair $(a,b) \in \mathcal{V} \times \mathcal{V}$, $a \neq b$, of $\tilde{\mathbf{\Theta}}_{\text{wAvg}}$, which is denoted by $\tilde{\mathbf{\Theta}}_{ab,\text{wAvg}}$, we have

$$\frac{\sqrt{nK}}{\sqrt{\sum_{k=1}^K \hat{\sigma}_{ab,k}^2}}(\tilde{\mathbf{\Theta}}_{ab,\text{wAvg}} - \mathbf{\Theta}_{ab}) = \mathbf{W}'_{ab,\mathbf{\Theta}} + \mathbf{R}'_{ab,\mathbf{\Theta}},$$

where $\mathbf{W}'_{ab,\mathbf{\Theta}}$ converges weakly to $\mathcal{N}(0,1)$ and $|\mathbf{R}'_{ab,\mathbf{\Theta}}| = o_p(1)$.

c) If K and the maximum node degree of $\mathbf{\Theta}$ grow at the rates $K = O\big(n_\dagger^{4/3}/n\big)$ and $d_1^{3/2} = o\big(n_\dagger^{1/3}/\log(p)\big)$, in the spirit of the special case (5.3), for every pair $(a,b) \in \mathcal{V} \times \mathcal{V}$, $a \neq b$, of $\tilde{\mathbf{\Theta}}_{\text{sAvg}}$, which is denoted by $\tilde{\mathbf{\Theta}}_{ab,\text{sAvg}}$, we have

$$\frac{K^{3/2}}{\sqrt{(\sum_{k=1}^K \hat{\sigma}_{ab,k}^2)(\sum_{k=1}^K 1/n_k)}}(\tilde{\mathbf{\Theta}}_{ab,\text{sAvg}} - \mathbf{\Theta}_{ab}) = \mathbf{W}''_{ab,\mathbf{\Theta}} + \mathbf{R}''_{ab,\mathbf{\Theta}},$$

where $\mathbf{W}''_{ab,\mathbf{\Theta}}$ converges weakly to $\mathcal{N}(0,1)$ and $|\mathbf{R}''_{ab,\mathbf{\Theta}}| = o_p(1)$.

The proof is given in Appendix A.6.

Remark 5. By a similar argument to that of Remark 4, it can be shown that the asymptotic variances of $\tilde{\mathbf{\Theta}}_{ab,\text{owAvg}}$ and $\tilde{\mathbf{\Theta}}_{ab,\text{wAvg}}$ are both equal to σ_{ab}^2, where we recall that $\sigma_{ab}^2 = \mathbf{\Theta}_{aa}\mathbf{\Theta}_{bb} + \mathbf{\Theta}_{ab}^2$. Additionally, by rewriting Theorem 3 for the full sample estimator, which uses the entire dataset of size n, it can be shown that the asymptotic variance of the debiased full estimator is also equal to σ_{ab}^2. Therefore, we can deduce that both the optimally weighted and the sample size weighted distributed estimators are asymptotically as efficient as the debiased full sample estimator, which implies that the efficiency loss from the distributed setting is asymptotically zero.

According to the asymptotic normal distribution of the proposed estimators, one can construct confidence intervals and perform hypothesis testing for the elements of the coefficient matrix $\mathbf{\Gamma}$ and of the precision matrix $\mathbf{\Theta}$. The $(1-\alpha)100\%$ asymptotic confidence intervals for a general quantity of interest Δ, namely $\mathbf{\Theta}_{ab}$ or $\text{vec}(\mathbf{\Gamma})_a$ in this paper, using the optimally weighted, weighted and simple average estimators can be constructed as

$$\tilde{\Delta}_{\text{owAvg}} \pm \Phi^{-1}(1-\alpha/2)/\sqrt{\sum_{k=1}^K n_k/\hat{\sigma}_k^2}, \tag{5.10}$$

$$\tilde{\Delta}_{\text{wAvg}} \pm \Phi^{-1}(1-\alpha/2)\sqrt{(\sum_{k=1}^{K}\hat{\sigma}_k^2)/(nK)}, \tag{5.11}$$

$$\tilde{\Delta}_{\text{sAvg}} \pm \Phi^{-1}(1-\alpha/2)\sqrt{(\sum_{k=1}^{K}\hat{\sigma}_k^2)(\sum_{k=1}^{K}1/n_k)/K^3}, \tag{5.12}$$

where $\Phi^{-1}(1-\alpha/2)$ is the $(1-\alpha/2)$-th quantile of the standard normal distribution and $\hat{\sigma}_k^2$ is the k-th consistent estimator of σ^2. Substituting $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a$, $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{wAvg}})_a$ and $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{sAvg}})_a$, respectively in (5.10), (5.11) and (5.12) and then replacing the estimated variance $\hat{\sigma}_k^2$ by $[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}$, one can construct $(1-\alpha)100\%$ asymptotic confidence intervals for the a-th element, $a = 1,\ldots,qp$, of the vectorized from of the coefficient matrix $\mathbf{\Gamma}$. Similarly, substituting $\tilde{\mathbf{\Theta}}_{ab,\text{owAvg}}$, $\tilde{\mathbf{\Theta}}_{ab,\text{wAvg}}$ and $\tilde{\mathbf{\Theta}}_{ab,\text{sAvg}}$, respectively in (5.10), (5.11) and (5.12) and then replacing the estimated variance $\hat{\sigma}_k^2$ by $\hat{\sigma}_{ab,k}^2 = \hat{\mathbf{\Theta}}_{aa,k}\hat{\mathbf{\Theta}}_{bb,k} + \hat{\mathbf{\Theta}}_{ab,k}^2$, the $(1-\alpha)100\%$ asymptotic confidence intervals for every pair $(a,b) \in \mathcal{V}\times\mathcal{V}$, $a \neq b$, of the precision matrix $\mathbf{\Theta}$ can be constructed. Moreover, by substituting the same quantities in

$$|\tilde{\Delta}_{\text{owAvg}}| > \Phi^{-1}(1-\alpha/2)/\sqrt{\sum_{k=1}^{K} n_k/\hat{\sigma}_k^2},$$

$$|\tilde{\Delta}_{\text{wAvg}}| > \Phi^{-1}(1-\alpha/2)\sqrt{(\sum_{k=1}^{K}\hat{\sigma}_k^2)/(nK)},$$

$$|\tilde{\Delta}_{\text{sAvg}}| > \Phi^{-1}(1-\alpha/2)\sqrt{(\sum_{k=1}^{K}\hat{\sigma}_k^2)(\sum_{k=1}^{K}1/n_k)/K^3},$$

the rejection regions at level α for the hypothesis tests $H_{0,a} : \text{vec}(\mathbf{\Gamma})_a = 0$ vs $H_{1,a} : \text{vec}(\mathbf{\Gamma})_a \neq 0$, where $a = 1,\ldots,pq$ and $H_{0,ab} : \mathbf{\Theta}_{ab} = 0$ vs $H_{1,ab} : \mathbf{\Theta}_{ab} \neq 0$, where $(a,b) \in \mathcal{V}\times\mathcal{V}, a \neq b$ can be constructed.

As it was mentioned, the distributed estimator has an efficiency loss approaching zero as the sample size grows. This allows us to compare its incurred loss with that of the practically unavailable, full-sample debiased estimator. Considering the definition of $\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a$, one has

$$\begin{aligned}
&\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a - \text{vec}(\mathbf{\Gamma})_a \\
&= \frac{\sum_{k=1}^{K} n_k/[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^{K} n_k/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}} \times \sum_{k=1}^{K} \frac{\sqrt{n_k}[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{n[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\mathbf{T}_{a,k} \\
&\quad + \frac{\sum_{k=1}^{K} n_k/[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^{K} n_k/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}} \times \sum_{k=1}^{K} \frac{\sqrt{n_k}[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{n[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\mathbf{R}_{a,k,\mathbf{\Gamma}}.
\end{aligned}$$

Since $[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa} \xrightarrow{p} 1$ and $\sum_{k=1}^{K}(n_k/[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa})/\sum_{k=1}^{K}(n_k/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}) \xrightarrow{p} 1$ as $K\to\infty$ and $n_k\to\infty$, $k = 1,\ldots,K$,

using similar arguments to the proof of Theorem 2, we can write

$$\begin{aligned}\|\tilde{\mathbf{\Gamma}}_{\text{owAvg}} - \mathbf{\Gamma}\|_\infty &= \max_{a\in\{1,\ldots,qp\}} |\text{vec}(\tilde{\mathbf{\Gamma}}_{\text{owAvg}})_a - \text{vec}(\mathbf{\Gamma})_a| \\ &= O_p\big(K \max\{\sqrt{d_2 \log(pq)/n}, s_2\sqrt{\log(q)\log(pq)}/n\}\big), \quad (5.13)\end{aligned}$$

where the last equality is deduced using (3.4), (A.5) and (B.21). By using the full sample data in Theorem 2, the convergence rate of the debiased full estimator, call it $\hat{\mathbf{\Gamma}}_F^d$, is of order

$$\|\hat{\mathbf{\Gamma}}_F^d - \mathbf{\Gamma}\|_\infty = O_p\big(\max\{\sqrt{d_2 \log(pq)/n}, s_2\sqrt{\log(q)\log(pq)}/n\}\big). \quad (5.14)$$

Combining the triangle inequality with (5.13) and (5.14), it is deduced that

$$\|\tilde{\mathbf{\Gamma}}_{\text{owAvg}} - \hat{\mathbf{\Gamma}}_F^d\|_\infty = O_p\big(K \max\{\sqrt{d_2 \log(pq)/n}, s_2\sqrt{\log(q)\log(pq)}/n\}\big),$$

which implies that the convergence rate of the distance between the distributed estimator and the full one is equal to the rate of the distance between the distributed estimator and the true matrix $\mathbf{\Gamma}$. As such, it is noteworthy that $\tilde{\mathbf{\Gamma}}_{\text{owAvg}}$ not only follows an asymptotic normal distribution, but also approximates the debiased full estimator $\hat{\mathbf{\Gamma}}_F^d$ well, and it exhibits a similar statistical error as $\hat{\mathbf{\Gamma}}_F^d$ does, as long as the number of machines K is not too large.

By a similar argument, one can derive the convergence rate of $\tilde{\mathbf{\Theta}}_{\text{owAvg}}$ to be of the order

$$\begin{aligned}\|\tilde{\mathbf{\Theta}}_{\text{owAvg}} - \mathbf{\Theta}\|_\infty = O_p\big(K \max\{&d_1\sqrt{\log(p)/n}, d_1^{3/2}\log(p)/n, \\ &d_1^2(\log(p))^{3/2}/(n\sqrt{n_\dagger}), d_1 s_2 \log(pq)/n\}\big), \quad (5.15)\end{aligned}$$

which is obtained by combining the definition of $\tilde{\mathbf{\Theta}}_{\text{owAvg}}$ with the rate in (4.5), the upper bound on $\|\mathbf{W}_k\|_\infty$ from Remark 3 and (A.9) from Appendix A.3. Comparing this convergence rate with the one from Remark 3 for the full sample data estimator, call it $\hat{\mathbf{\Theta}}_F^d$, which is of the order

$$\begin{aligned}\|\hat{\mathbf{\Theta}}_F^d - \mathbf{\Theta}\|_\infty = O_p\big(\max\{&d_1\sqrt{\log(p)/n}, d_1^{3/2}\log(p)/n, \\ &d_1^2(\log(p)/n)^{3/2}, d_1 s_2 \log(pq)/n\}\big),\end{aligned}$$

one can achieve the same conclusion as for the estimation of $\mathbf{\Gamma}$, which implies that if K is not too large, $\tilde{\mathbf{\Theta}}_{\text{owAvg}}$ attains a similar statistical error as the debiased full estimator $\hat{\mathbf{\Theta}}_F^d$.

Remark 6. One can show that the convergence rate of the optimally weighted average estimator in a zero-mean model is of the form

$$\begin{aligned}\|\tilde{\mathbf{\Theta}}_{\text{owAvg}} - \mathbf{\Theta}\|_\infty = O_p\big(K \max\{&d_1\sqrt{\log(p)/n}, d_1^{3/2}\log(p)/n, \\ &d_1^2(\log(p))^{3/2}/(n\sqrt{n_\dagger})\}\big).\end{aligned}$$

The reader can refer to [28] for more details. Comparing this bound with the one presented in (5.15), it is observed that the difference in the convergence

rates between a covariate adjusted Gaussian graphical model and a zero-mean Gaussian graphical model is in the term $d_1 s_2 \log(pq)/n$. However, when considering the condition $s_2 = o\big(n_\dagger^{\pi_3}/\log(pq)\big)$, $0 < \pi_3 \leqslant 1/6$, the cardinality of the non-zero entries of $\mathbf{\Gamma}$ does not grow fast, hence $\mathbf{\Gamma}$ will be much sparser than $\mathbf{\Theta}$, and as a result, the term $d_1 s_2 \log(pq)/n$ will be dominated by the other terms in the bound (5.15).

In the next section, the statistical error and coverage probability of estimators are compared from a finite sample perspective.

6. Simulation study

In this section, we examine empirically the performance of our proposed estimators. To this end, we followed the simulation setup of [38] for generating sparse matrices $\mathbf{\Gamma}$ and $\mathbf{\Theta}$.

First, to generate the precision matrix $\mathbf{\Theta}$, we randomly generated a link between all pairs $(a, b) \in \mathcal{V} \times \mathcal{V}, a \neq b$, with probability of connection of 0.01. Then, the corresponding entry in the precision matrix is generated uniformly from $[-1, -0.5] \bigcup [0.5, 1]$, for each link. After that, for each row, each entry except the diagonal one is divided by the sum of the absolute values of the off-diagonal entries multiplied by 1.5. Finally, the matrix is symmetrized and the diagonal entries are fixed to 1. To generate the regression coefficient matrix, we first generated a sparse indicator matrix with non-zero elements having a probability of 0.01 of occurring for every pair $(a, b); a = 1, \ldots, q, b = 1, \ldots, p$. Then, corresponding to the non-zero entries of this indicator matrix, we generated uniformly the entries of $\mathbf{\Gamma}$ from $[-1, -\nu_m] \bigcup [\nu_m, 1]$, where ν_m is the minimum absolute non-zero value of the generated precision matrix. To generate a dataset, we first generated $\dot{\mathbf{X}} = (X^1, \ldots, X^q)^\top$ from a q-dimensional Gaussian distribution with mean vector zero and covariance matrix $\mathbf{Q}$. We used the Toeplitz structure $\varrho^{|a-b|}$, $a, b \in \{1, \ldots, q\}$ with $\varrho = 0.9$ to generate the covariance matrix $\mathbf{Q}$. Finally, given $\dot{\mathbf{X}} = \dot{\mathbf{x}}$, we generated $\dot{\mathbf{Y}}$ from a p-dimensional Gaussian distribution with mean $\mathbf{\Gamma}^\top \dot{\mathbf{x}}$ and covariance matrix $\mathbf{\Theta}^{-1}$.

To conduct the simulation, we set $n = 25000$ and 50000 and the number of machines to $K = 5, 10$ and 20. To show the performance of the distributed estimator in the unbalanced setting, we considered the following splitting procedure. Suppose that among all available machines, two of them are powerful. The first one is the most powerful one and $(55 - K)\%$ of the dataset is distributed on this machine. The second one is less powerful than the first one and $(60 - K)\%$ of the remaining dataset is distributed on this machine. The remaining dataset is distributed roughly equally on the remaining machines. By considering this splitting procedure and setting the number of responses to $p = 1100$ and the number of predictors to $q = 550$, for some sub-samples we have high-dimensionality. For example, when $n = 25000$ and $K = 10$, we have $p > n_k$, $\forall k = 3, \ldots, 10$ and when $K = 20$, we have $p, q > n_k$, $\forall k = 3, \ldots, 20$. Moreover, when $n = 50000$ and $K = 20$, we have $p > n_k$, $\forall k = 3, \ldots, 20$. To compare the performance of the distributed estimators, we considered two

types of estimators: debiased (which are non-sparse) and sparse. The debiased estimators consist of:

1) (Full) A debiased estimator based on the full non-distributed data.
2) (sAvg) An estimator based on splitting the data and averaging directly the debiased estimators from each machine.
3) (wAvg) An estimator based on splitting the data and taking the weighted average of the debiased estimators from each machine, where the weight for the k-th sub-sample is set to (n_k/n).
4) (Top1) The estimator produced by the most powerful machine which takes $(55-K)\%$ of dataset. Since estimation on each machine is consistent and asymptotically normal, investigating the performance on the first machine which takes most of the dataset, is relevant.

The sparse competitors, which are shown respectively by SFull, SsAvg, SwAvg, STop1, are obtained in the same way as estimators in 1)–4) but without the debiasing step. A comparison with the full estimator reveals how much the performance deteriorates due to splitting the data, while a comparison with the simple and sample size weighted average estimators has the purpose of evaluating if indeed the proposed owAvg estimator is better equipped to tackle unbalanced settings due to a more appropriate weighting. A comparison with the Top1 estimator has the purpose to evaluate if the remaining $(K-1)$ machines which account for $(45+K)\%$ of the original data are still able to produce informative estimates even though they receive low amounts of data. In this study, the tuning parameters in (3.1), (4.1) and (A.1), which are needed to obtain $\mathbf{M}_k$, see Appendix A.1, are set to $\rho_k = \rho = \sqrt{\log(pq)/n}$, $\lambda_k = \lambda = \sqrt{\log(p)/n}$ and $\tilde{\rho}_{j,k} = \tilde{\rho} = \sqrt{\log(q)/n}$ for all simulation runs. All simulation results are calculated as averages over $R = 500$ different repetitions.

To compare the performance of the estimators, we used the Frobenius norm between the estimated matrix for each competitor and the true matrix from the data generating process. The results for the Frobenius norm and their standard deviation (between parentheses) for $n = 50000$ are presented in Table 1 for each competitor, separately on the active and the non-active sets. Note that the active sets on $\boldsymbol{\Theta}$ and $\boldsymbol{\Gamma}$ are indexed by S_1 and S_2, respectively. The results for $n = 25000$ are similar and are presented in Table 4 in Appendix C.

From Table 1, it is observed that the performance of the proposed debiased owAvg estimator is similar to that of the non-distributed, full estimator in terms of Frobenius norm. With increasing K from 5 to 20, the norm of the distributed estimator stays relatively constant. This suggests that by splitting observations in combination with the proposed aggregation, one might not lose much information. On the other hand, wAvg is quite sensitive to the number of machines and with increasing K, its norm performance deteriorates. Especially when $K = 20$, the difference relative to the full one is large on the non-active set. The worst estimator between the debiased ones is sAvg which imposes the same weights on all sub-samples and it is observed that its norm increases substantially, which suggests that it is highly sensitive to K, as opposed to the distributed owAvg estimator. Furthermore, the Frobenius norm of Top1 is much larger than that

TABLE 1

Average and standard deviation (between parentheses) of the Frobenius norm over 500 repetitions on the active and non-active sets, for the proposed estimators and different competitors, when $n = 50000$.

			Active set				Non-active set			
			K					K		
			1	5	10	20	1	5	10	20
$\boldsymbol{\Theta}$	Debiased	Full	0.73 (.01)				4.75 (.01)			
		owAvg		0.74 (.00)	0.78 (.01)	0.89 (.01)		4.86 (.00)	5.13 (.01)	5.59 (.01)
		Top1		1.00 (.01)	1.05 (.01)	1.19 (.01)		6.60 (.01)	6.95 (.01)	7.85 (.01)
		sAvg		1.07 (.01)	1.81 (.02)	2.94 (.02)		7.00 (.01)	10.44 (.05)	12.96 (.03)
		wAvg		0.75 (.00)	0.84 (.01)	1.32 (.01)		4.89 (.00)	5.42 (.01)	6.74 (.01)
	Sparse	SFull	2.03 (.01)				.38 (.00)			
		STop1		2.13 (.01)	2.15 (.01)	2.20 (.01)		1.02 (.01)	1.19 (.01)	1.69 (.01)
		SsAvg		2.02 (.01)	1.82 (.01)	1.49 (.01)		3.12 (.00)	5.06 (.02)	6.06 (.01)
		SwAvg		1.99 (.01)	1.90 (.01)	1.68 (.01)		1.44 (.00)	1.95 (.00)	2.78 (.00)
$\boldsymbol{\Gamma}$	Debiased	Full	1.09 (.01)				10.85 (.01)			
		owAvg		1.11 (.01)	1.15 (.01)	1.23 (.01)		11.10 (.01)	11.44 (.01)	12.24 (.01)
		Top1		1.55 (.01)	1.63 (.01)	1.86 (.02)		15.43 (.02)	16.29 (.02)	18.54 (.02)
		sAvg		1.57 (.01)	2.02 (.02)	2.23 (.02)		15.67 (.02)	20.15 (.03)	22.23 (.03)
		wAvg		1.11 (.01)	1.16 (.01)	1.29 (.01)		11.12 (.01)	11.59 (.01)	12.89 (.02)
	Sparse	SFull	4.74 (.00)				1.93 (.00)			
		STop1		4.74 (.00)	4.74 (.00)	4.75 (.00)		1.99 (.00)	2.00 (.00)	2.04 (.00)
		SsAvg		4.75 (.00)	4.36 (.13)	3.97 (.03)		2.04 (.00)	2.27 (.05)	2.65 (.02)
		SwAvg		4.74 (.00)	4.61 (.04)	4.40 (.01)		1.93 (.00)	1.90 (.01)	1.93 (.00)

of the centralized full estimator, which implies that by considering just the first machine with the largest amount of data and disregarding the remaining machines one loses information as this strategy does not provide an accurate

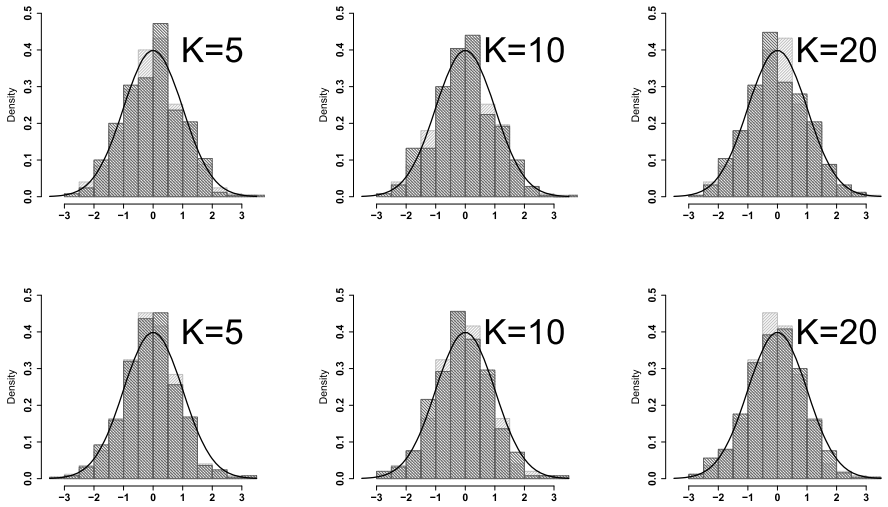


FIG 1. *Histograms of the normalized debiased full (light gray bars) and of the distributed estimator (dark gray bars), respectively, when* $n = 50000$, $(a, b) = (1, 3)$ *and* $K = 5$, 10 *or* 20. *From top to bottom, the figures present the asymptotic distributions for the estimation of* $\mathbf{\Gamma}$ *(top) and* $\mathbf{\Theta}$ *(bottom).*

estimate. Moreover, as it is expected, the performance of the sparse estimators is much better than the performance of the debiased estimators on the non-active set only, as they shrink most of the elements to zero. However, their errors are much larger than the errors of the debiased estimators on the active set as they do not correct for the bias.

Due to the asymptotic distribution of the proposed estimators, investigating their inferential properties is also of interest. However, since there is no distributional result available for the sparse estimators, it is not possible to perform inference using these estimators. Figure 1 shows the normalized distribution of the proposed debiased optimally weighted estimator and of the full non-distributed estimator for both $\mathbf{\Gamma}$ and $\mathbf{\Theta}$, respectively from top to bottom. As an illustrative example, these figures are reported for $(a, b) = (1, 3)$, others are available from the authors, but are similar. The light gray histograms correspond to the normalized distribution of the full estimators which are obtained from (3.4) and (4.6), by setting $K = 1$, for $\mathbf{\Gamma}$ and $\mathbf{\Theta}$, respectively. The dark gray histograms correspond to the asymptotic distributions of (5.8) and (5.9), respectively. The presented histograms confirm the asymptotic normal distribution of the proposed estimators and its similarity to the full one.

Using the asymptotic distribution of the estimators, the coverage probabilities and the length of the confidence intervals are presented in Table 2 at significance level $\alpha = .05$. Here, we explain the procedure for computing the empirical coverage probability for the elements of $\mathbf{\Theta}$. The same procedure is applied to compute the empirical coverage probability and the length of confidence intervals for the elements of $\mathbf{\Gamma}$. Similarly to [17], the empirical probability that the true parameter

TABLE 2
Average coverage probability and average length of the confidence intervals over 500 repetitions for the proposed estimators and different competitors, when $n = 50000$.

			Avg.Cov				Avg.Len			
			K				K			
			1	5	10	20	1	5	10	20
$\boldsymbol{\Theta}$	Active set	Full	.93				.02			
		owAvg		.93	.93	.92		.02	.02	.02
		Top1		.94	.94	.94		.02	.03	.02
		sAvg		.92	.85	.71		.02	.03	.04
		wAvg		.94	.96	.91		.02	.02	.03
	Non-active set	Full	.93				.02			
		owAvg		.95	.95	.95		.02	.02	.02
		Top1		.95	.95	.95		.02	.03	.05
		sAvg		.94	.92	.92		.02	.03	.04
		wAvg		.95	.97	.98		.02	.02	.03
$\boldsymbol{\Gamma}$	Active set	Full	.95				.06			
		owAvg		.95	.95	.95		.06	.06	.06
		Top1		.95	.95	.95		.09	.08	.09
		sAvg		.95	.94	.93		.09	.10	.10
		wAvg		.96	.96	.97		.06	.06	.07
	Non-active set	Full	.96				.08			
		owAvg		.95	.95	.95		.06	.06	.06
		Top1		.95	.95	.95		.09	.08	.09
		sAvg		.95	.94	.93		.09	.10	.10
		wAvg		.96	.96	.97		.06	.06	.07

$\boldsymbol{\Theta}_{ab}$ is included in the confidence interval is defined as $\hat{\mathbb{P}}_{ab} = \#\{\boldsymbol{\Theta}_{ab} \in \mathrm{CI}_{ab,r}\}/R$, where R is the number of repetitions in the simulation, $\mathrm{CI}_{ab,r}$ is the estimated confidence interval for $\boldsymbol{\Theta}_{ab}$ at the r-th repetition, and $\#$ denotes the number of times for which the true parameter $\boldsymbol{\Theta}_{ab}$ belongs to the confidence interval. After obtaining $\hat{\mathbb{P}}_{ab}$ for all $(a, b) \in \mathcal{V} \times \mathcal{V}, a \neq b$, the average coverage probability on the active set S_1 is obtained as $\mathrm{Avg.Cov}_{S_1} = (1/s_1) \sum_{(a,b)\in S_1} \hat{\mathbb{P}}_{ab}$, where s_1 is the number of active components. Similar computations have been implemented for obtaining the estimated coverage probability over the non-active set S_1^c.

From Table 2, it is observed that the performance of the owAvg estimator is close to the full one on both the active and non-active sets, as the coverage probabilities are close to the nominal level of 95%. Moreover, the average lengths are relatively low and are stable with increasing K. The results for Top1 are also close to the nominal level, but its length is slightly larger than that of the full and of the distributed estimator. However, in Table 1, we observed that its Frobenius norm performance is far away from the norm of the full estimator making it a less interesting alternative. The coverage probability of sAvg is low in some cases, especially in estimating $\boldsymbol{\Theta}$ on the active set. The length of its confidence interval is also relatively large, especially when estimating $\boldsymbol{\Gamma}$. The

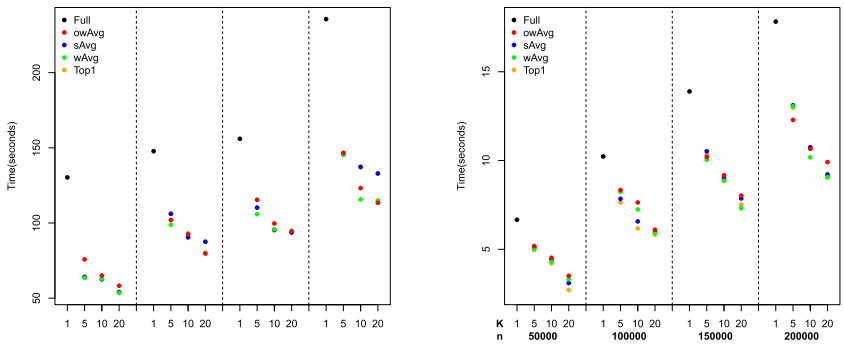


FIG 2. *Running time in seconds for the full and proposed estimators in estimating* $\mathbf{\Gamma}$ *(left) and* $\mathbf{\Theta}$ *(right), when* $p = 1100$ *and* $q = 550$. *The regularization parameters for the distributed estimators are considered as* $\lambda_k = \sqrt{\log(p)/n_k}$, $\rho_k = \sqrt{\log(pq)/n_k}$ *and* $\tilde{\rho}_k = \sqrt{\log(q)/n_k}$.

performance of wAvg is generally better than that of sAvg but over-coverage is observed. More than that, the length of its confidence interval is not stable and it increases with increasing K. The results for $n = 25000$ are similar and they are presented in Table 5 in Appendix C.

Another quantity which is important to keep track of, is the running time. In this paper, it is considered as the maximum running time among all parallel jobs plus the time to combine the results. These results are shown in Figure 2 for the debiased estimators of $\mathbf{\Gamma}$ and $\mathbf{\Theta}$ for different sample sizes from $n = 50000$ to 200000 and $K = 5, 10, 20$. Not surprisingly, the running times of the sparse estimators were at lower values than the running times of the debiased ones as they do not involve the debiasing step in the estimation procedure, and they are not reported in this figure. It is observed from Figure 2 that for any fixed sample size, the computation time of the distributed estimators is less than that of the full one and as expected, it decreases with increasing K. This running time is quite close for all owAvg, wAvg, sAvg and Top1 estimators. This behavior is the same for both estimators of $\mathbf{\Gamma}$ and $\mathbf{\Theta}$ and shows, as expected, the efficiency of the proposed estimators in terms of computation time.

7. Real data example

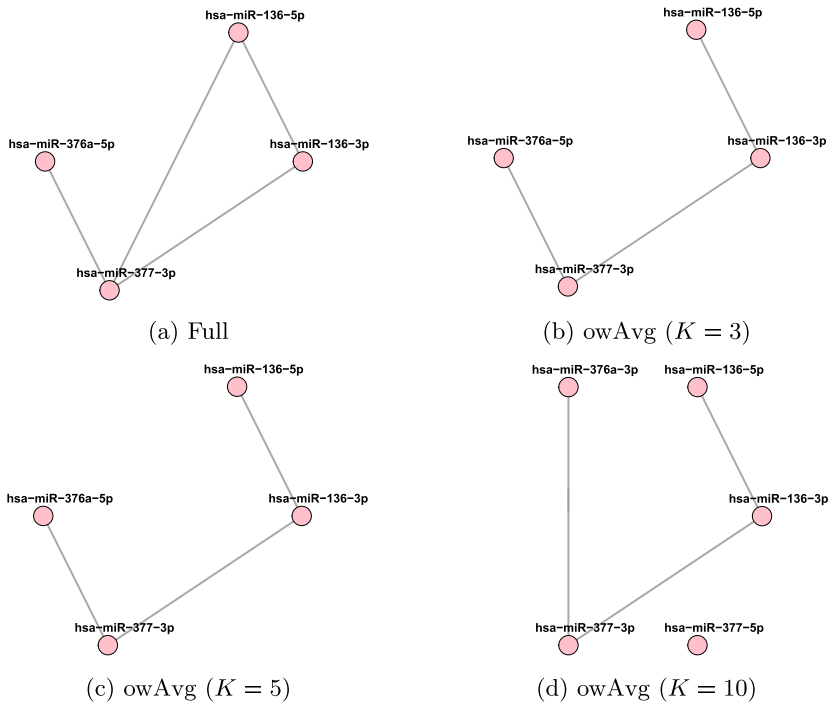
To explore the performance of the proposed methodology, we used the Pan-Cancer dataset from The Cancer Genome Atlas (TCGA) project (available at https://xenabrowser.net/datapages/) that fits perfectly the motivation setup presented in Section 1. This project was started in 2006 and in 10 years time, TCGA network investigators had characterized the molecular landscape of tumors from more than 11000 patients across 33 cancer types. This particular TCGA molecular dataset has been studied in multiple works to understand the cancer biology, including those of glioblastoma multiforme (GBM), ovarian, breast, lung, prostate, bladder and others (see for instance, [1, 23, 27]).

TABLE 3
Significant and common pairs between four competitors and the estimator using the full dataset. For all methods a Bonferroni correction is applied.

		Significant pairs				Percentage of common pairs with Full		
			K			K		
		1	3	5	10	3	5	10
$\mathbf{\Gamma}$	Full	964						
	owAvg		897	832	828	83	78	71
	Top1		300	290	295	26	25	19
	sAvg		817	466	234	76	43	23
	wAvg		888	948	1334	84	79	81
$\mathbf{\Theta}$	Full	6564						
	owAvg		6168	5917	5535	90	86	81
	Top1		3095	2966	2655	47	45	40
	sAvg		5234	2923	1914	78	44	29
	wAvg		6120	5463	4169	89	81	63

Although the estimation of the graph structure of genes can be effective in identifying the associated genetic variants, external covariates such as single nucleotide polymorphisms may affect their structure. As such, we applied the proposed conditional multivariate regression to regress 743 microRNA mature strand expressions on 27147 tumor gene-level copy numbers and to model how the gene expressions regulate the microRNA expressions. Estimation of the underlying graph among the microRNAs is also of interest. We further selected 1164 tumor gene expressions with empirical variance greater than 0.3, and the final dataset consists of $n = 9986$ subjects having both $q = 1164$ covariates and $p = 743$ responses.

To compare the performance of the proposed method, we split the dataset on $K = 3, 5$ and 10 different machines with the same splitting setting as in the simulation study in Section 6. Tuning parameters are also fixed as in the simulation study with $\alpha = 5\%$ and a Bonferroni correction is applied for multiple testing. The number of significant pairs and the percentage of the common pairs between the distributed procedures and the full one are presented in Table 3. It is observed that when estimating $\mathbf{\Gamma}$, the owAvg and wAvg estimators identify more common non-zero coefficients with the full estimator, while sAvg and Top1 are further away from the full estimator. When $K = 3$, sAvg is close to the full one, but with increasing K, it grows further apart, such that when $K = 10$, it identified only 234 coefficients of which 23% of them are common with the full estimator. The same holds for Top1 which identified much less coefficients compared to the full, owAvg and wAvg estimators. We conclude that there are more common edges between the graphs estimated by the distributed owAvg method and the full one, while the least similar competitors are Top1 and sAvg. With increasing K, the percentage of common edges tends to decrease for all competitors due to the loss of information incurred by splitting data on more machines.



(a) Full (b) owAvg ($K = 3$)

(c) owAvg ($K = 5$) (d) owAvg ($K = 10$)

FIG 3. *The estimated sub-graph between relevant genes for the GBM cancer. For all methods a Bonferroni correction is applied.*

As it is mentioned in [36], the genes hsa-miR-136, hsa-miR-376a and hsa-miR-377 are important genes in identifying GBM cancer. Figure 3 presents the estimated sub-graph between these genes using the owAvg and Full procedures after a Bonferroni correction. It is observed that owAvg procedure could identify a similar sub-graph to the full one. A similar sub-graph was also identified by the wAvg procedure. However, when $K = 10$, the identified sub-graph by wAvg procedure contained only two edges. The sAvg and Top1 procedures missed more edges, and they could identify only one edge namely between hsa-miR-136-3p and hsa-miR-136-5p. The sparse graphical Lasso estimator which completely ignores the impact of covariates on the mean structure of the responses is also considered as a competitor and with this method we identified 37376 edges suggesting that many estimated edges might in fact be false positive edges.

Afterwards, to investigate the performance of the estimators for completely independent variables, we permuted randomly sample data for each variable, thus breaking up the correlation structure in order to construct a dataset with mutually independent variables. As such, we expect that there are no selected variables in the estimated coefficient matrix and zero off-diagonal elements in the estimated precision matrix. By fitting separately the proposed owAvg estimator and all competitors on the dataset with $K = 3, 5$ and 10, we identified

no edges in the estimated precision matrix which confirms this assertion. However, a couple of non-zero coefficients were wrongly identified in the estimated coefficient matrix. The owAvg and sAvg estimators, both identified around 10 non-zero coefficients, while wAvg identified 84, which indicates more false positive discoveries produced by this estimator.

8. Discussion

Splitting a dataset on multiple locations with different sizes is an inevitable method to tackle the problem of large scale datasets which cannot be read nor stored in one single location. Separate analyses at multiple locations are also of contemporary interest due to today's security and privacy concerns. The essential step in distributed problems is choosing how to aggregate different estimators to a final one.

In this paper, aggregated estimators are introduced for the coefficient matrix in the multivariate regression models and the precision matrix corresponding to the graph structure of the response vector. To build these estimators, first debiased quantities were provided on each machine and then, they were pooled together to create the final estimators by maximizing a pseudo log-likelihood function which is constructed using the asymptotic distribution of the debiased estimators. Two special cases of the aggregated estimators, including simple and weighted averages, were provided under the known variance assumption. Statistical guarantees and the asymptotic distribution of the aggregated estimators were investigated under sparsity conditions and a growing number of machines as a function of the sample size.

It is shown that the aggregated estimators are asymptotically as efficient as the debiased, full non-distributed estimators and confirm the asymptotic negligibility of the efficiency loss in the distributed procedure. Moreover, based on the provided convergence rates, it is deduced that the aggregated estimators exhibit a similar statistical error as the debiased full estimator as long as the number of machine is not too large. In the estimation of the coefficient matrix, it is shown that as the number of machines grows at the rate $K = O\big(n^{1/4}/(\sqrt{\log(pq)\log(q)}\max\{s_2, \sqrt{d_2}\})\big)$, the final estimator is consistent and asymptotically normal. This growth rate adjusts to $K = O\big(n^{1/3}/(d_1\log(p))\big)$, in estimating the precision matrix. Statistical inference was also proposed based on the asymptotic normal distribution of the aggregated estimators. These distributions are valid as long as the number of machines grows at the mentioned rates.

The finite sample performance of the proposed estimators was evaluated with a simulation study where it was observed that the estimators produced competitive results relative to the non-distributed estimators that use the entire data. Since we perform estimation by distributing the computational load across multiple machines, not surprisingly the computational time comparison favors our novel estimator. Moreover, this estimator performed substantially better than the simple average-based estimator in terms of accuracy. It was also ob-

served that the coverage probabilities of the distributed estimators are close to those of the non-distributed estimators. This points to the fact that in practice, performing distributed estimation across multiple machines in our unbalanced framework induces a minimal loss in performance relative to models using all the data in a centralized location.

Appendix A: Proofs of the main theorems and lemmas

A.1. Proof of Theorem 1

Before starting the proof of Theorem 1, we provide a short description of the nodewise Lasso method from [34] using the k-th sub-sample, which is needed later in the proof.

For every $j \in \{1, \ldots, q\}$, use Lasso for the regression problem $\mathbf{X}_{j,k}$ against $\mathbf{X}_{-j,k}$, where $\mathbf{X}_{j,k}$ and $\mathbf{X}_{-j,k}$ are, the j-th column and the $n_k \times (q-1)$ dimensional sub-matrix of $\mathbf{X}_k$ obtained by removing the j-th column, respectively. Formally,

$$\hat{\boldsymbol{\eta}}_{j,k} = \arg\min_{\boldsymbol{\eta} \in \mathbb{R}^{q-1}} \left\{ \frac{1}{2n_k} \|\mathbf{X}_{j,k} - \mathbf{X}_{-j,k}\boldsymbol{\eta}\|_2^2 + \tilde{\rho}_{j,k}\|\boldsymbol{\eta}\|_1 \right\}, \tag{A.1}$$

with components $\hat{\boldsymbol{\eta}}_{j,k} = \{\hat{\eta}_{(j,j'),k}; j' = 1, \ldots, q, j' \neq j\}$, where $\hat{\eta}_{(j,j'),k}$ is the estimated regression coefficient associated to the j'-th column of $\mathbf{X}_k$ when column j is the response vector. Define

$$\hat{\boldsymbol{\Psi}}_k = \begin{pmatrix} 1 & -\hat{\eta}_{(1,2),k} & \cdots & -\hat{\eta}_{(1,q),k} \\ -\hat{\eta}_{(2,1),k} & 1 & \cdots & -\hat{\eta}_{(2,q),k} \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\eta}_{(q,1),k} & -\hat{\eta}_{(q,2),k} & \cdots & 1 \end{pmatrix}$$

and write

$$\hat{\boldsymbol{\Upsilon}}_k^2 = \operatorname{diag}(\hat{\tau}_{1,k}^2, \ldots, \hat{\tau}_{q,k}^2), \quad \hat{\tau}_{j,k}^2 = \|\mathbf{X}_{j,k} - \mathbf{X}_{-j,k}\hat{\boldsymbol{\eta}}_{j,k}\|_2^2/n_k + \tilde{\rho}_{j,k}\|\hat{\boldsymbol{\eta}}_{j,k}\|_1,$$

and finally, set

$$\mathbf{M}_k = \left[\hat{\boldsymbol{\Upsilon}}_k^2\right]^{-1}\hat{\boldsymbol{\Psi}}_k.$$

Now, to prove Gaussianity in Theorem 1, we mention that $\boldsymbol{\xi}_k$ is an $n_k \times p$ matrix with independent rows and distributed as $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$. As such, the random vector $\operatorname{vec}(\boldsymbol{\xi}_k)$ is distributed as a pn_k-dimensional Gaussian vector with mean zero and covariance matrix $\boldsymbol{\Sigma} \otimes \mathbf{I}_{n_k}$, which is a $pn_k \times pn_k$ matrix. Due to the properties of the $\operatorname{vec}(\cdot)$ function,

$$\mathbf{T}_k := \operatorname{vec}(\mathbf{M}_k\mathbf{X}_k^\top\boldsymbol{\xi}_k/\sqrt{n_k}) = (1/\sqrt{n_k})\big(\mathbf{I}_p \otimes (\mathbf{M}_k\mathbf{X}_k^\top)\big)\operatorname{vec}(\boldsymbol{\xi}_k).$$

As such, given $\mathbf{X}_k$, the random vector $\mathbf{T}_k$ is a linear transformation of $\operatorname{vec}(\boldsymbol{\xi}_k)$ and it is thus a pq-dimensional Gaussian vector with mean vector zero and covariance matrix

$$\mathbb{V}\mathrm{ar}(\mathbf{T}_k|\mathbf{X}_k) = (1/n_k)(\mathbf{I}_p \otimes (\mathbf{M}_k\mathbf{X}_k^\top))(\boldsymbol{\Sigma} \otimes \mathbf{I}_{n_k})(\mathbf{I}_p \otimes (\mathbf{X}_k\mathbf{M}_k^\top))$$

$$= \boldsymbol{\Sigma} \otimes (\mathbf{M}_k \mathbf{C}_k \mathbf{M}_k^\top).$$

To show negligibility of the remainder term $\mathbf{R}_{k,\boldsymbol{\Gamma}}$ in (3.3), we have

$$\begin{aligned} \|\mathbf{R}_{k,\boldsymbol{\Gamma}}\|_\infty &\leqslant \sqrt{n_k}\|\mathbf{I}_q - \mathbf{M}_k\mathbf{C}_k\|_\infty \|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_\infty \\ &\leqslant \sqrt{n_k}\|\mathbf{I}_q - \mathbf{M}_k\mathbf{C}_k\|_\infty \|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_1. \end{aligned} \tag{A.2}$$

Using the KKT conditions, [34] showed that $\|\mathbf{C}_k\mathbf{M}_{j,k}^\top - \mathbf{e}_j\|_\infty \leqslant \tilde{\rho}_{j,k}/\hat{\tau}_{j,k}^2$, where $\mathbf{M}_{j,k}$ is the j-th row of $\mathbf{M}_k$ and $\mathbf{e}_j$ is the j-th unit column vector with 1 at the j-th position and zero everywhere else. Under the assumptions (A1) and (A2) and considering the maximum row sparsity $d_2 = o(\sqrt{n_\dagger}/\log(q))$ in Lemma 5.3 of [34], we have

$$\max_{j\in\{1,\ldots,q\}} 1/\hat{\tau}_{j,k}^2 = O_p(1).$$

Therefore, by choosing uniformly $\tilde{\rho}_{j,k} \asymp \sqrt{\log(q)/n_k}$ for each $j = 1,\ldots,q$, we get

$$\|\mathbf{C}_k\mathbf{M}_k^\top - \mathbf{I}_q\|_\infty = \max_j \|\mathbf{C}_k\mathbf{M}_{j,k} - \mathbf{e}_j\|_\infty = O_p(\sqrt{\log(q)/n_k}). \tag{A.3}$$

Substituting (A.3) and (B.15) from Lemma 5 (see Appendix B) in (A.2), we get

$$\|\mathbf{R}_{k,\boldsymbol{\Gamma}}\|_\infty = O_p\big(s_2\sqrt{\log(q)\log(pq)/n_k}\big).$$

Finally, by considering $s_2 = o\big(n_\dagger^{\pi_1}/(\log(q)\log(pq))\big)$, $0 < \pi_1 \leqslant 1/2$, the required result follows. □

A.2. Proof of Theorem 2

Using (3.3), we have that,

$$\|\hat{\boldsymbol{\Gamma}}_k^d - \boldsymbol{\Gamma}\|_\infty \leqslant \|\mathrm{vec}(\mathbf{M}_k\mathbf{X}_k^\top\boldsymbol{\xi}_k)\|_\infty/n_k + \|\mathbf{R}_{k,\boldsymbol{\Gamma}}\|_\infty/\sqrt{n_k}. \tag{A.4}$$

Due to the properties of the vec(·) function,

$$\|\mathrm{vec}(\mathbf{M}_k\mathbf{X}_k^\top\boldsymbol{\xi}_k)\|_\infty/n_k \leqslant |||\mathbf{I}_p \otimes \mathbf{M}_k|||_\infty \|\mathrm{vec}(\mathbf{X}_k^\top\boldsymbol{\xi}_k)\|_\infty/n_k = O_p(\sqrt{d_2\log(pq)/n_k}), \tag{A.5}$$

where the last equality is obtained by (i) working on the event $\mathcal{F}_k(n_k,p,q)$ with $\rho_k \asymp \sqrt{\log(pq)/n_k}$, and (ii) by the same argument as in the proof of Lemma 5.4 from [34], as

$$|||\mathbf{I}_p \otimes \mathbf{M}_k|||_\infty = |||\mathbf{M}_k|||_\infty = \max_{j\in\{1,\ldots,q\}} \|\mathbf{M}_{j,k}\|_1 = O_p(\sqrt{d_2}),$$

where $\mathbf{M}_{j,k}$ is the j-th row of $\mathbf{M}_k$. Under assumptions (A1) and (A2) and using Theorem 1, by substituting (3.4) and (A.5) in (A.4), the result in (3.5) follows directly. □

A.3. Proof of Lemma 1

Before starting the proof of this Lemma, we provide some technical assumptions on the sample covariance matrix $\mathbf{C}_k$, which are needed later in the proof. For more information on these assumptions, the reader can refer to [38].

(C1) There exists an $\alpha_2 \in (0, 1]$, such that

$$\sup_{b\in\{1,\ldots,p\}} \|\!|(\mathbf{C}_k)_{S_2^c(b)S_2(b)}\big[(\mathbf{C}_k)_{S_2(b)S_2(b)}\big]^{-1}|\!\|_\infty \leqslant 1-\alpha_2,$$

where $(\mathbf{C}_k)_{S_2^c(b)S_2(b)}$ is a sub-matrix of $\mathbf{C}_k$ whose rows and columns are indexed by the elements of $S_2^c(b)$ and $S_2(b)$, respectively, where $S_2^c(b)$ is the complement set of $S_2(b)$, defined in Section 2. As it is mentioned in [38], this condition is the matrix version of the irrepresentability condition used in the ℓ_1 penalized regression setting of [41].

(C2) There exists a constant $C_{\max}$, such that the largest eigenvalue

$$\Lambda_{\max}\left(\big[(\mathbf{C}_k\otimes\mathbf{I}_p)_{S_2S_2}\big]^{-1}(\mathbf{C}_k\otimes\boldsymbol{\Sigma})_{S_2S_2}\big[(\mathbf{C}_k\otimes\mathbf{I}_p)_{S_2S_2}\big]^{-1}\right)\leqslant C_{\max}.$$

(C3) For all $n_k > 0$, the largest eigenvalue of $\mathbf{C}_k$ has a common upper bound Λ_3, that is $\Lambda_{\max}(\mathbf{C}_k)\leqslant\Lambda_3$.

Now to prove Lemma 1, using (4.4), we have

$$\begin{aligned}\|\mathbf{R}_{k,\boldsymbol{\Theta}}\|_\infty &= \sqrt{n_k}\|-(\hat{\boldsymbol{\Theta}}_k\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\mathbf{I}_p)(\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta})-(\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta})\mathbf{W}_k'\boldsymbol{\Theta}-\boldsymbol{\Theta}\mathbf{W}_k''\boldsymbol{\Theta}\|_\infty\\ &\leqslant \sqrt{n_k}\|\hat{\boldsymbol{\Theta}}_k\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\mathbf{I}_p\|_\infty\|\!|\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta}|\!\|_\infty+\sqrt{n_k}\|\!|\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta}|\!\|_\infty\|\mathbf{W}_k'\boldsymbol{\Theta}\|_\infty\\ &\quad+\sqrt{n_k}\|\boldsymbol{\Theta}\mathbf{W}_k''\boldsymbol{\Theta}\|_\infty.\end{aligned}\tag{A.6}$$

To find an upper bound on $\|\hat{\boldsymbol{\Theta}}_k\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\mathbf{I}_p\|_\infty$, we write

$$\begin{aligned}\|\hat{\boldsymbol{\Theta}}_k\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\mathbf{I}_p\|_\infty &= \|\hat{\boldsymbol{\Theta}}_k\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\hat{\boldsymbol{\Theta}}_k\boldsymbol{\Sigma}+\hat{\boldsymbol{\Theta}}_k\boldsymbol{\Sigma}-\mathbf{I}_p\|_\infty\\ &= \|\hat{\boldsymbol{\Theta}}_k(\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\boldsymbol{\Sigma})+\hat{\boldsymbol{\Theta}}_k\boldsymbol{\Sigma}-\boldsymbol{\Theta}\boldsymbol{\Sigma}+\boldsymbol{\Theta}\boldsymbol{\Sigma}-\mathbf{I}_p\|_\infty\\ &= \|\hat{\boldsymbol{\Theta}}_k(\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\boldsymbol{\Sigma})+(\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta})\boldsymbol{\Sigma}+\boldsymbol{\Theta}\boldsymbol{\Sigma}\\ &\quad-\boldsymbol{\Theta}\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}+\boldsymbol{\Theta}\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\mathbf{I}_p\|_\infty\\ &= \|\hat{\boldsymbol{\Theta}}_k(\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\boldsymbol{\Sigma})+(\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta})\boldsymbol{\Sigma}-\boldsymbol{\Theta}(\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\boldsymbol{\Sigma})\\ &\quad+\boldsymbol{\Theta}\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}-\boldsymbol{\Theta}\boldsymbol{\Sigma}\|_\infty\\ &\leqslant \|\!|\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta}|\!\|_\infty\|\mathbf{W}_k'\|_\infty+\|\hat{\boldsymbol{\Theta}}_k-\boldsymbol{\Theta}\|_\infty\|\!|\boldsymbol{\Sigma}|\!\|_\infty+\|\boldsymbol{\Theta}\mathbf{W}_k'\|_\infty.\end{aligned}\tag{A.7}$$

Substituting (A.7) in (A.6), we have

$$\|\mathbf{R}_{k,\boldsymbol{\Theta}}\|_\infty\leqslant\sqrt{n_k}\big\{2\|\mathbf{R}^1_{k,\boldsymbol{\Theta}}\|_\infty+\|\mathbf{R}^2_{k,\boldsymbol{\Theta}}\|_\infty+\|\mathbf{R}^3_{k,\boldsymbol{\Theta}}\|_\infty+\|\mathbf{R}^4_{k,\boldsymbol{\Theta}}\|_\infty\big\},\tag{A.8}$$

where

$$\begin{aligned}
\|\mathbf{R}^1_{k,\boldsymbol{\Theta}}\|_\infty &= \|\boldsymbol{\Theta}\mathbf{W}'_k\|_\infty \|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|_\infty, \\
\|\mathbf{R}^2_{k,\boldsymbol{\Theta}}\|_\infty &= \|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|^2_\infty \|\mathbf{W}'_k\|_\infty, \\
\|\mathbf{R}^3_{k,\boldsymbol{\Theta}}\|_\infty &= \|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|_\infty \|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|_\infty \kappa_{\boldsymbol{\Sigma}}, \\
\|\mathbf{R}^4_{k,\boldsymbol{\Theta}}\|_\infty &= \|\boldsymbol{\Theta}\mathbf{W}''_k\boldsymbol{\Theta}\|_\infty.
\end{aligned}$$

Under assumption (B1) from Section 4 for the matrix $\boldsymbol{\Theta}$, it holds that

$$\|\boldsymbol{\Theta}\|_\infty = \max_{a\in\mathcal{V}} \|\boldsymbol{\Theta}_a\|_1 \leqslant \sqrt{d_1}\Lambda_{\max}(\boldsymbol{\Theta}) = O(\sqrt{d_1}), \tag{A.9}$$

where $\boldsymbol{\Theta}_a$ is the a-th row of $\boldsymbol{\Theta}$. Under (B2) and the additional assumptions (C1)–(C3), the conditions of result 1 from Theorem 2 of [38] are fulfilled, and for the k-th sub-sample we have

$$\|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|_\infty = O_p\left(\{16\sqrt{2}(1+4\gamma^2)(1+8/\alpha_1)\max_a \boldsymbol{\Sigma}_{aa}\kappa_{\mathbf{H}}\}\sqrt{\frac{\log(4p^\tau)}{n_k}}\right), \tag{A.10}$$

where $\max_a \boldsymbol{\Sigma}_{aa}$ is the maximal diagonal element of the covariance matrix $\boldsymbol{\Sigma}$, $\tau > 2$ is a constant and γ is a common sub-Gaussian parameter for Gaussian random variables $\varepsilon^1, \ldots, \varepsilon^p$, where $\gamma = O(1)$. Moreover, based on result 2 of the aforementioned theorem, the edge set $\mathcal{E}(\hat{\boldsymbol{\Theta}}_k)$, i.e. the edge set created based on the estimated $\hat{\boldsymbol{\Theta}}_k$, is a subset of the true edge set $\mathcal{E}(\boldsymbol{\Theta})$ with high probability. As such, $\hat{\boldsymbol{\Theta}}_k$ has at most d_1 non-zero entries per row, and we get

$$\begin{aligned}
\|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|_\infty = \max_a \sum_{b=1}^{p} |\hat{\boldsymbol{\Theta}}_{ab,k} - \boldsymbol{\Theta}_{ab}| &= \max_a \sum_{b=1}^{d_1} |\hat{\boldsymbol{\Theta}}_{ab,k} - \boldsymbol{\Theta}_{ab}| \\
&\leqslant \max_a \sum_{b=1}^{d_1} \max_b |\hat{\boldsymbol{\Theta}}_{ab,k} - \boldsymbol{\Theta}_{ab}| \\
&= d_1 \|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|_\infty,
\end{aligned}$$

where $\hat{\boldsymbol{\Theta}}_{ab,k}$ is the (a,b)-th element of $\hat{\boldsymbol{\Theta}}_k$. Thus, using (A.10),

$$\|\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\|_\infty \leqslant O_p\left(d_1\{16\sqrt{2}(1+4\gamma^2)(1+8/\alpha_1)\max_a \boldsymbol{\Sigma}_{aa}\kappa_{\mathbf{H}}\}\sqrt{\frac{\log(4p^\tau)}{n_k}}\right).$$

Therefore in (A.8), we have

$$\|\mathbf{R}_{k,\boldsymbol{\Theta}}\|_\infty \leqslant 5\sqrt{n_k}\max\left\{\|\mathbf{R}^1_{k,\boldsymbol{\Theta}}\|_\infty, \|\mathbf{R}^2_{k,\boldsymbol{\Theta}}\|_\infty, \|\mathbf{R}^3_{k,\boldsymbol{\Theta}}\|_\infty, \|\mathbf{R}^4_{k,\boldsymbol{\Theta}}\|_\infty\right\}, \tag{A.11}$$

where

$$\|\mathbf{R}^1_{k,\boldsymbol{\Theta}}\|_\infty \leqslant \sqrt{d_1}\Lambda_{\max}(\boldsymbol{\Theta})\|\mathbf{W}'_k\|_\infty$$

$$\times O_p\Big(d_1\big\{16\sqrt{2}(1+4\gamma^2)(1+8/\alpha_1)\max_a \boldsymbol{\Sigma}_{aa}\kappa_{\mathbf{H}}\big\}\sqrt{\frac{\log(4p^\tau)}{n_k}}\Big),$$

$$\|\mathbf{R}^2_{k,\boldsymbol{\Theta}}\|_\infty \leqslant \|\mathbf{W}'_k\|_\infty O_p\Big(d_1^2\big\{16\sqrt{2}(1+4\gamma^2)(1+8/\alpha_1)\max_a \boldsymbol{\Sigma}_{aa}\kappa_{\mathbf{H}}\big\}^2\frac{\log(4p^\tau)}{n_k}\Big),$$

$$\|\mathbf{R}^3_{k,\boldsymbol{\Theta}}\|_\infty \leqslant \kappa_{\boldsymbol{\Sigma}} O_p\Big(d_1\big\{16\sqrt{2}(1+4\gamma^2)(1+8/\alpha_1)\max_a \boldsymbol{\Sigma}_{aa}\kappa_{\mathbf{H}}\big\}^2\frac{\log(4p^\tau)}{n_k}\Big).$$

To obtain an upper bound on $\|\mathbf{W}'_k\|_\infty$, under assumptions (C1)–(C3), and using Lemma 2 of [38], we can write

$$\|\mathbf{W}'_k\|_\infty = O_p\Big(\sqrt{\frac{\log(4p^\tau)}{C_2 n_k}}\Big),$$

where $C_2 = [128(1+4\gamma^2)^2\max_a^2 \boldsymbol{\Sigma}_{aa}]^{-1}$. Moreover,

$$\|\mathbf{R}^4_{k,\boldsymbol{\Theta}}\|_\infty = \|\boldsymbol{\Theta}\big\{(\mathbf{Y}_k - \mathbf{X}_k\hat{\boldsymbol{\Gamma}}_k)^\top(\mathbf{Y}_k - \mathbf{X}_k\hat{\boldsymbol{\Gamma}}_k)/n_k - \boldsymbol{\xi}_k^\top\boldsymbol{\xi}_k/n_k\big\}\boldsymbol{\Theta}\|_\infty.$$

Adding and subtracting $\mathbf{X}_k\boldsymbol{\Gamma}$ to the first term in $\mathbf{R}^4_{k,\boldsymbol{\Theta}}$, we get that

$$\|\mathbf{R}^4_{k,\boldsymbol{\Theta}}\|_\infty \leqslant \|\boldsymbol{\Theta}\|^2_\infty\Big\{2\|(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top\mathbf{X}_k^\top\boldsymbol{\xi}_k\|_\infty/n_k + \|\mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\|_F^2/n_k\Big\}.$$

Under assumption (B1) and by conditioning on the event $\mathcal{F}_k(n_k, p, q)$, we have

$$\|\mathbf{R}^4_{k,\boldsymbol{\Theta}}\|_\infty \leqslant d_1\big\{\rho_k\|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_1 + \|\mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\|_F^2/n_k\big\}.$$

Recall that S_2 and $S_2(b)$ denote the support of the coefficient matrix $\boldsymbol{\Gamma}$ and its b-th column, $b = 1, \ldots, p$, with cardinalities s_2 and $s_2(b)$, respectively. Under assumption (C1), by a similar argument as in the proof of Theorem 6.1 of [5], one can show that for the fixed design matrix $\mathbf{X}_k$,

$$\rho_k\|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_1 + \|\mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\|_F^2/n_k \leqslant 4\rho_k^2 s_2/\phi^2.$$

where $\phi^2 \in (0, \infty)$ is the compatibility constant which has a similar role to μ_b from the RE condition (B.6) for the fixed design matrix $\mathbf{X}_k$. It is noteworthy that, as it is shown in Theorem 7.2 of [5], under the fixed design setting, the irrepresentability condition (C1) is enough to reach the compatibility condition and the restricted eigenvalue condition (B.6) is not needed. The reader can refer to Chapter 7 of [5] for more details. Since $\phi^2 \in (0, \infty)$, there exists $L = O(1)$, such that $1/\phi^2 \leqslant L$. Combining this with $\rho_k \asymp \sqrt{\log(pq)/n_k}$, and substituting in $\mathbf{R}^4_{k,\boldsymbol{\Theta}}$, we get

$$\|\mathbf{R}^4_{k,\boldsymbol{\Theta}}\|_\infty = O_p(d_1 s_2 \log(pq)/n_k).$$

Substituting the obtained bounds in (A.11) and considering $\kappa_{\boldsymbol{\Sigma}} = O(1)$, $\kappa_{\mathbf{H}} = O(1)$, $1/\alpha_1 = O(1)$, and $\gamma = O(1)$, we get

$$\|\mathbf{R}_{k,\boldsymbol{\Theta}}\|_\infty = O_p\Big(\sqrt{n_k}\max\big\{d_1^{3/2}\log(p)/n_k, d_1^2(\log(p)/n_k)^{3/2}, d_1\log(p)/n_k, d_1 s_2\log(pq)/n_k\big\}\Big),$$

and under the sparsity assumptions $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$ and $s_2 = o(n_\dagger^{\pi_3}/\log(pq))$, $0 < \pi_3 \leqslant 1/6$ the result $\|\mathbf{R}_{k,\boldsymbol{\Theta}}\|_\infty = o_p(1)$ follows. □

A.4. Proof of Theorem 3

The proof of this theorem is an extension of the proof of Theorem 1 from [17] by considering a non-zero mean structure. Under the assumptions of Lemma 1 from the main text, it is shown that $\|\mathbf{R}_{k,\boldsymbol{\Theta}}\|_\infty = o_p(1)$. As such, using (4.3), we have

$$\begin{aligned}\sqrt{n_k}(\hat{\boldsymbol{\Theta}}^d_{ab,k} - \boldsymbol{\Theta}_{ab}) &= -\sqrt{n_k}\{\boldsymbol{\Theta}\mathbf{W}_k\boldsymbol{\Theta}\}_{ab} + o_p(1)\\ &= -\frac{1}{\sqrt{n_k}}\sum_{l=1}^{n_k}\big(\boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\boldsymbol{\Theta}_b\\ &\quad - \boldsymbol{\Theta}_{ab}\big) + o_p(1), \qquad \text{(A.12)}\end{aligned}$$

where $\dot{\mathbf{Y}}_{l,k} \in \mathbb{R}^p$ and $\dot{\mathbf{X}}_{l,k} \in \mathbb{R}^q$ are the l-th row of the sub-matrices $\mathbf{Y}_k$ and $\mathbf{X}_k$, respectively, and $\boldsymbol{\Theta}_a$ and $\boldsymbol{\Theta}_b$ are p-dimensional vectors coming from the a-th row and the b-th row of $\boldsymbol{\Theta}$. The second equality in (A.12) follows directly since $\boldsymbol{\Theta}\mathbf{W}_k\boldsymbol{\Theta} = \boldsymbol{\Theta}\hat{\boldsymbol{\Sigma}}_{k,\boldsymbol{\Gamma}}\boldsymbol{\Theta} - \boldsymbol{\Theta}$. To show the normality of the term in (A.12), define $\mathbf{Z}_{ab,l,k} := \boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\boldsymbol{\Theta}_b - \boldsymbol{\Theta}_{ab}$, $l = 1, \ldots, n_k$. Then,

$$\begin{aligned}\mathbb{E}(\mathbf{Z}_{ab,l,k}) &= \boldsymbol{\Theta}_a^\top\mathbb{E}\big\{(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\big\}\boldsymbol{\Theta}_b - \boldsymbol{\Theta}_{ab}\\ &= \mathbf{e}_a^\top\boldsymbol{\Theta}\mathbf{e}_b - \boldsymbol{\Theta}_{ab} = 0,\end{aligned}$$

where $\mathbf{e}_b$ is a p-dimensional unit column vector of zeros with one at position b and similarly for $\mathbf{e}_a$. On the other hand, since $\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k}$ is a Gaussian random vector, the variance

$$\sigma^2_{ab} = \mathbb{V}\mathrm{ar}(\mathbf{Z}_{ab,l,k}) = \mathbb{V}\mathrm{ar}\big(\boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\boldsymbol{\Theta}_b\big)$$

is finite. Denote by $\mathcal{Z}_{n_k} = \sum_{l=1}^{n_k}\mathbf{Z}_{ab,l,k}$. Then, $z^2_{n_k} := \mathbb{V}\mathrm{ar}(\mathcal{Z}_{n_k}) = n\sigma^2_{ab}$. Dividing (A.12) by $\sigma_{ab} > 0$, we get

$$\sqrt{n_k}(\hat{\boldsymbol{\Theta}}^d_{ab,k} - \boldsymbol{\Theta}_{ab})/\sigma_{ab} = \mathcal{Z}_{n_k}/z_{n_k} + o_p(1)/\sigma_{ab}.$$

It is enough to show that $\mathcal{Z}_{n_k}/z_{n_k} \xrightarrow{d} \mathcal{N}(0,1)$, where $\xrightarrow{d}$ denotes convergence in distribution. By substituting $\mathbf{Z}_{ab,l,k}$ in the proof of Theorem 1 from [17], with the same argument, the normality of $\mathcal{Z}_{n_k}/z_{n_k}$ follows. The reader can refer to [17] for more details.

To show $\sigma^2_{ab} = \boldsymbol{\Theta}_{aa}\boldsymbol{\Theta}_{bb} + \boldsymbol{\Theta}^2_{ab}$, under the multivariate Gaussian distribution of $\dot{\boldsymbol{\varepsilon}}_{l,k}$, which is the l-th row of $\boldsymbol{\xi}_k$, we have $\dot{\boldsymbol{\varepsilon}}_{l,k} = \dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k} \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$, and as such $\boldsymbol{\Theta}(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k}) \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{\Theta})$. Using a similar argument as in the proof of Lemma 2 from [17], it holds that $\sigma^2_{ab} = \boldsymbol{\Theta}_{aa}\boldsymbol{\Theta}_{bb} + \boldsymbol{\Theta}^2_{ab}$ and $1/\sigma_{ab} = O(1)$. □

A.5. Proof of Theorem 4

a) The proof of this theorem relies on Lemma 7, which shows the negligibility of the remainder term $\mathbf{R}_{a,\boldsymbol{\Gamma}}$ as $n_\dagger$ and K grow. To show the asymptotic normality

of $\mathbf{W}_{a,\mathbf{\Gamma}}$, define

$$\zeta_a := \sum_{k=1}^{K} \sqrt{\frac{n_k[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{n[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \times \frac{\mathbf{T}_{a,k}}{\sqrt{[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}},$$

where $\mathbf{T}_{a,k}$ is the a-th element of $\mathbf{T}_k$ from (3.3). As $\sqrt{\frac{\sum_{k=1}^K n_k/[\mathbf{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^K n_k/[\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \xrightarrow{p} 1$, see the proof of Lemma 7, using Slutsky's theorem, it is enough to show that ζ_a converges in distribution to $\mathcal{N}(0,1)$. Defining $\zeta_a' := \sum_{k=1}^K \sqrt{\frac{n_k}{n}} \times \frac{\mathbf{T}_{a,k}}{\sqrt{[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}$, we show in Lemma 8 that $|\zeta_a - \zeta_a'| \xrightarrow{p} 0$ as $K\to\infty$ and $n_k\to\infty$, $k=1,\dots,K$, and then convergence in distribution of ζ_a follows by convergence in distribution of ζ_a'.

Now to show the asymptotic normal distribution of ζ_a', denoting by $\mathbf{Z}_{a,k} := \sqrt{\frac{n_k}{n}} \times \frac{\mathbf{T}_{a,k}}{\sqrt{[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}$, we use the Lindeberg theorem (see for instance, Theorem 2.1 in Chapter 7 of [15]). We have

$$\begin{aligned}\mathbb{E}(\mathbf{Z}_{a,k}) &= \mathbb{E}\left\{\mathbb{E}\left(\sqrt{\frac{n_k}{n}} \times \frac{\mathbf{T}_{a,k}}{\sqrt{[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \mid \mathbf{X}_k\right)\right\} \\ &= \mathbb{E}\left\{\sqrt{\frac{n_k}{n}} \times \frac{1}{\sqrt{[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}\mathbb{E}\big(\mathbf{T}_{a,k} \mid \mathbf{X}_k\big)\right\} = 0,\end{aligned}$$

where the last equality is deduced from the conditional normal distribution of $\mathbf{T}_{a,k}$ shown in Theorem 1. Moreover,

$$\sum_{k=1}^{K}\mathbb{E}(\mathbf{Z}_{a,k}^2) = \sum_{k=1}^{K}\mathbb{E}\left\{\frac{n_k}{n} \times \frac{1}{[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\mathbb{E}\big(\mathbf{T}_{a,k}^2 \mid \mathbf{X}_k\big)\right\} = \sum_{k=1}^{K}\frac{n_k}{n} = 1,$$

where the second equality is deduced using the conditional variance of $\mathbf{T}_{a,k}$, which is equal to $[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}$. Now to check the Lindeberg condition, for every $\epsilon>0$, we can write

$$\begin{aligned}&\mathbb{E}\big(\mathbf{Z}_{a,k}^2\mathbb{I}(|\mathbf{Z}_{a,k}|>\epsilon) \mid \mathbf{X}_k\big) \\ &= \frac{n_k}{n[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\mathbb{E}\left\{\mathbf{T}_{a,k}^2\mathbb{I}\big(|\mathbf{T}_{a,k}|>\epsilon\sqrt{\frac{n[\mathbf{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}{n_k}}\big) \mid \mathbf{X}_k\right\},\end{aligned} \tag{A.13}$$

where $\mathbb{I}(\cdot)$ is the indicator function. Following [17], p. 1223, for a random variable X and a positive constant a, one can write

$$\mathbb{E}\big(X^2\mathbb{I}(|X|>a)\big) = a^2\mathbb{P}(|X|>a) + 2\int_a^\infty u\mathbb{P}(|X|>u)\,du.$$

Applying this equality to (A.13), we get

$$\begin{aligned}
&\mathbb{E}\big(\mathbf{Z}_{a,k}^2\mathbb{I}(|\mathbf{Z}_{a,k}|>\epsilon)\mid \mathbf{X}_k\big)\\
&\quad=\epsilon^2\mathbb{P}\bigg(|\mathbf{T}_{a,k}|>\epsilon\sqrt{\frac{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}{n_k}}\mid \mathbf{X}_k\bigg)\\
&\qquad+\frac{2n_k}{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\int_{\epsilon\sqrt{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}/n_k}}^{\infty}u\mathbb{P}(|\mathbf{T}_{a,k}|>u\mid\mathbf{X}_k)du.
\end{aligned}\tag{A.14}$$

Applying the concentration inequality $\mathbb{P}(|X-\mu|>t)\leqslant 2e^{-\frac{t^2}{2\sigma^2}}$, $t\in\mathbb{R}$, for a normal random variable X with mean μ and variance σ^2, we get

$$\mathbb{P}\bigg(|\mathbf{T}_{a,k}|>\epsilon\sqrt{\frac{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}{n_k}}\mid\mathbf{X}_k\bigg)\leqslant 2e^{-\frac{n\epsilon^2}{2n_k}}.$$

In the integral, by substituting $t:=\frac{\sqrt{n_k}u}{\epsilon\sqrt{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}$, we have

$$\begin{aligned}
&\int_{\epsilon\sqrt{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}/n_k}}^{\infty}u\mathbb{P}(|\mathbf{T}_{a,k}|>u\mid\mathbf{X}_k)\,du\\
&=\frac{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}\epsilon^2}{n_k}\int_1^\infty t\mathbb{P}\bigg(|\mathbf{T}_{a,k}|>\epsilon t\sqrt{\frac{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}{n_k}}\mid\mathbf{X}_k\bigg)dt\\
&\leqslant\frac{n[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}\epsilon^2}{n_k}\int_1^\infty 2te^{-\frac{n\epsilon^2t^2}{2n_k}}\,dt=2[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}e^{-\frac{n\epsilon^2}{2n_k}}.
\end{aligned}$$

As such, in (A.14),

$$\mathbb{E}\big(\mathbf{Z}_{a,k}^2\mathbb{I}(|\mathbf{Z}_{a,k}|>\epsilon)\mid\mathbf{X}_k\big)\leqslant 2e^{-\frac{n\epsilon^2}{2n_k}}\{\epsilon^2+2n_k/n\}.$$

Thus, in the Lindeberg condition, we get

$$\begin{aligned}
&\lim_{K\to\infty}\lim_{\substack{n_k\to\infty\\k=1,\dots,K}}\sum_{k=1}^K\mathbb{E}\Big\{\mathbf{Z}_{a,k}^2\mathbb{I}\big(|\mathbf{Z}_{a,k}|>\epsilon\big)\Big\}\\
&\quad=\lim_{K\to\infty}\lim_{\substack{n_k\to\infty\\k=1,\dots,K}}\sum_{k=1}^K\mathbb{E}\Big\{\mathbb{E}\big(\mathbf{Z}_{a,k}^2\mathbb{I}(|\mathbf{Z}_{a,k}|>\epsilon)\mid\mathbf{X}_k\big)\Big\}\\
&\quad\leqslant\lim_{K\to\infty}\lim_{\substack{n_k\to\infty\\k=1,\dots,K}}\sum_{k=1}^K 2e^{-\frac{n\epsilon^2}{2n_k}}\{\epsilon^2+2n_k/n\}\\
&\quad\leqslant\lim_{K\to\infty}\lim_{\substack{n_k\to\infty\\k=1,\dots,K}}\sum_{k=1}^K 2e^{-\frac{Kn_t\epsilon^2}{2n}}\{\epsilon^2+2n_k/n\}
\end{aligned}$$

$$= \lim_{K\to\infty} \left\{ \left(\lim_{\substack{n_k\to\infty \\ k=1,\ldots,K}} 2e^{-\frac{Kn_\dagger\epsilon^2}{2n}} \right) \left(\lim_{\substack{n_k\to\infty \\ k=1,\ldots,K}} \sum_{k=1}^{K} \{\epsilon^2 + 2n_k/n\} \right) \right\}$$
$$\leqslant \lim_{K\to\infty} \big(2\epsilon^2 + 4\big) K e^{-Kc_\dagger\epsilon^2/2} = 0,$$

where the last inequality is deduced due to the fact that $n_k < n$, $k = 1, \ldots, K$, and the last equality follows by l'Hôpital's rule.

b) By basic algebra it can be shown that

$$\frac{\sqrt{nK}}{\sqrt{\sum_{k=1}^{K}[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \big(\mathrm{vec}(\tilde{\boldsymbol{\Gamma}}_{\mathrm{wAvg}})_a - \mathrm{vec}(\boldsymbol{\Gamma})_a\big) = \mathbf{W}'_{a,\boldsymbol{\Gamma}} + \mathbf{R}'_{a,\boldsymbol{\Gamma}},$$

where

$$\mathbf{W}'_{a,\boldsymbol{\Gamma}} = \sqrt{\frac{K[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^{K}[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \sum_{k=1}^{K} \sqrt{\frac{n_k}{n}} \frac{1}{\sqrt{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}} \mathbf{T}_{a,k},$$
$$\mathbf{R}'_{a,\boldsymbol{\Gamma}} = \sqrt{\frac{K[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^{K}[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \sum_{k=1}^{K} \sqrt{\frac{n_k}{n}} \frac{1}{\sqrt{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}} \mathbf{R}_{a,k,\boldsymbol{\Gamma}},$$

where $\mathbf{T}_{a,k}$ and $\mathbf{R}_{a,k,\boldsymbol{\Gamma}}$ are respectively the a-th element of $\mathbf{T}_k$ and $\mathbf{R}_{k,\boldsymbol{\Gamma}}$ defined in (3.3). Consider the random variable $\zeta_a = \frac{1}{\sqrt{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}} \sum_{k=1}^{K} \sqrt{\frac{n_k}{n}} \mathbf{T}_{a,k}$. In part a), it is shown that $\zeta'_a := \sum_{k=1}^{K} \sqrt{\frac{n_k}{n}} \times \frac{1}{\sqrt{[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \mathbf{T}_{a,k}$ converges in distribution to $\mathcal{N}(0,1)$. On the other hand, one can easily show that $|\zeta_a - \zeta'_a| = O_p\big(Kd_2\sqrt{\log(pq)\log(q)/n}\big)$, and then the $o_p(1)$ result follows by the mentioned sparsity condition on d_2 and with $K = O\big(n^{1/3}/(\sqrt{\log(pq)\log(q)}\max\{s_2, d_2\})\big)$. As such, the asymptotic normality of ζ_a follows directly. Moreover, the $o_p(1)$ result of the remainder term $\mathbf{R}'_{a,\boldsymbol{\Gamma}}$ is reached by the same technique as in part a).

c) By basic algebra it can be shown that

$$\frac{K^{3/2}}{\sqrt{(\sum_{k=1}^{K}[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa})(\sum_{k=1}^{K} 1/n_k)}} \big(\mathrm{vec}(\tilde{\boldsymbol{\Gamma}}_{\mathrm{sAvg}})_a - \mathrm{vec}(\boldsymbol{\Gamma})_a\big)$$
$$= \mathbf{W}''_{a,\boldsymbol{\Gamma}} + \mathbf{R}''_{a,\boldsymbol{\Gamma}},$$

where

$$\mathbf{W}''_{a,\boldsymbol{\Gamma}} = \frac{\sqrt{K[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}}{\sqrt{\sum_{k=1}^{K}[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \times \frac{1}{\sqrt{([\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa})(\sum_{k=1}^{K} 1/n_k)}}$$
$$\times \sum_{k=1}^{K} \frac{1}{\sqrt{n_k}} \mathbf{T}_{a,k},$$
$$\mathbf{R}''_{a,\boldsymbol{\Gamma}} = \frac{\sqrt{K[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}}{\sqrt{\sum_{k=1}^{K}[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}} \times \frac{1}{\sqrt{([\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa})(\sum_{k=1}^{K} 1/n_k)}}$$

$$\times \sum_{k=1}^{K} \frac{1}{\sqrt{n_k}} \mathbf{R}_{a,k,\boldsymbol{\Gamma}}.$$

Considering $\zeta_a := \frac{1}{\sqrt{([\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa})(\sum_{k=1}^{K} 1/n_k)}} \sum_{k=1}^{K} \frac{1}{\sqrt{n_k}} \mathbf{T}_{a,k}$ and $\zeta'_a := \sum_{k=1}^{K} \frac{1}{\sqrt{n_k[\boldsymbol{\Sigma}\otimes(\mathbf{M_k}\mathbf{C_k}\mathbf{M_k}^\top)]_{aa}(\sum_{k=1}^{K} 1/n_k)}} \mathbf{T}_{a,k}$, by similar techniques as in part a), using the Lindeberg theorem, one can show that ζ'_a is asymptotically normal with mean zero and variance 1. Moreover, $|\zeta_a - \zeta'_a| = O_p\big(d_2\sqrt{nK\log(pq)\log(q)}/n_\dagger\big)$ and $|\mathbf{R}''_{a,\boldsymbol{\Gamma}}| = O_p\big(s_2\sqrt{nK\log(pq)\log(q)}/n_\dagger\big)$, which are $o_p(1)$ under the mentioned sparsity conditions and with $\sqrt{K} = O(n_\dagger^{1/3}/(\sqrt{\log(pq)\log(q)}\max\{s_2, d_2\}))$. □

A.6. Proof of Theorem 5

a) The proof of this theorem relies on Lemma 9 from Section B, which shows the negligibility of the remainder term $\mathbf{R}_{ab,\boldsymbol{\Theta}}$ as $n_\dagger$ and K both grow. To show the asymptotic normality, define

$$\zeta_{ab} := \sum_{k=1}^{K} \frac{1}{\sqrt{n}\sigma_{ab}} \times \frac{\sigma^2_{ab}}{\hat{\sigma}^2_{ab,k}} \sum_{l=1}^{n_k} \big(\boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\boldsymbol{\Theta}_b - \boldsymbol{\Theta}_{ab}\big),$$

where $\dot{\mathbf{Y}}_{l,k}$ and $\dot{\mathbf{X}}_{l,k}$ are the l-th row, $l = 1, \ldots, n_k$, of $\mathbf{Y}_k$ and $\mathbf{X}_k$, respectively. Similarly to the proof of Theorem 4, by defining the sequence

$$\zeta'_{ab} := \sum_{k=1}^{K} \frac{1}{\sqrt{n}\sigma_{ab}} \sum_{l=1}^{n_k} \big(\boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\boldsymbol{\Theta}_b - \boldsymbol{\Theta}_{ab}\big),$$

we have

$$\zeta'_{ab} = \frac{1}{\sqrt{n}\sigma_{ab}} \sum_{l=1}^{n} \big(\boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_l - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_l)(\dot{\mathbf{Y}}_l - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_l)^\top\boldsymbol{\Theta}_b - \boldsymbol{\Theta}_{ab}\big),$$

where $\dot{\mathbf{Y}}_l \in \mathbb{R}^p$ and $\dot{\mathbf{X}}_l \in \mathbb{R}^q$ are the l-th sample, $l = 1, \ldots, n$, of $\dot{\mathbf{Y}}$ and $\dot{\mathbf{X}}$, respectively. The sequence ζ'_{ab} converges to $\mathcal{N}(0,1)$ as it is shown in Theorem 2 of [17]. As such, we only need to show that $|\zeta_{ab} - \zeta'_{ab}| \xrightarrow{p} 0$ as $K \to \infty$ and $n_k \to \infty$, $k = 1, \ldots, K$, and then convergence in distribution of ζ'_{ab} yields convergence in distribution of ζ_{ab}. As it is shown in the proof of Lemma 9, the term $(1/\hat{\sigma}^2_{ab,k}) = O_p(1)$. Using Remark 2, we have

$$\begin{aligned}\big|\zeta_{ab} - \zeta'_{ab}\big| \leqslant \sum_{k=1}^{K} \frac{1}{\sqrt{n}\sigma_{ab}} \times O_p\big(\max\{\log(p)/n_k, \sqrt{d_1\log(p)/n_k}\}\big) \\ \times \left|\sum_{l=1}^{n_k} \big(\boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\boldsymbol{\Theta}_b - \boldsymbol{\Theta}_{ab}\big)\right|. \quad \text{(A.15)}\end{aligned}$$

Due to the definition of $\mathbf{W}_k$ in Section 4, we have that $\big|\sum_{l=1}^{n_k}\big(\boldsymbol{\Theta}_a^\top(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})(\dot{\mathbf{Y}}_{l,k} - \boldsymbol{\Gamma}^\top\dot{\mathbf{X}}_{l,k})^\top\boldsymbol{\Theta}_b - \boldsymbol{\Theta}_{ab}\big)\big| = n_k\big|\{\boldsymbol{\Theta}\mathbf{W}_k\boldsymbol{\Theta}\}_{ab}\big|$. Under assumption (B2) and using Lemma 2 of [21], for which the conditions are fulfilled, we have

$$n_k\big|\{\boldsymbol{\Theta}\mathbf{W}_k\boldsymbol{\Theta}\}_{ab}\big| \leqslant n_k\|\boldsymbol{\Theta}\mathbf{W}_k\boldsymbol{\Theta}\|_\infty \leqslant n_k|||\boldsymbol{\Theta}|||_\infty^2\|\mathbf{W}_k\|_\infty \leqslant d_1\sqrt{n_k\log(p)}.$$

Substituting this bound in (A.15), we get

$$|\zeta_{ab} - \zeta'_{ab}| \leqslant O_p\bigg(\frac{K}{\sqrt{n}}\max\big\{d_1(\log(p))^{3/2}/\sqrt{n_\dagger}, d_1^{3/2}\log(p)\big\}\bigg),$$

and under the conditions $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$ and $K = O\big(n^{1/3}/(d_1\log(p))\big)$, the result $|\zeta_{ab} - \zeta'_{ab}| = o_p(1)$ follows.

b) Similarly to a).

c) By a similar technique as for the proof of part c) of Theorem 4, it can be shown that

$$|\mathbf{R}''_{ab,\boldsymbol{\Theta}}| = O_p\bigg(\sqrt{nK}\max\big\{d_1^{3/2}\log(p)/n_\dagger, d_1^2(\log(p)/n_\dagger)^{3/2}, d_1 s_2\log(pq)/n_\dagger\big\}\bigg),$$

which is $o_p(1)$ under the sparsity conditions $d_1^{3/2} = o\big(n_\dagger^{1/3}/\log(p)\big)$ and $s_2 = o\big(n_\dagger^{\pi_3}/\log(pq)\big)$, $0 < \pi_3 \leqslant 1/6$, and $K = O(n_\dagger^{4/3}/n)$. The proof of asymptotic normality is similar to the one from part c) of Theorem 4, and we do not repeat it here. □

Appendix B: Some technical lemmas and their proofs

Lemma 2. Consider the regression model (2.2) with random noise matrix $\boldsymbol{\xi}_k$ and random Gaussian design matrix $\mathbf{X}_k$. By choosing $\rho_{0,k} = (\max_{1\leqslant a\leqslant q}\sqrt{[\mathbf{C}_k]_{aa}}) \times (\max_{1\leqslant b\leqslant p}\sqrt{\boldsymbol{\Sigma}_{bb}})\sqrt{A\log(pq)/n_k}$, where $[\mathbf{C}_k]_{aa}$ is the a-th diagonal element of $\mathbf{C}_k = \mathbf{X}_k^\top\mathbf{X}_k/n_k$, and $A > 4$ is a universal constant for all $k = 1, \ldots, K$, we have

$$\mathbb{P}\big(\mathcal{F}_k(n_k, p, q)\big) \geqslant 1 - 2/(pq)^{A/2-1}.$$

Proof. According to the definition of the elementwise ℓ_∞ norm, the event $\mathcal{F}_k$ is equivalent to

$$\mathcal{F}_k(n_k, p, q) = \bigg\{\max_{1\leqslant a\leqslant q}\max_{1\leqslant b\leqslant p}\bigg|\sum_{l=1}^{n_k}\varepsilon_{l,k}^b X_{l,k}^a\bigg|/n_k \leqslant \rho_{0,k}\bigg\},$$

where $X_{l,k}^a$ and $\varepsilon_{l,k}^b$ are the a-th and the b-th components of the vectors $\dot{\mathbf{X}}_{l,k}$ and $\dot{\boldsymbol{\varepsilon}}_{l,k}$ which are the l-th row of $\mathbf{X}_k$ and $\boldsymbol{\xi}_k$, respectively. Conditioning on $\mathbf{X}_k$, the random variable $\sum_{l=1}^{n_k}\varepsilon_{l,k}^b X_{l,k}^a$ is a linear transformation of $\varepsilon_{1,k}^b, \ldots, \varepsilon_{n_k,k}^b$ and it is normally distributed with mean zero and variance $n_k\boldsymbol{\Sigma}_{bb}[\mathbf{C}_k]_{aa}$. Defining

the random variable $V_{ab,k} := \sum_{l=1}^{n_k} \varepsilon_{l,k}^b X_{l,k}^a / \sqrt{n_k \boldsymbol{\Sigma}_{bb}[\mathbf{C}_k]_{aa}}$, we get $V_{ab,k} \mid \mathbf{X}_k \sim \mathcal{N}(0,1)$. With our choice of $\rho_{0,k}$, we have,

$$\mathbb{P}\Big(\max_{\substack{1\leqslant a\leqslant q \\ 1\leqslant b\leqslant p}} \big| \sum_{l=1}^{n_k} \varepsilon_{l,k}^b X_{l,k}^a \big| / n_k \geqslant \rho_{0,k} \Big) = \int \mathbb{P}\Big(\max_{\substack{1\leqslant a\leqslant q \\ 1\leqslant b\leqslant p}} \Big| \sum_{l=1}^{n_k} \varepsilon_{l,k}^b X_{l,k}^a \Big| / n_k \geqslant \rho_{0,k} \mid \mathbf{X}_k \Big) f_{\mathbf{X}_k}(\mathbf{x}_k)\, d\mathbf{x}_k. \tag{B.1}$$

To have a bound on $\mathbb{P}\big(\max_{\substack{1\leqslant a\leqslant q \\ 1\leqslant b\leqslant p}} \big| \sum_{l=1}^{n_k} \varepsilon_{l,k}^b X_{l,k}^a \big| / n_k \geqslant \rho_{0,k} \mid \mathbf{X}_k \big)$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\max_{\substack{1\leqslant a\leqslant q \\ 1\leqslant b\leqslant p}} \Big| \sum_{l=1}^{n_k} \varepsilon_{l,k}^b X_{l,k}^a \Big| / n_k \geqslant \rho_{0,k} \mid \mathbf{X}_k \Big) \\
&= \mathbb{P}\Big(\frac{\max_{\substack{1\leqslant a\leqslant q \\ 1\leqslant b\leqslant p}} \big| \sum_{l=1}^{n_k} \varepsilon_{l,k}^b X_{l,k}^a \big|}{n_k (\max_{1\leqslant a\leqslant q} \sqrt{[\mathbf{C}_k]_{aa}})(\max_{1\leqslant b\leqslant p} \sqrt{\boldsymbol{\Sigma}_{bb}})} > \sqrt{A \log(pq)/n_k} \mid \mathbf{X}_k \Big) \\
&\leqslant \mathbb{P}\Big(\max_{\substack{1\leqslant a\leqslant q \\ 1\leqslant b\leqslant p}} \Big| \frac{\sum_{l=1}^{n_k} \varepsilon_{l,k}^b X_{l,k}^a}{\sqrt{n_k \boldsymbol{\Sigma}_{bb}[\mathbf{C}_k]_{aa}}} \Big| > \sqrt{A \log(pq)} \mid \mathbf{X}_k \Big) \\
&\leqslant \sum_{a=1}^{q} \sum_{b=1}^{p} \mathbb{P}\big(|V_{ab,k}| \geqslant \sqrt{A \log(pq)} \mid \mathbf{X}_k \big).
\end{aligned} \tag{B.2}$$

To obtain a bound on $\mathbb{P}\big(|V_{ab,k}| \geqslant \sqrt{A \log(pq)} \mid \mathbf{X}_k \big)$, using the Gaussian concentration inequality $\mathbb{P}(|X - \mu| > \sigma t) \leqslant 2\exp(-t^2/2)$, $t \in \mathbb{R}$, for a normal random variable X with mean μ and variance σ^2, we get

$$\mathbb{P}\big(|V_{ab,k}| \geqslant \sqrt{A \log(pq)} \mid \mathbf{X}_k \big) \leqslant 2\exp\{-A \log(pq)/2\} = \frac{2}{(pq)^{A/2}}. \tag{B.3}$$

Substituting (B.3) in (B.2), and then in (B.1), the result of this Lemma follows directly. □

Lemma 3. *Under the assumptions of Lemma 2, and by considering the universal constants $0 < c_1 \leqslant c_2 < \infty$, such that $c_1 \leqslant \sup_{1\leqslant k\leqslant K} [\mathbf{C}_k]_{aa} \leqslant c_2$, for $a = 1, \ldots, q$, it holds that*

$$\mathbb{P}\Big(\bigcap_{k=1}^{K} \mathcal{F}_k(n_k, p, q) \Big) \geqslant 1 - \frac{2K}{(pq)^{A/2-1}},$$

and if K grows at the rate $K = o\big((pq)^{A/2-1}\big)$, then this probability tends to one with increasing n.

Proof. We have

$$\mathbb{P}\Big(\bigcap_{k=1}^{K} \mathcal{F}_k(n_k, p, q) \Big) = 1 - \mathbb{P}\Big(\big(\bigcap_{k=1}^{K} \mathcal{F}_k(n_k, p, q) \big)^c \Big) = 1 - \mathbb{P}\Big(\bigcup_{k=1}^{K} \mathcal{F}_k^c(n_k, p, q) \Big). \tag{B.4}$$

We have as well that

$$\mathbb{P}\Big(\bigcup_{k=1}^{K}\mathcal{F}_k^c(n_k,p,q)\Big)\leqslant\sum_{k=1}^{K}\mathbb{P}\big(\mathcal{F}_k^c(n_k,p,q)\big)\leqslant\sum_{k=1}^{K}\frac{2}{(pq)^{A/2-1}}=\frac{2K}{(pq)^{A/2-1}},\tag{B.5}$$

where the second inequality follows using Lemma 2. By substituting (B.5) in (B.4), the result of this Lemma follows directly. □

Before providing Lemmas 4 and 5, we need some preliminary notation. The results are provided under the Restricted Eigenvalue (RE) condition. The following definition from [31] defines the RE condition for the population covariance matrix and is essentially equivalent to the RE condition of [4]. For a given subset $G\subset\{1,\ldots,q\}$ with cardinality $g=\#G$ and a constant $\delta\geqslant 1$, define the set

$$C(G;\delta):=\big\{\boldsymbol{\nu}\in\mathbb{R}^q:\quad\|\boldsymbol{\nu}_{G^c}\|_1\leqslant\delta\|\boldsymbol{\nu}_G\|_1\big\},$$

where $\boldsymbol{\nu}_G$ is the restriction of vector $\boldsymbol{\nu}$ to G and has zeros outside the set G. The symbol G^c denotes the complement set of G.

Definition 1 ([31])**.** A deterministic covariance matrix $\mathbf{Q}$ satisfies the RE condition over G of order g, if there exist constants $(\delta,\mu)\in[1,\infty)\times(0,\infty)$ such that

$$\|\mathbf{Q}^{1/2}\boldsymbol{\nu}\|_2\geqslant\mu\|\boldsymbol{\nu}\|_2,\quad\text{for all }\ \boldsymbol{\nu}\in C(G;\delta),$$

where $\mathbf{Q}^{1/2}$ is the square root matrix of $\mathbf{Q}$.

Note that Definition 1 provides a lower bound on the ℓ_2 norm of the product between $\mathbf{Q}^{1/2}$ and the vector $\boldsymbol{\nu}$. As in the multivariate regression we have a matrix of coefficients not a vector anymore, we need to provide a lower bound on the product of $\mathbf{Q}^{1/2}$ and each column of the coefficient matrix. To this end, consider an arbitrary matrix $\mathbf{V}\in\mathbb{R}^{q\times p}$, and denote its vectorized form with $\boldsymbol{\nu}:=\mathrm{vec}(\mathbf{V})$ such that $\boldsymbol{\nu}=(\boldsymbol{\nu}_{(1)}^\top,\ldots,\boldsymbol{\nu}_{(p)}^\top)^\top\in\mathbb{R}^{qp}$, where $\boldsymbol{\nu}_{(b)}\in\mathbb{R}^q$ is the b-th column of $\mathbf{V}$, $b=1,\ldots,p$. To simplify the notation and having one index for the elements of $\boldsymbol{\nu}$, denote the index set of $\boldsymbol{\nu}_{(b)}$ as $\{q(b-1)+1,\ldots,qb\}$, $b=1,\ldots,p$. Consider $G_b\subset\{q(b-1)+1,\ldots,qb\}$ with cardinality $g_b=\#G_b$ and complement set G_b^c. For a constant $\delta_b\geqslant 1$, define the set

$$C(G_b;\delta_b):=\big\{\boldsymbol{\nu}_{(b)}\in\mathbb{R}^q:\quad\|[\boldsymbol{\nu}_{(b)}]_{G_b^c}\|_1\leqslant\delta_b\|[\boldsymbol{\nu}_{(b)}]_{G_b}\|_1\big\},$$

where $[\boldsymbol{\nu}_{(b)}]_{G_b}$ is the restriction of $\boldsymbol{\nu}_{(b)}$ to G_b which has zeros outside the subset G_b.

Definition 2. A deterministic covariance matrix $\mathbf{Q}$ satisfies the multivariate restricted eigenvalue (MRE) condition over $G=\{G_1,\ldots,G_p\}$ of order $g=\sum_{b=1}^p g_b$, if there exist constants $(\delta_b,\mu_b)\in[1,\infty)\times(0,\infty)$, such that for every $b=1,\ldots,p$,

$$\|\mathbf{Q}^{1/2}\boldsymbol{\nu}_{(b)}\|_2\geqslant\mu_b\|\boldsymbol{\nu}_{(b)}\|_2,\quad\text{for all}\quad\boldsymbol{\nu}_{(b)}\in C(G_b;\delta_b).$$

Definition 3. The sample covariance matrix $\mathbf{C}_k = \mathbf{X}_k^\top \mathbf{X}_k/n_k$ of the design matrix $\mathbf{X}_k$ satisfies the MRE condition over $G = \{G_1, \ldots, G_p\}$ of order $g = \sum_{b=1}^p g_b$, if there exist constants $(\delta_b, \mu_b) \in [1, \infty) \times (0, \infty)$, such that for every $b = 1, \ldots, p$,

$$\boldsymbol{\nu}_{(b)}^\top \mathbf{C}_k \boldsymbol{\nu}_{(b)} \equiv \|\mathbf{X}_k \boldsymbol{\nu}_{(b)}\|_2^2/n_k \geqslant \mu_b^2 \|\boldsymbol{\nu}_{(b)}\|_2^2, \quad \text{for all} \quad \boldsymbol{\nu}_{(b)} \in C(G_b; \delta_b). \tag{B.6}$$

To introduce Lemma 4, let $S_2(b)$ be the support of the b-th column of $\boldsymbol{\Gamma}$. By vectorizing $\boldsymbol{\Gamma}$ and rewriting its columns as explained for the general matrix $\mathbf{V}$, we have $S_2(b) \subset \{q(b-1)+1, \ldots, qb\}$, with cardinality $s_2(b) = \#S_2(b)$ and complement set $S_2^c(b)$. For a constant $\delta_b \geqslant 1$, denote the set

$$C(S_2(b); \delta_b) := \big\{\boldsymbol{\nu}_{(b)} \in \mathbb{R}^q : \quad \|[\boldsymbol{\nu}_{(b)}]_{S_2^c(b)}\|_1 \leqslant \delta_b \|[\boldsymbol{\nu}_{(b)}]_{S_2(b)}\|_1\big\},$$

where $[\boldsymbol{\nu}_{(b)}]_{S_2(b)}$ is the restriction of $\boldsymbol{\nu}_{(b)}$ to $S_2(b)$ which has zeros outside the subset $S_2(b)$.

Lemma 4. Consider the regression model (2.2) for the k-th sub-sample with random Gaussian design $\mathbf{X}_k$ which has independent and identical $\mathcal{N}_q(\mathbf{0}, \mathbf{Q})$ rows, and denote the maximum diagonal element of $\mathbf{Q}$ by $\mathbf{Q}_{\max}$. Suppose that $\mathbf{Q}$ satisfies the MRE condition over S_2 of order s_2 for all vectors in $C(S_2(b); \delta_b)$ with parameters (δ_b, μ_b), $b = 1, \ldots, p$. On the event $\mathcal{F}_k(n_k, p, q)$, for some positive constants c, c' and c'', if the sub-sample size n_k, $k = 1, \ldots, K$, satisfies

$$n_k \geqslant c'' \frac{\mathbf{Q}_{\max}^2 (1+\delta_{\max})^2}{\mu_{\min}^2} s_2 \log(q), \tag{B.7}$$

where $\mu_{\min} = \min_{1 \leqslant b \leqslant p} \mu_b$ and $\delta_{\max} = \max_{1 \leqslant b \leqslant p} \delta_b$, then we have

$$\frac{\|\mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\|_F^2}{n_k} \geqslant (\mu_{\min}/8)^2 \|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_F^2, \tag{B.8}$$

with probability at least $1 - c' \exp(-cn_k)$.

Proof. In general, consider the matrix $\mathbf{V} \in \mathbb{R}^{q \times p}$ with the b-th column $\boldsymbol{\nu}_{(b)}$, $b = 1, \ldots, p$, and its vectorized form as $\boldsymbol{\nu} \in \mathbb{R}^{qp}$. We have

$$\frac{\|\mathbf{X}_k \mathbf{V}\|_F}{\sqrt{n_k}} = \frac{\|\mathrm{vec}(\mathbf{X}_k \mathbf{V})\|_2}{\sqrt{n_k}} = \frac{\|(\mathbf{I}_p \otimes \mathbf{X}_k)\boldsymbol{\nu}\|_2}{\sqrt{n_k}} = \sqrt{\frac{\sum_{b=1}^p \|\mathbf{X}_k \boldsymbol{\nu}_{(b)}\|_2^2}{n_k}}. \tag{B.9}$$

Using Theorem 1 of [31], under the Gaussianity of the design matrix $\mathbf{X}_k$, we have

$$\frac{\|\mathbf{X}_k \boldsymbol{\nu}_{(b)}\|_2}{\sqrt{n_k}} \geqslant \frac{1}{4}\|\mathbf{Q}^{1/2} \boldsymbol{\nu}_{(b)}\|_2 - 9\mathbf{Q}_{\max} \sqrt{\frac{\log(q)}{n_k}} \|\boldsymbol{\nu}_{(b)}\|_1, \quad \text{for all} \quad \boldsymbol{\nu}_{(b)} \in \mathbb{R}^q,$$

with probability at least $1 - c' \exp(-cn_k)$. Using the relation between ℓ_1 and ℓ_2 norms, for all vectors $\boldsymbol{\nu}_{(b)} \in C(S_2(b); \delta_b)$ with parameters (μ_b, δ_b), $b = 1, \ldots, p$, we have

$$\|\boldsymbol{\nu}_{(b)}\|_1 = \|[\boldsymbol{\nu}_{(b)}]_{S_2(b)}\|_1 + \|[\boldsymbol{\nu}_{(b)}]_{S_2^c(b)}\|_1 \leqslant (1+\delta_b)\|[\boldsymbol{\nu}_{(b)}]_{S_2(b)}\|_1$$

$$\leqslant (1+\delta_b)\sqrt{s_2(b)}\|\boldsymbol{\nu}_{(b)}\|_2. \tag{B.10}$$

Under the MRE condition of $\mathbf{Q}$ for all vectors $\boldsymbol{\nu}_{(b)} \in C(S_2(b);\delta_b)$ with parameters (μ_b,δ_b), $b=1,\ldots,p$, together with (B.10) and the fact that $s_2(b) \leqslant s_2$, $b=1,\ldots,p$, we get

$$\frac{\|\mathbf{X}_k\boldsymbol{\nu}_{(b)}\|_2^2}{n_k} \geqslant \left\{\frac{1}{4}\mu_{\min} - 9\mathbf{Q}_{\max}(1+\delta_{\max})\sqrt{\frac{s_2\log(q)}{n_k}}\right\}^2 \|\boldsymbol{\nu}_{(b)}\|_2^2. \tag{B.11}$$

Substituting (B.11) in (B.9),

$$\frac{\|\mathbf{X}_k\mathbf{V}\|_F}{\sqrt{n_k}} \geqslant \left\{\frac{1}{4}\mu_{\min} - 9\mathbf{Q}_{\max}(1+\delta_{\max})\sqrt{\frac{s_2\log(q)}{n_k}}\right\}\sqrt{\sum_{b=1}^{p}\|\boldsymbol{\nu}_{(b)}\|_2^2}. \tag{B.12}$$

Applying the lower bound (B.7) in (B.12) for some constant c'', we get

$$\|\mathbf{X}_k\mathbf{V}\|_F/\sqrt{n_k} \geqslant (\mu_{\min}/8)\|\mathbf{V}\|_F. \tag{B.13}$$

Note that using (B.16) from the proof of Lemma 5, we have

$$\|\hat{\boldsymbol{\Gamma}}_{S_2^c,k} - \boldsymbol{\Gamma}_{S_2^c}\|_1 \leqslant 4\|\hat{\boldsymbol{\Gamma}}_{S_2,k} - \boldsymbol{\Gamma}_{S_2}\|_1, \tag{B.14}$$

where $\boldsymbol{\Gamma}_{S_2}$ and $\hat{\boldsymbol{\Gamma}}_{S_2,k}$ are the matrices $\boldsymbol{\Gamma}$ and $\hat{\boldsymbol{\Gamma}}_k$ which have zeros outside the set S_2, respectively. By the same argument as in the proof of Lemma 5 for univariate linear regression (see for example, [4]), a similar inequality to (B.14) can be written for the b-th column of $\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}$ and it can be deduced that the b-th column of $\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}$ is in $C(S_2(b);\delta_b)$. As such, by replacing $\mathbf{V}$ with $\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}$ in (B.13), the result in (B.8) follows directly. □

The following Lemma provides an ℓ_1-error bound on $\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}$, by a similar approach to that of [8] for the Dantzig selector.

Lemma 5. Under the conditions of Lemma 4 and the lower bound (B.7) on the sub-sample size n_k, with probability at least $1 - c'\exp(-cn_k)$, we have

$$\|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_1 = O_p\big(s_2\sqrt{\log(pq)/n_k}\big). \tag{B.15}$$

Proof. Consider the regression model (2.2) and recall the Lasso solution (3.1). Due to the fact that $\hat{\boldsymbol{\Gamma}}_k$ minimizes the loss in (3.1),

$$\|\mathbf{X}_k(\boldsymbol{\Gamma} - \hat{\boldsymbol{\Gamma}}_k)\|_F^2/(2n_k) + \rho_k\|\hat{\boldsymbol{\Gamma}}_k\|_1 \leqslant \operatorname{trace}\big((\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top\mathbf{X}_k^\top\boldsymbol{\xi}_k\big)/n_k + \rho_k\|\boldsymbol{\Gamma}\|_1,$$

where

$$\begin{aligned}\operatorname{trace}\big((\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top\mathbf{X}_k^\top\boldsymbol{\xi}_k\big) \leqslant \Big|\operatorname{trace}\big((\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top\mathbf{X}_k^\top\boldsymbol{\xi}_k\big)\Big| &= \Big|\big(\operatorname{vec}(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\big)^\top\operatorname{vec}(\mathbf{X}_k^\top\boldsymbol{\xi}_k)\Big| \\ &\leqslant \|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_1\|\operatorname{vec}(\mathbf{X}_k^\top\boldsymbol{\xi}_k)\|_\infty,\end{aligned}$$

where the last inequality holds due to Hölder's inequality. Now, on the event $\mathcal{F}_k(n_k, p, q)$, by a similar argument as in the univariate regression (see for example [5], chapter 6), we get

$$\frac{\|\mathbf{X}_k(\mathbf{\Gamma} - \hat{\mathbf{\Gamma}}_k)\|_F^2}{2n_k} + \frac{\rho_k \|\hat{\mathbf{\Gamma}}_k - \mathbf{\Gamma}\|_1}{2} \leqslant 2\rho_k \|\hat{\mathbf{\Gamma}}_{S_2,k} - \mathbf{\Gamma}_{S_2}\|_1, \tag{B.16}$$

where $\mathbf{\Gamma}_{S_2}$ and $\hat{\mathbf{\Gamma}}_{S_2,k}$ are the matrices $\mathbf{\Gamma}$ and $\hat{\mathbf{\Gamma}}_k$ which have zeros outside the set S_2, respectively. Due to the relation between the elementwise ℓ_1 and Frobenius norms of a matrix, we have

$$\|\hat{\mathbf{\Gamma}}_{S_2,k} - \mathbf{\Gamma}_{S_2}\|_1 \leqslant \sqrt{s_2}\|\hat{\mathbf{\Gamma}}_k - \mathbf{\Gamma}\|_F. \tag{B.17}$$

Substituting (B.17) in (B.16) and then applying Lemma 4 under the lower bound (B.7) on the sub-sample size n_k, we get

$$\frac{\|\mathbf{X}_k(\mathbf{\Gamma} - \hat{\mathbf{\Gamma}}_k)\|_F^2}{2n_k} + \frac{\rho_k \|\hat{\mathbf{\Gamma}}_k - \mathbf{\Gamma}\|_1}{2} \leqslant 2\rho_k \sqrt{s_2}\left(\frac{8}{\mu_{\min}}\right)\frac{\|\mathbf{X}_k(\hat{\mathbf{\Gamma}}_k - \mathbf{\Gamma})\|_F}{\sqrt{n_k}},$$

with probability at least $1 - c' \exp(-cn_k)$, and

$$\|\mathbf{X}_k(\mathbf{\Gamma} - \hat{\mathbf{\Gamma}}_k)\|_F^2/n_k \leqslant 16\rho_k^2 s_2 (8/\mu_{\min})^2, \quad \|\hat{\mathbf{\Gamma}}_k - \mathbf{\Gamma}\|_1 \leqslant 16\rho_k s_2 (8/\mu_{\min})^2. \tag{B.18}$$

Since $\mu_{\min} \in (0, \infty)$, there exists $L = O(1)$ such that $(8/\mu_{\min})^2 \leqslant L$, and by considering $\rho_k \asymp \sqrt{\log(pq)/n_k}$, the result of (B.15) follows. □

Lemma 6. Consider the regression model (2.2) where $\mathbf{X}_k$ satisfies assumptions (A1) and (A2). Suppose that the eigenvalues of $\mathbf{\Sigma}$, the covariance matrix of the noise, are bounded from below and above. On the event $\mathcal{F}_k(n_k, p, q)$ with regularization parameter $\rho_k \asymp \sqrt{\log(pq)/n_k}$, we have

$$\|\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} - \mathbf{\Sigma}\|_\infty = O_p\big(\max\{\sqrt{\log(p)/n_k}, s_2 \log(pq)/n_k\}\big), \tag{B.19}$$

and under the sparsity condition $s_2 = o\big(n_\dagger^{\pi_1}/(\log(q)\log(pq))\big)$, $0 < \pi_1 \leqslant 1/2$, we get $\|\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} - \mathbf{\Sigma}\|_\infty = o_p(1)$.

Proof. Recalling that $\boldsymbol{\xi}_k = \mathbf{Y}_k - \mathbf{X}_k\mathbf{\Gamma}$, by adding and subtracting $\boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k/n_k$ to $\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} - \mathbf{\Sigma}$, we get

$$\|\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} - \mathbf{\Sigma}\|_\infty \leqslant \|\boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k/n_k - \mathbf{\Sigma}\|_\infty + \|\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} - \boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k/n_k\|_\infty. \tag{B.20}$$

Under the boundedness of the eigenvalues of $\mathbf{\Sigma}$ from below and above, using Lemma 2 of [21],

$$\|\boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k/n_k - \mathbf{\Sigma}\|_\infty = O_p(\sqrt{\log(p)/n_k}). \tag{B.21}$$

To find an upper bound on the second term of (B.20), by adding and subtracting $\mathbf{X}_k\mathbf{\Gamma}$ to $\mathbf{Y}_k - \mathbf{X}_k\hat{\mathbf{\Gamma}}_k$, we get

$$\|\hat{\mathbf{\Sigma}}_{k,\hat{\mathbf{\Gamma}}_k} - \boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k/n_k\|_\infty$$

$$= \|\big(\boldsymbol{\xi}_k - \mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\big)^\top \big(\boldsymbol{\xi}_k - \mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\big)/n_k - \boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k/n_k\|_\infty$$
$$\leqslant 2\|(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top \mathbf{X}_k^\top \boldsymbol{\xi}_k/n_k\|_\infty + \|(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top \mathbf{X}_k^\top \mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})/n_k\|_\infty. \quad \text{(B.22)}$$

Using Lemma 5, on the event $\mathcal{F}_k(n_k, p, q)$,

$$2\|(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top \mathbf{X}_k^\top \boldsymbol{\xi}_k/n_k\|_\infty \leqslant 2\|\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma}\|_1 \|\mathbf{X}_k^\top \boldsymbol{\xi}_k\|_\infty/n_k = O_p(s_2 \log(pq)/n_k). \quad \text{(B.23)}$$

On the other hand,

$$\|(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})^\top \mathbf{X}_k^\top \mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})/n_k\|_\infty \leqslant \|\mathbf{X}_k(\hat{\boldsymbol{\Gamma}}_k - \boldsymbol{\Gamma})\|_F^2/n_k = O_p(s_2 \log(pq)/n_k), \quad \text{(B.24)}$$

where the last equality is due to the fact that in (B.18), $\mu_{\min} \in (0, \infty)$ and $\rho_k \asymp \sqrt{\log(pq)/n_k}$. Substituting (B.23) and (B.24) in (B.22) and then substituting (B.22) and (B.21) in (B.20), the result in (B.19) follows. □

Lemma 7. Consider the regression model (2.2) where $\mathbf{X}_k$ satisfies assumptions (A1) and (A2), and consider the maximum row sparsity of $\mathbf{Q}^{-1}$ as $d_2 = o(\sqrt{n_\dagger}/\log(q))$. Moreover, consider the coefficient matrix $\boldsymbol{\Gamma}$ with sparsity condition $s_2 = o\big(n_\dagger^{\pi_1}/(\log(pq)\log(q))\big)$, $0 < \pi_1 \leqslant 1/2$. Suppose that the event $\mathcal{F}_k(n_k, p, q)$ holds jointly in $k = 1, \ldots, K$. Moreover, suppose that $\hat{\boldsymbol{\Gamma}}_k^d$, $k = 1, \ldots, K$, is the k-th debiased estimator in (3.3) with tuning parameter $\rho_k \asymp \sqrt{\log(pq)/n_k}$ and $\tilde{\boldsymbol{\Gamma}}_{\text{owAvg}}$ is the pooled estimator in (5.4). Let $n_k/n \to c_k \in (0, 1)$ as n_k grows, such that $\lim_{K\to\infty} \sum_{k=1}^K c_k = 1$, and consider the remainder term $\mathbf{R}_{a,\boldsymbol{\Gamma}}$ defined in (5.5). Then, it follows that

$$|\mathbf{R}_{a,\boldsymbol{\Gamma}}| = O_p\big(K s_2 \sqrt{\log(pq)\log(q)/n}\big), \quad \text{(B.25)}$$

and if K grows at the rate $K = O\big(n^{1/4}/(\sqrt{\log(pq)\log(q)}\max\{s_2, \sqrt{d_2}\})\big)$, we have $|\mathbf{R}_{a,\boldsymbol{\Gamma}}| = o_p(1)$.

Proof. First note that using Lemma 6, $|[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} - \boldsymbol{\Sigma}]_{aa}| = O_p\big(\max\{\sqrt{\log(p)/n_k}, s_2 \log(pq)/n_k\}\big)$, where we recall that $\mathbf{A}_{aa}$ is the a-th diagonal element of an arbitrary matrix $\mathbf{A}$. Moreover, under the sparsity condition $d_2 = o(\sqrt{n_\dagger}/\log(q))$, with tuning parameter $\tilde{\rho}_{j,k} \asymp \sqrt{\log(q)/n_k}$ in the nodewise Lasso regression (A.1), using Lemma 5.4 of [34], we get $|[\mathbf{M}_k \mathbf{C}_k \mathbf{M}_k^\top - \mathbf{Q}^{-1}]_{aa}| = O_p(\sqrt{d_2 \log(q)/n_k})$. As such,

$$\big|[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k \mathbf{C}_k \mathbf{M}_k^\top)]_{aa} - [\boldsymbol{\Sigma} \otimes \mathbf{Q}^{-1}]_{aa}\big|$$
$$= O_p\big(\max\{\sqrt{d_2 \log(q)/n_k}, \sqrt{\log(p)/n_k}, s_2 \log(pq)/n_k\}\big). \quad \text{(B.26)}$$

It can be shown that

$$\left|\frac{1}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k \mathbf{C}_k \mathbf{M}_k^\top)]_{aa}} - \frac{1}{[\boldsymbol{\Sigma} \otimes \mathbf{Q}^{-1}]_{aa}}\right|$$
$$= O_p\big(\max\{\sqrt{d_2 \log(q)/n_k}, \sqrt{\log(p)/n_k}, s_2 \log(pq)/n_k\}\big). \quad \text{(B.27)}$$

As such,

$$\begin{aligned}
&\left|\sum_{k=1}^{K}\left\{\frac{n_k}{n}\times\frac{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\right\}-1\right| \\
&\quad=[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}\left|\sum_{k=1}^{K}\frac{n_k}{n}\left\{\frac{1}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}-\frac{1}{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}\right\}\right| \\
&\quad\leqslant[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}\sum_{k=1}^{K}\frac{n_k}{n}\left|\frac{1}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}-\frac{1}{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}\right| \\
&\quad=O_p\left(K\max\left\{\sqrt{d_2\log(q)/n},\sqrt{\log(p)/n},s_2\log(pq)/\sqrt{nn_\dagger}\right\}\right),
\end{aligned}$$

where the last equality holds using (B.27), and the fact that $n_k \leqslant n$, $k = 1,\ldots,K$. As such, $\left|\sum_{k=1}^{K}\left\{\frac{n_k}{n}\times\frac{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\right\}-1\right| = o_p(1)$ as K grows at the rate $K = O\big(n^{1/4}/\big(\sqrt{\log(q)\log(pq)}\max\{s_2,\sqrt{d_2}\}\big)\big)$, and as a result $\frac{\sum_{k=1}^{K}n_k/[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}{\sum_{k=1}^{K}n_k/[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}\xrightarrow{p}1$. By considering the continuous map $g(x) = 1/\sqrt{x}$, the sequence $\sqrt{\frac{\sum_{k=1}^{K}n_k/[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{\sum_{k=1}^{K}n_k/[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}$ converges in probability to 1 as K and n_k, $k = 1,\ldots,K$, grow. As such, due to the definition of $\mathbf{R}_{a,\boldsymbol{\Gamma}}$, it is enough to show the bound (B.25) for $\sum_{k=1}^{K}\frac{\sqrt{n_k[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}}{\sqrt{n[\hat{\boldsymbol{\Sigma}}_k\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}\mathbf{R}_{a,k,\boldsymbol{\Gamma}}$, with $\mathbf{R}_{a,k,\boldsymbol{\Gamma}}$ the a-th element, $a = 1,\ldots,qp$, of $\mathbf{R}_{k,\boldsymbol{\Gamma}}$ from (3.3).

We first show that $1/[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa} = O_p(1)$. Note that $[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}$ is nothing else than the a-th element of the outer product of the diagonal elements of $\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}$ and $\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top$. Thus, it is just needed to find a lower bound for the diagonal elements of these two matrices. By construction, the diagonal elements of $\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}$ are always positive. Combining Theorem 2.2 of [34] and the reversed triangle inequality, for each $a \in \{1,\ldots,q\}$ we get

$$|\mathbf{Q}_{aa}^{-1}| - |[\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}| \leqslant |\mathbf{Q}_{aa}^{-1} - [\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}| = o_p(1). \tag{B.28}$$

Since $\mathbf{Q}^{-1}$ is positive definite, using assumption (A2), we get $\mathbf{Q}_{aa}^{-1} \geqslant \Lambda_{\min}(\mathbf{Q}^{-1}) > 1/\Lambda_1$. As such, for sufficiently large n_k, using (B.28) we have that $|[\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}| > 0$. Thus, for every $a,b,c \in \{1,\ldots,q\}$, there exists a bound J_k such that

$$\frac{1}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}} \leqslant \frac{1}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}]_{bb}|[\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{cc}|} \leqslant \frac{1}{J_k} = O_p(1). \tag{B.29}$$

By the fact that $[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}$ does not grow with n based on the boundedness assumption of the diagonal entries of $\boldsymbol{\Sigma}$ and $\mathbf{Q}^{-1}$, using Theorem 1 and combining it with (B.29), we have that

$$\Big|\sum_{k=1}^{K}\frac{\sqrt{n_k[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}}{\sqrt{n}[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\mathbf{R}_{a,k,\boldsymbol{\Gamma}}\Big| \leqslant \sum_{k=1}^{K}\sqrt{n_k/n}\big|\mathbf{R}_{a,k,\boldsymbol{\Gamma}}\big|\times O_p(1)$$
$$\leqslant O_p(Ks_2\sqrt{\log(q)\log(pq)/n}).$$

Considering the sparsity conditions $s_2 = o\big(n_\dagger^{\pi_1}/(\log(pq)\log(q))\big)$, $0<\pi_1\leqslant 1/2$, $d_2 = o(\sqrt{n_\dagger}/\log(q))$ and $K = O\big(n^{1/4}/(\sqrt{\log(pq)\log(q)}\max\{s_2,\sqrt{d_2}\})\big)$, the $o_p(1)$ result follows immediately. □

Lemma 8. Under the assumptions of Theorem 4, by considering

$$\zeta_a = \sum_{k=1}^{K}\sqrt{\frac{n_k[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{n[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}\times\frac{\mathbf{T}_{a,k}}{\sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}},$$

where $\mathbf{T}_{a,k}$ is the a-th element of $\mathbf{T}_k$ from (3.3), and

$$\zeta_a' = \sum_{k=1}^{K}\sqrt{\frac{n_k}{n}}\times\frac{\mathbf{T}_{a,k}}{\sqrt{[\boldsymbol{\Sigma}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}},$$

we have $|\zeta_a-\zeta_a'|\xrightarrow{p}0$, as $K\to\infty$ and $n_k\to\infty$, $k=1,\dots,K$.

Proof. Denoting the sequence $\zeta_a'' = \sum_{k=1}^{K}\sqrt{\frac{n_k}{n}}\times\frac{\mathbf{T}_{a,k}}{\sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}}$, we show that $|\zeta_a-\zeta_a''|\xrightarrow{p}0$ and $|\zeta_a''-\zeta_a'|\xrightarrow{p}0$, and then automatically, $|\zeta_a-\zeta_a'|\xrightarrow{p}0$. We have

$$\begin{aligned}
|\zeta_a-\zeta_a''| &\leqslant \sum_{k=1}^{K}\sqrt{\frac{n_k}{n}}\Big|\sqrt{\frac{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}}}-1\Big|\times\frac{|\mathbf{T}_{a,k}|}{\sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}}}\\
&= \sum_{k=1}^{K}\sqrt{\frac{n_k}{n}}\frac{\big|\sqrt{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}-\sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}}\big|}{\sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}}}\\
&\quad\times\frac{|\mathbf{T}_{a,k}|}{\sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}}}\\
&\leqslant O_p(1)\sum_{k=1}^{K}\sqrt{\frac{n_k}{n}}\Big|\sqrt{[\boldsymbol{\Sigma}\otimes\mathbf{Q}^{-1}]_{aa}}-\sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top]_{aa}}\Big|\times|\mathbf{T}_{a,k}|,
\end{aligned}$$

where the last inequality follows by the fact that $1/[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k}\otimes(\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa} = O_p(1)$ using (B.29). Moreover, using (B.26) we get

$$|\zeta_a-\zeta_a''|\leqslant\sum_{k=1}^{K}\sqrt{\frac{n_k}{n}}|\mathbf{T}_{a,k}|\times O_p\big(\max\{\sqrt{d_2\log(q)/n_k},\sqrt{\log(p)/n_k},s_2\log(pq)/n_k\}\big). \tag{B.30}$$

On the other hand,

$$\mathbf{T}_k = \text{vec}(\mathbf{M}_k\mathbf{X}_k^\top\boldsymbol{\xi}_k/\sqrt{n_k}) = (\mathbf{I}_p \otimes \mathbf{M}_k)\text{vec}(\mathbf{X}_k^\top\boldsymbol{\xi}_k)/\sqrt{n_k},$$

and working on the event $\mathcal{F}_k(n_k, p, q)$, by considering $\rho_k \asymp \sqrt{\log(pq)/n_k}$,

$$\|\mathbf{T}_k\|_\infty \leqslant \sqrt{n_k}O_p(\sqrt{\log(pq)/n_k})|||\mathbf{I}_p \otimes \mathbf{M}_k|||_\infty. \tag{B.31}$$

As was shown in the proof of Lemma 1,

$$|||\mathbf{I}_p \otimes \mathbf{M}_k|||_\infty = |||\mathbf{M}_k|||_\infty = \max_{j\in\{1,\ldots,q\}} \|\mathbf{M}_{j,k}\|_1 = O_p(\sqrt{d_2}),$$

where $\mathbf{M}_{j,k}$ is the j-th row of $\mathbf{M}_k$. As such, in (B.31), we get $\|\mathbf{T}_k\|_\infty \leqslant O_p(\sqrt{d_2\log(pq)})$. Substituting this result in (B.30), we have

$$\begin{aligned}
|\zeta_a - \zeta_a''| &\leqslant \sum_{k=1}^K \sqrt{\frac{n_k}{n}} O_p\big(\sqrt{d_2\log(pq)} \times \max\{\sqrt{d_2\log(q)/n_k}, \sqrt{\log(p)/n_k}, \\
&\qquad s_2\log(pq)/n_k\}\big) \\
&\leqslant O_p\big(K\sqrt{d_2\log(pq)/n} \times \max\{\sqrt{d_2\log(q)}, \sqrt{\log(p)}, \\
&\qquad s_2\log(pq)/\sqrt{n_\dagger}\}\big),
\end{aligned}$$

where by considering the sparsity conditions $s_2 = o\big(n_\dagger^{\pi_1}/(\log(pq)\log(q))\big)$, $0 < \pi_1 \leqslant 1/2$, $d_2 = o(\sqrt{n_\dagger}/\log(q))$ and $K = O\big(n^{1/4}/(\sqrt{\log(pq)\log(q)} \times \max\{s_2, \sqrt{d_2}\})\big)$, the $o_p(1)$ result follows immediately.

Now, to show $|\zeta_a'' - \zeta_a'| \xrightarrow{p} 0$, by the same argument,

$$\begin{aligned}
|\zeta_a'' - \zeta_a'| &\leqslant \sum_{k=1}^K \sqrt{\frac{n_k}{n}} \Big|\sqrt{[\boldsymbol{\Sigma} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}} - \sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\Big| \\
&\qquad \times |\mathbf{T}_{a,k}| \\
&\leqslant \sum_{k=1}^K \sqrt{\frac{n_k}{n}} \Big\{\Big|\sqrt{[\boldsymbol{\Sigma} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}} - \sqrt{[\boldsymbol{\Sigma} \otimes \mathbf{Q}^{-1}]_{aa}}\Big| \\
&\qquad + \Big|\sqrt{[\boldsymbol{\Sigma} \otimes \mathbf{Q}^{-1}]_{aa}} - \sqrt{[\hat{\boldsymbol{\Sigma}}_{k,\hat{\boldsymbol{\Gamma}}_k} \otimes (\mathbf{M}_k\mathbf{C}_k\mathbf{M}_k^\top)]_{aa}}\Big|\Big\} |\mathbf{T}_{a,k}|.
\end{aligned}$$

Similarly to the proof of $|\zeta_a - \zeta_a''| \xrightarrow{p} 0$, by considering the mentioned sparsity conditions, we conclude that $|\zeta_a'' - \zeta_a'| \xrightarrow{p} 0$. □

Lemma 9. Consider the regression model (2.2), and suppose that assumptions (B1)–(B2) and (C1)–(C3) from Appendix A.3 hold. Consider the coefficient matrix $\boldsymbol{\Gamma}$ with sparsity condition $s_2 = o(n_\dagger^{\pi_3}/\log(pq))$, $0 < \pi_3 \leqslant 1/6$, and the precision matrix $\boldsymbol{\Theta}$ with maximum node degree $d_1^{3/2} = o(\sqrt{n_\dagger}/\log(p))$. Suppose that the event $\mathcal{F}_k(n_k, p, q)$ holds jointly in $k = 1, \ldots, K$. Moreover, suppose that $\hat{\boldsymbol{\Theta}}_k^d$, $k = 1, \ldots, K$, is the k-th debiased estimator in (4.3) with tuning parameter

$\lambda_k \asymp \sqrt{\log(p)/n_k}$ and $\tilde{\boldsymbol{\Theta}}_{\text{owAvg}}$ is the pooled estimator in (5.6). Consider the remainder term $\mathbf{R}_{ab,\boldsymbol{\Theta}}$ defined in (5.7). Then it follows that

$$|\mathbf{R}_{ab,\boldsymbol{\Theta}}| = O_p\left(\frac{K}{\sqrt{n}} \max\left\{d_1^{3/2}\log(p), d_1^2(\log(p))^{3/2}/\sqrt{n_\dagger}, d_1 s_2 \log(pq)\right\}\right), \quad \text{(B.32)}$$

and if K grows as $K = O\big(n^{1/3}/(d_1\log(p))\big)$, we have $|\mathbf{R}_{ab,\boldsymbol{\Theta}}| = o_p(1)$.

Proof. By a similar argument as in the proof of Lemma 7 and using Remark 2, $\big|\frac{\sum_{k=1}^K n_k/\hat{\sigma}^2_{ab,k}}{\sum_{k=1}^K n_k/\sigma^2_{ab}} - 1\big| = O_p\big(K\max\big\{\log(p)/n, \sqrt{d_1\log(p)/n}\big\}\big)$, which is of order $o_p(1)$ as K grows at the rate $K = O\big(n^{1/3}/(d_1\log(p))\big)$. As a result, by considering the continuous map $g(x) = 1/\sqrt{x}$, the sequence $\sqrt{\frac{\sum_{k=1}^K n_k/\sigma^2_{ab}}{\sum_{k=1}^K n_k/\hat{\sigma}^2_{ab,k}}}$ converges in probability to 1 as K and n_k, $k = 1,\ldots,K$, grow. As such, we just need to show the boundedness of $\sum_{k=1}^K \frac{\sqrt{n_k}\sigma_{ab}}{\sqrt{n}\hat{\sigma}^2_{ab,k}} \times \mathbf{R}_{ab,k,\boldsymbol{\Theta}}$. Due to the positive definiteness of the graphical Lasso estimator defined in (4.1), there exists a positive constant L_k such that with high probability $\hat{\sigma}^2_{ab,k} \geqslant \Lambda^2_{\min}(\hat{\boldsymbol{\Theta}}_k) > L_k$, where $\Lambda_{\min}(\hat{\boldsymbol{\Theta}}_k)$ is the minimum eigenvalue of $\hat{\boldsymbol{\Theta}}_k$ and then the term $1/\hat{\sigma}^2_{ab,k} = O_p(1)$. Moreover, using (4.5),

$$\left|\sum_{k=1}^K \frac{\sqrt{n_k}\sigma_{ab}}{\sqrt{n}\hat{\sigma}^2_{ab,k}}\mathbf{R}_{ab,k,\boldsymbol{\Theta}}\right| \leqslant \sum_{k=1}^K \frac{1}{\sqrt{n}} O_p\big(\max\{d_1^{3/2}\log(p), d_1^2(\log(p))^{3/2}/\sqrt{n_\dagger}, d_1 s_2\log(pq)\}\big),$$

and the result in (B.32) follows directly. By considering the mentioned sparsity and $K = O\big(n^{1/3}/(d_1\log(p))\big)$, one can reach the $o_p(1)$ rate. □

Appendix C: Extra simulation results

TABLE 4

Average and standard deviation (between parentheses) of the Frobenius norm over 500 repetitions on the active and non-active sets, for the proposed estimators and different competitors, when $n = 25000$.

			Active set				Non-active set			
					K				K	
			1	5	10	20	1	5	10	20
$\boldsymbol{\Theta}$	Debiased	Full	1.00				6.50			
			(.01)				(.01)			
		owAvg		1.03	1.08	1.18		6.72	7.15	7.79
				(.01)	(.01)	(.01)		(.01)	(.01)	(.01)
		Top1		1.39	1.46	1.19		9.12	9.60	7.85
				(.01)	(.01)	(.01)		(.01)	(.01)	(.01)
		sAvg		1.59	2.88	3.95		10.27	16.91	20.15
				(.01)	(.02)	(.03)		(.03)	(.02)	(.03)
		wAvg		1.04	1.25	1.81		6.80	8.00	10.10
				(.01)	(.01)	(.01)		(.01)	(.01)	(.01)
	Sparse	SFull	2.74				.35			
			(.01)				(.00)			
		STop1		2.82	2.84	2.88		1.31	1.56	2.28
				(.01)	(.01)	(.01)		(.01)	(.01)	(.01)
		SsAvg		2.54	2.29	2.04		4.29	7.37	8.83
				(.01)	(.01)	(.01)		(.01)	(.01)	(.01)
		SwAvg		2.59	2.45	2.22		1.93	2.77	4.01
				(.01)	(.01)	(.01)		(.00)	(.00)	(.01)
$\boldsymbol{\Gamma}$	Debiased	Full	1.55				15.43			
			(.01)				(.02)			
		owAvg		1.62	1.74	1.55		16.18	17.30	14.28
				(.02)	(.02)	(.01)		(.02)	(.02)	(.02)
		Top1		2.21	2.34	2.67		22.08	23.33	26.65
				(.02)	(.02)	(.03)		(.03)	(.03)	(.03)
		sAvg		2.42	3.91	7.09		24.17	39.04	70.18
				(.02)	(.04)	(1.45)		(.03)	(.06)	(14.36)
		wAvg		1.63	1.90	3.38		16.30	18.93	33.51
				(.02)	(.02)	(.61)		(.02)	(.02)	(6.03)
	Sparse	SFull	4.79				1.82			
			(.00)				(.00)			
		STop1		4.80	4.80	4.80		1.91	1.93	1.99
				(.00)	(.00)	(.00)		(.00)	(.00)	(.01)
		SsAvg		4.67	4.23	4.17		2.03	2.82	3.11
				(.12)	(.02)	(.02)		(.02)	(.01)	(.01)
		SwAvg		4.75	4.59	4.51		1.82	1.84	1.98
				(.04)	(.01)	(.01)		(.02)	(.00)	(.00)

TABLE 5
Average coverage probability and average length of the confidence intervals over 500 repetitions for the proposed estimators and different competitors, when $n = 25000$.

			Avg.Cov				Avg.Len			
			K				K			
			1	5	10	20	1	5	10	20
Θ	Active set	Full	.94				.02			
		owAvg		.94	.94	.95		.02	.03	.03
		Top1		.94	.94	.94		.03	.04	.03
		sAvg		.92	.89	.90		.04	.06	.08
		wAvg		.95	.98	.98		.03	.04	.06
	Non-active set	Full	.95				.02			
		owAvg		.95	.96	.96		.02	.03	.03
		Top1		.96	.96	.95		.03	.04	.03
		sAvg		.94	.94	.97		.04	.06	.08
		wAvg		.97	.98	.98		.03	.04	.06
Γ	Active set	Full	.95				.08			
		owAvg		.95	.95	.92		.08	.09	.07
		Top1		.95	.95	.95		.11	.12	.14
		sAvg		.94	.92	.90		.12	.17	.27
		wAvg		.96	.97	.98		.09	.12	.20
	Non-active set	Full	.95				.08			
		owAvg		.95	.95	.94		.08	.09	.07
		Top1		.95	.95	.95		.11	.12	.14
		sAvg		.94	.92	.90		.12	.17	.27
		wAvg		.96	.97	.98		.09	.12	.20

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