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Applying monoid duality to a double contact process*

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Abstract

In this paper we use duality techniques to study a coupling of the well-known *contact process* (CP) and the *annihilating branching process*. As the latter can be seen as a *cancellative* version of the contact process, we rebrand it as the *cancellative contact process* (cCP). Our process of interest will consist of two components, the first being a CP and the second being a cCP. We call this process the *double contact process* (2CP) and prove that it has (depending on the model parameters) at most one invariant law under which ones are present in both processes. In particular, we can choose the model parameters in such a way that CP and cCP are monotonely coupled. In this case also the above mentioned invariant law will have the property that, under it, ones (modeling "infected individuals") can only be present in the cCP at sites where there are also ones in the CP. Along the way we extend the dualities for Markov processes discovered in our paper "Commutative monoid duality" to processes on infinite state spaces so that they, in particular, can be used for interacting particle systems.

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1 Introduction

1.1 Aim of the paper

After having identified in [10] a class of duality functions based on commutative monoids, our aim for this present paper is to apply one of those dualities to a specific process. To do so we couple the contact process with its cancellative version, the process formerly known as the annihilating branching process. The considerations in [10] indicate that this coupled process has a self-duality that we use here to characterise all invariant laws of the process.

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To use the dualities discovered in [10], we first have to generalise the techniques presented in [10] to infinite state spaces. This is done in Section 2 and is one of the main contributions of the present paper.

Additionally, in Section 3, we give precise definitions and some first results towards the goal of characterising all duality functions of the type considered in [10] that determine the law of a process uniquely. This was posed as an open problem in [10, Section 1.5].

1.2 Contact processes

We set $T := \{0, 1\}$ and let $\mathcal{T}$ denote the space of all functions $x : \mathbb{Z}^d \to T$. Moreover, we let $\vee$ and $\oplus$ denote the binary operators on T defined by the addition tables

$$\begin{array}{c|cc} \vee & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}, \qquad \begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}.$$

In words, this says that for $a, b \in T$ the quantity $a \vee b$ is the maximum of a and b and $a \oplus b$ is the sum of a and b modulo 2. For all $i, j \in \mathbb{Z}^d$, we define "infection maps" $\mathtt{inf}^*_{ij} : \mathcal{T} \to \mathcal{T}$ ($* \in \{\vee, \oplus\}$) and a "death map" $\mathtt{dth}_i : \mathcal{T} \to \mathcal{T}$ as follows:

$$\mathtt{inf}^*_{ij}(x)(k) := \begin{cases} x(i) * x(j) & \text{if } k = j, \\ x(k) & \text{else,} \end{cases}, \qquad \mathtt{dth}_i(x)(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{else.} \end{cases} \tag{1.1}$$

We say that $i, j \in \mathbb{Z}^d$ are nearest neighbours and write $i \sim j$ if $\|i - j\|_1 = 1$. We define formal generators

$$G_* f(x) := \lambda \sum_{i,j \in \mathbb{Z}^d : i \sim j} \big\{ f\big(\mathtt{inf}^*_{ij}(x)\big) - f\big(x\big) \big\} + \delta \sum_{i \in \mathbb{Z}^d} \big\{ f\big(\mathtt{dth}_i(x)\big) - f\big(x\big) \big\} \tag{1.2}$$

for $* \in \{\vee, \oplus\}$, where $\lambda, \delta \geq 0$ are model parameters. It is well-known (compare [15, Theorem 4.30]) that continuous functions that depend only on finitely many coordinates form a core for the generator G_* ($* \in \{\vee, \oplus\}$). In words, we can describe the dynamics of the process generated by G_* ($* \in \{\vee, \oplus\}$) as follows:

- At each site $i \in \mathbb{Z}^d$ sit two "exponential clocks", one with rate $2d\lambda$ for *reproduction* and one with rate δ for *death*.
- If the clock for reproduction at site $i \in \mathbb{Z}$ rings, the corresponding individual *reproduces* by choosing a neighbouring site j uniformly at random and adding its local state to the local state at j, where addition has to be interpreted in the sense of the operator $*$.
- If the "death clock" at site i rings, individual i *dies* which means that its local state is replaced by 0, regardless of its previous value.

The process $C = (C_t)_{t \geq 0}$ with generator $G_\vee$ is the well-known contact process on $\mathbb{Z}^d$ with infection rate λ and death rate δ (introduced in [4]). We denote this process shortly as CP(λ, δ). The process $D = (D_t)_{t \geq 0}$ with generator $G_\oplus$ was introduced as the annihilating branching process in [2]. We refer to it as the cancellative contact process (cCP(λ, δ)) to stress the similarity of the two processes, which differ only in the type of operator used in the definition of the infection maps $\mathtt{inf}^*_{ij}$ ($* \in \{\vee, \oplus\}$).

To speak about the long-time behaviour of the CP and the cCP we define shift operators $\theta_i : \mathcal{T} \to \mathcal{T}$ by

$$(\theta_i x)(j) := x(j - i) \qquad \big(i, j \in \mathbb{Z}^d,\ x \in \mathcal{T}\big). \tag{1.3}$$

We say that a probability measure μ on $\mathcal{T}$ is *shift-invariant* if $\mu = \mu \circ \theta_i^{-1}$ ($i \in \mathbb{Z}^d$). For $a \in T$ we let $\underline{a}$ denote the constant configuration $\underline{a}(i) := a$ ($i \in \mathbb{Z}^d$). We say that a distribution μ on $\mathcal{T}$ is *non-trivial* if $\mu(\{\underline{0}\}) = 0$.

It is well-known [15, Theorem 6.35] that the $\mathrm{CP}(\lambda, \delta)$ with $\lambda + \delta > 0$ started in a non-trivial shift-invariant distribution converges weakly to a (time-) invariant distribution $\bar{\nu}$ called the *upper invariant law* of the contact process. Similarly, it is known [2, Theorem 1.2 & Theorem 1.3] that the $\mathrm{cCP}(\lambda, \delta)$ with $\lambda + \delta > 0$ started in a non-trivial shift-invariant distribution converges weakly to an invariant distribution $\dot{\nu}$, that we call, in accordance with [12], the *odd upper invariant law* of the cancellative contact process.

Letting $\delta_{\underline{0}}$ denote the Dirac measure concentrated on the "all 0" configuration $\underline{0}$, $\bar{\nu}$ and $\dot{\nu}$ may or may not differ from $\delta_{\underline{0}}$ depending on the choice of the model parameters λ and δ. For a $\mathrm{CP}(\lambda, \delta)$ ($\lambda + \delta > 0$) there exists a critical value $\lambda_{\mathrm{CP}} = \lambda_{\mathrm{CP}}(d) \in (0, \infty)$ (dependent on the dimension d) such that $\bar{\nu} \neq \delta_{\underline{0}}$ if and only if $\lambda/\delta > \lambda_{\mathrm{CP}}$ [7, Chapter IV.1], [1]. Here and in the following we set $x/0 = \infty$ for $x \in (0, \infty)$. For the cCP we can define $\lambda^{\pm}_{\mathrm{cCP}} = \lambda^{\pm}_{\mathrm{cCP}}(d)$ as

$$\begin{aligned}
\lambda^{-}_{\mathrm{cCP}} &:= \inf\{\lambda \geq 0 : \text{the odd upper invariant law of the cCP}(\lambda, 1) \text{ does not equal } \delta_{\underline{0}}\}, \\
\lambda^{+}_{\mathrm{cCP}} &:= \sup\{\lambda \geq 0 : \text{the odd upper invariant law of the cCP}(\lambda, 1) \text{ equals } \delta_{\underline{0}}\}.
\end{aligned}$$

It is known that $\lambda^{+}_{\mathrm{cCP}} < \infty$ ([2, Theorem 1.1] & Proposition 5.1 below). By coupling the CP and cCP in such a way that infections and deaths only occur in both processes simultaneously (see below) one shows that $\lambda_{\mathrm{CP}} \leq \lambda^{-}_{\mathrm{cCP}}$. Thus, it is established that

$$0 < \lambda_{\mathrm{CP}} \leq \lambda^{-}_{\mathrm{cCP}} \leq \lambda^{+}_{\mathrm{cCP}} < \infty.$$

Simulations suggest that $\lambda^{-}_{\mathrm{cCP}} = \lambda^{+}_{\mathrm{cCP}}$ and $\lambda_{\mathrm{CP}} < \lambda^{-}_{\mathrm{cCP}}$ in all dimensions. The first assertion is a long-standing open problem that due to the non-monotone nature of the process seems very difficult. Using the bound $\lambda_{\mathrm{CP}}(1) \leq 1.942$, proved in [8], and the following result we can conclude the latter assertion at least in dimension one.

Proposition 1.1 (Lower bound for $\boldsymbol{\lambda^{-}_{\mathrm{cCP}}(1)}$)**.** *One has* $\lambda^{-}_{\mathrm{cCP}}(1) \geq 2$*.*

As the methods in the proof of Proposition 1.1 (to be found in Section 5) are essentially one-dimensional in nature, it is not clear how to generalise the result to higher dimensions.

1.3 The double contact process

We will be interested in a joint process, consisting of a CP and a cCP, that are coupled in such a way that some of the infections and deaths happen for both processes at the same times. Our motivation to study this coupled process comes primarily from the theoretical side of view. The duality techniques explored below are by no means restricted to this one process. In particular, further couplings of "classic" interacting particle systems can be studied in a similar way. However, in order to prevent the reader from getting lost in abstract statements, we stick to this one process. Further details regarding additional applications of monoid duality to similar processes are given within the text below, in particular at the end of Section 2 and below Proposition 3.2.

Informally, the coupled process of interest will behave in the following way. With rates $\lambda, \delta \geq 0$ infections and deaths, respectively, happen simultaneously for the CP and the cCP. With rates $\lambda_{\vee}, \delta_{\vee} \geq 0$ they only happen for the CP and with rates $\lambda_{\oplus}, \delta_{\oplus} \geq 0$ only for the cCP.

It will be helpful to write the generator of the coupled process in a form similar to (1.2). To achieve this, we define $U := T \times T = \{0, 1\} \times \{0, 1\}$. In parallel to the above we denote by $\mathcal{U}$ the space of all functions $x = (x_1, x_2) : \mathbb{Z}^d \to U$ and for each $i, j \in \mathbb{Z}^d$, we define infection maps $\texttt{INF}_{ij}, \texttt{inf}^1{}_{ij}, \texttt{inf}^2{}_{ij} : \mathcal{U} \to \mathcal{U}$ and death maps $\texttt{DTH}_i, \texttt{dth}^1{}_i, \texttt{dth}^2{}_i : \mathcal{U} \to \mathcal{U}$ as

$$\begin{aligned}
&\texttt{INF}_{ij}(x) := (\texttt{inf}^{\vee}_{ij}(x_1), \texttt{inf}^{\oplus}_{ij}(x_2)), && \texttt{DTH}_i(x) := (\texttt{dth}_i(x_1), \texttt{dth}_i(x_2)),\\
&\texttt{inf}^1{}_{ij}(x) := (\texttt{inf}^{\vee}_{ij}(x_1), x_2), && \texttt{dth}^1{}_i(x) := (\texttt{dth}_i(x_1), x_2),\\
&\texttt{inf}^2{}_{ij}(x) := (x_1, \texttt{inf}^{\oplus}_{ij}(x_2)), && \texttt{dth}^2{}_i(x) := (x_1, \texttt{dth}_i(x_2)), \qquad (x = (x_1, x_2) \in \mathcal{U}),
\end{aligned} \tag{1.4}$$

where the maps on the right hand sides are the maps from (1.1). We then define the generator $G_{\veebar}$ as

$$\begin{aligned}
G_{\veebar} f(x) := \lambda &\sum_{i,j \in \mathbb{Z}^d : i \sim j} \big\{ f\big(\texttt{INF}_{ij}(x)\big) - f\big(x\big)\big\} + \delta \sum_{i \in \mathbb{Z}^d} \big\{ f\big(\texttt{DTH}_i(x)\big) - f\big(x\big)\big\}\\
&+ \lambda_{\vee} \sum_{i,j \in \mathbb{Z}^d : i \sim j} \big\{ f\big(\texttt{inf}^1{}_{ij}(x)\big) - f\big(x\big)\big\} + \delta_{\vee} \sum_{i \in \mathbb{Z}^d} \big\{ f\big(\texttt{dth}^1{}_i(x)\big) - f\big(x\big)\big\}\\
&+ \lambda_{\oplus} \sum_{i,j \in \mathbb{Z}^d : i \sim j} \big\{ f\big(\texttt{inf}^2{}_{ij}(x)\big) - f\big(x\big)\big\} + \delta_{\oplus} \sum_{i \in \mathbb{Z}^d} \big\{ f\big(\texttt{dth}^2{}_i(x)\big) - f\big(x\big)\big\},
\end{aligned} \tag{1.5}$$

where $\lambda, \delta, \lambda_{\vee}, \delta_{\vee}, \lambda_{\oplus}, \delta_{\oplus} \geq 0$ are model parameters. Standard results [15, Theorem 4.30] tell us that the process $X = (X^1, X^2) = (X^1_t, X^2_t)_{t \geq 0}$ with generator $G_{\veebar}$ is (like C and D before) well-defined. For later use, letting

$$\mathcal{U}_{\rm fin} := \big\{ x = (x_1, x_2) \in \mathcal{U} : |\{k \in \mathbb{Z}^d : (x_1(k), x_2(k)) \neq (0,0)\}| < \infty \big\} \tag{1.6}$$

denote the set of finite configurations, one has, by Theorem 2.7 below, for all choices of model parameters that

$$X_0 \in \mathcal{U}_{\rm fin} \qquad \text{implies} \qquad X_t \in \mathcal{U}_{\rm fin} \qquad (t \geq 0) \quad \text{almost surely.} \tag{1.7}$$

We call X the *double contact process* and denote it shortly as $\mathrm{2CP}(\lambda, \delta, \lambda_{\vee}, \delta_{\vee}, \lambda_{\oplus}, \delta_{\oplus})$. If X is a $\mathrm{2CP}(\lambda, \delta, \lambda_{\vee}, \delta_{\vee}, \lambda_{\oplus}, \delta_{\oplus})$, then X^1 is a $\mathrm{CP}(\lambda + \lambda_{\vee}, \delta + \delta_{\vee})$ and X^2 is a $\mathrm{cCP}(\lambda + \lambda_{\oplus}, \delta + \delta_{\oplus})$.

In particular, if $\lambda = \delta = 0$, then X^1 and X^2 are independent processes. On the other extreme, if $\delta_{\vee} = \lambda_{\vee} = \delta_{\oplus} = \lambda_{\oplus} = 0$, then X^1 and X^2 are fully coordinated in the sense that their infections and deaths happen at the same times. An interesting consequence of this choice of parameters is that the CP stochastically dominates the cCP. The first part of the following lemma says that this holds a bit more generally: if $\delta_{\vee} = \lambda_{\oplus} = 0$ and the process is started in an initial state such that the CP dominates the cCP, then it follows from the definition of the maps in (1.4) that this order is preserved by the evolution.

Lemma 1.2 (Special choice of parameters)**.** *Assume that $X = (X^1, X^2) = (X^1_t, X^2_t)_{t \geq 0}$ is a $\mathrm{2CP}(\lambda, \delta, \lambda_{\vee}, \delta_{\vee}, \lambda_{\oplus}, \delta_{\oplus})$ with $\delta_{\vee} = \lambda_{\oplus} = 0$. Then*

$$X^1_0(k) \geq X^2_0(k) \quad (k \in \mathbb{Z}^d) \quad \textit{implies} \quad X^1_t(k) \geq X^2_t(k) \quad (k \in \mathbb{Z}^d,\ t \geq 0). \tag{1.8}$$

In this paper we are interested in the long-time behaviour of the 2CP started in a shift-invariant distribution. With a slight abuse of notation we define shift operators $\theta_i : \mathcal{U} \to \mathcal{U}$ by applying the operators from (1.3) in both coordinates. As for distributions on $\mathcal{T}$ above we say that a probability measure μ on $\mathcal{U}$ is shift-invariant if $\mu = \mu \circ \theta_i^{-1}$ ($i \in \mathbb{Z}^d$). Moreover, we say that a distribution μ on $\mathcal{U}$ is non-trivial if

$$\mu\big(\{\underline{(0,0)}\}\big) = 0,$$

where also for $a \in U$ the configuration $\underline{a} \in \mathcal{U}$ is defined as $\underline{a}(i) := a$. We set

$$\begin{aligned}\mathcal{U}_{(0,*)} &:= \big\{x = (x_1, x_2) \in \mathcal{U} : x_1 = \underline{0}\big\},\\ \mathcal{U}_{(*,0)} &:= \big\{x = (x_1, x_2) \in \mathcal{U} : x_2 = \underline{0}\big\},\\ \mathcal{U}_{\text{mix}} &:= \mathcal{U} \setminus (\mathcal{U}_{(0,*)} \cup \mathcal{U}_{(*,0)}).\end{aligned}$$

The known results for CP and cCP imply that the 2CP $X = (X_t)_{t\geq 0} = (X^1_t, X^2_t)_{t\geq 0}$ started in a non-trivial shift-invariant distribution on $\mathcal{U}_{(*,0)}$ converges weakly to $\bar{\nu} \otimes \delta_{\underline{0}}$. Analogously, the 2CP started in a non-trivial shift-invariant distribution on $\mathcal{U}_{(0,*)}$ converges weakly to $\delta_{\underline{0}} \otimes \dot{\nu}$. If X is started in a non-trivial shift-invariant distribution on $\mathcal{U}_{\text{mix}}$, then the laws of X^1_t and X^2_t individually converge weakly as $t \to \infty$ to $\bar{\nu}$ and $\dot{\nu}$, respectively. However, as a measure on a product space is in general not determined by its marginals, the long-time behaviour of the joint law of $X_t = (X^1_t, X^2_t)$ is less straightforward.

A priori there might, for example, exist an increasing sequence $(t_n)_{n\in\mathbb{N}}$ so that the sequence of laws of $(X_{t_n})_{n\in\mathbb{N}}$ has several cluster points all having the marginal distributions $\bar{\nu}$ and $\dot{\nu}$, respectively. Or the law of X might converge weakly to different distributions depending on where on $\mathcal{U}_{\text{mix}}$ its initial law is supported. We will show that the behaviour outlined in the last two sentences does not occur.

Theorem 1.3 (Joint invariant law)**.** *Let $X = (X^1, X^2) = (X^1_t, X^2_t)_{t\geq 0}$ be a 2CP with parameters $\lambda, \delta, \lambda_\vee, \delta_\vee, \lambda_\oplus, \delta_\oplus \geq 0$ so that $\lambda + \lambda_\vee + \delta + \delta_\vee > 0$ and $\lambda + \lambda_\oplus + \delta + \delta_\oplus > 0$. Then X has an invariant law ν so that if X is started in a shift-invariant initial law that is concentrated on $\mathcal{U}_{\text{mix}}$, then*

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu, \tag{1.9}$$

i.e. the law of X converges weakly to ν.

Note that (1.9) implies that ν is (as $\bar{\nu}$ and $\dot{\nu}$) shift-invariant. In the special case that $\delta_\vee = \lambda_\oplus = 0$, corresponding to the monotone coupling of CP and cCP, one has that

$$\nu\big(\big\{x \in \mathcal{U} : \exists\, k \in \mathbb{Z}^d : x(k) = (0,1)\big\}\big) = 0,$$

as we can chose a shift-invariant initial law that is concentrated on $\mathcal{U}_{\text{mix}}$ with the above property. This property is then preserved by the dynamics. One example of such an initial law would be the Dirac measure concentrated on $\underline{(1,1)}$. Thus, as long as the initial distribution of this special 2CP is shift-invariant and concentrated on $\mathcal{U}_{\text{mix}}$, the law of this 2CP converges weakly to a monotonically coupled law, no matter how high the density of $(0,1)$s was in the initial distribution.

Taking into account our earlier remarks about initial laws on $\mathcal{U}_{(0,*)}$ and $\mathcal{U}_{(*,0)}$, one can conclude (compare [15, Corollary 6.39]) that all shift-invariant invariant laws of the 2CP are convex combinations of $\delta_{\underline{0}} \otimes \delta_{\underline{0}}$, $\bar{\nu} \otimes \delta_{\underline{0}}$, $\delta_{\underline{0}} \otimes \dot{\nu}$ and ν.

1.4 Duality

The main tool to prove Theorem 1.3 will be duality. It is well-known that the CP is self-dual in the sense of additive systems duality [5, Section 7b]. Similarly, the cCP is self-dual in the sense of cancellative systems duality [2, Proposition 1.1]. This suggests that the 2CP should also possess a self-duality.

To present a complete picture we repeat the definitions of the additive and the cancellative duality function. Analogously to [10] we define $\psi_1, \psi_2 : T \times T \to T$ as

$$\begin{pmatrix}\psi_1(0,0) & \psi_1(0,1)\\ \psi_1(1,0) & \psi_1(1,1)\end{pmatrix} = \begin{pmatrix}\psi_2(0,0) & \psi_2(0,1)\\ \psi_2(1,0) & \psi_2(1,1)\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{1.10}$$

in parallel to (1.6) we set

$$\mathcal{T}_{\text{fin}} := \big\{x \in \mathcal{T} : |\{k \in \mathbb{Z}^d : x(k) \neq 0\}| < \infty\big\} \tag{1.11}$$

and, for $x, y \in \mathcal{T}$ so that either $x \in \mathcal{T}_{\text{fin}}$ or $y \in \mathcal{T}_{\text{fin}}$, we define

$$\boldsymbol{\psi}_1(x,y) := \bigvee_{k\in\mathbb{Z}^d} \psi_1\big(x(k), y(k)\big) \qquad \text{and} \qquad \boldsymbol{\psi}_2(x,y) := \bigoplus_{k\in\mathbb{Z}^d} \psi_2\big(x(k), y(k)\big), \tag{1.12}$$

where $\vee$ and $\oplus$ are the operators defined at the beginning of Section 1.2 corresponding to taking the maximum and addition modulo 2, respectively. Since either $x \in \mathcal{T}_{\text{fin}}$ or $y \in \mathcal{T}_{\text{fin}}$, $\psi_i(x(k), y(k)) = 0$ $(i = 1, 2)$ for all but finitely many $k \in \mathbb{Z}^d$ and hence the expressions are well-defined. Fix $\lambda, \delta \geq 0$, let $(X^1_t)_{t\geq 0}$ denote the CP(λ, δ), and let $(X^2_t)_{t\geq 0}$ denote the cCP(λ, δ). Following [15, Lemma 6.6 and Lemma 6.11] the self-dualities of the contact process and the cancellative contact process can be written as

$$\mathbb{E}^x\big[\boldsymbol{\psi}_i(X^i_t, y)\big] = \mathbb{E}^y\big[\boldsymbol{\psi}_i(x, X^i_t)\big] \qquad (x \in \mathcal{T},\ y \in \mathcal{T}_{\text{fin}},\ t \geq 0,\ i = 1, 2), \tag{1.13}$$

where $\mathbb{E}^z$ denotes expectation with respect to the law of the process $(X^i_t)_{t\geq 0}$ $(i = 1, 2)$ started in the initial state $z \in \{x, y\}$, i.e. $X^i_0 = z$. In general, throughout this paper, we write $\mathbb{P}^z$ and $\mathbb{E}^z$ to denote the law and expectation of a Markov process $Z = (Z_t)_{t\geq 0}$ started in the initial state $Z_0 = z$.

We will prove a similar self-duality for the 2CP. The first step is to find the right duality function. To this aim, we rewrite the duality functions $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2$ in (1.12) in such a way that the operators $\vee$ and $\oplus$ are replaced by the product in $\mathbb{R}$. For this purpose, we define maps $\gamma_i : T \to \mathbb{R}$ $(i = 1, 2)$ by

$$\gamma_1(0) = 1, \quad \gamma_1(1) = 0 \qquad \text{and} \qquad \gamma_2(0) = 1, \quad \gamma_2(1) = -1. \tag{1.14}$$

Then it is easy to check that $\gamma_1(a \vee b) = \gamma_1(a) \cdot \gamma_1(b)$ and $\gamma_2(a \oplus b) = \gamma_2(a) \cdot \gamma_2(b)$ $(a, b \in T)$. We define, again for $x, y \in \mathcal{T}$ so that either $x \in \mathcal{T}_{\text{fin}}$ or $y \in \mathcal{T}_{\text{fin}}$,

$$\boldsymbol{\psi}_{\text{add}}(x,y) := \gamma_1\big(\boldsymbol{\psi}_1(x,y)\big) \qquad \text{and} \qquad \boldsymbol{\psi}_{\text{canc}}(x,y) := \gamma_2\big(\boldsymbol{\psi}_2(x,y)\big). \tag{1.15}$$

One then readily checks that

$$\boldsymbol{\psi}_{\text{add}}(x,y) = \prod_{k\in\mathbb{Z}^d} \gamma_1\big(\psi_1(x(k), y(k))\big) \qquad \text{and} \qquad \boldsymbol{\psi}_{\text{canc}}(x,y) = \prod_{k\in\mathbb{Z}^d} \gamma_2\big(\psi_2(x(k), y(k))\big),$$

where the product is the usual product in $\mathbb{R}$. As γ_1 and γ_2 are bijections from T to $\{0, 1\}$ resp. to $\{-1, 1\}$, (1.13) remains true if we replace $\boldsymbol{\psi}_1$ by $\boldsymbol{\psi}_{\text{add}}$ and $\boldsymbol{\psi}_2$ by $\boldsymbol{\psi}_{\text{canc}}$.

We now define, for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{U}$ so that either $x \in \mathcal{U}_{\text{fin}}$ or $y \in \mathcal{U}_{\text{fin}}$,

$$\boldsymbol{\psi}(x,y) := \boldsymbol{\psi}_{\text{add}}(x_1, y_1)\boldsymbol{\psi}_{\text{canc}}(x_2, y_2).$$

One then checks that

$$\boldsymbol{\psi}(x,y) = \prod_{k\in\mathbb{Z}^d} \psi\big(x(k), y(k)\big), \tag{1.16}$$

where

$$\psi\big(x(k), y(k)\big) = \gamma_1\big(\psi_1(x_1(k), y_1(k))\big)\gamma_2\big(\psi_2(x_2(k), y_2(k))\big), \tag{1.17}$$

i.e. $\psi : U \times U \to \{-1, 0, 1\}$ is defined as

$$\begin{pmatrix} \psi((0,0),(0,0)) & \psi((0,0),(0,1)) & \psi((0,0),(1,0)) & \psi((0,0),(1,1)) \\ \psi((0,1),(0,0)) & \psi((0,1),(0,1)) & \psi((0,1),(1,0)) & \psi((0,1),(1,1)) \\ \psi((1,0),(0,0)) & \psi((1,0),(0,1)) & \psi((1,0),(1,0)) & \psi((1,0),(1,1)) \\ \psi((1,1),(0,0)) & \psi((1,1),(0,1)) & \psi((1,1),(1,0)) & \psi((1,1),(1,1)) \end{pmatrix} := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}. \tag{1.18}$$

The basis of the present paper is the following duality relation.

Proposition 1.4 (Basic duality relation). *For $\lambda, \delta, \lambda_\vee, \delta_\vee, \lambda_\oplus, \delta_\oplus \geq 0$ let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ both be a $2\mathrm{CP}(\lambda, \delta, \lambda_\vee, \delta_\vee, \lambda_\oplus, \delta_\oplus)$. Fixing a $t \geq 0$ one can almost surely construct X and Y on a common probability space in such a way that for every $s \in [0,t]$ the random variables X_s and Y_{t-s} are independent and*

$$[0,t] \ni s \mapsto \boldsymbol{\psi}(X_s, Y^-_{t-s})$$

is constant, where $Y^- = (Y^-_t)_{t\geq 0}$ is the càglàd modification of Y, i.e. it is left-continuous with right limits but coincides almost everywhere with Y, which is càdlàg, i.e. right-continuous with left limits.

In fact, in the following we only need equality in expectation, i.e. that

$$\mathbb{E}\big[\boldsymbol{\psi}(X_s, Y_{t-s})\big] = \mathbb{E}\big[\boldsymbol{\psi}(X_u, Y_{t-u})\big] \tag{1.19}$$

for all $s, u \in [0,t]$. Here the symbol $\mathbb{E}$ denotes expectation with respect to the probability measure of the underlying probability space on which both X and Y are constructed. In particular, setting $s = t$ and $u = 0$ and restricting ourselves to the case that $Y_0 = y$ and $X_0 = x$ are deterministic, this is a relation of the form (1.13), but with the cancellative and additive duality functions $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ replaced by the new duality function $\boldsymbol{\psi}$. Note that, by (1.7) and the assumption that either $X_0 \in \mathcal{U}_{\text{fin}}$ or $Y_0 \in \mathcal{U}_{\text{fin}}$, the expression $\boldsymbol{\psi}(X_s, Y_{t-s})$ is well-defined for all $s \in [0,t]$. The following lemma highlights the strength of the duality relation (1.19).

Lemma 1.5 (The duality is informative). *If X and X' are $\mathcal{U}$-valued random variables such that*

$$\mathbb{E}\big[\boldsymbol{\psi}(X, y)\big] = \mathbb{E}\big[\boldsymbol{\psi}(X', y)\big]$$

for all $y \in \mathcal{U}_{\text{fin}}$, then X and X' are equal in distribution.

In particular, the duality function $\boldsymbol{\psi}$ characterises the invariant law ν from Theorem 1.3 in the following way.

Proposition 1.6 (Characterisation of the invariant law). *The invariant law ν from Theorem 1.3 is uniquely characterised by the relation*

$$\int \boldsymbol{\psi}(x,y)\, \mathrm{d}\nu(x) = \mathbb{P}^y\Big[\exists\, t \geq 0 : X_t = \underline{(0,0)}\Big] \qquad \big(y \in \mathcal{U}_{\text{fin}}\big).$$

1.5 Outline

The paper is structured as follows. In Section 2 we provide a proof for Proposition 1.4. If fact, we prove in Theorem 2.6 a generalisation of Proposition 1.4 that is independent of our process of interest, so that it can directly be applied to further processes. Section 3 deals with the proof of Lemma 1.5. Also here we prove in Proposition 3.2 a generalisation of Lemma 1.5. Additionally, towards the goal of classifying the dualities found in [10] regarding their ability to determine laws of processes uniquely, we introduce two notions, namely the notions of *weak informativeness* and *informativeness*, and show that they basically coincide in our setup. In Section 4 we prove Theorem 1.3 and Proposition 1.6. As Proposition 1.1 is independent of the monoid dualities from [10], we prove it last. Its proof is found in Section 5. Finally, in Appendix A we show how Lemma 4.2, an auxiliary result we use for the proof of Theorem 2.6, follows from a corollary from [12]. As this corollary is stated in [12] in a rather general form, we decided to repeat the definitions from [12], slightly reformulate the result and move this discussion to the appendix.

2 Monoid duality for interacting particle systems

In [10] a duality theory is developed for Markov processes with state space of the form S^Λ where S is a finite commutative monoid and Λ is a finite set. Here we generalise this to countable Λ which allows us to define duality relations for interacting particle systems on countable lattices. For the special cases of additive and cancellative dualities infinite Λ have already been treated in [15, Chapter 6.6 & Chapter 6.7].

We start by extending the concept of duality between monoids (i.e. semigroups with a neutral element) presented in [10] to monoids that carry a topology. We say that a monoid $(M,+)$ is a *topological monoid* if it is equipped with a topology so that the map $M \times M \ni (x,y) \mapsto x+y \in M$ is continuous, where $M \times M$ is equipped with the product topology. For a second topological monoid $(N,+)$ we denote by $\mathcal{H}(M,N)$ the space of all continuous monoid homomorphisms, i.e. continuous functions from M to N that preserve the operation and map the neutral element of M to the neutral element of N. Throughout this paper we always equip finite and countable monoids with the discrete topology, so that every finite or countable monoid is a topological monoid. This makes every function between two finite or countable monoids continuous. Thus, if N and M are finite, the space $\mathcal{H}(M,N)$ defined above coincides with the space of all monoid homomorphisms (called $\mathcal{H}(M,N)$ in [10]).

Let M_1, M_2 and N be topological monoids. We say that M_1 is N*-dual* to M_2 with *duality function* ψ if the following conditions are satisfied:

(i) $\psi(x_1,y) = \psi(x_2,y)$ for all $y \in M_2$ implies $x_1 = x_2$ $(x_1,x_2 \in M_1)$,
(ii) $\mathcal{H}(M_1,N) = \{\psi(\,\cdot\,,y) : y \in M_2\}$,
(iii) $\psi(x,y_1) = \psi(x,y_2)$ for all $x \in M_1$ implies $y_1 = y_2$ $(y_1,y_2 \in M_2)$,
(iv) $\mathcal{H}(M_2,N) = \{\psi(x,\,\cdot\,) : x \in M_1\}$.

As we equip finite monoids with the discrete topology, the definition above coincides with the definition of duality between monoids from [10] if M_1, M_2 and N are finite.

Repeating the definition from [10], for arbitrary spaces $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ we say that the map $m : \mathcal{X} \to \mathcal{X}$ is *dual* to the map $\hat{m} : \mathcal{Y} \to \mathcal{Y}$ with respect to the *duality function* $\psi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ if

$$\psi(m(x),y) = \psi(x,\hat{m}(y)) \qquad (x \in \mathcal{X},\ y \in \mathcal{Y}).$$

In parallel to [15] we say that a map $m : \mathcal{X} \to \mathcal{X}$ *preserves* a set $\mathcal{H}$ of functions from $\mathcal{X}$ to $\mathcal{Y}$ if

$$f \circ m \in \mathcal{H} \quad \text{for all} \quad f \in \mathcal{H}.$$

The following proposition is the analogue of [10, Proposition 5], that is formulated for dualities between monoids without attached topologies.

Proposition 2.1 (Maps having a dual)**.** *Let S, R and T be commutative topological monoids such that S is T-dual to R with duality function ψ. Then a map $m : S \to S$ has a dual map $\hat{m} : R \to R$ with respect to ψ if and only if m preserves $\mathcal{H}(S,T)$. The dual map $\hat{m}$, if it exists, is unique and preserves $\mathcal{H}(R,T)$.*

Proof. If $m : S \to S$ preserves $\mathcal{H}(S,T)$, then, by property (ii) from the definition of duality, for all $y \in R$ one has $\psi(m(\,\cdot\,),y) \in \mathcal{H}(S,T)$. Applying property (ii) again, it follows that there exists an $\hat{m}(y) \in R$ such that $\psi(m(\,\cdot\,),y) = \psi(\,\cdot\,,\hat{m}(y))$. Property (iii) from the definition of duality implies that $\hat{m}(y)$ is unique. This shows that m has a unique dual map $\hat{m} : R \to R$ if m preserves $\mathcal{H}(S,T)$.

On the other hand, if $m : S \to S$ has a dual map $\hat{m} : R \to R$, then $\psi(m(\,\cdot\,),y) = \psi(\,\cdot\,,\hat{m}(y))$, i.e. m preserves $\{\psi(\,\cdot\,,y) : y \in R\}$. By property (ii) from the definition of duality m then also preserves $\mathcal{H}(S,T)$. This finishes the proof that $m: S \to S$ has a dual map $\hat{m}: R \to R$ if and only if m preserves $\mathcal{H}(S,T)$.

Finally, if $\hat{m}$ exists, then it has m as a dual map with respect to $\psi^\dagger : R \times S \to T$ defined as $\psi^\dagger(y,x) := \psi(x,y)$ $(y \in R,\ x \in S)$, and the previously proved statement implies that $\hat{m}$ has to preserve $\mathcal{H}(R,T)$. □

Clearly, any $m \in \mathcal{H}(S,S)$ preserves $\mathcal{H}(S,T)$. Conversely, if the assumptions on S, T and R from Proposition 2.1 are satisfied and $m: S \to S$ preserves $\mathcal{H}(S,T)$, then the proof of [10, Proposition 5] shows that $m: S \to S$ has to be a monoid homomorphism. However, while duality implies that $\psi(m(\,\cdot\,), y)$ is continuous for all $y \in R$, we do not know if m itself always has to be continuous.

We are especially interested in countable products of topological monoids as we will view state spaces of an interacting particle system as such products. Let, throughout this section, Λ be a countable set. For a topological monoid M with $|M| \geq 2$ we equip M^Λ with the product topology, making this uncountable monoid a topological monoid. We define the countable sub-monoid $M^\Lambda_{\text{fin}} \subset M^\Lambda$ as

$$M^\Lambda_{\text{fin}} := \{x \in M^\Lambda : |\{i \in \Lambda : x(i) \neq 0\}| < \infty\},$$

where 0 denotes the neutral element of M. As in Section 1.2, we denote by $\underline{0}$ the constant configuration with $\underline{0}(i) = 0$ for all $i \in \Lambda$ that is the neutral element of M^Λ_{fin} and M^Λ.

Before we investigate duality between such “product monoids” we collect some definitions and results for general product spaces from [15] that we will need in the following. Let L and V be arbitrary spaces. For a function $f: L^\Lambda \to V$ we say that $j \in \Lambda$ is *f-relevant* if

$$\exists\, x_1, x_2 \in L^\Lambda : f(x_1) \neq f(x_2) \text{ but } x_1(k) = x_2(k)\ \forall k \neq j.$$

We set

$$\mathcal{R}(f) := \{j \in \Lambda : j \text{ is } f\text{-relevant}\}$$

and cite the following result [15, Lemma 4.13].

Lemma 2.2 (Continuous maps)**.** *Let L and V be finite sets equipped with the discrete topology. A map $f: L^\Lambda \to V$ is continuous with respect to the product topology if and only if the following two conditions hold:*

(i) *$\mathcal{R}(f)$ is finite.*
(ii) *If $x_1, x_2 \in L^\Lambda$ satisfy $x_1(j) = x_2(j)$ for all $j \in \mathcal{R}(f)$, then $f(x_1) = f(x_2)$.*

Let L be finite. For any map $\mathtt{m}: L^\Lambda \to L^\Lambda$ and $i \in \Lambda$ we define $\mathtt{m}[i]: L^\Lambda \to L$ as

$$\mathtt{m}[i](x) := \mathtt{m}(x)(i) \qquad (x \in L^\Lambda).$$

Moreover, we let

$$\mathcal{D}(\mathtt{m}) := \{i \in \Lambda : \exists\, x \in L^\Lambda : \mathtt{m}[i](x) \neq x(i)\}.$$

We say that a map $\mathtt{m}: L^\Lambda \to L^\Lambda$ is *local* if

(i) $\mathtt{m}$ is continuous and (ii) $\mathcal{D}(\mathtt{m})$ is finite.

For a finite monoid M we denote by $\mathcal{H}_{\text{loc}}(M^\Lambda, M^\Lambda)$ the space of all maps $\mathtt{m} \in \mathcal{H}(M^\Lambda, M^\Lambda)$ that are local. As we equip, according to our conventions, M^Λ with the product topology and M with the discrete one, $\mathtt{m}: M^\Lambda \to M^\Lambda$ is local if and only if $\mathcal{D}(\mathtt{m})$ is finite and $\mathtt{m}[j]$

satisfies the conditions of Lemma 2.2 for all $j \in \Lambda$. Note that every $\mathtt{m} \in \mathcal{H}_{\rm loc}(M^\Lambda, M^\Lambda)$ maps $M^\Lambda_{\rm fin}$ into itself.

Let throughout the rest of this section $(S, \odot)$, $(R, \boxdot)$ and $(T, \otimes)$ be commutative finite (and hence topological) monoids and assume that S is T-dual to R with duality function $\psi : S \times R \to T$. We denote all three neutral elements by 0 and define $\Psi : S^\Lambda \times R^\Lambda_{\rm fin} \to T$ by

$$\Psi(\mathbf{x}, \mathbf{y}) := \bigotimes_{i\in\Lambda} \psi\big(\mathbf{x}(i), \mathbf{y}(i)\big) \qquad \big(\mathbf{x} \in S^\Lambda,\ \mathbf{y} \in R^\Lambda_{\rm fin}\big). \tag{2.1}$$

Note that Ψ is well-defined as for all but finitely many $i \in \Lambda$ one has $y(i) = 0$ and $\psi(\,\cdot\,, 0) = o$ due to property (iv) of the definition of duality, where $o : S \to T$ is the function that is constantly 0. In general, for all monoids M and N, let $\mathrm{id} \in \mathcal{H}(M, M)$ denote the identity and $o \in \mathcal{H}(M, N)$ the function constantly 0. Using Lemma 2.2 we can prove the following.

Proposition 2.3 (Duality on product spaces)**.** *Let S, R, T be finite commutative monoids. If S is T-dual to R with duality function ψ, then S^Λ is T-dual to $R^\Lambda_{\rm fin}$ with duality function Ψ.*

Proof. The properties (i) and (iii) from the definition of duality follow directly from the corresponding properties of the duality between S and R. To be more precise, assuming that $\Psi(\mathbf{x}_1, \mathbf{y}) = \Psi(\mathbf{x}_2, \mathbf{y})$ for all $\mathbf{y} \in R^\Lambda_{\rm fin}$ in particular implies for $i \in \Lambda$ and $y \in R$ that

$$\psi(\mathbf{x}_1(i), y) = \Psi(\mathbf{x}_1, y^i) = \Psi(\mathbf{x}_2, y^i) = \psi(\mathbf{x}_2(i), y),$$

where $y^i \in R^\Lambda_{\rm fin}$ is defined as

$$y^i(j) = \begin{cases} y & \text{if } j = i, \\ 0 & \text{else}, \end{cases} \qquad (j \in \Lambda). \tag{2.2}$$

Hence, the fact that S is T-dual to R implies that $\mathbf{x}_1(i) = \mathbf{x}_2(i)$ for all $i \in \Lambda$ and thus $\mathbf{x}_1 = \mathbf{x}_2$. Property (iii) follows in the same way.

The fact that $\Psi(\,\cdot\,, \mathbf{y})$ and $\Psi(\mathbf{x}, \,\cdot\,)$ are monoid homomophisms for all $\mathbf{y} \in R^\Lambda_{\rm fin}$ and for all $\mathbf{x} \in S^\Lambda$, respectively, are implied by properties (ii) and (iv) of the duality between S and R and the definition of Ψ. As $R^\Lambda_{\rm fin}$ is countable this implies $\Psi(\mathbf{x}, \,\cdot\,) \in \mathcal{H}(R^\Lambda_{\rm fin}, T)$. For $\mathbf{y} \in R^\Lambda_{\rm fin}$ we have that $\mathcal{R}\big(\Psi(\,\cdot\,, \mathbf{y})\big) = \{j \in \Lambda : \mathbf{y}(j) \neq 0\}$, so $\Psi(\,\cdot\,, \mathbf{y})$ satisfies the conditions of Lemma 2.2 and hence also $\Psi(\,\cdot\,, \mathbf{y}) \in \mathcal{H}(S^\Lambda, T)$.

To prove the implication $\subset$ in property (iv) from the definition of duality, assume that $g \in \mathcal{H}(R^\Lambda_{\rm fin}, T)$. Then using (2.2), for each $i \in \Lambda$, we define $g_i : R \to T$ as $g_i(y) := g(y^i)$ $(i \in \Lambda)$. The fact that $g \in \mathcal{H}(R^\Lambda_{\rm fin}, T)$ directly implies that $g_i \in \mathcal{H}(R, T)$, and property (iv) of the duality between S and R implies that there exists an $x_i \in S$ such that $g_i = \psi\big(x_i, \,\cdot\,\big)$. Defining $\mathbf{x} \in S^\Lambda$ by $\mathbf{x}(i) := x_i$, one has for $\mathbf{y} \in R^\Lambda_{\rm fin}$ that

$$\begin{aligned} g(\mathbf{y}) = g\left(\boxdot_{i:\mathbf{y}(i)\neq 0} \mathbf{y}(i)^i\right) &= \bigotimes_{i:\mathbf{y}(i)\neq 0} g\big(\mathbf{y}(i)^i\big) = \bigotimes_{i:\mathbf{y}(i)\neq 0} g_i\big(\mathbf{y}(i)\big) = \bigotimes_{i:\mathbf{y}(i)\neq 0} \psi\big(x_i, \mathbf{y}(i)\big) \\ &= \Psi(\mathbf{x}, \mathbf{y}), \end{aligned}$$

which finishes the proof of property (iv) from the definition of duality.

Lastly, we prove the implication $\subset$ in property (ii) from the definition of duality. We assume that $f \in \mathcal{H}(S^\Lambda, T)$. Then Lemma 2.2 implies that there exists a finite set $\Delta \subset \Lambda$ such that f only depends on the coordinates in Δ. Letting for $\mathbf{x} \in S^\Lambda$ the restriction $\mathbf{x}_\Gamma$ to some set $\Gamma \subset \Lambda$ be defined as

$$\mathbf{x}_\Gamma(j) := \begin{cases} \mathbf{x}(j) & \text{if } j \in \Gamma, \\ 0 & \text{else}, \end{cases} \qquad (j \in \Lambda),$$

we see that

$$f(\mathbf{x}) = f\big(\mathbf{x}_{\Delta^c} \odot \mathbf{x}_\Delta\big) = f\big(\mathbf{x}_{\Delta^c}\big) \otimes \bigotimes_{i\in\Delta} f\big(\mathbf{x}(i)^i\big),$$

where $x^i \in S^\Lambda_{\text{fin}}$ is defined as $y^i \in R^\Lambda_{\text{fin}}$ in (2.2). But as f does not depend on Δ^c we conclude that

$$f\big(\mathbf{x}_{\Delta^c}\big) = f\big(\underline{0}_{\Delta^c}\big) = f(\underline{0}) = 0.$$

Analogously to above we can now define $\mathbf{y} \in R^\Lambda_{\text{fin}}$ by $\mathbf{y}(i) := y_i$ for $i \in \Delta$ and $\mathbf{y}(i) := 0$ for $i \in \Delta^c$, where $y_i \in R$ satisfies $f\big(\mathbf{x}(i)^i\big) = \psi(\mathbf{x}(i), y_i)$ independent of the value of $\mathbf{x}(i)$. Then $f = \Psi\big(\,\cdot\,, \mathbf{y}\big)$, which finishes the proof of property (ii) from the definition of duality and thus the proof is complete. $\square$

Having proved the duality between S^Λ and R^Λ_{fin}, Proposition 2.1 and the remarks below it imply that every $m \in \mathcal{H}(S^\Lambda, S^\Lambda)$ has a unique dual map with respect to Ψ. In fact, using the definition of duality and the properties of the product topology it is easy to see that $m : S^\Lambda \to S^\Lambda$ has a unique dual map with respect to Ψ if and only if $m \in \mathcal{H}(S^\Lambda, S^\Lambda)$.

However, it is not clear how to compute the dual map of $m \in \mathcal{H}(S^\Lambda, S^\Lambda)$ in general, so we will focus on local monoid homomorphisms, for which we will be able to compute the dual maps explicitly. The following lemma generalises [10, Lemma 7] to infinite Λ.

Lemma 2.4 (Local monoid homomorphisms)**.** *Let $(S, \odot)$ be a finite monoid. Let $M = (M_{ij})_{i,j\in\Lambda}$ be an infinite matrix with values in $\mathcal{H}(S, S)$ such that the set*

$$\Delta := \big\{(i,j) \in \Lambda^2 : i \neq j,\ M_{ij} \neq o\big\} \cup \big\{(i,i) \in \Lambda^2 : M_{ii} \neq \text{id}\big\} \tag{2.3}$$

is finite. Then setting

$$\mathtt{m}[j](\mathbf{x}) := \bigodot_{i\in\Lambda} M_{ij}\big(\mathbf{x}(i)\big) \qquad \big(j \in \Lambda,\ \mathbf{x} \in S^\Lambda\big) \tag{2.4}$$

defines a map $\mathtt{m} \in \mathcal{H}_{\text{loc}}(S^\Lambda, S^\Lambda)$. Conversely, each $\mathtt{m} \in \mathcal{H}_{\text{loc}}(S^\Lambda, S^\Lambda)$ is of this form.

Proof. First assume that $\mathtt{m}$ is of the form (2.4). Then $\mathtt{m}$ is well-defined as Δ from (2.3) is finite. As M takes values in $\mathcal{H}(S,S)$ it follows readily that $\mathtt{m}[j] \in \mathcal{H}(S^\Lambda, S)$ for all $j \in \Lambda$, thus $\mathtt{m} \in \mathcal{H}(S^\Lambda, S^\Lambda)$. Let $j \in \Lambda$. One sees that

$$\mathcal{R}(\mathtt{m}[j]) = \begin{cases} \{i \in \Lambda \setminus \{j\} : (i,j) \in \Delta\} \cup \{j\} & \text{if } M_{jj} \neq o, \\ \{i \in \Lambda \setminus \{j\} : (i,j) \in \Delta\} & \text{if } M_{jj} = o. \end{cases}$$

In both cases $\mathcal{R}(\mathtt{m}[j])$ satisfies the conditions of Lemma 2.2. Additionally

$$\mathcal{D}(\mathtt{m}) = \{j \in \Lambda : \exists\, i \in \Lambda : (i,j) \in \Delta\}$$

is finite and it follows that $\mathtt{m}$ is local, so $\mathtt{m} \in \mathcal{H}_{\text{loc}}(S^\Lambda, S^\Lambda)$.

Now assume that $\mathtt{m} \in \mathcal{H}_{\text{loc}}(S^\Lambda, S^\Lambda)$. In particular, one has that $\mathtt{m}[j] : S^\Lambda \to S$ is continuous for all $j \in \Lambda$ by the properties of the product topology. Moreover, $\mathcal{D}(\mathtt{m}) \subset \Lambda$ is finite and, by definition, for $j \in \mathcal{D}(\mathtt{m})^c$ one has $\mathtt{m}[j](\mathbf{x}) = \mathbf{x}(j)$ for all $\mathbf{x} \in S^\Lambda$. Due to Lemma 2.2, for each $j \in \mathcal{D}(\mathtt{m})$ the set $\mathcal{R}(\mathtt{m}[j])$ is finite and we can identify $\mathtt{m}[j]$ with a map $\mathtt{m}[j]|_{\mathcal{R}(\mathtt{m}[j])} : S^{\mathcal{R}(\mathtt{m}[j])} \to S$. By [10, Lemma 7] there exists a vector $M^j = (M^j_i)_{i\in\mathcal{R}(\mathtt{m}[j])}$ with coordinates in $\mathcal{H}(S,S)$ such that

$$\mathtt{m}[j]|_{\mathcal{R}(\mathtt{m}[j])}(\mathbf{x}) = \bigodot_{i\in\mathcal{R}(\mathtt{m}[j])} M^j_i\big(\mathbf{x}(i)\big) \qquad \big(\mathbf{x} \in S^{\mathcal{R}(\mathtt{m}[j])}\big).$$

Defining now $M = (M_{ij})_{i,j\in\Lambda}$ as

$$M_{ij} := \begin{cases} M_i^j & \text{if } j \in \mathcal{D}(\mathtt{m}), i \in \mathcal{R}(\mathtt{m}[j]), \\ \mathrm{id} & \text{if } i = j \notin \mathcal{D}(\mathtt{m}), \\ o & \text{else}, \end{cases}$$

gives a representation of $\mathtt{m}[j]$ for all $j \in \Lambda$ as in (2.4) with the property that the set Δ from (2.3) is finite. This completes the proof. □

As already claimed, with the help of the above lemma we can compute the dual function of each $m \in \mathcal{H}_{\mathrm{loc}}(S^\Lambda, S^\Lambda)$.

Proposition 2.5 (Dual local homomorphisms). *Let S, R, T be finite commutative monoids so that S is T-dual to R with duality function ψ. For each $\mathtt{m} \in \mathcal{H}_{\mathrm{loc}}(S^\Lambda, S^\Lambda)$ there exists a map $\hat{\mathtt{m}} \in \mathcal{H}_{\mathrm{loc}}(R^\Lambda, R^\Lambda)$ so that the restriction of $\hat{\mathtt{m}}$ to R^Λ_{fin} is the unique dual map of $\mathtt{m}$ with respect to the duality function Ψ from (2.1). If $M = (M_{ij})_{i,j\in\Lambda}$ denotes the matrix from Lemma 2.4 such that (2.4) holds, then $\hat{\mathtt{m}}$ is given via*

$$\hat{\mathtt{m}}[i](\mathbf{y}) = \boxdot_{j\in\Lambda} \widehat{M}_{ij}\big(\mathbf{y}(j)\big) \qquad (i \in \Lambda,\, \mathbf{y} \in R^\Lambda), \tag{2.5}$$

where, for $i, j \in \Lambda$, $\widehat{M}_{ij} \in \mathcal{H}(R, R)$ is the (unique) dual map of $M_{ij} \in \mathcal{H}(S, S)$ with respect to the duality function ψ.

Proof. Let $\mathbf{x} \in S^\Lambda$, $\mathbf{y} \in R^\Lambda_{\mathrm{fin}}$ and let $\hat{\mathtt{m}}$ be defined via (2.5). Note that $\hat{\mathtt{m}}$ indeed maps R^Λ_{fin} into itself as Δ from (2.3) is finite for $\mathtt{m}$ and the (unique) dual maps of $o, \mathrm{id} \in \mathcal{H}(S, S)$ with respect to ψ are $o \in \mathcal{H}(R, R)$ and $\mathrm{id} \in \mathcal{H}(R, R)$, respectively. Moreover, Lemma 2.4 implies that $\hat{\mathtt{m}} \in \mathcal{H}_{\mathrm{loc}}(R^\Lambda, R^\Lambda)$. We compute that

$$\begin{aligned}
\Psi(\mathtt{m}(\mathbf{x}), \mathbf{y}) &= \bigotimes_{j\in\Lambda} \psi\left(\bigodot\nolimits_{i\in\Lambda} M_{ij}(\mathbf{x}(i)),\, \mathbf{y}(j)\right) = \bigotimes_{i,j\in\Lambda} \psi\big(M_{ij}(\mathbf{x}(i)),\, \mathbf{y}(j)\big) \\
&= \bigotimes_{i,j\in\Lambda} \psi\big(\mathbf{x}(i),\, \widehat{M}_{ij}(\mathbf{y}(j))\big) = \bigotimes_{i\in\Lambda} \psi\left(\mathbf{x}(i), \boxdot\nolimits_{j\in\Lambda} \widehat{M}_{ij}(\mathbf{y}(j))\right) \\
&= \Psi(\mathbf{x}, \hat{\mathtt{m}}(\mathbf{y})).
\end{aligned}$$

Uniqueness of the dual map follows directly from property (iii) of the duality between S^Λ and R^Λ_{fin} established in Proposition 2.3. □

We are now ready to apply the non-probabilistic results above to Markov processes. Let S, R and T still be the finite monoids from above and let $\mathcal{G}$ be a countable collection of maps in $\mathcal{H}_{\mathrm{loc}}(S^\Lambda, S^\Lambda)$. We are considering two formal Markov generators G and $\widehat{G}$ defined as

$$Gf(\mathbf{x}) := \sum_{\mathtt{m}\in\mathcal{G}} r_{\mathtt{m}}\big(f(\mathtt{m}(\mathbf{x})) - f(\mathbf{x})\big) \qquad (\mathbf{x} \in S^\Lambda), \tag{2.6}$$

and

$$\widehat{G}g(\mathbf{y}) := \sum_{\mathtt{m}\in\mathcal{G}} r_{\mathtt{m}}\big(g(\hat{\mathtt{m}}(\mathbf{y})) - g(\mathbf{y})\big) \qquad (\mathbf{y} \in R^\Lambda_{\mathrm{fin}}), \tag{2.7}$$

where $\hat{\mathtt{m}}$ denotes the dual map of $\mathtt{m} \in \mathcal{G}$ from Proposition 2.5 and $(r_{\mathtt{m}})_{\mathtt{m}\in\mathcal{G}}$ are non-negative rates. We assume that G satisfies the summability condition

$$\sup_{i\in\Lambda} \sum_{\substack{\mathtt{m}\in\mathcal{G} \\ \mathcal{D}(\mathtt{m})\ni i}} r_{\mathtt{m}}\big(|\mathcal{R}(\mathtt{m}[i])| + 1\big) < \infty. \tag{2.8}$$

Under this condition we can almost surely construct a unique interacting particle system $X = (X_t)_{t\geq 0}$ with generator G on S^Λ (see [15, Theorem 4.30]). It turns out (see Theorem 2.7 below) that this condition moreover already implies that there exists a non-explosive Markov chain $(Y_t)_{t\geq 0}$ with generator $\widehat{G}$ on the countable state space $R^\Lambda_{\rm fin}$. We want to prove the following generalisation of Proposition 1.4.

Theorem 2.6 (Pathwise monoid duality). *Let S, R and T be finite commutative monoids so that S is T-dual to R with duality function ψ. Let G and $\widehat{G}$ be the generators from (2.6) and (2.7) defined via $\mathcal{G}$, a countable collection of maps in $\mathcal{H}_{\rm loc}(S^\Lambda, S^\Lambda)$ and their unique dual maps from Proposition 2.5. Assume that G satisfies (2.8). Fixing a $T \geq 0$, one can almost surely construct $X = (X_t)_{t\geq 0}$, the process with generator G, and $Y = (Y_t)_{t\geq 0}$, the process with generator $\widehat{G}$, in such a way that for every $t \in [0,T]$ the random variables X_t and Y_{T-t} are independent and*

$$[0,T] \ni t \mapsto \Psi\big(X_t, Y^-_{T-t}\big) \tag{2.9}$$

is constant, where $Y^- = (Y^-_t)_{t\geq 0}$ is the càglàd modification of Y.

By definition, we say that X and Y are *pathwise dual* if they can be constructed in such a way that (2.9) is satisfied. To prove the above result we cite general theory from [15].

Let L and V be arbitrary finite sets and let $\mathcal{Y}$ be an arbitrary countable set. As always, we equip L^Λ with the product topology and $\mathcal{Y}$ with the discrete one. Let $\varphi : L^\Lambda \times \mathcal{Y} \to V$ be a function. Let $\mathcal{H}$ be a countable collection of local maps $\mathtt{m} : L^\Lambda \to L^\Lambda$ and assume that every $m \in \mathcal{H}$ has a unique dual map $\hat{\mathtt{m}} : \mathcal{Y} \to \mathcal{Y}$ with respect to φ. Let $(r_{\mathtt{m}})_{\mathtt{m}\in\mathcal{H}}$ be non-negative rates and define formal generators H and $\widehat{H}$ in parallel to (2.6) and (2.7) with $\mathcal{G}$ replaced by $\mathcal{H}$. Let ω denote a Poisson point set on $\mathcal{H} \times \mathbb{R}$ with intensity measure $\rho(\{m\} \times A) := r_m \ell(A)$ $(m \in \mathcal{H},\ A \in \mathcal{B}(\mathbb{R}))$, where ℓ denotes the Lebesgue measure. Under condition (2.8), [15, Theorem 6.16] says that we can almost surely define stochastic flows[1] $(\mathbf{X}^+_{s,u})_{s\leq u}$ and $(\mathbf{X}^-_{s,u})_{s\leq u}$ of random continuous maps from L^Λ to itself so that, for $s \leq u$, $\omega^+_{s,u} := \{(m,t) \in \omega : t \in (s,u]\}$ and $\omega^-_{s,u} := \{(m,t) \in \omega : t \in [s,u)\}$,

$$\mathbf{X}^\pm_{s,u}(x) = \lim_{\omega_n \uparrow \omega^\pm_{s,u}} \mathbf{X}^{\omega_n}_{s,u}(x) \qquad (x \in L^\Lambda) \tag{2.10}$$

pointwise, where $(\omega_n)_n$ is an arbitrary increasing sequence of finite subsets of $\omega^\pm_{s,u}$ whose union is $\omega^\pm_{s,u}$, and $\mathbf{X}^{\omega_n}_{s,u}$ is the concatenation of all maps in ω_n (ordered by the time coordinate t).

Let $\widehat{\mathcal{H}} := \{\hat{m} : m \in \mathcal{H}\}$ and let $\hat{\omega}$ be defined by

$$\hat{\omega} := \{(\hat{m}, -t) : (m,t) \in \omega\}.$$

Then $\hat{\omega}$ is a Poisson point set on $\widehat{\mathcal{H}} \times \mathbb{R}$ with intensity measure $\hat{\rho}(\{\hat{m}\} \times A) := r_m \ell(A)$ and analogously to above we can almost surely define stochastic flows $(\mathbf{Y}^+_{s,u})_{s\leq u}$ and $(\mathbf{Y}^-_{s,u})_{s\leq u}$ of random continuous maps from $\mathcal{Y}$ to itself so that, for $s \leq u$, $\mathbf{Y}^+_{s,u}$ and $\mathbf{Y}^-_{s,u}$ correspond to pointwise limits as in (2.10), replacing $\omega^+_{s,u}$ by $\hat{\omega}^+_{s,u} = \{(\hat{m},t) : (m,t) \in \omega^-_{-u,-s}\}$ and $\omega^-_{s,u}$ by $\hat{\omega}^-_{s,u} = \{(\hat{m},t) : (m,t) \in \omega^+_{-u,-s}\}$. The next statement follows from [15, Theorem 6.20]. Recall that a continuous-time Markov chain is called *non-explosive* if, for any initial state x, the probability that the chain stated in x jumps infinitely often up to a finite time $t > 0$ is zero (compare [11, Chapter 2.7]).

Theorem 2.7 (Pathwise dual of an IPS). *Assume that the function $\varphi : L^\Lambda \times \mathcal{Y} \to V$ is continuous if $L^\Lambda \times \mathcal{Y}$ is equipped with the product topology, and that it satisfies property*

[1] By definition, $(\mathbf{Z}_{s,u})_{s\leq u}$ is a stochastic flow if $\mathbf{Z}_{s,s}$ is the identity map for all $s \in \mathbb{R}$ and if $\mathbf{Z}_{t,u} \circ \mathbf{Z}_{s,t} = \mathbf{Z}_{s,u}$ $(s \leq t \leq u)$.

(iii) of the definition of duality, i.e. that $\varphi(x,y_1)=\varphi(x,y_2)$ *for all* $x\in L^\Lambda$ *implies* $y_1=y_2$ *(*$y_1,y_2\in\mathcal{Y}$*). Further assume that* H *satisfies (2.8). Then there exists a continuous-time Markov chain with generator* $\widehat{H}$ *that is non-explosive. Moreover, constructing* $(\mathbf{X}^\pm_{s,u})_{s\le u}$ *and* $(\mathbf{Y}^\pm_{s,u})_{s\le u}$ *as above,*

$$\varphi\big(\mathbf{X}^\pm_{s,u}(x),y\big)=\varphi\big(x,\mathbf{Y}^\mp_{-u,-s}(y)\big) \tag{2.11}$$

holds almost surely simultaneously for all $s\le u$*,* $x\in L^\Lambda$ *and* $y\in\mathcal{Y}$*.*

If two stochastic flows satisfy (2.11) for all $s\le u$ and for all x and y, we say that they are *dual*. Theorem 2.6 follows now almost directly from Theorem 2.7.

Proof of Theorem 2.6. First note that Proposition 2.3 and the definition of the product topology imply that, by property (ii) of the definition of duality, Ψ from (2.1) is also continuous as a function from $S^\Lambda\times R^\Lambda_{\text{fin}}$ to T. Proposition 2.5 and Theorem 2.7 then show that we can, almost surely, construct stochastic flows $(\mathbf{X}^\pm_{s,u})_{s\le u}$ and $(\mathbf{Y}^\pm_{s,u})_{s\le u}$ corresponding to the maps in $\mathcal{G}$ as in Theorem 2.7.

Fix now $T\ge 0$ and choose a random variable X_0 on S^Λ and a random variable Y_0 on R^Λ_{fin}, both independent of $(\mathbf{X}^+_{s,u})_{s\le u}$ and $(\mathbf{Y}^-_{s,u})_{s\le u}$. Setting

$$X_t:=\mathbf{X}^+_{0,t}(X_0)\quad\text{and}\quad Y_t:=\mathbf{Y}^+_{-T,t-T}(Y_0)\qquad(t\ge 0)$$

yields by [15, Proposition 2.9 & Theorem 4.20] and Theorem 2.7 a Markov process $X=(X_t)_{t\ge0}$ with generator G and a non-explosive continuous-time Markov chain $Y=(Y_t)_{t\ge0}$ with generator $\widehat{G}$. By the construction in [15, Section 6.4] defining $Y_t^-:=\mathbf{Y}^-_{-T,t-T}(Y_0)$ for $t\ge 0$ gives the càglàd modification $Y^-:=(Y_t^-)_{t\ge0}$ of Y. Using the duality of the stochastic flows, i.e. (2.11), one then has for all $s,u\in\mathbb{R}$ satisfying $0\le s\le u\le T$ that

$$\begin{aligned}\Psi\big(X_s,Y^-_{T-s}\big)&=\Psi\big(\mathbf{X}^+_{0,s}(X_0),\mathbf{Y}^-_{-T,-s}(Y_0)\big)=\Psi\big(\mathbf{X}^+_{0,s}(X_0),\mathbf{Y}^-_{-u,-s}\circ\mathbf{Y}^-_{-T,-u}(Y_0)\big)\\&=\Psi\big(\mathbf{X}^+_{s,u}\circ\mathbf{X}^+_{0,s}(X_0),\mathbf{Y}^-_{-T,-u}(Y_0)\big)=\Psi\big(\mathbf{X}^+_{0,u}(X_0),\mathbf{Y}^-_{-T,-u}(Y_0)\big)\\&=\Psi\big(X_u,Y^-_{T-u}\big),\end{aligned}$$

i.e. the function in (2.9) is constant, and the proof is complete. □

Applying the general theory to the 2CP we prove Proposition 1.4.

Proof of Proposition 1.4. We equip $U=\{0,1\}\times\{0,1\}$ with $\veebar$, the product operator of $\vee$ and $\oplus$ from the beginning of Section 1.2, i.e. $(x,y)\veebar(v,w):=(x\vee v,y\oplus w)$ for $(x,y),(v,w)\in U$. This gives the addition table

$\veebar$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(1,0)	(1,1)
(1,1)	(1,1)	(1,0)	(1,1)	(1,0)

.

Then $U=(U,\veebar)$ is indeed a monoid. Next one computes $\mathcal{H}(U,U)$ and $\mathcal{H}(U,M)$, with $M:=(\{-1,0,1\},\cdot)$, where $\cdot$ denotes the usual multiplication in $\mathbb{R}$. To compute $\mathcal{H}(U,U)=\{(o,o),(o,\text{id}),(\text{id},o),(\text{id},\text{id})\}$ one can apply [10, Lemma 6], noting that $U=M_1\times M_2$, where $M_1:=(\{0,1\},\vee)$ and $M_2:=(\{0,1\},\oplus)$, and checking that $\mathcal{H}(M_i,M_j)=\{o,\text{id}\}$ if $i=j$ and $=\{o\}$ if $i\ne j$ ($i,j\in\{1,2\}$). To compute $\mathcal{H}(U,M)$ one can apply the same result, computing first $\mathcal{H}(M_1,M)=\{1,\gamma_1\}$ and $\mathcal{H}(M_2,M)=\{1,\gamma_2\}$, where 1 is the function constantly 1, and γ_1 and γ_2 are the functions from (1.14). Using the definition of duality one then confirms that U is M-dual to itself with respect to ψ from (1.18).

Having computed $\mathcal{H}(U,U)$ one directly concludes that all its maps are self-dual as o and id are always self-dual. All maps in (1.4) (that are used in the definition of $G_{\veebar}$ in (1.5)) can be written as in (2.4) with Δ from (2.3) finite, so Lemma 2.4 implies that they are elements of $\mathcal{H}_{\mathrm{loc}}(U^\Lambda, U^\Lambda)$, with $\Lambda = \mathbb{Z}^d$ and U^Λ being, as always, equipped with the product topology. Proposition 2.5 shows that $G_{\veebar}$ can play the role of both G and $\widehat{G}$ from (2.6) and (2.7). One quickly verifies that (2.8) holds and the claim follows from Theorem 2.6. □

One can check that the monoid U is isomorphic to M_{23} from [10, Appendix A.1] and the monoid $M = (\{-1,0,1\}, \cdot\,)$ is isomorphic to M_5 from [10, Section 5.1]. The function ψ is denoted in [10] as ψ_{235} and the fact that U is M-dual to itself can be found in the table in [10, Appendix A.2]. The fact that $\mathcal{H}(U,U) = \{(o,o),(o,\mathrm{id}),(\mathrm{id},o),(\mathrm{id},\mathrm{id})\}$ is, by [10, Proposition 4], encoded in the duality function ψ_{23} from [10, Appendix A.2].

Instead of using, as we did in Proposition 1.4, as a local state space the monoid $U = M_1 \times M_2$ ($M_1 = (\{0,1\},\vee)$ and $M_2 := (\{0,1\},\oplus)$ as in the proof above) one can also consider interacting particle systems that have $V := M_1 \times M_1$ or $W := M_2 \times M_2$ as their local state space. It follows from [10, Proposition 8] that V is M_1-dual to itself while W is M_2-dual to itself. Note that M_1 and M_2 coincide with M_1 and M_2 from [10]. Continuing as in the proof above, one can also prove the self-duality of two coupled CPs or two coupled cCPs given via a generator as in (1.5) but using in (1.4) only infection maps with the superscript $\vee$ or only infection maps with the superscript $\oplus$, respectively. However, $\mathcal{H}_{\mathrm{loc}}(V^\Lambda, V^\Lambda)$ is the set of local additive maps on $(M_1 \times M_1)^\Lambda$ and $\mathcal{H}_{\mathrm{loc}}(W^\Lambda, W^\Lambda)$ is the set of local calcellative maps on $(M_2 \times M_2)^\Lambda$, so in this case one just arrives at the well-known additive and cancellative dualities for interacting particle systems that are defined on an extended lattice, where each site in the original lattice Λ has been replaced by two new sites, that correspond to the two copies of M_1 or M_2, respectively. By contrast, the duality in Proposition 1.4 is not covered by known results about additive or cancellative duality.

3 Informativeness and representations

In this subsection Lemma 1.5 is proved. In fact, as already stated in the outline, we are going to prove a more general result and we are going to investigate the open task to classify the monoid dualities from [10] that determine the law of processes uniquely. Let, as in the section above, $(S,\odot)$, $(R,\boxdot)$ and $(T,\otimes)$ be commutative finite monoids and assume that S is T-dual to R with duality function $\psi : S \times R \to T$. Let Λ be countable, let $\mathbb{V}$ be a finite dimensional real or complex vector space and let V be an arbitrary measurable space.

Towards the goal of classification we give the following definitions. For an arbitrary index set I we call a family $(f_i)_{i\in I}$ of measurable functions $f_i : S^\Lambda \to \mathbb{V}$ *distribution determining* if, for two random variables X and X' on S^Λ,

$$\mathbb{E}[f_i(X)] = \mathbb{E}[f_i(X')] \quad \forall i \in I \qquad \text{implies} \qquad X \overset{d}{=} X',$$

where $\overset{d}{=}$ denotes equality in distribution. Similarly, we call a family $(g_i)_{i\in I}$ of measurable functions $g_i : S^\Lambda \to V$ *weakly distribution determining* if

$$g_i(X) \overset{d}{=} g_i(X') \quad \forall i \in I \qquad \text{implies} \qquad X \overset{d}{=} X'.$$

The first of the two definition is already widely used (compare [15]), while the second one we introduce here newly.

A family $(f_i)_{i\in I}$ of functions $f_i : S^\Lambda \to \mathbb{V}$ that is distribution determining is clearly also weakly distribution determining. The reverse implication is not true in general, but holds in the following special case. Recall that $v_1,\dots,v_n \in \mathbb{V}$ are called *affinely independent* if

$$\sum_{k=1}^{n} \lambda_k v_k = 0 \text{ with scalars } \lambda_1,\dots,\lambda_n \text{ s.t. } \sum_{k=1}^{n} \lambda_k = 0 \quad \text{implies} \quad \lambda_1 = \ldots = \lambda_n = 0.$$

Proposition 3.1 (Equality of notions)**.** *Let $(f_i)_{i\in I}$ be a family of functions $f_i : S^\Lambda \to \{v_1,\dots,v_n\} \subset \mathbb{V}$. If $v_1,\dots,v_n$ are affinely independent, then $(f_i)_{i\in I}$ is distribution determining if and only if it is weakly distribution determining.*

Proof. Comparing the definitions it suffices to show for fixed $i \in I$ that, under the assumption of the proposition, $\mathbb{E}[f_i(X)] = \mathbb{E}[f_i(X')]$ implies $f_i(X) \overset{d}{=} f_i(X')$. As the set $\{v_1,\dots,v_n\}$ is finite, the condition $\mathbb{E}[f_i(X)] = \mathbb{E}[f_i(X')]$ is equivalent to writing

$$\sum_{k=1}^{n} v_k \big(\mathbb{P}[f_i(X) = v_k] - \mathbb{P}[f_i(X') = v_k]\big) = 0.$$

But as $v_1,\dots,v_n$ are affinely independent, then also

$$\mathbb{P}[f_i(X) = v_k] - \mathbb{P}[f_i(X') = v_k] = 0 \qquad (k = 1,\dots,n),$$

i.e. $f_i(X)$ and $f_i(X')$ are equal in distribution. □

Let now $\Psi : S^\Lambda \times R^\Lambda_{\text{fin}} \to T$ be the function from (2.1). In parallel to [15] we say that Ψ is *weakly informative* if

$$\big(\Psi(\,\cdot\,,\mathbf{y})\big)_{\mathbf{y}\in R^\Lambda_{\text{fin}}} \tag{3.1}$$

is weakly distribution determining. If the monoid T is also a subset of a real or complex vector space, we say that Ψ is *informative* if the functions in (3.1) are distribution determining. We prove the following result.

Proposition 3.2 (Informativeness of Ψ)**.** *Under the assumptions of this subsection Ψ is informative if T is a sub-monoid of $(\mathbb{C},\,\cdot\,)$, where $\cdot$ denotes the usual multiplication.*

It is easy to see that all finite sub-monoids of $(\mathbb{C},\,\cdot\,)$ (apart from $(\{0\},\,\cdot\,)$) consist of the multiplicative group of n-th roots of unity for some $n \in \mathbb{N}$, either with or without an added 0. Those with cardinality up to four are named $M_0, M_1, M_2, M_5, M_7, M_{18}$ and M_{26} in our paper [10], so by Proposition 3.2 all duality functions from [10] that take values in these monoids are informative. In particular, setting $(T,\otimes) = (\{1,-1,0\},\,\cdot\,)$ and $(S,\odot) = (R,\boxdot) = (U,\veebar)$, Proposition 3.2 implies Lemma 1.5.

To prove Proposition 3.2 we use a Stone-Weierstrass argument. Let $\mathcal{C}(\mathcal{X},\mathcal{Y})$ denote the space of continuous functions from space $\mathcal{X}$ to space $\mathcal{Y}$. We say that $\mathcal{H} \subset \mathcal{C}(\mathcal{X},\mathcal{Y})$ *separates points* if for $x, x' \in \mathcal{X}$ with $x \neq x'$ there exists $f \in \mathcal{H}$ such that $f(x) \neq f(x')$. Moreover, we say that $\mathcal{G} \subset \mathcal{C}(\mathcal{X},\mathbb{C})$ is *self-adjoint* if $f \in \mathcal{G}$ implies $\overline{f} \in \mathcal{G}$, where $\overline{f}(x) := \overline{f(x)}$ ($x \in \mathcal{X}$), the complex conjugate of $f(x)$.

Lemma 3.3 (Application of Stone-Weierstrass)**.** *Let E be a compact metrizable space. Assume that $\mathcal{G} \subset \mathcal{C}(E,\mathbb{C})$ separates points and is closed under products. Then $\mathcal{G}$ is distribution determining.*

Proof. The statement with $\mathbb{C}$ replaced by $\mathbb{R}$ is proved in [15, Lemma 4.37]. Note that

$$\mathbb{E}\big[f(X)\big] = \mathbb{E}\big[f(X')\big] \quad \text{implies} \quad \mathbb{E}\left[\overline{f}(X)\right] = \mathbb{E}\left[\overline{f}(X')\right] \qquad (f \in \mathcal{G}), \tag{3.2}$$

as $\mathbb{E}\big[\overline{f}(X)\big] = \overline{\mathbb{E}\big[f(X)\big]}$, where X and X' are random variables on E. We can enlarge $\mathcal{G}$ with the constant function 1, take linear combinations and convex conjugates and receive an algebra $\mathcal{H} \supset \mathcal{G}$ that is closed under products, self-adjoint and separates points. If $\mathbb{E}\big[f(X)\big] = \mathbb{E}[f(X')]$ for all $f \in \mathcal{G}$ then also $\mathbb{E}\big[f(X)\big] = \mathbb{E}[f(X')]$ for all $f \in \mathcal{H}$ by the linearity of the integral and (3.2). We then can apply the complex version of the Stone-Weierstrass theorem and continue as in the proof of [15, Lemma 4.37]. □

Proof of Proposition 3.2. By definition, we have to prove that the family

$$\mathcal{G} := \big(\Psi(\,\cdot\,,\mathbf{y})\big)_{\mathbf{y}\in R^\Lambda_{\text{fin}}}$$

is distribution determining.

By Tychonoff's theorem (see, for example, [3, Theorem I.8.9]), the space S^Λ, equipped with the product topology, is compact. Moreover, the product topology is metrizable. For example, if $(a_i)_{i\in\Lambda}$ are are strictly positive constants such that $\sum_{i\in\Lambda} a_i < \infty$, then the metric d, defined via

$$d(\mathbf{x},\mathbf{x}') := \sum_{i\in\Lambda} a_i 1\!\!1_{\{\mathbf{x}(i)\neq\mathbf{x}'(i)\}}(\mathbf{x},\mathbf{x}') \qquad (\mathbf{x},\mathbf{x}' \in S^\Lambda)$$

generates the product topology. The fact that $\mathcal{G}$ is closed under products follows from the duality between S^Λ and R^Λ_{fin}: Property (i) in the definition of duality implies that

$$\Psi(\mathbf{x},\mathbf{y}_1)\Psi(\mathbf{x},\mathbf{y}_2) = \Psi(\mathbf{x},\mathbf{y}_1 \boxdot \mathbf{y}_2) \qquad (\mathbf{x}\in S^\Lambda,\ \mathbf{y}_1,\mathbf{y}_2 \in R^\Lambda_{\text{fin}}).$$

The fact that $\mathcal{G}$ separates points follows directly from property (ii) of the definition of (topological) duality. Applying Lemma 3.3 then yields Proposition 3.2. □

To further investigate the case in which the monoid T can not naturally be written as a sub-monoid of $(\mathbb{C},\,\cdot\,)$, we provide some additional notions. The reader that is just concerned with the 2CP may skip ahead to the next section.

Let $(\mathbb{A},+,\,\cdot\,)$ be a unital commutative algebra with unit I. A *multiplicative representation* of a commutative monoid $(M,+)$ with neutral element 0 is a map $\gamma : M \to \mathbb{A}$ so that $\gamma(x+y) = \gamma(x)\cdot\gamma(y)$ and $\gamma(0) = I$. Then $\gamma(M) = \{\gamma(x) : x \in M\}$ is a sub-monoid of $(\mathbb{A},\,\cdot\,)$ and $\gamma : M \to \gamma(M)$ is a monoid homomorphism. We say that γ is *faithful* if this map (with codomain $\gamma(M)$) is an isomorphism.

We again consider the function $\Psi : S^\Lambda \times R^\Lambda_{\text{fin}} \to T$ from (2.1). By Proposition 2.3, the assumptions on S, R and T stated at the beginning of this section imply that S^Λ is T-dual to R^Λ_{fin} with duality function Ψ. Let $\gamma : T \to \mathbb{A}$ be a faithful multiplicative representation. As usual, we equip the finite monoids T and $\gamma(T)$ with the discrete topology and it follows from the definition of duality that S^Λ is also $\gamma(T)$-dual to R^Λ_{fin} with duality function $\gamma\circ\Psi$. If Ψ is weakly informative and if the elements of $\gamma(T)$ are affinely independent, then Proposition 3.1 and the faithfulness of γ imply that $\gamma\circ\Psi$ is informative.

We say that γ is a *good multiplicative representation* of Ψ if γ is a faithful multiplicative representation of T and $\gamma\circ\Psi$ is informative. The next result states that we can always find such a good multiplicative representation of a weakly informative duality function.

Proposition 3.4 (Existence of good representations)**.** *Under the assumptions of this subsection there exist a finite dimensional real unital commutative algebra $\mathbb{A}$ and a faithful representation $\gamma : T \to \mathbb{A}$ such that $\gamma\circ\Psi$ is informative if Ψ is weakly informative.*

Proof. Let $\mathbb{R}^T$ be the space of all functions mapping from T to $\mathbb{R}$. The space $(\mathbb{R}^T,+)$, where $+$ denotes the usual (pointwise) sum of real-valued functions, is a finite dimensional real vector space on which we can define the product $*$ as

$$(g*h)(a) := \sum_{b,c\in T} g(b)h(c)1\!\!1_{\{a\}}(b\otimes c) \qquad \big(g,h\in\mathbb{R}^T,\ a\in T\big),$$

where the sum is the usual sum in $\mathbb{R}$ and $\mathbb{1}$ denotes the indicator function. One readily checks that this makes $(\mathbb{R}^T, +, *)$ a finite dimensional real unital algebra with unit $\mathbb{1}_{\{0\}}$. Defining $\gamma : T \to \mathbb{R}^T$ as $\gamma(a) = \mathbb{1}_{\{a\}}$ $(a \in T)$ then gives a faithful multiplicative representation of T and clearly the elements of $\gamma(T)$ are affinely independent. The claim then follows from Proposition 3.1 and the faithfulness of γ as stated above. □

By the above proposition we can reformulate the classification problem by asking to classify general duality functions (that do not map into sub-monoids of $(\mathbb{C}, \cdot)$) into the classes "weak informative" and "not weak informative". This remains an open problem.

We end this section with an additional observation. While $\mathbb{R}^T$ from the proof of Proposition 3.4 is a $|T|$-dimensional vector space, Proposition 3.2 implies that for large T also representations in lower dimensional spaces can be good, even if the elements of $\gamma(T)$ are not affinely independent. As it is in practice often easier to work in a lower dimensional space, there can exist "better" representations of weakly informative duality functions than the one from Proposition 3.4. In light of Proposition 3.2 one might even hope that γ is always a good representation of a weakly informative Ψ as long as γ is faithful. This, however, is not true and we provide a counterexample below.

Example 3.5 (Representations of $\boldsymbol{\psi}_{23}$)**.** We again consider the monoid $(U, \veebar)$ defined in Section 1.2. From [10, Appendix A.2] we know that there also exists the "local" duality function ψ_{23} mapping from $U \times U$ back into U. Reordering the elements of M_{23} as in the present paper (i.e. as in U) one has that

$$\psi_{23}(x, y) = \big(\psi_1(x_1, y_1), \psi_2(x_2, y_2)\big) \qquad \big(x = (x_1, x_2),\ y = (y_1, y_2) \in U\big),$$

where ψ_1 and ψ_2 are the "local" additive and cancellative duality function, defined in (1.10). It follows from (1.17) that

$$\psi_{23}(x, y) = \psi_{23}(v, w) \qquad \text{implies} \qquad \psi(x, y) = \psi(v, w) \qquad \big(x, y, v, w \in U\big). \tag{3.3}$$

We define a "global" duality function $\boldsymbol{\psi}_{23} : \mathcal{U} \times \mathcal{U}_{\text{fin}} \to U$ as in (1.16), but for ψ_{23} instead of ψ and with the "product" taken in U. It follows from (3.3) that for two random variables X, X' on $\mathcal{U}$ and for $y \in \mathcal{U}_{\text{fin}}$,

$$\boldsymbol{\psi}_{23}(X, y) \stackrel{d}{=} \boldsymbol{\psi}_{23}(X', y) \quad \text{implies} \quad \boldsymbol{\psi}(X, y) \stackrel{d}{=} \boldsymbol{\psi}(X', y),$$

and, due to the informativeness of $\boldsymbol{\psi}$, the duality function $\boldsymbol{\psi}_{23}$ is weakly informative. Defining now $\gamma : U \to \mathbb{R}^2$ as

$$\gamma(x) := \big(\gamma_1(x_1), \gamma_2(x_2)\big) \qquad (x = (x_1, x_2) \in U),$$

with γ_1, γ_2 defined in (1.14), yields a faithful multiplicative representation of U in $\mathbb{R}^2$, viewed as a unital algebra equipped with pointwise multiplication. However, γ is *not* a good representation of $\boldsymbol{\psi}_{23}$. For example, the random variables X, X' on $\mathcal{U}$ with

$$\mathbb{P}[X(i) = (0,0)] = \mathbb{P}[X'(i) = (0,0)] = 1 \text{ for } i \in \mathbb{Z}^d \setminus \{0\},$$

$$\mathbb{P}[X(0) = x] = \frac{1}{4} \text{ for all } x \in U \qquad \mathbb{P}[X'(0) = x] = \begin{cases} \frac{1}{2} & \text{if } x \in \{(0,0), (1,1)\}, \\ 0 & \text{else}, \end{cases}$$

show that $\gamma \circ \boldsymbol{\psi}_{23} : \mathcal{U} \times \mathcal{U}_{\text{fin}} \to \mathbb{R}^2$ is not informative. Here $0 \in \mathbb{Z}^d$ denotes the origin.

4 The main convergence result

In this section we prove Theorem 1.3 and Proposition 1.6. Recall that $\mathcal{T}$ denotes the space of all functions $z : \mathbb{Z}^d \to T = \{0,1\}$ and recall the definition of $\mathcal{T}_{\text{fin}}$ in (1.11). For $z \in \mathcal{T}$ we shortly write $|z| := |\{i \in \mathbb{Z}^d : z(i) = 1\}|$. We are going to use several auxiliary lemmas to prove Theorem 1.3. The first one is [15, Lemma 6.37]. The symbol $\wedge$ denotes the pointwise minimum, i.e. $(z_1 \wedge z_2)(i) = \min\{z_1(i), z_2(i)\}$ for $i \in \mathbb{Z}^d,\ z_1, z_2 \in \mathcal{T}$.

Lemma 4.1 (Non-zero intersection: CP)**.** *Let $Z = (Z_t)_{t\geq 0}$ be a $\text{CP}(\lambda, \delta)$ ($\lambda > 0,\ \delta \geq 0$) with non-trivial shift-invariant initial distribution. Given $\varepsilon > 0$, for each time $s > 0$ there exists an $N_{\text{CP}} \in \mathbb{N}$ such that for any $z \in \mathcal{T}$ with $|z| \geq N_{\text{CP}}$ one has*

$$\mathbb{P}\big(Z_s \wedge z = \underline{0}\big) \leq \varepsilon.$$

Additionally we are going to use the following application of [12, Corollary 9]. As [12, Corollary 9] is not stated in the most accessible form we devote Appendix A to showing how the result below follows from it. Instead of using the result below we could have also followed the strategy of the proof of [2, Theorem 1.2]. There the authors use the graphical representation of the cCP explicitly to work around the statement below.

Lemma 4.2 (Parity indeterminacy: cCP)**.** *Let $Z = (Z_t)_{t\geq 0}$ be a $\text{cCP}(\lambda, \delta)$ ($\lambda > 0,\ \delta \geq 0$) with non-trivial shift-invariant initial distribution. Given $\varepsilon > 0$, for each time $s > 0$ there exists an $N_{\text{cCP}} \in \mathbb{N}$ such that for any $z \in \mathcal{T}_{\text{fin}}$ with $|z| \geq N_{\text{cCP}}$ one has*

$$\left|\mathbb{P}\big[|Z_s \wedge z| \text{ is odd}\big] - \frac{1}{2}\right| \leq \varepsilon.$$

Finally, the following result extends [15, Lemma 6.36] and [2, Lemma 2.1].

Lemma 4.3 (Extinction or unbounded growth)**.** *Let $Z = (Z_t)_{t\geq 0}$ be either a $\text{CP}(\lambda, \delta)$ or a $\text{cCP}(\lambda, \delta)$ ($\lambda, \delta \geq 0,\ \lambda + \delta > 0$). For each $z \in \mathcal{T}_{\text{fin}}$ and $N \in \mathbb{N}$ one has*

$$\lim_{t\to\infty} \mathbb{P}^z[0 < |Z_t| < N] = 0. \tag{4.1}$$

Proof. If $z = \underline{0}$ the statement is trivial, so let $z \in \mathcal{T}_{\text{fin}} \setminus \{\underline{0}\}$. In the case $\lambda, \delta > 0$ [15, Lemma 6.36] and [2, Lemma 2.1] imply

$$\mathbb{P}^z\big[\exists\, t \geq 0 : Z_t = \underline{0} \text{ or } |Z_t| \to \infty \text{ as } t \to \infty\big] = 1 \tag{4.2}$$

for the CP and the cCP, respectively, and (4.2) clearly implies (4.1). In fact, the two proofs are just reformulations of each other, both based on Lévy's 0-1 law.

In the case $\lambda = 0,\ \delta > 0$ there is no difference between a CP and a cCP and

$$\mathbb{P}^z\big[\exists\, t \geq 0 : Z_t = \underline{0}\big] = \lim_{t\to\infty} \mathbb{P}^z\big[Z_t = \underline{0}\big] = \lim_{t\to\infty} \big(1 - e^{-\delta t}\big)^{|z|} = 1$$

since $\underline{0}$ is absorbing. This implies (4.2) and hence also (4.1).

In the case $\lambda > 0,\ \delta = 0$, and if Z is a CP, the function $t \mapsto |Z_t|$ is non-decreasing, hence it converges in $\mathbb{N} \cup \{\infty\}$. Let $N \in \mathbb{N}$. One has

$$\mathbb{P}^z\big[\lim\nolimits_{t\to\infty} |Z_t| \leq N\big] = 1 - \mathbb{P}^z\big[\exists\, t \geq 0 : |Z_t| > N\big] = 1 - \lim_{t\to\infty} \mathbb{P}^z[|Z_t| > N] = 0 \tag{4.3}$$

as choosing a suitable sequence of neighbours and neighbours of neighbours of the infected individuals in z yields that

$$\mathbb{P}^z[|Z_t| > N] \geq \Big(1 - \mathbb{1}_{\{|z| \leq N\}} e^{-\frac{\lambda t}{N+1-|z|}}\Big)^{N+1-|z|}$$

for $t > 0$. Here, in the case that $|z| \le N$, we have divided time into $N+1-|z|$ subintervals and used the fact that $1 - e^{-\lambda t}$ is the probability to infect a previously chosen neighbour of an infected individual during a time interval of length t. Finally, (4.3) implies that

$$\begin{aligned}\mathbb{P}^z\big[|Z_t| \to \infty \text{ as } t \to \infty\big] &= 1 - \mathbb{P}^z\big[\exists\, N \in \mathbb{N} : \lim_{t\to\infty} |Z_t| = N\big] \\ &\ge 1 - \sum_{N\in\mathbb{N}} \mathbb{P}^z\big[\lim_{t\to\infty} |Z_t| \le N\big] = 1,\end{aligned}$$

again implying (4.2) and hence also (4.1).

To treat the cCP in the case $\lambda > 0,\ \delta = 0$, we use [2, Theorem 1.3]. It says that a cCP$(1, 0)$, started in *any* initial state other than $\underline{0}$, converges weakly to the product law assigning probability 1/2 to both 0 and 1 at every node. By changing the time scale the same holds for a cCP$(\lambda, 0)$ with an arbitrary $\lambda > 0$. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Choose now an $M = M(N, \varepsilon) > N$ so that $p_N := \mathbb{P}[X \le N] < \varepsilon$ if X is a binomially distributed random variable with parameters $n = M$ and $p = 1/2$. Additionally, choose an arbitrary $x \in \mathcal{T}_{\text{fin}}$ with $|x| = M$. Then, by the weak convergence,

$$\limsup_{t\to\infty} \mathbb{P}^z\big[|Z_t| \le N\big] \le \lim_{t\to\infty} \mathbb{P}^z\big[|Z_t \wedge x| \le N\big] = p_N < \varepsilon,$$

implying $\lim_{t\to\infty} \mathbb{P}^z\big[|Z_t| \le N\big] = 0$ (i.e. convergence in probability to ∞). Thus (4.1) holds. □

Using the three lemmas above we are able to prove Theorem 1.3 and Proposition 1.6.

Proof of Theorem 1.3 and Proposition 1.6. Let $Y = (Y^1, Y^2) = (Y^1_t, Y^2_t)_{t\ge0}$ be a 2CP with the same parameters as the 2CP $X = (X^1, X^2) = (X^1_t, X^2_t)_{t\ge0}$ in the formulation of the theorem, but started in the deterministic state $y = (y_1, y_2) \in \mathcal{U}_{\text{fin}}$. Fix $t > 0$. Following [6, Proposition 1.4] we can construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there exist independent processes $\tilde{X} = (\tilde{X}_t)_{t\ge0}$ and $\tilde{Y} = (\tilde{Y}_t)_{t\ge0}$ whose finite dimensional distributions coincide with those of X and Y, respectively, and

$$\mathbb{E}\big[\boldsymbol{\psi}(\tilde{X}_s, \tilde{Y}_{t+1-s})\big] = \mathbb{E}\big[\boldsymbol{\psi}(\tilde{X}_u, \tilde{Y}_{t+1-u})\big] \tag{4.4}$$

holds for all $s, u \in [0, t+1]$, where $\mathbb{E}$ denotes taking expectation with respect to $\mathbb{P}$. Below, we drop the tildes from the notation. In contrast to $\mathbb{P}$, the symbol $\mathbb{P}^y$ denotes the law of Y started in $y \in \mathcal{U}_{\text{fin}}$. Due to the informativeness of $\boldsymbol{\psi}$ and the compactness of $\mathcal{U}$, the set $\mathcal{G}$ from the proof of Lemma 1.5 is also convergence determining, i.e. showing

$$\lim_{s\to\infty} \mathbb{E}\big[\boldsymbol{\psi}(X_s, y)\big] = \mathbb{P}^y\left[\exists\, s \ge 0 : Y_s = \underline{(0,0)}\right] \tag{4.5}$$

for all $y \in \mathcal{U}_{\text{fin}}$ implies (1.9) (compare [15, Lemma 4.38]). If $y = \underline{(0,0)}$, (4.5) follows trivially from the definition of $\boldsymbol{\psi}$, so assume $y \neq \underline{(0,0)}$. We set

$$\lambda_1 := \lambda + \lambda^{\vee}, \quad \delta_1 := \delta + \delta^{\vee}, \quad \lambda_2 := \lambda + \lambda^{\oplus}, \quad \delta_2 := \delta + \delta^{\oplus},$$

so that X^1 and Y^1 are both a CP(λ_1, δ_1), and X^2 and Y^2 are both a cCP(λ_2, δ_2). Assume, for now, that $\lambda_1, \lambda_2 > 0$, so that all three auxiliary lemmas above are applicable. Let $\varepsilon > 0$ be arbitrary. Choose N_{CP} and N_{cCP} as in Lemma 4.1 and Lemma 4.2 in dependence of the chosen ε, $s = 1$, and the model parameters. We have, using the duality equation

(4.4) and the law of total expectation, that

$$\begin{aligned}
&\mathbb{E}[\boldsymbol{\psi}(X_{t+1}, y)] \\
&= \mathbb{E}[\boldsymbol{\psi}(X_1, Y_t)] \\
&= \mathbb{E}\big[\boldsymbol{\psi}(X_1, Y_t) \mid Y_t^1 = Y_t^2 = \underline{0}\big]\mathbb{P}^y\big[Y_t^1 = Y_t^2 = \underline{0}\big] \\
&\quad + \mathbb{E}\big[\boldsymbol{\psi}(X_1, Y_t) \mid Y_t^1 = \underline{0},\ 0 < |Y_t^2| < N_{\mathrm{cCP}}\big] \underbrace{\mathbb{P}^y\big[Y_t^1 = \underline{0},\ 0 < |Y_t^2| < N_{\mathrm{cCP}}\big]}_{=:p_1(y,t)} \\
&\quad + \underbrace{\mathbb{E}\big[\boldsymbol{\psi}(X_1, Y_t) \mid Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\big]}_{=:E_1(y,t)} \mathbb{P}^y\big[Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\big] \\
&\quad + \mathbb{E}\big[\boldsymbol{\psi}(X_1, Y_t) \mid 0 < |Y_t^1| < N_{\mathrm{CP}}\big] \underbrace{\mathbb{P}^y\big[0 < |Y_t^1| < N_{\mathrm{CP}}\big]}_{=:p_2(y,t)} \\
&\quad + \underbrace{\mathbb{E}\big[\boldsymbol{\psi}(X_1, Y_t) \mid |Y_t^1| \geq N_{\mathrm{CP}}\big]}_{=:E_2(y,t)} \mathbb{P}^y\big[|Y_t^1| \geq N_{\mathrm{CP}}\big].
\end{aligned} \tag{4.6}$$

Depending on the choice of the model parameters and y, the deterministic initial state of Y, it might happen that some of the events on which we condition above have probability zero. The cases that either $y_1 = \underline{0}$ or $y_2 = \underline{0}$, or the monotonely coupled case $\delta_\vee = \lambda_\oplus = 0$ when y satisfies $y(i) \neq (0,1)$ for all $i \in \mathbb{Z}^d$ are such examples. In these cases we define the corresponding conditioned expectation (arbitrarily) to equal 1. As these conditioned expectations are then multiplied by 0, the lines in (4.6) where they occur drop out. For the remaining ones we can argue as below.

From the definition of $\boldsymbol{\psi}$ it is clear that $\mathbb{E}\big[\boldsymbol{\psi}(X_1, Y_t) \mid Y_t^1 = Y_t^2 = \underline{0}\big] = 1$ and

$$\mathbb{P}^y\big[Y_t^1 = Y_t^2 = \underline{0}\big] \nearrow \mathbb{P}^y\Big[\exists\, t \geq 0 : Y_t = \underline{(0,0)}\Big]$$

as $t \to \infty$. Moreover, Lemma 4.3 implies that

$$\lim_{t\to\infty} p_1(y,t) = \lim_{t\to\infty} p_2(y,t) = 0.$$

As in the proof of [15, Theorem 6.35] we use Lemma 4.1 to compute that

$$\begin{aligned}
|E_2(y,t)| &= \big|\mathbb{P}\big[\boldsymbol{\psi}(X_1, Y_t) = 1 \mid |Y_t^1| \geq N_{\mathrm{CP}}\big] - \mathbb{P}\big[\boldsymbol{\psi}(X_1, Y_t) = -1 \mid |Y_t^1| \geq N_{\mathrm{CP}}\big]\big| \\
&\leq \mathbb{P}\big[\boldsymbol{\psi}(X_1, Y_t) \neq 0 \mid |Y_t^1| \geq N_{\mathrm{CP}}\big] \\
&= \mathbb{P}\big[X_1^1 \wedge Y_t^1 = \underline{0} \mid |Y_t^1| \geq N_{\mathrm{CP}}\big] \leq \varepsilon
\end{aligned} \tag{4.7}$$

by the choice of N_{CP}. For $E_1(y,t)$ one has that

$$\begin{aligned}
E_1(y,t) &= 1 - 2\mathbb{P}\big[\boldsymbol{\psi}(X_1, Y_t) = -1 \mid Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\big] \\
&= 1 - 2\mathbb{P}\big[|X_1^2 \wedge Y_t^2| \text{ is odd} \bigm| Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\big]
\end{aligned}$$

and, due to the independence of X and Y, we can apply Lemma 4.2 and conclude that

$$|E_1(y,t)| \leq 2\varepsilon.$$

Plugging then back into (4.6) and computing the limit inferior and the limit superior, one concludes (4.5) as ε was arbitrary.

To finish the proof we consider the case that $\lambda_1 = 0$ and/or $\lambda_2 = 0$. By assumption, λ_i ($i \in \{1,2\}$) can only equal zero if $\delta_i > 0$. The idea is to still use (4.6), where we used $\lambda_1 > 0$ for the treatment of $E_2(y,t)$ and $\lambda_2 > 0$ for the treatment of $E_1(y,t)$. However, if $\lambda_1 = 0$, then Y^1 is a $\mathrm{CP}(0, \delta_1)$ with $\delta_1 > 0$, so the number of infected individuals can only decrease. Choosing $N_{\mathrm{CP}} := |y_1| + 1$ makes the line in (4.6) in which $E_2(y,t)$ appears vanish. Analogously, choosing $N_{\mathrm{cCP}} := |y_2| + 1$ makes the line in which $E_1(y,t)$ appears vanish if $\lambda_2 = 0$. For the rest of the terms one then can argue as above.

We conclude that in all cases (4.5) holds, thus also (1.9) as explained above. Lastly, it is well-known (compare [15, Lemma 4.40]) that (1.9) implies that ν is indeed invariant and the proof is complete. □

5 Survival

In this section we prove Proposition 1.1. Let $X = (X_t)_{t\geq 0}$ be a cCP and let $\delta_0 \in \mathcal{T}_{\mathrm{fin}}$ be the configuration that equals 1 only at the origin. We say that X *survives* if

$$\mathbb{P}^{\delta_0}\big[\exists\, t \geq 0 : X_t = \underline{0}\big] < 1.$$

The following result is known to hold for several processes. It is stated as [12, Lemma 1] for an important class of cancellative processes. However, the cCP does not fit into this class and the definition of survival in the cited paper slightly differs from the one we are using here, so we provide a short proof below. Recall that $\dot{\nu}$ is an invariant law of the $\mathrm{cCP}(\lambda,\delta)$ that is defined as the long-time limit law of the process started in a non-trivial shift-invariant distribution, which is known to exist for $\lambda+\delta > 0$ by [2, Theorem 1.2 & Theorem 1.3].

Proposition 5.1 (Survival of the cCP)**.** *One has $\dot{\nu} \neq \delta_{\underline{0}}$ if and only if the* cCP *survives.*

Proof. We prove this statement using ψ_{canc}, the (multiplicative representation of the) cancellative duality function defined in (1.15). It is well-known that ψ_{canc} is informative, a fact that also follows from Proposition 3.2. Let $X = (X_t)_{t\geq 0}$ be a $\mathrm{cCP}(\lambda,\delta)$ ($\lambda,\delta \geq 0,\ \lambda+\delta > 0$) and let $x \in \mathcal{T}_{\mathrm{fin}}$. If $\lambda,\delta > 0$, then [2, Theorem 1.2] implies that

$$\dot{\nu}\big(\{y : |x \wedge y| \text{ is odd}\}\big) = \frac{1}{2}\mathbb{P}^x\big[X_t \neq \underline{0}\ \forall t \geq 0\big]. \tag{5.1}$$

By the definition of ψ_{canc}, (5.1) is equivalent to

$$\int \psi_{\mathrm{canc}}(x,y)\,\mathrm{d}\dot{\nu}(y) = \mathbb{P}^x\big[\exists\, t \geq 0 : X_t = \underline{0}\big]. \tag{5.2}$$

Choosing $x = \delta_0$ implies that $\dot{\nu} \neq \delta_{\underline{0}}$ if X survives. On the other hand, if X does not survive and Y is a random variable with law $\dot{\nu}$, then (5.1) with $x = \delta_0$ implies that $\mathbb{P}[Y(0) = 0] = 1$ and the shift-invariance of $\dot{\nu}$ implies that $\mathbb{P}[Y(j) = 0] = 1$ for all $j \in \mathbb{Z}$. Hence $\dot{\nu} = \delta_{\underline{0}}$ as measures on $\mathcal{U}$ are characterised by their final dimensional marginals.

To complete the proof we consider the two special cases $\lambda = 0$ and $\delta = 0$. If $\lambda = 0$, then $\delta > 0$ and clearly X does not survive while $\dot{\nu} = \delta_{\underline{0}}$. If $\delta = 0$, then $\lambda > 0$ and X survives (one even has $\mathbb{P}^{\delta_0}[\exists\, t \geq 0 : X_t = \underline{0}] = 0$) and $\dot{\nu} \neq \delta_{\underline{0}}$ by [2, Theorem 1.3]. □

By Proposition 5.1, to prove Proposition 1.1, it suffices to show that the $\mathrm{cCP}(\lambda,\delta)$ does not survive when $\lambda \leq 2\delta$. Let now $d = 1$. Following [14] (compare the definition of L in [14, Section 2]), the idea for the proof of Proposition 1.1 is to construct a supermartingale applying Dynkin's formula to the function $g : \mathcal{T}_{\mathrm{fin}} \setminus \{\underline{0}\} \to \mathbb{N}_0$ defined as

$$g(x) := \max\{i \in \mathbb{Z} : x(i) = 1\} - \min\{i \in \mathbb{Z} : x(i) = 1\} \qquad (x \in \mathcal{T}_{\mathrm{fin}}). \tag{5.3}$$

In order to be able to apply Dynkin's formula one can "reduce" the cCP to a finite state space similarly as in [13, Proof of Lemma 3]. A full proof including the technical details is given below.

Proof of Proposition 1.1. Let $d = 1$ and assume that X is a cCP(λ, δ) with $\lambda \le 2\delta$. Using the g from (5.3) we define $f : \mathcal{T}_{\text{fin}} \to \mathbb{N}_0$ as

$$f(x) = \begin{cases} g(x) + 4 & \text{if } x \ne \underline{0}, \\ 0 & \text{else,} \end{cases} \qquad (x \in \mathcal{T}_{\text{fin}}).$$

One then has that $G_\oplus f(x) \le 0$ for all $x \in \mathcal{T}_{\text{fin}}$, where $G_\oplus$ denotes the generator of the cCP from (1.2). To see this we first look at $x_{101}, x_{11} \in \mathcal{T}_{\text{fin}}$ defined as

$$x_{101}(i) = \begin{cases} 1 & \text{if } i \in \{0, 2\}, \\ 0 & \text{else,} \end{cases} \qquad x_{11}(i) = \begin{cases} 1 & \text{if } i \in \{0, 1\}, \\ 0 & \text{else,} \end{cases} \qquad (x \in \mathbb{Z}).$$

In the configuration x_{101} the one at the origin reproduces with rate λ to the left, increasing the function f by one and it dies with rate δ, decreasing f by two. A reproduction to the right has no effect on f. By symmetry, an analogous statement holds for the one at $2 \in \mathbb{Z}$ so that $G_\oplus f(x_{101}) = 2\lambda - 4\delta$. For x_{11} on the other hand, a reproduction of the one at the origin to the right reduces f by one and its death reduces f by only one, while a reproduction to the left again increases f by one. Hence $G_\oplus f(x_{11}) = -2\delta$. Let now $x \in \mathcal{T}_{\text{fin}}$ be an arbitrary configuration with at least two ones. As f is shift-invariant, i.e. $f = f \circ \theta_i^{-1}$ for all $i \in \mathbb{Z}$, one has that $G_\oplus f(x) \le G_\oplus f(x_{101})$ if x has the form $010\ldots010$, $G_\oplus f(x) = G_\oplus f(x_{11})$ if x has the form $011\ldots110$ and $G_\oplus f(x) \le (G_\oplus f(x_{11}) + G_\oplus f(x_{101}))/2$ if x has the form $010\ldots110$ or $011\ldots010$. Note we had to use inequalities above as a death event of a one at the edge of a configuration reduces f by the number of zeros "to the inside" of this one, hence by at least two if there is a zero directly to the inside of the one. Finally we consider the special case $x = \delta_0$, in which with rate 2λ the lone individual reproduces (either to the left or to the right) and with rate δ it dies. Hence $G_\oplus f(\delta_0) = G_\oplus f(x_{101}) = 2\lambda - 4\delta$, which was the reason to add the 4 in the definition of f. This completes the argument that $\lambda \le 2\delta$ implies that $G_\oplus f(x) \le 0$ for all $x \in \mathcal{T}_{\text{fin}}$.

The rest of the proof is a standard argument from the theory of continuous-time Markov chains, but, for the sake of completeness, we state it completely. Let $N \in \mathbb{N}$ be arbitrary and set $\tau_N := \inf\{t \ge 0 : f(X_t) \ge N + 4\}$. We claim that $M^N = (M_t^N)_{t \ge 0}$ defined as

$$M_t^N := f(X_{t \wedge \tau_N}) - \int_0^{t \wedge \tau_N} G_\oplus f(X_s) \, \mathrm{d}s \qquad (t \ge 0)$$

is a martingale. Let

$$\mathcal{T}_N := \{x \in \mathcal{T}_{\text{fin}} : x(i) = 0 \text{ if } i \notin \{0, \ldots, N-1\}\} \cup \{x_N\},$$

where

$$x_N(i) := \begin{cases} 1 & \text{if } i \in \{0, N\}, \\ 0 & \text{else,} \end{cases} \qquad (i \in \mathbb{Z}).$$

By shifting every $x \in \mathcal{T}_{\text{fin}}$ so that its leftmost 1 lies at the origin we can construct a continuous-time Markov chain $Y = (Y_t)_{t \ge 0}$ on the finite state space $\mathcal{T}_N$ so that

$$M_t^N = f(Y_t) - \int_0^t G_\oplus f(Y_s) \, \mathrm{d}s \qquad (t \ge 0).$$

As a continuous-time Markov chain on a finite state space Y is a Feller process and Dynkin's formula implies that M^N is indeed a martingale.

As $G_\oplus f(x) \le 0$ for all $x \in \mathcal{T}_{\text{fin}}$ we conclude that $M^s = (f(X_{t \wedge \tau_N}))_{t \ge 0}$ is a uniformly integrable supermartingale and the martingale convergence theorem implies that M^s

converges almost surely and in L_1 to a random variable M_∞. The random variable M_∞ is supported on $\{0, N+4\}$ as $M_\infty \in \{1, \dots, N+3\}$ would imply that there exists a $t_0 \geq 0$ such that $M_t^s = M_{t_0}^s \in \{1, \dots, N+3\}$ for all $t \geq t_0$, which has probability zero. Hence

$$4 = \mathbb{E}^{\delta_0}[f(X_0)] \geq \mathbb{E}[M_\infty] = (N+4)(1 - \mathbb{P}(M_\infty = 0))$$

and we conclude that

$$\mathbb{P}^{\delta_0}(\exists\, t \geq 0 : X_t = \underline{0}) \geq \mathbb{P}^{\delta_0}(\exists\, t \leq \tau_N : X_t = \underline{0}) = \mathbb{P}(M_\infty = 0) \geq \frac{N}{N+4}.$$

As N was arbitrary it follows that $\mathbb{P}^{\delta_0}(\exists\, t \geq 0 : X_t = \underline{0}) = 1$ and Proposition 5.1 implies that $\dot{\nu} = \delta_{\underline{0}}$. This establishes that $\lambda_{\rm cCP} \geq 2$. □

A Parity indeterminacy

In this appendix we restate [12, Corollary 9] in a more accessible form and show how it can be derived from the somewhat less accessible formulation in [12]. Then we show how this result implies Lemma 4.2. Recall from Section 1.2 and Section 1.4 the definitions of the operator $\oplus$ (addition modulo 2), of $\mathcal{T}$, the space all functions from $\mathbb{Z}^d$ to $T = \{0,1\}$, of $\mathcal{T}_{\rm fin} \subset \mathcal{T}$, and of the cancellative duality function ψ_2. Let $\mathcal{A}$ be the set of all matrices of the form $A = (A(i,j))_{i,j \in \mathbb{Z}^d}$ with $A(i,j) \in \{0,1\}$ for all $i,j \in \mathbb{Z}^d$ and $\sum_{i,j} A(i,j) < \infty$. For $A \in \mathcal{A}$ and $x \in \mathcal{T}$, we define $Ax \in \mathcal{T}_{\rm fin}$, corresponding to the usual matrix-vector multiplication, as

$$Ax(i) := \bigoplus_{j \in \mathbb{Z}^d} \big(A(i,j) \cdot x(j)\big) \qquad (i \in \mathbb{Z}^d),$$

where $\cdot$ denotes the usual product in $\mathbb{R}$. Let $A^\dagger(i,j) := A(j,i)$ denote the adjoint of A. We will be interested in an interacting particle system $X = (X_t)_{t \geq 0}$ with state space $\mathcal{T}$, that jumps from its current state x as

$$x \mapsto x \oplus Ax \quad \text{with rate} \quad a(A), \tag{A.1}$$

where $(a(A))_{A \in \mathcal{A}}$ are non-negative rates and the operator $\oplus$ has to be interpreted in a pointwise sense, as well as the interacting particle system $Y = (Y_t)_{t \geq 0}$ that jumps as

$$y \mapsto y \oplus A^\dagger y \quad \text{with rate} \quad a(A).$$

In order for these interacting particle systems to be well-defined, we assume (compare [12, Condition (3.1)]) that

$$\sup_{i \in \mathbb{Z}^d} \sum_{A \in \mathcal{A}} a(A) \big|\{j : A(j,i) = 1\}\big| < \infty \quad \text{and} \quad \sup_{i \in \mathbb{Z}^d} \sum_{A \in \mathcal{A}} a(A) \big|\{j : A^\dagger(j,i) = 1\}\big| < \infty. \tag{A.2}$$

Recall from Section 4 that $|z| := |\{i \in \mathbb{Z}^d : z(i) = 1\}|$ ($z \in \mathcal{T}$). Indeed, it is not hard to verify that condition (A.2) guarantees that [15, Condition (4.15)] is satisfied both for the process X and the process Y, and hence both processes are well-defined by [15, Theorem 4.19 & Theorem 4.20]. Moreover, [15, Proposition 6.10] yields the duality relation

$$\mathbb{P}\big[|X_t Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 Y_t| \text{ is odd}\big] \qquad (t \geq 0) \tag{A.3}$$

whenever X and Y are independent and either $|X_0|$ or $|Y_0|$ is a.s. finite.

We will restate [12, Corollary 9], which gives sufficient conditions for the left-hand side of (A.3) to be close to $1/2$. We assume that the rates are translation invariant in the sense that

$$a(\theta_i A) = a(A) \qquad (i \in \mathbb{Z}^d,\ A \in \mathcal{A}), \tag{A.4}$$

where $\theta_i A$ denotes the "translated" matrix $(\theta_i A)(j,k) := A(j-i,k-i)$ $(j,k\in\mathbb{Z}^d)$. By definition, we say that a state $x\in\mathcal{T}$ is X*-nontrivial* if

$$\mathbb{P}^x\big[\big(X_t(i)\big)_{i\in\Delta}=\big(z(i)\big)_{i\in\Delta}\big]>0 \quad \text{for all } t>0, \text{ finite } \Delta\subset\mathbb{Z}^d, \text{ and } \big(z(i)\big)_{i\in\Delta}\in\{0,1\}^\Delta. \tag{A.5}$$

We fix a finite subset $\mathcal{B}\subset\mathcal{A}$ such that $a(B)>0$ for all $B\in\mathcal{B}$ and we define, for $x\in\mathcal{T}$,

$$\|x\|_{\mathcal{B}} := \big|\big\{i\in\mathbb{Z}^d : \exists\, y\in\mathcal{T} \text{ and } B\in\mathcal{B} \text{ s.t. } \boldsymbol{\psi}_2\big(x,(\theta_i B)y\big)=1\big\}\big|.$$

With these definitions, [12, Corollary 9] can be restated as follows. Recall the definition of the (pointwise) minimum operator $\wedge$ from Section 4.

Proposition A.1 (Parity indeterminacy)**.** *Let X be started in a shift-invariant initial law that is concentrated on X-nontrivial configurations. Then for each $\varepsilon>0$ and $t>0$, there exists an $N<\infty$ such that*

$$\left|\mathbb{P}\big[|X_t\wedge y| \text{ is odd}\big]-\frac{1}{2}\right|\le\varepsilon \tag{A.6}$$

for all $y\in\mathcal{T}_{\text{fin}}$ with $\|y\|_{\mathcal{B}}\ge N$.

Proof. This is a simple reformulation of [12, Corollary 9]. There, it is proved that if $y_n\in\mathcal{T}_{\text{fin}}$ satisfy $\|y_n\|_{\mathcal{B}}\to\infty$, then $\mathbb{P}\big[|X_t\wedge y_n| \text{ is odd}\big]\to\frac{1}{2}$. To see that this implies the claim of Proposition A.1, note that if the claim would be false, then there exists an $\varepsilon>0$ such that for all $n\ge1$ one can find $y_n\in\mathcal{T}_{\text{fin}}$ with $\|y_n\|_{\mathcal{B}}\ge n$ such that the left-hand side of (A.6) is $>\varepsilon$, contradicting [12, Corollary 9]. ☐

Applying Proposition A.1 to the cancellative contact process we obtain Lemma 4.2.

Proof of Lemma 4.2. We first show that the jump rates of the cancellative contact process can be cast in the form (A.1). Let $e_1,\dots,e_d\in\mathbb{Z}^d$ denote the unit vectors and let $0\in\mathbb{Z}^d$ denote the origin. For $1\le k\le d$, we define $I_k^\pm\in\mathcal{A}$ by $I_k^\pm(i,j):=1$ if $(i,j)=(\pm e_k,0)$ and $I_k^\pm(i,j):=0$ otherwise. Also, we define $D\in\mathcal{A}$ by $D(i,j):=1$ if $(i,j)=(0,0)$ and $D(i,j):=0$ otherwise. Finally, we define rates $\big(a(A)\big)_{A\in\mathcal{A}}$ by

$$a(\theta_i I_k^\pm):=\lambda \quad\text{and}\quad a(\theta_i D):=\delta \qquad (i\in\mathbb{Z}^d,\ 1\le k\le d),$$

and $a(A):=0$ in all other cases. Clearly, these rates are translation invariant in the sense of (A.4) and satisfy the summability condition (A.2). Also, a jump of the form $x\mapsto x\oplus(\theta_{-i}I_k^\pm)x$ corresponds to a jump of the form $x\mapsto \texttt{inf}^{\oplus}_{i,i\pm e_k}(x)$ in the notation of Section 1.2 and a jump of the form $x\mapsto x\oplus(\theta_{-i}D)x$ corresponds to a jump of the form $x\mapsto \texttt{dth}_i(x)$, so the process defined by these rates is a cCP(λ,δ). The claim of Lemma 4.2 will now follow from Proposition A.1 provided we show that: (i) each configuration $x\neq\underline{0}$ is X-nontrivial and: (ii) we can choose $\mathcal{B}$ such that $\|y\|_{\mathcal{B}}=|y|$.

We start by proving (ii). We set $\mathcal{B}:=\{I_1^+\}$, where I_1^+ as defined above is one of the matrices corresponding to an infection next to the origin. Then $a(I_1^+)=\lambda>0$. Moreover,

$$\boldsymbol{\psi}_2\big((\theta_{-i}I_1^+)x,y\big)=x(i)\cdot y(i+e_1)$$

and hence

$$y(i)=1 \quad\text{if and only if}\quad e_1-i\in\big\{i\in\mathbb{Z}^d : \exists\, x\in\mathcal{T} \text{ and } B\in\mathcal{B} \text{ s.t. } \boldsymbol{\psi}_2\big((\theta_i B)x,y\big)=1\big\},$$

which shows that $\|y\|_{\mathcal{B}}=|y|$.

It remains to prove (i). Fix $x\in\mathcal{T}\setminus\{\underline{0}\}$, a finite set $\Delta\subset\mathbb{Z}^d$, and $\big(z(i)\big)_{i\in\Delta}\in\{0,1\}^\Delta$. Using the fact that $x\neq\underline{0}$ and $\lambda>0$, in a finite number of infection steps, we can infect each site in $\Delta \cup \{i \in \mathbb{Z}^d : \exists j \in \Delta : j \sim i\}$. Starting with the sites in Δ with the highest graph distance to $\mathbb{Z}^d \setminus \Delta$, we then can remove the infection from all sites i such that $z(i) = 0$ only using further infections, proving that the probability in (A.5) is positive for each $t > 0$. $\square$

The true strength of Proposition A.1 lies in the fact that it can be applied even in situations where the definitions of X-nontriviality and the norm $\|y\|_{\mathcal{B}}$ are more complicated. In particular, [12, Theorem 3] is based on an application of Proposition A.1 in a situation where the X-nontrivial configurations are all $x \neq \underline{0}, \underline{1}$, and $\|y\|_{\mathcal{B}} = \big|\{(i,j) : |i-j| = 1,\ y(i) \neq y(j)\}\big|$.

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