

On the second moment method and RS phase of multi-species spherical spin glasses*

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Abstract

Excluding some special cases, computing the critical inverse-temperature β_c of a mixed p -spin spin glass model is a difficult task. The only known method to calculate its value for a general model requires the full power of the Parisi formula. On the other hand, an easy application of the second moment method to the partition function yields an explicit lower bound $\beta_m \leq \beta_c$ to the critical inverse-temperature. Interestingly, in the important case of the Sherrington-Kirkpatrick model $\beta_m = \beta_c$. In this work we consider the multi-species spherical mixed p -spin models without external field, and characterize by a simple condition the models for which the second moment method works in the whole replica symmetric phase, namely, models such that $\beta_m = \beta_c$. In particular, for those models we obtain the value of β_c .

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1 Introduction

The high-temperature phase of a mixed p -spin spin glass model (with no external field) consists of inverse-temperatures $\beta \leq \beta_c$ such that, for large N ,

$$\mathbb{E}F_{N,\beta} := \frac{1}{N} \mathbb{E} \log Z_{N,\beta} \approx \frac{1}{N} \log \mathbb{E} Z_{N,\beta}. \quad (1.1)$$

Computing the mean of the partition function $Z_{N,\beta}$ is trivial, and thus for β as above one has the mean of the free energy $F_{N,\beta}$ in the large N limit. Beyond the expression for the free energy, the system simplifies in this phase in several ways. Most importantly, no ‘replica symmetry breaking’ occurs — that is, independent samples from the Gibbs measure are typically roughly orthogonal to each other.

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Despite this, computing the critical value β_c is generally a difficult problem. For general mixed p -spin models, the only known method to achieve its value heavily relies on the Parisi formula [20, 21, 32], one of the deepest, most complicated results in mean-field spin glass theory. See the works of Chen [11] and Talagrand [31] where a characterization for sub-critical inverse-temperatures β is derived from the optimality criterion for the Parisi distribution, for (single-species) models with Ising and spherical spins respectively.

In sharp contrast to the usage of the Parisi formula in its extreme simplicity, an application of the second moment method to the partition function $Z_{N,\beta}$ very easily allows one to lower bound the critical inverse-temperature β_c . Interestingly, for the important SK model [22], the bound actually gives the correct critical value β_c . In this paper we focus on the question: when does the second moment work up to the critical β_c ? We answer this question for the multi-species spherical mixed p -spin models.

1.1 Definition of the model

Consider a finite set of species \mathcal{S} , which will be fixed throughout the paper. For each $N \geq 1$, we will suppose that

$$\{1, \dots, N\} = \bigcup_{s \in \mathcal{S}} I_s, \quad \text{for some disjoint } I_s.$$

The subsets I_s , of course, vary with N . Denoting $N_s := |I_s|$, we will assume that the proportion of each species converges

$$\lim_{N \rightarrow \infty} \frac{N_s}{N} = \lambda(s) \in (0, 1), \quad \text{for all } s \in \mathcal{S}.$$

Let $S(d) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = \sqrt{d}\}$ be the sphere of radius \sqrt{d} in dimension d . The *configuration space* of the multi-species spherical mixed p -spin model is

$$S_N = \{(\sigma_1, \dots, \sigma_N) \in \mathbb{R}^N : \forall s \in \mathcal{S}, (\sigma_i)_{i \in I_s} \in S(N_s)\}.$$

Denoting $\mathbb{Z}_+ := \{0, 1, \dots\}$ and $|p| := \sum_{s \in \mathcal{S}} p(s)$ for $p \in \mathbb{Z}_+^{\mathcal{S}}$, let

$$P = \{p \in \mathbb{Z}_+^{\mathcal{S}} : |p| \geq 2\}.$$

Given some nonnegative numbers $(\Delta_p)_{p \in P}$, define the *mixture polynomial* in $x = (x(s))_{s \in \mathcal{S}} \in \mathbb{R}^{\mathcal{S}}$,

$$\xi(x) = \sum_{p \in P} \Delta_p^2 \prod_{s \in \mathcal{S}} x(s)^{p(s)}. \tag{1.2}$$

We will assume that $\xi(1 + \epsilon) < \infty$ for some $\epsilon > 0$, where for $a \in \mathbb{R}$ we write $\xi(a)$ for the evaluation of ξ at the constant function $x \equiv a$.

The multi-species mixed p -spin *Hamiltonian* $H_N : S_N \rightarrow \mathbb{R}$ corresponding to the mixture ξ is given by

$$H_N(\boldsymbol{\sigma}) = \sqrt{N} \sum_{k=2}^{\infty} \sum_{i_1, \dots, i_k=1}^N \Delta_{i_1, \dots, i_k} J_{i_1, \dots, i_k} \sigma_{i_1} \cdots \sigma_{i_k}, \tag{1.3}$$

where J_{i_1, \dots, i_k} are i.i.d. standard normal variables and if $\#\{j \leq k : i_j \in I_s\} = p(s)$ for any $s \in \mathcal{S}$, then $\Delta_{i_1, \dots, i_k} = \Delta_{i_1, \dots, i_k}(N)$ is defined by

$$\Delta_{i_1, \dots, i_k}^2 = \Delta_p^2 \frac{\prod_{s \in \mathcal{S}} p(s)!}{|p|!} \prod_{s \in \mathcal{S}} N_s^{-p(s)}. \tag{1.4}$$

By a straightforward calculation, the covariance function of $H_N(\boldsymbol{\sigma})$ is given by

$$\mathbb{E}H_N(\boldsymbol{\sigma})H_N(\boldsymbol{\sigma}') = N\xi(R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')), \tag{1.5}$$

where we define the overlap vector

$$R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := (R_s(\boldsymbol{\sigma}, \boldsymbol{\sigma}'))_{s \in \mathcal{S}}, \quad R_s(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := N_s^{-1} \sum_{i \in I_s} \sigma_i \sigma'_i.$$

Identifying S_N with the product space $\prod_{s \in \mathcal{S}} S(N_s)$, let μ be the product of the uniform measures on each of the spheres $S(N_s)$. The *partition function* and *free energy* at inverse-temperature $\beta \geq 0$ are, respectively, defined by

$$Z_{N,\beta} := \int_{S_N} e^{\beta H_N(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}) \quad \text{and} \quad F_{N,\beta} := \frac{1}{N} \log Z_{N,\beta}. \tag{1.6}$$

The *Gibbs measure* is the random probability measure on S_N with density

$$\frac{dG_{N,\beta}}{d\mu}(\boldsymbol{\sigma}) = Z_{N,\beta}^{-1} e^{\beta H_N(\boldsymbol{\sigma})}.$$

If $|\mathcal{S}| = 1$, all the definitions above coincide with the usual (single-species) spherical mixed p -spin model.

By Jensen's inequality,

$$\mathbb{E}F_{N,\beta} = \frac{1}{N} \mathbb{E} \log Z_{N,\beta} \leq \frac{1}{N} \log \mathbb{E} Z_{N,\beta} = \frac{1}{2} \beta^2 \xi(1). \tag{1.7}$$

It is not difficult to check (also using Jensen's inequality, see Lemma 2.1 below) that for any $\beta' < \beta$,

$$\mathbb{E}F_{N,\beta} \leq \mathbb{E}F_{N,\beta'} + \frac{1}{2}(\beta^2 - \beta'^2)\xi(1).$$

Hence, there exists a critical inverse-temperature β_c such that

$$\lim_{N \rightarrow \infty} \mathbb{E}F_{N,\beta} = \frac{1}{2} \beta^2 \xi(1) \iff \beta \leq \beta_c. \tag{1.8}$$

1.2 The second moment method

By the Paley-Zygmund inequality, if we are able to show for some β that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} Z_{N,\beta}^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left((\mathbb{E} Z_{N,\beta})^2 \right) = \beta^2 \xi(1), \tag{1.9}$$

then for any $\theta \in (0, 1)$, $Z_{N,\beta} \geq \theta \mathbb{E} Z_{N,\beta}$ with probability not exponentially small in N . Combined with the well-known concentration of the free energy, (see e.g., [17, Theorem 1.2]) this easily implies that $\beta \leq \beta_c$.

By Fubini's theorem and symmetry we have that

$$\begin{aligned} \mathbb{E} Z_{N,\beta}^2 &= \int \int \mathbb{E} \exp \{ \beta H_N(\boldsymbol{\sigma}) + \beta H_N(\boldsymbol{\sigma}') \} d\mu(\boldsymbol{\sigma}) d\mu(\boldsymbol{\sigma}') \\ &= \int \exp \{ N\beta^2(\xi(1) + \xi(R(\boldsymbol{\sigma}, \boldsymbol{\sigma}_0))) \} d\mu(\boldsymbol{\sigma}), \end{aligned}$$

where $\boldsymbol{\sigma}_0 \in S_N$ is an arbitrary point. Using the coarea formula, one can then check that

$$\mathbb{E} Z_{N,\beta}^2 = \int_{[-1,1]^{\mathcal{S}}} \prod_{s \in \mathcal{S}} \frac{\omega_{N_s-1}}{\omega_{N_s}} (1-r(s)^2)^{\frac{N_s-3}{2}} e^{N\beta^2(\xi(1)+\xi(r))} dr,$$

where ω_d denotes the volume of the unit sphere in \mathbb{R}^d . Therefore

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} Z_{N,\beta}^2 = \beta^2 \xi(1) + \max_{r \in [0,1]^{\mathcal{S}}} f_\beta(r), \tag{1.10}$$

where

$$f_\beta(r) := \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda(s) \log(1 - r(s)^2) + \beta^2 \xi(r).$$

If we define the threshold inverse-temperature

$$\beta_m := \max \left\{ \beta \geq 0 : \max_{r \in [0,1]^{\mathcal{S}}} f_\beta(r) = f_\beta(0) = 0 \right\},$$

then (1.9) holds if and only if $\beta \leq \beta_m$, from which we have that $\beta_m \leq \beta_c$.

The second moment method similarly works for models with Ising spins up to a threshold β_m as above, if one appropriately modifies the logarithmic entropy term in the definition of $f_\beta(r)$. For the SK model, Talagrand used the method in [29, Section 2] to show that $\beta_m = 1/\sqrt{2}$ (the fact that $\beta_c \geq 1/\sqrt{2}$ was first proved in [1]). Exploiting special properties of the SK model, Comets proved in [14] that $\beta_c \leq 1/\sqrt{2}$ and therefore $\beta_c = \beta_m$. For the p -spin generalization of the SK models, Talagrand [30] used a truncated second moment argument to prove a lower bound for the critical β_c . Bolthausen [9] applied the second moment method conditional on an event related to the TAP equations to compute the free energy of the SK model with an external field at high-temperature.

In the context of the spherical models but in a different direction than the above, the second moment method was used in the study of critical points. In [4, 8, 15, 23, 28] it was applied to the complexity of critical points to prove its concentration around the mean [2, 3, 16].

1.3 Main results

Our main result is the following theorem. We will make the following assumption,

$$x \in [0, 1]^{\mathcal{S}} \setminus \{0\} \implies \xi(x) > 0. \tag{A}$$

The assumption will be used in the proofs to conclude that if $r \in [0, 1]^{\mathcal{S}}$ and $\xi(r) \approx 0$, then $r \approx 0$. In the single-species case $|\mathcal{S}| = 1$, which is also covered by our results, the assumption always holds.

Theorem 1.1. *Assuming (A), $\beta_m = \beta_c$ if and only if*

$$\left(\frac{d}{dr(s)} \frac{d}{dr(t)} f_{\beta_m}(0) \right)_{s,t \in \mathcal{S}} \text{ is a singular matrix.} \tag{1.11}$$

Since $\left(\frac{d}{dr(s)} f_\beta(0) \right)_{s \in \mathcal{S}} = 0$, the matrix in (1.11) determines the local behavior of $f_{\beta_m}(r)$ around the origin. Roughly speaking, the theorem says that the second moment method works up to the critical inverse-temperature if for β slightly above β_m the condition that $\max_{r \in [0,1]^{\mathcal{S}}} f_\beta(r) = 0$ is broken around $r = 0$.

The proof that $\beta_m < \beta_c$ if the matrix in (1.11) is regular is the easy part of the theorem. It will follow by showing that in this case $\beta_m < \tilde{\beta}_m \leq \beta_c$ for some other threshold $\tilde{\beta}_m$, which will arise from applying the second moment method to a random variable different from $Z_{N,\beta}$. The argument essentially uses the same idea as in the truncated second moment method used by Talagrand in [30].

To prove the main part of the theorem, concerning the case that the matrix (1.11) is singular, we will prove the following proposition.

Proposition 1.2. *Assuming (A), if for some β ,*

$$\left(\frac{d}{dr(s)}\frac{d}{dr(t)}f_\beta(0)\right)_{s,t\in\mathcal{S}} \text{ has a non-negative eigenvalue,} \tag{1.12}$$

then $\beta_c \leq \beta$.

For the spherical single-species mixed p -spin models, the Parisi formula for the limit of the free energy was proved by Talagrand [31] for models with even interactions, and later generalized to arbitrary mixtures by Chen [10]. For the multi-species spherical mixed p -spin models, the Parisi formula was recently proved by Bates and Sohn [6, 7], assuming that the mixture polynomial $\xi(x)$ is convex on $[0, 1]^{\mathcal{S}}$. For the proof of the Parisi formula for models with Ising spins, see the works of Talagrand and Panchenko for the single-species case [18, 32] and the work of Panchenko [19] for the multi-species SK model, where it was also assumed that $\xi(x)$ is convex.

For the single-species spherical models, Talagrand also proved in [31] the following characterization of sub-critical inverse-temperatures, using the Parisi formula: $\beta \leq \beta_c$ if and only if

$$\forall r \in [0, 1) : g_\beta(r) := \log(1 - r) + r + \beta^2 \xi(r) \leq 0. \tag{1.13}$$

Using that $\log(1 - r) = -\sum_{n \geq 1} r^n/n$, we have that in the single-species case

$$\forall r \in [0, 1) : f_\beta(r) = g_\beta(r) + \frac{r^3}{3} + \frac{r^5}{5} + \frac{r^7}{7} + \dots \tag{1.14}$$

Reassuringly, this is consistent with Theorem 1.1. Possibly, an analogue of the characterization of the high-temperature phase from [31] for multi-species models can be proved using the results of [6, 7], assuming the convexity of $\xi(x)$.

For single-species spherical models, one can deduce Theorem 1.1 from the characterization (1.13) and (1.14). Still, even in this setting, it is interesting to understand from basic principles rather than the Parisi formula when does a basic tool like the second moment method fails or succeeds. For the multi-species models, our main result allows one to compute β_c for a certain class of models. This result is new, as no analogue for the criterion from [31] is known in this case. In particular, this class includes models which do not satisfy the assumption that $\xi(x)$ is convex¹ as in the proof of the Parisi formula in the multi-species setting [5, 7, 19]. Finally, one of the main motivations for this work is that our results are crucial to [27] where we compute the free energy for pure multi-species spherical models using the TAP representation developed in [26] (in particular the result we prove in the Appendix).

It is well-known (see Section 3) that for any $\beta \leq \beta_c$ and $\epsilon > 0$, with probability going to 1 as $N \rightarrow \infty$,

$$F_{N,\beta} \approx \frac{1}{N} \log \int_{L_\beta(\epsilon)} e^{\beta H_N(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}),$$

where we define the subset

$$L_\beta(\epsilon) := \left\{ \boldsymbol{\sigma} \in S_N : \frac{1}{N} H_N(\boldsymbol{\sigma}) \in (\beta \xi(1) - \epsilon, \beta \xi(1) + \epsilon) \right\}.$$

Moreover, with high probability,

$$\frac{1}{N} \log \mu(L_\beta(\epsilon)) \approx \frac{1}{N} \log \mathbb{E} \mu(L_\beta(\epsilon)). \tag{1.15}$$

¹For example, if $\Delta_p > 0$ only when $|p| = 2$ then $\xi(x)$ is a quadratic function, from which one can verify that (1.11) holds. Of course, the coefficients Δ_p with $|p| = 2$ can be chosen so that $\xi(x)$ is not convex. If $f_\beta(r)$ has a unique maximum at $r = 0$, one can add to such a mixture positive small coefficients $\Delta_p > 0$ with $|p| \geq 3$ such that also after the addition the mixture satisfies (1.11).

For a point $\sigma \in S_N$, an overlap vector $r \in [-1, 1]^{\mathcal{S}}$ and width $\delta > 0$, define the subset

$$B(\sigma, r, \delta) := \left\{ \sigma' \in S_N : \forall s \in \mathcal{S}, |R_s(\sigma, \sigma') - r(s)| \leq \delta \right\}.$$

We will show using (1.15) that for most points σ in $L_\beta(\epsilon)$, the free energy on $B(\sigma, r, \delta)$, namely

$$\frac{1}{N} \log \int_{B(\sigma, r, \delta)} e^{\beta H_N(\sigma')} d\mu(\sigma'), \tag{1.16}$$

is close to its conditional expectation given $H_N(\sigma)$ (see Lemma 3.2). By estimating this conditional expectation, we will prove the following in Section 3.

Proposition 1.3. *Assume (A) and that $\Delta_p \neq 0$ for finitely many p . Let $\beta \leq \beta_c$, $t > 0$. Then for any $r \in [0, 1]^{\mathcal{S}}$ and sufficiently small $\delta, \epsilon > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\mu(L_\beta(\epsilon) \cap A_r(t)) \geq (1 - e^{-Nc}) \cdot \mu(L_\beta(\epsilon)) > 0 \right) = 1, \tag{1.17}$$

where $A_r(t) \subset S_N$ is the set of points σ such that

$$\left| \frac{1}{N} \log \int_{B(\sigma, r, \delta)} e^{\beta H_N(\sigma')} d\mu(\sigma') - \frac{1}{2} \beta^2 \xi(1) - f_\beta(r) \right| < \kappa(r^2) + t, \tag{1.18}$$

for some constant $c = c(t, \xi) > 0$ and function $\kappa(x) = \kappa(x, \xi, \beta)$ of $x \in [0, 1]^{\mathcal{S}}$ such that $\kappa(0) = 0$ and whose directional derivatives at the origin in any direction x are zero

$$\lim_{\epsilon \rightarrow 0^+} \frac{\kappa(\epsilon x)}{\epsilon} = 0. \tag{1.19}$$

Obviously, the free energy (1.16) on $B(\sigma, r, \delta)$ lower bounds the total free energy $F_{N, \beta}$. Hence, for $\beta \leq \beta_c$, the proposition in particular gives us a lower bound for the free energy $F_{N, \beta}$ by using only one point from the set $L_\beta(\epsilon) \cap A_r(t)$, with high probability. To prove Proposition 1.2, assuming (1.12) we will show that if there were some inverse-temperature $\beta < \beta' < \beta_c$, then this lower bound would imply that $\mathbb{E}F_{N, \beta'} > \frac{1}{2} \beta'^2 \xi(1)$ in contradiction to (1.7).

In Section 2 we prove Proposition 1.2 and Theorem 1.1, assuming Proposition 1.3 which is be proved in Section 3 that occupies the rest of the paper.

2 Proof of the main results assuming Proposition 1.3

In this section we prove Theorem 1.1 and Proposition 1.2, assuming Proposition 1.3 which we will prove in Section 3.

2.1 Proof of Proposition 1.2

We will first prove the proposition assuming that $\Delta_p \neq 0$ for only finitely many p , in which case we may use Proposition 1.3.

Write $f_\beta(r) = f_0(r) + \beta^2 \xi(r)$ where

$$f_0(r) = \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda(s) \log(1 - r(s)^2).$$

Note that for any r and real α ,

$$\frac{d}{d\alpha} \Big|_{\alpha=0} f_\beta(\alpha r) = \frac{d}{d\alpha} \Big|_{\alpha=0} f_0(\alpha r) = \frac{d}{d\alpha} \Big|_{\alpha=0} \xi(\alpha r) = 0$$

and

$$\frac{d^2}{d\alpha^2} \Big|_{\alpha=0} f_0(\alpha r) = - \sum_{s \in \mathcal{S}} \lambda(s) r(s)^2 < 0,$$

where as usual $(\alpha r)(s) = \alpha r(s)$. Therefore, for any $\beta < \beta'$,

$$\frac{d^2}{d\alpha^2} \Big|_{\alpha=0} f_\beta(\alpha r) \geq 0 \implies \frac{d^2}{d\alpha^2} \Big|_{\alpha=0} f_{\beta'}(\alpha r) > 0. \tag{2.1}$$

Let β be some inverse-temperature and assume that the matrix in (1.12) has some non-negative eigenvalue. Then there exists some r with $\|r\| = 1$ such that the inequality on the left-hand side of (2.1) holds. From (1.2), it is easy to see that we may assume that this r belongs to $[0, 1]^{\mathcal{S}}$.

Assume towards contradiction that $\beta < \beta_c$ and let $\beta' \in (\beta, \beta_c)$. Then with the same r ,

$$\frac{d}{d\alpha} \Big|_{\alpha=0} f_{\beta'}(\alpha r) = 0 \quad \text{and} \quad \frac{d^2}{d\alpha^2} \Big|_{\alpha=0} f_{\beta'}(\alpha r) > 0.$$

Hence, we may choose some small enough α and $t > 0$ such that

$$f_{\beta'}(\alpha r) > \kappa(\alpha^2 r^2) + 2t,$$

where $\kappa(x)$ is the function from Proposition 1.3.

By the latter proposition, for some small $\delta > 0$ with probability going to 1 as $N \rightarrow \infty$, there exists a point $\sigma \in S_N$ such that

$$F_{N,\beta'} \geq \frac{1}{N} \log \int_{B(\sigma, \alpha r, \delta)} e^{\beta' H_N(\sigma')} d\mu(\sigma') > \frac{1}{2} \beta'^2 \xi(1) + t.$$

Combined with the well-known concentration of the free energy (see [17, Theorem 1.2]), this contradicts (1.7). We therefore conclude that $\beta_c \leq \beta$.

It remains to prove the proposition in the case where infinitely many Δ_p are non-zero. Given some $\xi(x)$ consider the mixture $\bar{\xi}(X)$ obtained by replacing by zero all the coefficients Δ_p whenever $|p| \geq 3$. Note that the matrix in (1.12) is determined by the coefficients Δ_p with $|p| = 2$ only. Moreover, by the following lemma, the critical inverse-temperature of $\xi(x)$ is less than or equal to that of $\bar{\xi}(x)$. From this, the proposition follows also when infinitely many Δ_p are non-zero.

Lemma 2.1. *Suppose that*

$$\xi(x) = \sum_{p:|p|\geq 1} \Delta_p^2 \prod_{s \in \mathcal{S}} x(s)^{p(s)}, \quad \bar{\xi}(x) = \sum_{p:|p|\geq 1} \bar{\Delta}_p^2 \prod_{s \in \mathcal{S}} x(s)^{p(s)}$$

are two mixtures such that $\Delta_p^2 \geq \bar{\Delta}_p^2$ for any p . Let $F_{N,\beta}$ and $\bar{F}_{N,\beta}$ their corresponding free energies. Then

$$\lim_{N \rightarrow \infty} \mathbb{E} F_{N,\beta} = \frac{1}{2} \beta^2 \xi(1) \implies \lim_{N \rightarrow \infty} \mathbb{E} \bar{F}_{N,\beta} = \frac{1}{2} \beta^2 \bar{\xi}(1).$$

Proof. Define the mixture

$$\hat{\xi}(x) = \xi(x) - \bar{\xi}(x) = \sum_{p:|p|\geq 1} (\Delta_p^2 - \bar{\Delta}_p^2) \prod_{s \in \mathcal{S}} x(s)^{p(s)}.$$

Denote by $H_N(\sigma)$, $\bar{H}_N(\sigma)$ and $\hat{H}_N(\sigma)$ the Hamiltonians corresponding to $\xi(x)$, $\bar{\xi}(x)$ and $\hat{\xi}(x)$, respectively. Define $\bar{H}_N(\sigma)$ and $\hat{H}_N(\sigma)$ on the same probability space such that they are independent. Note that in distribution (as processes)

$$H_N(\sigma) = \bar{H}_N(\sigma) + \hat{H}_N(\sigma),$$

since the covariance functions and expectation of the Gaussian processes in both sides are equal.

By Jensen's inequality,

$$\begin{aligned} \mathbb{E} \log \int_{S_N} e^{\beta H_N(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}) &= \mathbb{E} \mathbb{E} \left(\log \int_{S_N} e^{\beta(\bar{H}_N(\boldsymbol{\sigma}) + \hat{H}_N(\boldsymbol{\sigma}))} d\mu(\boldsymbol{\sigma}) \mid (\bar{H}_N(\boldsymbol{\sigma}))_{\boldsymbol{\sigma} \in S_N} \right) \\ &\leq \mathbb{E} \log \int_{S_N} e^{\beta \bar{H}_N(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}) + \frac{1}{2} N \beta^2 \hat{\xi}(1). \end{aligned}$$

Hence, if $\lim_{N \rightarrow \infty} \mathbb{E} F_{N,\beta} = \frac{1}{2} \beta^2 \xi(1)$, then

$$\liminf_{N \rightarrow \infty} \mathbb{E} \bar{F}_{N,\beta} \geq \frac{1}{2} \beta^2 \xi(1) - \frac{1}{2} \beta^2 \hat{\xi}(1) = \frac{1}{2} \beta^2 \bar{\xi}(1).$$

The matching upper bound follows from (1.7). □

2.2 Proof of Theorem 1.1

Recall that $\beta_m \leq \beta_c$. If the matrix

$$\left(\frac{d}{dr(s)} \frac{d}{dr(t)} f_{\beta_m}(0) \right)_{s,t \in \mathcal{S}} \tag{2.2}$$

is singular, then by Proposition 1.2, $\beta_c \leq \beta_m$ and thus $\beta_c = \beta_m$.

Henceforth, assume that the matrix above is regular. We will prove that in this case there exists $\beta_m < \beta \leq \beta_c$. As mentioned in the introduction, the argument we use is essentially equivalent to the second moment method with truncation used in [30].

Fix some arbitrary $\gamma > 0$. For $\beta > 0$, consider the random variable

$$U_{N,\beta} := \mu \left(\left\{ \boldsymbol{\sigma} \in S_N : \left| \frac{1}{N} H_N(\beta) - \beta \xi(1) \right| < N^{-\gamma} \right\} \right).$$

It is easy to check that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} U_{N,\beta} = -\frac{1}{2} \beta^2 \xi(1),$$

and similarly to (1.10),

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} U_{N,\beta}^2 = -\beta^2 \xi(1) + \max_{r \in [0,1]^{\mathcal{S}}} \tilde{f}_{\beta}(r),$$

where

$$\tilde{f}_{\beta}(r) := \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda(s) \log(1 - r(s)^2) + \beta^2 \frac{\xi(1)\xi(r)}{\xi(1) + \xi(r)}.$$

Define the inverse-temperature

$$\tilde{\beta}_m := \max \left\{ \beta \geq 0 : \max_{r \in [0,1]^{\mathcal{S}}} \tilde{f}_{\beta}(r) = \tilde{f}_{\beta}(0) = 0 \right\}.$$

For any $\beta \leq \tilde{\beta}_m$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} U_{N,\beta}^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left((\mathbb{E} U_{N,\beta})^2 \right) = -\beta^2 \xi(1).$$

From the Paley–Zygmund inequality, for such β and any $\theta \in (0, 1)$, with probability not exponentially small in N , $U_{N,\beta} \geq \theta \mathbb{E} U_{N,\beta}$. On this event, $F_{N,\beta} \geq \frac{1}{2} \beta^2 \xi(1) + o(1)$. Hence, from the well-known concentration of the free energy and (1.7), $\beta_m \leq \beta_c$.

Note that $\beta_m \leq \tilde{\beta}_m$, since on $[0, 1]^{\mathcal{S}}$,

$$f_\beta(r) - \tilde{f}_\beta(r) = \beta^2 \frac{\xi(r)^2}{\xi(1) + \xi(r)} \geq 0. \tag{2.3}$$

Recall that we assume that the Hessian matrix (2.2) is regular. By the definition of β_m , since

$$\forall s \in \mathcal{S}, \quad \frac{d}{dr(s)} f_\beta(0) = 0, \tag{2.4}$$

all the eigenvalues of the Hessian (2.2) of $f_{\beta_m}(r)$ at $r = 0$ are strictly negative. Choose some $\alpha > 0$ such that they are all less than $-\alpha$. From continuity, the eigenvalues of the Hessian of $f_\beta(r)$ at $r = 0$ are less than $-\alpha/2$ for any $\beta \geq \beta_m$ close enough to β_m . Combined with (2.4), for such β , this implies that

$$\begin{aligned} \max_{r \in A \cap [0, 1]^{\mathcal{S}}} \tilde{f}_\beta(r) &\leq \max_{r \in A \cap [0, 1]^{\mathcal{S}}} f_\beta(r) = 0, \\ \max_{r \in \partial A \cap [0, 1]^{\mathcal{S}}} f_\beta(r) &\leq -\epsilon, \end{aligned} \tag{2.5}$$

for some open neighborhood A of $r = 0$ and $\epsilon > 0$ independent of β , where ∂A denotes the boundary of A .

For small enough $\delta > 0$ and any $\beta \geq \beta_m$ close enough to β_m ,

$$\max_{r \in [0, 1]^{\mathcal{S}} \setminus [0, 1 - \delta]^{\mathcal{S}}} \tilde{f}_\beta(r) \leq \max_{r \in [0, 1]^{\mathcal{S}} \setminus [0, 1 - \delta]^{\mathcal{S}}} f_\beta(r) < 0.$$

Define the set

$$K = \left\{ r \in [0, 1 - \delta]^{\mathcal{S}} \setminus A : f_{\beta_m}(r) \geq -\frac{1}{2}\epsilon \right\}.$$

K cannot be empty because this would mean that there is some $\beta \geq \beta_m$ for which $f_\beta(r) \leq 0$ for all $r \in [0, 1]^{\mathcal{S}}$.

Assume towards contradiction that $M := \min_{r \in K} \xi(r) = 0$. Since K is a closed set, the minimum is obtained at some point $r \in K$. Thus, there exists some $r \in K$ such that $f_{\beta_m}(r) \geq -\epsilon/2$ and $\xi(r) = 0$. Note that since $\xi(r) = 0$ there exists some non-empty $T \subset \mathcal{S}$ such that $s \in T \iff r(s) = 0$. Note that reducing the value of $r(s)$ for $s \notin T$ increases the value of $f_{\beta_m}(r)$. Hence, there is some $r' \in \partial A \cap [0, 1 - \delta]^{\mathcal{S}}$ for which $f_{\beta_m}(r) \geq -\epsilon/2$ in contradiction to (2.5). We conclude that $M > 0$, i.e. that for any $r \in [0, 1 - \delta]^{\mathcal{S}} \setminus A$, either $f_{\beta_m}(r) < -\frac{1}{2}\epsilon$ or $\xi(r) > M > 0$. Combining this with (2.3) one sees that for $\beta > \beta_m$ close enough to β_m ,

$$\max_{r \in [0, 1 - \delta]^{\mathcal{S}} \setminus A} \tilde{f}_\beta(r) < 0.$$

Hence, by combining the bounds above, such β satisfies $\beta \leq \tilde{\beta}_m \leq \beta_c$, which completes the proof. \square

3 Proof of Proposition 1.3

To prove Proposition 1.3, we will need the three auxiliary results below. The first is an elementary well known result about the volume of approximate level sets, or entropy for sub-critical β .

Lemma 3.1. *If $\beta \leq \beta_c$, for any $t > 0$, for small enough $\epsilon > 0$, there exists some $c > 0$ such that for large N ,*

$$\mathbb{P} \left(\left| \frac{1}{N} \log \mu(L_\beta(\epsilon)) + \frac{1}{2} \beta^2 \xi(1) \right| < t \right) > 1 - e^{-Nc}.$$

Define the random fields

$$\phi_{N,\beta}(\boldsymbol{\sigma}, r, \delta) = \frac{1}{N} \log \int_{B(\boldsymbol{\sigma}, r, \delta)} e^{\beta H_N(\boldsymbol{\sigma}')} d\mu(\boldsymbol{\sigma}')$$

and

$$\varphi_{N,\beta}(\boldsymbol{\sigma}, r, \delta) = \mathbb{E} \left(\phi_{N,\beta}(\boldsymbol{\sigma}, r, \delta) \mid H_N(\boldsymbol{\sigma}) \right), \tag{3.1}$$

and the random set

$$D_N(r, \delta, t) = \{ \boldsymbol{\sigma} \in \mathbb{S}^{N-1} : |\phi_{N,\beta}(\boldsymbol{\sigma}, r, \delta) - \varphi_{N,\beta}(\boldsymbol{\sigma}, r, \delta)| > t \}.$$

Proposition 1.3 concerns the volume of points in $L_\beta(\epsilon)$ such that $\phi_{N,\beta}(\boldsymbol{\sigma}, r, \delta)$ is close to a certain value. The next lemma shows that $\phi_{N,\beta}(\boldsymbol{\sigma}, r, \delta)$ and $\varphi_{N,\beta}(\boldsymbol{\sigma}, r, \delta)$ are close to each other on $L_\beta(\epsilon)$, up to a subset of small volume. It will allow us to work with the conditional expectation $\varphi_{N,\beta}(\boldsymbol{\sigma}, r, \delta)$, which only depends on the value of the Hamiltonian at $\boldsymbol{\sigma}$.

Lemma 3.2. *Suppose that $\beta \leq \beta_c$. For any $r \in (-1, 1)^\mathcal{J}$ and positive δ and t , for large N ,*

$$\mathbb{P} \left(\frac{1}{N} \log \mu(L_\beta(\epsilon) \cap D_N(r, \delta, t)) \geq -\frac{1}{2} \beta^2 \xi(1) + \beta \epsilon - \frac{\epsilon^2}{2\xi(1)} - \frac{t^2}{8\xi(1)} \right) \leq e^{-\frac{Nt^2}{10\xi(1)}}.$$

The main ingredient in the proof of Proposition 1.3 is the following estimate on the conditional expectation $\varphi_{N,\beta}(\boldsymbol{\sigma}, r, \delta)$.

Proposition 3.3. *Assume (A) and that $\Delta_p \neq 0$ for finitely many values of p and $\beta \leq \beta_c$. Then, for any $r \in [0, 1)^\mathcal{J}$, almost surely,*

$$\limsup_{N \rightarrow \infty} \sup_{\boldsymbol{\sigma} \in S_N} \left| \frac{\beta}{N} \frac{\xi(r)}{\xi(1)} H_N(\boldsymbol{\sigma}) - \beta^2 \xi(r) + f_\beta(r) + \frac{1}{2} \beta^2 \xi(1) - \varphi_{N,\beta}(\boldsymbol{\sigma}, r, \delta) \right| \leq \beta \delta c_\xi + \kappa(r^2), \tag{3.2}$$

for some constant c_ξ depending only on ξ and function $\kappa(x) = \kappa(x, \xi, \beta)$ as in Proposition 1.3.

Next we will prove Proposition 1.3, assuming the three results above. They will be proved in the following subsections. Let $t, \epsilon, \delta > 0$, $r \in [0, 1)^\mathcal{J}$ and $\alpha \in (0, 1)$. Set $c = \frac{1}{3} \frac{(t/2)^2}{8\xi(1)}$. By Lemmas 3.1 and 3.2, if $\epsilon > 0$ is small enough, for some $a > 0$ and large enough N , with probability at least $1 - e^{-Na}$,

$$\begin{aligned} \frac{1}{N} \log \mu(L_\beta(\epsilon)) &> -\frac{1}{2} \beta^2 \xi(1) - c, \\ \frac{1}{N} \log \mu(L_\beta(\epsilon) \cap D_N(r, \delta, t/2)) &< -\frac{1}{2} \beta^2 \xi(1) - 3c. \end{aligned}$$

On this event,

$$\frac{\mu(L_\beta(\epsilon) \setminus D_N(r, \delta, t/2))}{\mu(L_\beta(\epsilon))} > 1 - e^{-Nc}.$$

On $L_\beta(\epsilon)$,

$$\left| \frac{\beta}{N} \frac{\xi(r)}{\xi(1)} H_N(\boldsymbol{\sigma}) - \beta^2 \xi(r) \right| \leq \beta \epsilon.$$

By Proposition 3.3, a.s.,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sup_{\boldsymbol{\sigma} \in S_N \setminus D_N(r, \delta, t/2)} \left| \frac{\beta}{N} \frac{\xi(r)}{\xi(1)} H_N(\boldsymbol{\sigma}) - \beta^2 \xi(r) + f_\beta(r) + \frac{1}{2} \beta^2 \xi(1) \right. \\ \left. - \varphi_{N,\beta}(\boldsymbol{\sigma}, r, \delta) \right| \leq \beta \delta c_\xi + \kappa(r^2) + \frac{1}{2} t, \end{aligned}$$

for c_ξ and $\kappa(x)$ as in the proposition.

Hence, a.s.,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sup_{\sigma \in L_\beta(\epsilon) \setminus D_N(r, \delta, t/2)} \left| f_\beta(r) + \frac{1}{2} \beta^2 \xi(1) - \phi_{N, \beta}(\sigma, r, \delta) \right| \\ < \beta \delta c_\xi + \beta^2 \kappa(r^2) + \frac{1}{2} t + \beta \epsilon, \end{aligned}$$

and thus, for small enough ϵ and δ ,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(L_\beta(\epsilon) \setminus D_N(r, \delta, t/2) \subset A_r(t) \right) = 1.$$

This completes the proof of Proposition 1.3. It remains to prove the three results above.

3.1 Proof of Lemma 3.1

Suppose that $\beta \leq \beta_c$ and let $\epsilon, t > 0$ be arbitrary numbers. Using Fubini's theorem, one sees that for small enough $\epsilon > 0$,

$$\frac{1}{N} \log \mathbb{E} \mu(L_\beta(\epsilon)) = -\frac{1}{2} \beta^2 \xi(1) + \frac{t}{2} + o(1). \tag{3.3}$$

By Markov's inequality,

$$\mathbb{P} \left(\frac{1}{N} \log \mu(L_\beta(\epsilon)) > -\frac{1}{2} \beta^2 \xi(1) + t \right) < e^{-\frac{Nt}{2} + o(N)}.$$

It remains to show that for some $c > 0$ and large N ,

$$\mathbb{P} \left(\frac{1}{N} \log \mu(L_\beta(\epsilon)) < -\frac{1}{2} \beta^2 \xi(1) - t \right) < e^{-Nc}.$$

For $\beta < \beta_c$ close enough to β_c ,

$$\mathbb{P} \left(\frac{1}{N} \log \mu(L_{\beta_c}(\epsilon)) < -\frac{1}{2} \beta_c^2 \xi(1) - t \right) < \mathbb{P} \left(\frac{1}{N} \log \mu(L_\beta(\epsilon/2)) < -\frac{1}{2} \beta^2 \xi(1) - t \right),$$

hence it will be enough to prove the inequality for $\beta < \beta_c$.

Assume towards contradiction that for any $c > 0$, for some large as we wish N ,

$$\mathbb{P} \left(\frac{1}{N} \log \mu(L_\beta(\epsilon)) < -\frac{1}{2} \beta^2 \xi(1) - t \right) > e^{-Nc}. \tag{3.4}$$

Let β_0 be some inverse-temperature. On the event in (3.4),

$$\begin{aligned} \frac{1}{N} \log \int_{L_\beta(\epsilon)} e^{\beta_0 H_N(\sigma)} d\mu(\sigma) &\leq -\frac{1}{2} \beta^2 \xi(1) - t + \beta_0 (\beta \xi(1) + \epsilon) \\ &< \frac{1}{2} \beta_0^2 \xi(1) - \frac{t}{2}, \end{aligned} \tag{3.5}$$

where the second inequality holds if we assume that β_0 is close enough to β and ϵ is sufficiently small.

Assume in addition that $\beta_0 \in (\beta - \epsilon/\xi(1), \beta + \epsilon/\xi(1))$. Then,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \int_{S_N \setminus L_\beta(\epsilon)} e^{\beta_0 H_N(\sigma)} d\mu(\sigma) = \max_{y: |y - \beta \xi(1)| \geq \epsilon} \left\{ -\frac{y^2}{2\xi(1)} + \beta_0 y \right\} < \frac{1}{2} \beta_0^2 \xi(1).$$

By Markov's inequality, for some $a, b > 0$ which depend on β, β_0, ϵ and $\xi(1)$, for large N ,

$$\mathbb{P}\left(\frac{1}{N} \log \int_{S_N \setminus L_\beta(\epsilon)} e^{\beta_0 H_N(\sigma)} d\mu(\sigma) > \frac{1}{2} \beta_0^2 \xi(1) - a\right) < e^{-Nb}. \quad (3.6)$$

Since we assumed that $\beta < \beta_c$, we may also assume that $\beta_0 < \beta_c$, and thus $\lim_{N \rightarrow \infty} \mathbb{E} F_{N, \beta_0} = \frac{1}{2} \beta_0^2 \xi(1)$.

Combining the above, we have that for some N as large as we wish, both (3.5) and the complement of the event in (3.6) occur simultaneously with probability at least $e^{-Nc} - e^{-Nb}$. For any small $c > 0$, for such N we therefore have that

$$\mathbb{P}(F_{N, \beta_0} < \mathbb{E} F_{N, \beta_0} - \min\{t/4, a/2\}) > \frac{1}{2} e^{-Nc}, \quad (3.7)$$

in contradiction to the well-known concentration of the free energy (see e.g. [17, Theorem 1.2]). This completes the proof. \square

3.2 Proof of Lemma 3.2

Let $\sigma \in S_N$ be an arbitrary point. Conditional on $H_N(\sigma)$ the value at σ , the Hamiltonian is a Gaussian field whose variance is bounded by the variance before conditioning $N\xi(1)$. Hence, from the well-known concentration of the free energy, for any $x \in \mathbb{R}$,

$$\mathbb{P}\left(\sigma \in D_N(r, \delta, t) \mid H_N(\sigma) = x\right) \leq 2e^{-\frac{Nt^2}{4\xi(1)}}.$$

Therefore, from Fubini's Theorem, as $N \rightarrow \infty$,

$$\begin{aligned} \mathbb{E}\mu\left(D_N(r, \delta, t) \cap L_\beta(\epsilon)\right) &= \mathbb{P}\left(\sigma \in D_N(r, \delta, t) \cap L_\beta(\epsilon)\right) \\ &\leq 2e^{-\frac{Nt^2}{4\xi(1)}} \mathbb{P}\left(\sigma \in L_\beta(\epsilon)\right) = \exp\left(-N\left(\frac{t^2}{4\xi(1)} + \frac{1}{2}\beta^2\xi(1) - \beta\epsilon + \frac{\epsilon^2}{2\xi(1)}\right) + o(N)\right). \end{aligned}$$

By Markov's inequality,

$$\mathbb{P}\left(\mu\left(D_N(r, \delta, t) \cap L_\beta(\epsilon)\right) \geq e^{-N\left(\frac{t^2}{8\xi(1)} + \frac{1}{2}\beta^2\xi(1) - \beta\epsilon + \frac{\epsilon^2}{2\xi(1)}\right)}\right) \leq e^{-\frac{Nt^2}{8\xi(1)} + o(N)}. \quad \square$$

3.3 Proof of Proposition 3.3

The proof will be based on the three lemmas below which will be proved in the next subsections. Define

$$B(\sigma, r) := B(\sigma, r, 0) = \left\{ \sigma' \in S_N : \forall s \in \mathcal{S}, R_s(\sigma, \sigma') = r(s) \right\}.$$

Note that we may identify $B(\sigma, r)$ with the product spheres, one for each $s \in \mathcal{S}$, of codimension 1 in $S(N_s)$. Endow each of those spheres with the uniform probability measure and let $\nu = \nu_{\sigma, r}$ denote the product measure on $B(\sigma, r)$. Similarly to $\varphi_{N, \beta}(\sigma, r, \delta)$, (see (3.1)) define

$$\varphi_{N, \beta}(\sigma, r) = \mathbb{E}\left(\frac{1}{N} \log \int_{B(\sigma, r)} e^{\beta H_N(\sigma')} d\nu(\sigma') \mid H_N(\sigma)\right).$$

Lemma 3.4. *Let $\beta \geq 0$ and $r \in [0, 1]^\mathcal{S}$. For large enough N , almost surely,*

$$\sup_{\sigma \in S_N} \left| \varphi_{N, \beta}(\sigma, r, \delta) - \varphi_{N, \beta}(\sigma, r) - \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda(s) \log(1 - r(s)^2) \right| \leq \delta c_\xi \beta \left(\max_{\sigma \in S_N} \frac{|H_N(\sigma)|}{N} + 1 \right), \quad (3.8)$$

where $c_\xi > 0$ is a constant that only depends on ξ .

For $r \in [0, 1]^{\mathcal{S}}$, define

$$\tilde{\xi}_r(x) = \xi((1 - r^2)x + r^2) - \xi(r^2), \tag{3.9}$$

Here, all operations between functions $\mathcal{S} \rightarrow \mathbb{R}$ are performed elementwise, for example, $((1 - r^2)x)(s) := (1 - r(s)^2)x(s)$. Explicitly,

$$\begin{aligned} \tilde{\xi}_r(x) &= \sum_{p: |p| \geq 2} \Delta_p^2 \left(\prod_{s \in \mathcal{S}} ((1 - r(s)^2)x(s) + r(s)^2)^{p(s)} - \prod_{s \in \mathcal{S}} r(s)^{2p(s)} \right) \\ &= \sum_{p: |p| \geq 1} \Delta_{p,r}^2 \prod_{s \in \mathcal{S}} x(s)^{p(s)}, \end{aligned} \tag{3.10}$$

where

$$\Delta_{p,r}^2 := \sum_{p' \geq p} \Delta_{p'}^2 \prod_{s \in \mathcal{S}} \binom{p'(s)}{p(s)} (1 - r(s)^2)^{p(s)} r(s)^{2(p'(s) - p(s))},$$

where we write $p' \geq p$ if $p'(s) \geq p(s)$ for all $s \in \mathcal{S}$.

Remark 3.5. In the Introduction we defined the multi-species mixtures (1.2) with coefficients for p with $|p| \geq 2$ and their corresponding Hamiltonians in (1.3). Of course, one may consider mixtures with non-zero coefficients also for p with $|p| = 1$, for which the summation in the definition of the corresponding Hamiltonian in (1.3) starts from $k = 1$.

Note that $\tilde{\xi}_r(x)$ is a mixture as in the remark above and let $\tilde{H}_N^r(\sigma)$ be the corresponding Hamiltonian. We remark that the same mixture has been considered in several previous works in the study of the Gibbs measure [8, 24] and in the context of the TAP approach [12, 13, 25, 26].

Lemma 3.6. *Let $\beta \geq 0$ and $r \in [0, 1]^{\mathcal{S}}$. Then, almost surely,*

$$\lim_{N \rightarrow \infty} \sup_{\sigma \in S_N} \left| \varphi_{N,\beta}(\sigma, r) - \frac{\beta}{N} \frac{\xi(r)}{\xi(1)} H_N(\sigma) - \frac{1}{N} \mathbb{E} \log \int_{S_N} e^{\beta \tilde{H}_N^r(\sigma')} d\mu(\sigma') \right| = 0. \tag{3.11}$$

The last lemma we need approximates the free energy of $\tilde{H}_N^r(\sigma)$, for small r .

Lemma 3.7. *Assume (A) and that $\Delta_p \neq 0$ for finitely many values of p and $\beta \leq \beta_c$. Then for any $r \in [0, 1]^{\mathcal{S}}$,*

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \mathbb{E} \log \int_{S_N} e^{\beta \tilde{H}_N^r(\sigma')} d\mu(\sigma') - \frac{1}{2} \beta^2 \xi(1) \right| \leq \kappa(r^2), \tag{3.12}$$

where $\kappa(x) = \kappa(x, \xi, \beta)$ is a function as in Proposition 1.3.

Suppose that $\xi(x)$ satisfies the assumption in (A) and let $\beta \leq \beta_c$. Combining the three lemmas above, we have that for $r \in [0, 1]^{\mathcal{S}}$, almost surely,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sup_{\sigma \in S_N} \left| \varphi_{N,\beta}(\sigma, r, \delta) - \left(\frac{\beta}{N} \frac{\xi(r)}{\xi(1)} H_N(\sigma) + \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda(s) \log(1 - r(s)^2) + \frac{1}{2} \beta^2 \xi(1) \right) \right| \\ \leq \delta c_\xi \beta \left(\max_{\sigma \in S_N} \frac{|H_N(\sigma)|}{N} + 1 \right) + \beta^2 \kappa(r^2), \end{aligned}$$

where c_ξ and $\kappa(x)$ are as in the lemmas. By [26, Lemma 25], for some constant $C_\xi > 0$ that depends only on ξ ,

$$\mathbb{E} \max_{\sigma \in S_N} \frac{|H_N(\sigma)|}{N} \leq C_\xi. \tag{3.13}$$

By the Borell-TIS inequality and the Borel-Cantelli lemma, almost surely,

$$\limsup_{N \rightarrow \infty} \max_{\sigma \in S_N} \frac{|H_N(\sigma)|}{N} \leq 2C_\xi.$$

Proposition 3.3 follows by combining the above. It remains to prove the three lemmas above. This will be done in Subsections 3.4-3.6 below. \square

3.4 Proof of Lemma 3.4

Fix some $\sigma_* \in S_N$. Since the Hamiltonian $H_N(\sigma)$ is a Gaussian process, it can be decomposed as

$$H_N(\sigma) = \hat{H}_N(\sigma) + \eta(\sigma)H_N(\sigma_*), \tag{3.14}$$

where $\eta(\sigma)H_N(\sigma_*) = \mathbb{E}(H_N(\sigma) | H_N(\sigma_*))$ with

$$\eta(\sigma) := \frac{\mathbb{E}(H_N(\sigma)H_N(\sigma_*))}{\mathbb{E}(H_N(\sigma_*)^2)} = \frac{\xi(R(\sigma, \sigma_*))}{\xi(1)}$$

and $\hat{H}_N(\sigma)$ is a centered Gaussian process, independent of $H_N(\sigma_*)$, with covariance function

$$\begin{aligned} \frac{1}{N} \mathbb{E}(\hat{H}_N(\sigma)\hat{H}_N(\sigma')) &= \frac{1}{N} \mathbb{E}(H_N(\sigma)H_N(\sigma')) - \frac{1}{N} \frac{\mathbb{E}(H_N(\sigma)H_N(\sigma_*))\mathbb{E}(H_N(\sigma')H_N(\sigma_*))}{\mathbb{E}(H_N(\sigma_*)^2)} \\ &= \xi(R(\sigma, \sigma')) - \frac{\xi(R(\sigma, \sigma_*))\xi(R(\sigma', \sigma_*))}{\xi(1)}. \end{aligned}$$

Note that, since $\eta(\sigma) = \frac{\xi(r)}{\xi(1)}$ on $B(\sigma_*, r)$,

$$\varphi_{N,\beta}(\sigma_*, r) = \mathbb{E}\left(\frac{1}{N} \log \int_{B(\sigma_*, r)} e^{\beta \hat{H}_N(\sigma)} d\nu(\sigma)\right) + \frac{\beta}{N} \frac{\xi(r)}{\xi(1)} H_N(\sigma_*). \tag{3.15}$$

For any $\sigma \in B(\sigma_*, r, \delta)$,

$$|\eta(\sigma) - \eta(\sigma_*)| \leq \max_{t: t(s) \in [r(s) - \delta, r(s) + \delta]} \left| \frac{\xi(t)}{\xi(1)} - \frac{\xi(r)}{\xi(1)} \right| \leq \frac{\delta}{\xi(1)} \sum_{s \in \mathcal{S}} \frac{d}{dx(s)} \xi(1) =: c_\xi \delta,$$

and therefore

$$\left| \varphi_{N,\beta}(\sigma_*, r, \delta) - \mathbb{E}\left(\frac{1}{N} \log \int_{B(\sigma_*, r, \delta)} e^{\beta \hat{H}_N(\sigma)} d\mu(\sigma)\right) - \frac{\beta}{N} \frac{\xi(r)}{\xi(1)} H_N(\sigma_*) \right| \leq \frac{\beta}{N} c_\xi \delta |H_N(\sigma_*)|.$$

Hence, to prove the lemma it will be enough to show that for some $c > 0$ depending only on ξ , for large N ,

$$\begin{aligned} &\left| \mathbb{E} \frac{1}{N} \log \int_{B(\sigma_*, r)} e^{\beta \hat{H}_N(\sigma)} d\nu(\sigma) + \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda(s) \log(1 - r(s)^2) \right. \\ &\quad \left. - \mathbb{E} \frac{1}{N} \log \int_{B(\sigma_*, r, \delta)} e^{\beta \hat{H}_N(\sigma)} d\mu(\sigma) \right| < \delta c \beta. \end{aligned}$$

Since the variance of $\hat{H}_N(\sigma)$ is bounded uniformly in σ by $N\xi(1)$, the variance of the unconditional Hamiltonian, from the concentration of the free energies around their mean (see [17, Theorem 1.2]) it will be enough to show that

$$\begin{aligned} &\left| \frac{1}{N} \log \int_{B(\sigma_*, r)} e^{\beta \hat{H}_N(\sigma)} d\nu(\sigma) + \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda(s) \log(1 - r(s)^2) \right. \\ &\quad \left. - \frac{1}{N} \log \int_{B(\sigma_*, r, \delta)} e^{\beta \hat{H}_N(\sigma)} d\mu(\sigma) \right| < \delta c \beta \end{aligned} \tag{3.16}$$

with probability that goes to 1 as $N \rightarrow \infty$, for c as above.

By [26, Lemma 25], for any $C > 0$, for some $L > 0$ that depends only on ξ , with probability at least $1 - e^{-NC}$, for all $\sigma, \sigma' \in S_N$,

$$\frac{1}{N} |H_N(\sigma) - H_N(\sigma')| \leq L \max_{s \in \mathcal{S}} \sqrt{R_s(\sigma - \sigma', \sigma - \sigma')}.$$

For any $\sigma, \sigma' \in S_N$,

$$|\eta(\sigma) - \eta(\sigma')| \leq \frac{1}{\xi(1)} \sum_{s \in \mathcal{S}} \frac{d}{dx(s)} \xi(1) \cdot \sqrt{R_s(\sigma - \sigma', \sigma - \sigma')}.$$

Hence, from (3.14) and the fact that $H_N(\sigma_*)$ is a Gaussian variable with zero mean and variance $N\xi(1)$, with the probability going to 1 as $N \rightarrow \infty$, for all $\sigma, \sigma' \in S_N$,

$$\frac{1}{N} |\hat{H}_N(\sigma) - \hat{H}_N(\sigma')| \leq 2L \max_{s \in \mathcal{S}} \sqrt{R_s(\sigma - \sigma', \sigma - \sigma')}.$$

On this event, (3.16) holds since by the co-area formula,

$$\begin{aligned} & \int_{B(\sigma_*, r, \delta)} e^{\beta \hat{H}_N(\sigma)} d\mu(\sigma) \\ &= \int_{B(\sigma_*, r)} \int_{T(\sigma, \delta)} e^{\beta \hat{H}_N(\sigma')} \prod_{s \in \mathcal{S}} \frac{\omega_{N_s-1}}{\omega_{N_s}} \left(1 - R_s(\sigma', \sigma')\right)^{\frac{N_s-2}{2}} d\rho(\sigma') d\nu(\sigma) \end{aligned}$$

where ω_d denotes the volume of the unit sphere in \mathbb{R}^d , ρ denotes the volume measure corresponding to the Riemannian metric on $T(\sigma, \delta)$ induced by the Euclidean structure in \mathbb{R}^N , and

$$T(\sigma, \delta) := \left\{ \sigma' \in B(\sigma_*, r, \delta) : \frac{P_s(\sigma')}{R_s(\sigma', \sigma')} = \frac{P_s(\sigma)}{R_s(\sigma, \sigma)}, \forall s \in \mathcal{S} \right\},$$

where $P_s(\sigma)$ is the orthogonal projection of $(\sigma_i)_{i \in I_s}$ to the orthogonal space to $(\sigma_{*,i})_{i \in I_s}$. □

3.5 Proof of Lemma 3.6

Fix some $r \in [0, 1]^{\mathcal{S}}$. Recall the decomposition (3.14) of $H_N(\sigma)$. In light of (3.15), we need to prove that

$$\lim_{N \rightarrow \infty} \left| \frac{1}{N} \mathbb{E} \log \int_{B(\sigma_*, r)} \exp(\beta \hat{H}_N(\sigma)) d\nu(\sigma) - \frac{1}{N} \mathbb{E} \log \int_{S_N} \exp(\beta \tilde{H}_N^r(\sigma)) d\mu(\sigma) \right| = 0, \tag{3.17}$$

for some arbitrary $\sigma_* \in S_N$.

For any $\sigma = (\sigma_i)_{i=1}^N \in B(\sigma_*, r)$, define $\tilde{\sigma} = (\tilde{\sigma}_i)_{i=1}^N \in B(\sigma_*, 0) \subset S_N$ by

$$\tilde{\sigma}_i := \sqrt{\frac{1}{1 - r(s)^2}} (\sigma_i - r(s)\sigma_{*,i}), \quad \text{if } i \in I_s, \tag{3.18}$$

and define the Hamiltonian $\bar{H}_N(\tilde{\sigma}) = \hat{H}_N(\sigma)$ on $B(\sigma_*, 0)$. Then, for any two points $\tilde{\sigma}^1, \tilde{\sigma}^2$ from $B(\sigma_*, 0)$, by a straightforward calculation,

$$\frac{1}{N} \mathbb{E} (\bar{H}_N(\tilde{\sigma}^1) \bar{H}_N(\tilde{\sigma}^2)) = \tilde{\xi}_r(R(\tilde{\sigma}^1, \tilde{\sigma}^2)) + \xi(r^2) - \frac{\xi(r)^2}{\xi(1)}, \tag{3.19}$$

where the mixture $\tilde{\xi}_r(x)$ is defined in (3.9).

Extend the Hamiltonian \bar{H}_N from $B(\sigma_*, 0)$ to a centered Gaussian field on S_N whose covariance is given by (3.19). Of course,

$$\begin{aligned} \frac{1}{N} \mathbb{E} \log \int_{B(\sigma_*, r)} \exp(\beta \hat{H}_N(\sigma)) d\nu(\sigma) &= \frac{1}{N} \mathbb{E} \log \int_{B(\sigma_*, 0)} \exp(\beta \bar{H}_N(\sigma)) d\nu(\sigma) \\ &= \frac{1}{N} \mathbb{E} \log \int_{S_N} \exp(\beta \bar{H}_N(\sigma)) d\mu(\sigma) + o_N(1). \end{aligned} \tag{3.20}$$

Since $A(r)^2 := \xi(r^2) - \frac{\xi(r)^2}{\xi(1)} \geq 0$, we may write the Hamiltonian \bar{H}_N as

$$\bar{H}_N(\boldsymbol{\sigma}) = \tilde{H}_N^r(\boldsymbol{\sigma}) + \sqrt{N}A(r)X,$$

where X is a standard Gaussian variable independent of the Hamiltonian $\tilde{H}_N^r(\boldsymbol{\sigma})$ with mixture $\tilde{\xi}_r(x)$. Obviously,

$$\frac{1}{N} \mathbb{E} \log \int_{S_N} \exp\left(\beta(\tilde{H}_N^r(\boldsymbol{\sigma}) + \sqrt{N}A(r)X)\right) d\mu(\boldsymbol{\sigma}) = \frac{1}{N} \mathbb{E} \log \int_{S_N} \exp\left(\beta\tilde{H}_N^r(\boldsymbol{\sigma})\right) d\mu(\boldsymbol{\sigma}). \tag{3.21}$$

Combining (3.20) and (3.21) proves (3.17), and completes the proof. \square

3.6 Proof of Lemma 3.7

Let $\beta \leq \beta_c$ and $r \in [0, 1]^{\mathcal{S}}$. By (1.7),

$$\frac{1}{N} \mathbb{E} \log \int_{S_N} e^{\beta\tilde{H}_N^r(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}) \leq \frac{1}{2}\beta^2\tilde{\xi}_r(1) \leq \frac{1}{2}\beta^2\xi(1),$$

thus we only need to prove the lower bound

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \log \int_{S_N} e^{\beta\tilde{H}_N^r(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}) \geq \frac{1}{2}\beta^2\xi(1) - \kappa(r^2), \tag{3.22}$$

for $\kappa(x)$ as in the statement of the lemma.

Given $x \in [0, 1]^{\mathcal{S}}$ and $t > 0$, by setting $r = \sqrt{tx}$ (where the square root is applied elementwise $\sqrt{tx}(s) := \sqrt{tx(s)}$), (3.22) becomes

$$\liminf_{N \rightarrow \infty} \mathbb{E} \tilde{F}_{N,\beta}(t) \geq \frac{1}{2}\beta^2\xi(1) - \kappa(tx),$$

where we define

$$\tilde{F}_{N,\beta}(t) = \frac{1}{N} \log \int_{S_N} e^{\beta\tilde{H}_N^{\sqrt{tx}}(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}).$$

Recall that, in distribution,

$$\tilde{H}_N^{\sqrt{tx}}(\boldsymbol{\sigma}) = \sum_{p:|p|\geq 1} \Delta_{p,\sqrt{tx}} H_{N,p}(\boldsymbol{\sigma}),$$

where the coefficients $\Delta_{p,\sqrt{tx}}$ are as in (3.10) and $H_{N,p}(\boldsymbol{\sigma})$ are the pure p -spin models with mixture $\prod_{s \in \mathcal{S}} x(s)^{p(s)}$ which we assume to be independent. Define

$$\bar{H}_N^{t,x}(\boldsymbol{\sigma}) = \sum_{p:|p|\geq 1} \Delta_{p,t,x} H_{N,p}(\boldsymbol{\sigma}), \quad \text{where} \quad \Delta_{p,t,x} = \Delta_p + t \cdot \frac{d}{d\epsilon} \Big|_{\epsilon=0} \Delta_{p,\sqrt{\epsilon x}}.$$

Since we assume that Δ_p for finitely many values of p , for small enough $t \geq 0$, $\Delta_{p,t,x} \geq 0$ for all p .

Of course, for some constants C_p and $C := \sum_p C_p < \infty$ that depend on ξ and x , for small t ,

$$|\Delta_{p,t,x}^2 - \Delta_{p,\sqrt{tx}}^2| \leq C_p t^2$$

and by Gaussian integration by parts

$$|\bar{F}_{N,\beta}(t) - \tilde{F}_{N,\beta}(t)| \leq Ct^2,$$

where we define

$$\bar{F}_{N,\beta}(t) = \frac{1}{N} \log \int_{S_N} e^{\beta\bar{H}_N^{t,x}(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}).$$

Hence, it will be enough to prove that

$$\liminf_{N \rightarrow \infty} \mathbb{E} \bar{F}_{N,\beta}(t) \geq \frac{1}{2} \beta^2 \xi(1) - \kappa(tx), \tag{3.23}$$

for an appropriate function $\kappa(x)$.

In order to be able to invoke Talagrand’s positivity principle [33, 17], we will add a perturbation to the Hamiltonian. Its definition is taken from [19] where Panchenko introduced a multi-species version of the Ghirlanda-Guerra identities which induce the positivity of overlaps, and showed that they are satisfied in the presence of the perturbation Hamiltonian. The results of [19] concern the multi-species SK model, but they are general and also cover the multi-species spherical models. The proofs for the spherical case have been worked out in [7].

Let \mathscr{W} be a countable dense subset of $[0, 1]^{\mathscr{S}}$. For any $p \geq 1$ and vector

$$w = (w_s)_{s \in \mathscr{S}} \in \mathscr{W}$$

define $s_i(w) = \sqrt{w_s}$ for $i \in I_s$ and $s \in \mathscr{S}$, and consider the Hamiltonian

$$h_{N,w,p}(\boldsymbol{\sigma}) = \frac{1}{N^{\frac{p}{2}}} \sum_{1 \leq i_1, \dots, i_p \leq N} g_{i_1, \dots, i_p}^{w,p} \sigma_{i_1} s_{i_1}(w) \cdots \sigma_{i_p} s_{i_p}(w),$$

where $g_{i_1, \dots, i_p}^{w,p}$ are i.i.d. standard Gaussian variables, independent for all combinations of indices $p \geq 1$ and $1 \leq i_1, \dots, i_p \leq N$.

Consider some one-to-one function $j : \mathscr{W} \rightarrow \mathbb{N}$. Let $y = (y_{w,p})_{w \in \mathscr{W}, p \geq 1}$ be i.i.d. random variables uniform in the interval $[1, 2]$ and independent of all other variables. Define the Hamiltonian

$$h_N(\boldsymbol{\sigma}) = \sum_{w \in \mathscr{W}} \sum_{p \geq 1} 2^{-j(w)-p} y_{w,p} h_{N,w,p}(\boldsymbol{\sigma}).$$

Let γ be an arbitrary number in $(0, 1/2)$ and set $s_N = N^\gamma$. For $t \in [0, 1]$, define

$$H_N^{t,x}(\boldsymbol{\sigma}) = \bar{H}_N^{t,x}(\boldsymbol{\sigma}) + s_N h_N(\boldsymbol{\sigma})$$

and

$$F_{N,\beta}(t) = \frac{1}{N} \log \int_{S_N} e^{\beta H_N^{t,x}(\boldsymbol{\sigma})} d\mu(\boldsymbol{\sigma}).$$

Conditional on $y = (y_{w,p})_{w \in \mathscr{W}, p \geq 1}$, $h_N(\boldsymbol{\sigma})$ is a Gaussian process with variance bounded by 4 (see e.g. [19, (26)]). Hence, using Jensen’s inequality, we have that

$$\left| F_{N,\beta}(t) - \bar{F}_{N,\beta}(t) \right| \leq \frac{4\beta^2 s_N^2}{N}. \tag{3.24}$$

Since $s_N^2/N \rightarrow 0$, (3.23) will follow if we prove that

$$\liminf_{N \rightarrow \infty} \mathbb{E} F_{N,\beta}(t) \geq \frac{1}{2} \beta^2 \xi(1) - \kappa(tx), \tag{3.25}$$

For $t = 0$, $\bar{F}_{N,\beta}(0) = F_{N,\beta}$. Therefore, since $\beta \leq \beta_c$, (3.23) and (3.25) hold with $\kappa(0) = 0$. Since $\Delta_{p,t,x}$ are affine in t , using Hölder’s inequality one can check that $F_{N,\beta}(t)$ is a convex function of t (and thus has one-sided derivatives). Hence, in order to show that there exists a function $\kappa(x)$ which satisfies (1.19) and (3.25), it will be enough to show that

$$\lim_{N \rightarrow \infty} \frac{d^+}{dt} \Big|_{t=0} \mathbb{E} F_{N,\beta}(t) = 0, \tag{3.26}$$

where $\frac{d^+}{dt} \Big|_{t=0}$ denotes the derivative from the right at 0.

Let $G_{N,\beta}^0$ be the Gibbs measure

$$dG_{N,\beta}^0(\boldsymbol{\sigma}) = \frac{e^{\beta H_N^{0,x}(\boldsymbol{\sigma})}}{\int_{S_N} e^{\beta H_N^{0,x}(\boldsymbol{\sigma}')} d\mu(\boldsymbol{\sigma}')} d\mu(\boldsymbol{\sigma})$$

corresponding to

$$H_N^{0,x}(\boldsymbol{\sigma}) \stackrel{d}{=} H_N(\boldsymbol{\sigma}) + s_N h_N(\boldsymbol{\sigma}). \tag{3.27}$$

For any function $f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n)$ of n points from S_N denote

$$\langle f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n) \rangle = \int_{S_N^n} f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n) d(G_{N,\beta}^0)^{\otimes n},$$

where $(G_{N,\beta}^0)^{\otimes n}$ denotes the n -fold product measure of $G_{N,\beta}^0$ with itself.

Note that

$$\frac{d}{dt} \Big|_{t=0} \mathbb{E} F_{N,\beta}(t) = \frac{1}{N} \mathbb{E} \left\langle \beta \sum_{p:|p|\geq 1} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \Delta_{p,\sqrt{\epsilon x}} \cdot H_{N,p}(\boldsymbol{\sigma}) \right\rangle. \tag{3.28}$$

By Gaussian integration by parts [17, Lemma 1.1], the right-hand side of (3.28) is equal to

$$\mathbb{E} \langle C(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^1) - C(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \rangle, \tag{3.29}$$

where

$$\begin{aligned} C(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) &:= \frac{\beta^2}{N} \sum_{p:|p|\geq 1} \left(\Delta_p \frac{d}{d\epsilon} \Big|_{\epsilon=0} \Delta_{p,\sqrt{\epsilon x}} \right) \cdot \mathbb{E} \left(H_{N,p}(\boldsymbol{\sigma}^1) H_{N,p}(\boldsymbol{\sigma}^2) \right) \\ &= \frac{1}{2} \beta^2 \eta_x(R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)), \end{aligned}$$

where we denote $\partial_s \xi(r) = \frac{d}{dr(s)} \xi(r)$ and for $z : [-1, 1]^{\mathcal{S}} \rightarrow \mathbb{R}$,

$$\eta_x(z) := \frac{d}{d\epsilon} \Big|_{\epsilon=0} \tilde{\xi}_{\sqrt{\epsilon x}}(z) = \sum_{s \in \mathcal{S}} \left(\partial_s \xi(z) x(s) (1 - z(s)) - \partial_s \xi(0) x(s) \right).$$

Combining the above, we obtain that

$$\lim_{N \rightarrow \infty} \frac{d}{dt} \Big|_{t=0} \mathbb{E} F_{N,\beta}(t) = \frac{1}{2} \beta^2 \left(\eta_x(1) - \lim_{N \rightarrow \infty} \mathbb{E} \langle \eta_x(R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)) \rangle \right). \tag{3.30}$$

Note that since $\partial_s \xi(0) = 0$,

$$\eta_x(1) = \eta_x(0) = 0. \tag{3.31}$$

Hence, to complete the proof of Lemma 3.7 it will be enough to show that for any $\epsilon > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\langle \mathbf{1} \left\{ \max_{s \in \mathcal{S}} |R_s(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)| < \epsilon \right\} \right\rangle = 0, \tag{3.32}$$

where $\mathbf{1}\{A\}$ is the indicator of an event A .

Note that since $\beta \leq \beta_c$,²

$$\begin{aligned} \beta \xi(1) &= \frac{d}{d\beta} \lim_{N \rightarrow \infty} \mathbb{E} F_{N,\beta}(0) = \lim_{N \rightarrow \infty} \frac{d}{d\beta} \mathbb{E} F_{N,\beta}(0) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \langle H_N^{0,x}(\boldsymbol{\sigma}) \rangle \\ &= \beta \left(\xi(1) - \lim_{N \rightarrow \infty} \mathbb{E} \langle \xi(R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)) \rangle \right), \end{aligned}$$

²For $\beta = \beta_c$ the derivative from the left of $\lim_{N \rightarrow \infty} \mathbb{E} F_{N,\beta}$ is $\beta_c \xi(1)$. Since $\lim_{N \rightarrow \infty} \mathbb{E} F_{N,\beta}$ is convex in β , its derivative from the right exists and is bounded from below by the derivative from the left. The latter is bounded by $\beta_c \xi(1)$, by (1.7). Hence the derivative at β_c exists and is equal to $\beta_c \xi(1)$.

where the limit and derivative may be interchanged by the convexity of $\beta \mapsto \mathbb{E}F_{N,\beta}(0) = \mathbb{E}F_{N,\beta}$, and the last equality follows from Gaussian integration by parts.

Thanks to the perturbation term $s_N h_N(\boldsymbol{\sigma})$ in (3.6), from Lemma 3.3 of [7], for any $\epsilon > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\langle \mathbf{1} \left\{ \min_{s \in \mathcal{S}} R_s(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) < -\epsilon \right\} \right\rangle = 0.$$

Combining the above with the assumption (A) one easily concludes (3.32), which completes the proof. \square

Appendix

In this appendix we explain how the following result follows by slightly modifying our proofs. Here we denote by $\langle \cdot \rangle$ averaging by the Gibbs measure $G_{N,\beta}$ corresponding directly to $H_N(\boldsymbol{\sigma})$,

$$dG_{N,\beta}(\boldsymbol{\sigma}) = \frac{e^{\beta H_N(\boldsymbol{\sigma})}}{\int_{S_N} e^{\beta H_N(\boldsymbol{\sigma}')} d\mu(\boldsymbol{\sigma}')} d\mu(\boldsymbol{\sigma}).$$

Lemma 3.8. *Let $H_N(\boldsymbol{\sigma})$ be a model with mixture $\xi(x) = \sum_{p \in P} \Delta_p^2 \prod_{s \in \mathcal{S}} x(s)^{p(s)}$ such that $\Delta_p \neq 0$ for finitely many p and let $\beta \leq \beta_c$. Suppose that for any $\epsilon > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\langle \mathbf{1} \left\{ \max_{s \in \mathcal{S}} |R_s(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)| < \epsilon \right\} \right\rangle = 0. \tag{3.33}$$

Then, the matrix

$$\left(\frac{d}{dr(s)} \frac{d}{dr(t)} f_\beta(0) \right)_{s,t \in \mathcal{S}} \tag{3.34}$$

is negative semi-definite.

Note that above instead of assuming (A) as we did in the main results, we assume (3.33). This result is crucial to the analysis of the TAP representation of the multi-species pure p -spin models in [27], where indeed we need to deal with a model which for some values of p may not satisfy (A).

First we explain how the conclusion of Proposition 1.3 follows in the setting of Lemma 3.8. The only place we used the assumption of (A) in the proof of Proposition 1.3 is the very last step in the proof of Lemma 3.7, to prove (3.32). So we only need to explain how to prove the conclusion (3.12) of Lemma 3.7. By the same argument as in the proof of the latter lemma, the lemma follows if we can prove (3.23). By the argument we used around (3.26), to prove (3.23) it is enough to show that

$$\lim_{N \rightarrow \infty} \frac{d^+}{dt} \Big|_{t=0} \mathbb{E} \bar{F}_{N,\beta}(t) = 0. \tag{3.35}$$

As in (3.30),

$$\lim_{N \rightarrow \infty} \frac{d^+}{dt} \Big|_{t=0} \mathbb{E} \bar{F}_{N,\beta}(t) = \frac{1}{2} \beta^2 \left(\eta_x(1) - \lim_{N \rightarrow \infty} \mathbb{E} \langle \eta_x(R(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)) \rangle \right), \tag{3.36}$$

where now the averaging is w.r.t. the Gibbs measure corresponding to the Hamiltonian without perturbation, which have the same law as $G_{N,\beta}$. Thus, (3.35) follows from (3.33) and (3.31). This proves the conclusion of Proposition 1.3 in the setting of Lemma 3.8.

Now assume towards contradiction that the matrix (3.34) has a positive eigenvalue. Then for some r , which we can choose to be in $[0, 1)^{\mathcal{S}}$,

$$\frac{d}{d\alpha} \Big|_{\alpha=0} f_\beta(\alpha r) = 0 \quad \text{and} \quad \frac{d^2}{d\alpha^2} \Big|_{\alpha=0} f_\beta(\alpha r) > 0.$$

By the same argument as in the proof of Proposition 1.2, this leads to a contradiction to the fact that $\beta \leq \beta_c$, from which we conclude that (3.34) is negative semi-definite.

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