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Generalized BSDE and reflected BSDE with random time horizon*

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Abstract

Motivated by structural, reduced-form and hybrid models of the third party and counterparty credit risk, we study a generalized backward stochastic differential equations (BSDE) up to a random time horizon ϑ, which is not a stopping time with respect to a reference filtration. In contrast to the existing literature in the area of credit risk modeling, we do not impose specific assumptions on the random time ϑ and we study the existence of solutions to BSDE and reflected BSDE with a random time horizon through the method of reduction. For this purpose, we also examine BSDE and reflected BSDE with a làdlàg driver where the driver is allowed to have a finite number of jumps overlapping with jumps of the martingale part. Theoretical results are illustrated by particular instances of a random time and explicit BSDEs in either the Brownian or Brownian-Poisson filtration.

Keywords: BSDE; reflected BSDE; credit risk; random time; enlargement of filtration.
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1 Introduction

Our work is motivated by the arbitrage-free pricing of European and American style contracts in models with the third party and counterparty credit risk and, more generally, problems of mitigation of financial and insurance risks triggered by an extraneous event. Since our goal is to provide a comprehensive mathematical framework for financial models outlined in Section 3.2, we study BSDEs and reflected BSDEs (RBSDEs) up to a

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finite random time horizon ϑ while making virtually no assumptions about a random time ϑ, which is not an $\mathbb{F}$-stopping time with respect to a reference filtration $\mathbb{F}$. In contrast to the existing literature (see, e.g., Ankirchner et al. [6], Kharroubi and Lim [36], Crépey and Song [11, 12], Dumitrescu et al. [15, 16, 17], Grigorova et al. [27] and Kim et al. [34]), we do not make any of simplifying assumptions frequently encountered in works on the theory of progressive enlargement of filtration, such as: the immersion hypothesis, Jacod's equivalence hypothesis, the condition **(C)** of continuity of all $\mathbb{F}$-martingales, or the condition **(A)** of avoidance of all $\mathbb{F}$-stopping times by ϑ. We only postulate that the *Azéma supermartingale* of ϑ with respect to $\mathbb{F}$ (see Definition 2.2) is a strictly positive process, although we also show that this assumption can be relaxed and thus our results apply to a larger class of random times (see the class $\mathcal{K}$ in Section 3.3). For a more detailed account of relation of our results to the existing financial literature (in particular, to the concept of the *invariance time*, which was introduced by Crépey and Song [12, 13]), the reader is referred to Section 4.3.

Stimulated by the paper by Choulli et al. [9] on the martingale representation theorem in the progressive enlargement of a given filtration $\mathbb{F}$ with observations of a random time ϑ, which is henceforth denoted as $\mathbb{G}$, we study $\mathbb{G}$-adapted BSDEs and $\mathbb{G}$-adapted RBSDEs with an $\mathcal{F}_\vartheta$-measurable terminal value at a random horizon ϑ. The crucial difference between $\mathbb{G}$ BSDEs and $\mathbb{G}$ RBSDEs introduced in Definitions 3.1 and 3.3, respectively, and various classes of BSDEs previously studied in the existing literature is that the driver is assumed to be a làglàd process and the integrand against the pure jump martingale $m^{\mathbb{G}}$ given by equation (2.2) is assumed to be $\mathbb{F}$-optional, rather than $\mathbb{G}$-predictable or, equivalently, $\mathbb{F}$-predictable.

Our main purpose is to apply the *method of reduction* to study the existence and construction of a solution to $\mathbb{G}$ BSDE and $\mathbb{G}$ RBSDE given by equations (3.2) and (3.3), respectively. It should be acknowledged that the idea of reduction of a BSDE in a given filtration to a more tractable BSDE in a shrunken filtration has already been explored in papers by Crépey and Song [11, 12] and Kharroubi and Lim [36] but the authors of these papers worked under simplifying assumptions about their setup and have not examined reflected BSDEs related to American style options with the counterparty credit risk.

The first main contribution of this work is that we demonstrate that the idea of reduction can also be applied to the $\mathbb{G}$ RBSDE (3.3). To be more precise, we show that the $\mathbb{G}$ RBSDE with $\mathcal{F}_\vartheta$-measurable terminal value can be solved up to a random horizon ϑ by first solving the corresponding reduced $\mathbb{F}$ RBSDE and then constructing a solution to the original $\mathbb{G}$ RBSDE by combining a solution to the $\mathbb{F}$ RBSDE with an appropriate adjustment to the terminal value at time ϑ. In particular, we analyze in detail the required adjustment at ϑ when the driver of the $\mathbb{G}$ RBSDE is a discontinuous làdlàg process. Furthermore, since the simplifying conditions **(C)** and **(A)** of continuity of all $\mathbb{F}$-martingales and avoidance of all $\mathbb{F}$-stopping times by ϑ are not imposed, we allow the reference filtration $\mathbb{F}$ to support discontinuous martingales and, in addition, we also cover the situation where a random time ϑ may overlap $\mathbb{F}$-stopping times (see Aksamit et al. [3]).

As a consequence, unlike in previous works, the reduced $\mathbb{F}$ BSDE and $\mathbb{F}$ RBSDE obtained in our setup have the property that the driver and the martingale part may share common jumps. Hence a BSDE now carries an additional constraint (as, e.g., in Peng and Xu [45] who dealt with constrained BSDEs in a different context), which is directly related to the jump of the driver. We stress that the reduced $\mathbb{F}$ BSDE and $\mathbb{F}$ RBSDE have a fairly general form that was not well studied in the existing literature. Hence, as a second contribution, we show how to construct a solution to the reduced $\mathbb{F}$ BSDE for which the driver and the martingales appearing in a BSDE may have a finite number of common jumps.

Our approach relies on a detailed analysis of the appropriate intermediate BSDE with a làglàd driver. To support our method, we also show that a solution to an intermediate BSDE with a làglàd driver can be obtained by adapting the existing results from Essaky et al. [20] and Ren and El Otmani [48] who studied a particular class of BSDEs with a continuous driver. Notice that our arguments do not hinge on solving directly the $\mathbb{G}$ BSDE (or the $\mathbb{G}$ RBSDE) through a fixed point theorem under appropriate assumptions on the solution space, the generator and the driver. Instead, our aim is to show that one can reduce the $\mathbb{G}$ BSDE to a more manageable $\mathbb{F}$ BSDE, which can be solved by making use of the solution to an intermediate BSDEs with làglàd driver. Then we show that the solution of the intermediate BSDE with làglàd driver can be obtained by a careful analysis of jumps, as in Confortola et al. [10], Essaky et al. [20] and Klimsiak et al. [35], and making use of a large variety of existing results on BSDEs and RBSDEs with a continuous driver (see, for instance, [18, 20, 22, 23, 44, 48]) to deal with solutions on stochastic intervals between successive jumps. For further applications of our results, the reader is referred to the follow-up work by Li et al. [38].

The structure of the paper is as follows. We first introduce in Section 2 the setup and notation and we recall some auxiliary results from the theory of progressive enlargement of filtration (see, e.g., Aksamit and Jeanblanc [4] and Jeanblanc and Li [32]).

In Section 3, in view of recent works on RBSDEs with irregular barriers (see, e.g., Grigorova et al. [28, 29] and Klimsiak et al. [35]) and to demonstrate the generality of our methodology, we introduce in Definition 3.1 the notion of the làglàd $\mathbb{G}$ BSDE (see also Definition 3.3 for the làglàd $\mathbb{G}$ RBSDE). We show in Section 3.1 how the reward process can be reduced and we elaborate in Section 3.2 on relationships between our setup and techniques used in credit risk modeling. We then discuss in Sections 3.3 and 3.4 some possible extensions of the setup introduced in Assumption 3.1.

Section 4 is devoted to the issues of reduction of the $\mathbb{G}$ BSDE (3.2) and subsequently also a method for construction of its solution. We first show in Proposition 4.9 that the $\mathbb{G}$ BSDE can be effectively reduced to coupled equations in the filtration $\mathbb{F}$. Next, Proposition 4.12 makes it clear that a solution to the $\mathbb{G}$ BSDE (3.2) can be constructed by first solving the constrained $\mathbb{F}$ BSDE (4.13)–(4.14). Finally, to examine the existence of a solution to the constrained $\mathbb{F}$ BSDE (4.13)–(4.14), we first prove that the stronger constrained $\mathbb{F}$ BSDE (4.15)–(4.16) can be transformed into the constrained $\mathbb{F}$ BSDE (4.17)–(4.18), which in turn is more tractable and whose solution can be used to resolve the problem of well-posedness of the coupled equations (4.15)–(4.16). Concrete situations where the constrained BSDE (4.15)–(4.16) possesses a unique solution are studied in Section 4.4 for the Brownian filtration (see Proposition 4.13).

In Section 5, we are concerned with analogous issues for $\mathbb{G}$ RBSDEs and we first show that the method of reduction can be used to reduce the $\mathbb{G}$ RBSDE to the $\mathbb{F}$ RBSDE. As shown in Proposition 5.1, the main new feature in the reflected case is that the $\mathbb{G}$-predictable reflection can be uniquely reduced to the $\mathbb{F}$-predictable reflection, which is required to meet the appropriately modified Skorokhod conditions. We then show in Proposition 5.3 that, in principle, a solution to the $\mathbb{G}$ RBSDE can be constructed from a solution to the reduced $\mathbb{F}$ RBSDE. The existence of a solution to the $\mathbb{G}$ RBSDE in the Brownian case is studied in Section 5.3 where Proposition 5.4 offers sufficient conditions for the existence of a solution to the $\mathbb{F}$ RBSDE in the case of the Brownian filtration $\mathbb{F}$.

In Section 6, we deviate from the setup studied in Sections 4 and 5 and, for given a filtration $\mathbb{F}$, we focus on the BSDE (6.1) and the RBSDE (6.12), which share the key feature that the driver is làglàd and its jumps may overlap the jumps of the driving martingale. Even when the driver is càdlàg, there is apparently a gap in the existing literature on BSDEs when the driver may share jumps with the driving martingale and thus we develop a jump-adapted method to solve BSDEs of such a general form.

Our approach in Section 6 hinges on two steps. We first show, through a careful analysis of right-hand jumps, that the problem of solving the BSDE (6.1) on the whole interval $[\![0, \tau]\!]$ can be addressed by solving a recursive system of càdlàg BSDEs (6.2) and then stitching together the solutions to that system. In the second step, we show in Proposition 6.4 that a solution to a càdlàg BSDE can be obtained from a solution of an intermediate làglàd BSDE (6.6), which in turn can be handled by solving a recursive system of càdlàg BSDEs (6.8) with a continuous driver, which are given on intervals defined by the right-hand jumps of a làglàd BSDE and, once again, appropriately aggregating these solutions. We argue that a reduction to the case of a continuous driver is important since it allow us to use existing results on the well-posedness of BSDEs with a continuous driver. Concrete instances of our approach in a Brownian-Poisson filtration are presented in Examples 6.6 and 6.7. We conclude the paper by showing that an analogous method can be used to study the existence of a solution to the RBSDE (6.12) with a làglàd driver using results for RBSDEs with a continuous driver. The main difference here is that we need to analyze the adjustment to the reflection process at the right-hand jumps and provide a rigorous check that the appropriate Skorokhod conditions are satisfied. The main result, Proposition 6.8, is illustrated by an explicit example in a Poisson filtration (Example 6.9). Finally, some auxiliary results are collected in the appendix.

2 Setup and notation

Regarding the background knowledge, for the general theory of stochastic processes, we refer to He et al. [30] and the reader interested in stochastic calculus for optional semimartingales is referred to Gal'čuk [25]. For more details on the theory of random times and enlargement of filtration with applications to problems arising in financial mathematics (such as credit risk modeling or insider trading), the interested reader may consult the monograph by Aksamit and Jeanblanc [4] and the recent paper by Jeanblanc and Li [32]. We start by introducing the notation and recalling some fundamental concepts associated with modeling of a random time and the associated notion of the progressive enlargement of a reference filtration. We assume that a strictly positive and finite random time ϑ, which is defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, as well as some *reference filtration* $\mathbb{F}$ are given. Then the enlarged filtration $\mathbb{G}$ is defined as the *progressive enlargement* of $\mathbb{F}$ by observations of ϑ (see, e.g., [4]) and thus a random time ϑ, which is not necessarily an $\mathbb{F}$-stopping time and belongs to the set of all finite $\mathbb{G}$-stopping times, denoted as $\widehat{\mathcal{T}}$. We emphasize that the filtrations $\mathbb{F}$ and $\mathbb{G}$ are henceforth supposed to satisfy the usual conditions of $\mathbb{P}$-completeness and right-continuity.

We will use the following notation for classes of processes adapted to the filtration $\mathbb{F}$:

- $\mathcal{O}(\mathbb{F})$, $\mathcal{P}(\mathbb{F})$, $\overline{\mathcal{P}}(\mathbb{F})$ and $\mathcal{P}r(\mathbb{F})$ are the classes of all real-valued, $\mathbb{F}$-optional, $\mathbb{F}$-predictable, $\mathbb{F}$-strongly predictable and $\mathbb{F}$-progressively measurable processes, respectively;
- $\mathcal{O}_d(\mathbb{F})$, $\mathcal{P}_d(\mathbb{F})$, $\overline{\mathcal{P}}_d(\mathbb{F})$ and $\mathcal{P}r_d(\mathbb{F})$ are the classes of all $\mathbb{R}^d$-valued, $\mathbb{F}$-optional, $\mathbb{F}$-predictable, $\mathbb{F}$-strongly predictable and $\mathbb{F}$-progressively measurable processes, respectively;
- $\mathcal{M}(\mathbb{F})$ (respectively, $\mathcal{M}_{loc}(\mathbb{F})$) is the class of all $\mathbb{F}$-martingales (respectively, $\mathbb{F}$-local martingales);
- $\mathcal{M}^{\vartheta}(\mathbb{F})$ (respectively, $\mathcal{M}^{\vartheta}_{loc}(\mathbb{F})$) is the class of all $\mathbb{F}$-martingales (respectively, $\mathbb{F}$-local martingales), which are stopped at ϑ.

A stochastic process X with sample paths possessing right-hand limits is said to be $\mathbb{F}$-*strongly predictable* if it is $\mathbb{F}$-predictable and the process X_+ is $\mathbb{F}$-optional (Definition 1.1 in [25]). An analogous notation is used for various classes of $\mathbb{G}$-adapted processes.

For instance, $\mathcal{P}(\mathbb{G})$ denotes the class of all $\mathbb{G}$-predictable processes, $\mathcal{M}^{\vartheta}_{loc}(\mathbb{G})$ is the class of all $\mathbb{G}$-local martingales, which are stopped at the random time ϑ, etc.

In order to simplify the notation, we denote by $X \bullet Y$ the usual Itô stochastic integral of X with respect to a (càdlàg) semimartingale Y, that is, $(X \bullet Y)_t := \int_{]\!]0,t]\!]} X_s\, dY_s$, while we also write $(X \star Y)_t := \int_{[\![0,t[\![} X_s\, dY_s$ so that the process $X \star Y$ is left-continuous as the integration is done over the interval $[\![0,t[\![$. Due to the potential presence of a jump of Y at time zero, we have that $(X \star Y)_t = (X \bullet Y)_{t-} + X_0 \Delta Y_0$ where, by the usual convention, $Y_{0-} = 0$ so that $\Delta Y_0 = Y_0$.

Let us recall from Gal'čuk [25] the notation pertaining to a pathwise decomposition of a làdlàg process. If C is an $\mathbb{F}$-adapted, làdlàg process, then we write $C = C^c + C^d + C^g$ where the process C^c is continuous, the càdlàg process C^d equals $C^d_t := \sum_{0 \leq s \leq t}(C_s - C_{s-})$ and the càglàd process C^g is given by $C^g_t := \sum_{0 \leq s < t}(C_{s+} - C_s)$. This also means that $C = C^r + C^g$ where the càdlàg process C^r satisfies $C^r = C - C^g = C^c + C^d$. Notice that if C is a càglàd process, then manifestly $C^d = 0$ and thus $C^r = C^c$ is a continuous process. Similarly, if C is a càdlàg process, then $C^g = 0$ and thus $C = C^r$. For the sake of convenience, we denote by C^g_+ the càdlàg version of the càglàd process C^g.

For a fixed random time ϑ, we define the *indicator process* $A \in \mathcal{O}(\mathbb{G})$ by $A := \mathbb{1}_{[\![\vartheta,\infty[\![}$ so that $A_t = \mathbb{1}_{\{\vartheta \leq t\}}$ for all $t \in \mathbb{R}_+$ and we denote by A^p (respectively, A^o) the dual $\mathbb{F}$-predictable projection (respectively, the dual $\mathbb{F}$-optional projection) of A. The BMO $\mathbb{F}$-martingales m and n associated with A^o and A^p, respectively, are defined as follows.

Definition 2.1. Let $m_t := \mathbb{E}(A^o_\infty \,|\, \mathcal{F}_t)$ so that $m_\infty = A^o_\infty$ and let $n_t := \mathbb{E}(A^p_\infty \,|\, \mathcal{F}_t)$ so that $n_\infty = A^p_\infty$.

As in Azéma [7], we introduce the $\mathbb{F}$-supermartingales G and $\widetilde{G}$ associated with ϑ.

Definition 2.2. The càdlàg process $G \in \mathcal{O}(\mathbb{F})$ given by the equality $G_t := \mathbb{P}(\vartheta > t \,|\, \mathcal{F}_t)$ is called the *Azéma supermartingale* of ϑ with respect to $\mathbb{F}$. The làdlàg process $\widetilde{G} \in \mathcal{O}(\mathbb{F})$ given by the equality $\widetilde{G}_t := \mathbb{P}(\vartheta \geq t \,|\, \mathcal{F}_t)$ is called the *Azéma optional supermartingale* of ϑ with respect to $\mathbb{F}$.

Notice that $G = {}^o(\mathbb{1}_{[\![0,\vartheta[\![}) = {}^o(1 - A)$ and $\widetilde{G} = {}^o(\mathbb{1}_{[\![0,\vartheta]\!]}) = {}^o(1 - A_-)$. For the reader's convenience, we recall some important properties of Azéma supermartingales G and $\widetilde{G}$ (see, e.g., Aksamit and Jeanblanc [4]).

Lemma 2.3. *(i) We have that $G = n - A^p = m - A^o$ and $\widetilde{G} = m - A^o_-$ and thus*

$$G_t = \mathbb{E}(A^p_\infty - A^p_t \,|\, \mathcal{F}_t) = \mathbb{E}(A^o_\infty - A^o_t \,|\, \mathcal{F}_t), \quad \widetilde{G}_t = \mathbb{E}(A^o_\infty - A^o_{t-} \,|\, \mathcal{F}_t).$$

(ii) The processes G and $\widetilde{G}$ satisfy $\widetilde{G}_- = G_-$ and $\widetilde{G}_+ = G_+ = G$.

(iii) The inequality $\widetilde{G} \geq G$ holds and the equalities $\widetilde{G} - G = {}^o(\mathbb{1}_{[\![\vartheta]\!]}) = \Delta A^o$ and $\widetilde{G} - G_- = \Delta m$ are valid.

The equality $G = n - A^p$ gives the Doob-Meyer decomposition in the filtration $\mathbb{F}$ of the bounded $\mathbb{F}$-supermartingale G. From the classical theory of enlargement of filtration, it is well known that the $\mathbb{G}$-martingale $n^{\mathbb{G}}$ from the Doob-Meyer decomposition in the filtration $\mathbb{G}$ of the bounded $\mathbb{G}$-submartingale A can be represented as follows

$$n^{\mathbb{G}} := A - \mathbb{1}_{]\!]0,\vartheta]\!]} G^{-1}_- \bullet A^p = A - \mathbb{1}_{]\!]0,\vartheta]\!]} \bullet \Gamma \tag{2.1}$$

where the $\mathbb{F}$-*predictable hazard process* of ϑ equals $\Gamma := G^{-1}_- \bullet A^p$ (Definition 1.6 in Jeanblanc and Li [32]). Furthermore, it was shown in Choulli et al. [9] (see Theorem 2.3 therein) that the following process $m^{\mathbb{G}}$ is a $\mathbb{G}$-martingale with the integrable variation

$$m^{\mathbb{G}} := A - \mathbb{1}_{]\!]0,\vartheta]\!]} \widetilde{G}^{-1} \bullet A^o = A - \mathbb{1}_{]\!]0,\vartheta]\!]} \bullet \widetilde{\Gamma} \tag{2.2}$$

where the $\mathbb{F}$-*optional hazard process* of ϑ is defined by $\widetilde{\Gamma} := \widetilde{G}^{-1} \bullet A^o$ (Definition 2.8 in Jeanblanc and Li [32]). The processes $n^{\mathbb{G}}$ and $m^{\mathbb{G}}$ are known to belong to the class $\mathcal{M}(\mathbb{G})$

but their properties are markedly different. In particular, $n^{\mathbb{G}}$ is not necessarily a *pure default martingale* (Definition 2.2 in Choulli et al. [9]) whereas $m^{\mathbb{G}}$ has that property.

We will also make use of the following general result due to Aksamit et al. [2].

Proposition 2.4. *Let the $\mathbb{F}$-stopping time $\widetilde{\eta}$ be given by $\widetilde{\eta} := \inf\{t \in \mathbb{R}_+ \mid \widetilde{G}_{t-} > \widetilde{G}_t = 0\}$. If M is an $\mathbb{F}$-local martingale, then the process*

$$M^{\vartheta} - \mathbb{1}_{[\![0,\vartheta]\!]}\widetilde{G}^{-1} \bullet [M,m] + \mathbb{1}_{[\![0,\vartheta]\!]} \bullet \big(\Delta M_{\widetilde{\eta}} \mathbb{1}_{[\![\widetilde{\eta},\infty[\![}\big)^p \tag{2.3}$$

is a $\mathbb{G}$-local martingale stopped at ϑ.

In particular, if $\widetilde{G}$ is a strictly positive process, then we set, for any $\mathbb{F}$-local martingale M,

$$\widetilde{M} := M - \widetilde{G}^{-1} \bullet [M,m] \tag{2.4}$$

so that the process $\widetilde{M}^{\vartheta}$ is a $\mathbb{G}$-local martingale stopped at ϑ. If, in addition, all $\mathbb{F}$-martingales are continuous (that is, if the condition **(C)** is satisfied – for instance, when $\mathbb{F}$ is a Brownian filtration), then $\mathcal{O}(\mathbb{F}) = \mathcal{P}(\mathbb{F})$ and thus the equalities $\widetilde{G} = G_-$ and $A^o = A^p$ are valid so that also $m^{\mathbb{G}} = n^{\mathbb{G}}$. Then equality (2.4) becomes $\widetilde{M} = M - G^{-1} \bullet \langle M, n\rangle$.

Lemma 2.5. *If M is a uniformly integrable $\mathbb{F}$-martingale, then the process*

$$M_t^{\vartheta -} - \int_{]\!]0,t]\!]} G_s^{-1}\, d[M,n]_s^{\vartheta -}$$

is a $\mathbb{G}$-local martingale.

Proof. Let H be a bounded $\mathbb{G}$-predictable process and h its bounded $\mathbb{F}$-predictable reduction on $[\![0,\vartheta]\!]$. By using the dual $\mathbb{F}$-predictable projection of A, we obtain

$$\mathbb{E}((H \bullet \mathbb{1}_{[\![0,\vartheta[\![} M)_\infty) = \mathbb{E}((h \bullet M)_{\vartheta -}) = \mathbb{E}(((h \bullet M)_- \bullet A^p)_\infty) = -\mathbb{E}(((h \bullet M)_- \bullet G)_\infty).$$

The integration by parts formula and Yœurp's lemma yield

$$\begin{aligned} -\mathbb{E}(((h \bullet M)_- \bullet G)_\infty) &= \mathbb{E}((G_- \bullet (h \bullet M))_\infty) + \mathbb{E}([h \bullet M, G]_\infty) \\ &= \mathbb{E}((h \bullet [M,n])_\infty) = \mathbb{E}((HG^{-1}\mathbb{1}_{[\![0,\vartheta[\![} \bullet [M,n])_\infty) \end{aligned}$$

where we have used the fact that $H = h$ on $[\![0,\vartheta]\!]$ and $\{\vartheta > t\} \subset \{G_t > 0\}$. □

3 Generalized BSDE and RBSDE with random time horizon

Our study of various generalized BSDEs and RBSDEs is conducted within the following setup.

Assumption 3.1. We assume that we are given the following objects:

(i) a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$;

(ii) a random time ϑ such that the Azéma supermartingale G (hence also G_- and the Azéma optional supermartingale $\widetilde{G}$) is a strictly positive process;

(iii) the class of all finite $\mathbb{G}$-stopping times $\widehat{\mathcal{T}}$ where $\mathbb{G}$ denotes the progressive enlargement of $\mathbb{F}$ with a random time ϑ;

(iv) the bounded processes $X, R \in \mathcal{O}(\mathbb{F})$, which are used to define the bounded *reward process* $\widehat{X} \in \mathcal{O}(\mathbb{G})$ through the following expression

$$\widehat{X} := X\mathbb{1}_{[\![0,\vartheta[\![} + R_{\vartheta}\mathbb{1}_{[\![\vartheta,\infty[\![}; \tag{3.1}$$

(v) a real-valued $\mathbb{G}$-martingale $m^{\mathbb{G}}$ associated with ϑ and given by (2.2);

(vi) an $\mathbb{R}^d$-valued, $\mathbb{F}$-local martingale M, which is assumed to have the predictable representation property (PRP) for the filtration $\mathbb{F}$;

(vii) an $\mathbb{R}^k$-valued, $\mathbb{G}$-adapted process $\widehat{D} = (\widehat{D}^1, \widehat{D}^2, \dots, \widehat{D}^k)$ where $\widehat{D}^i$ is a linear combination of a làdlàg $\mathbb{G}$-strongly predictable process of finite variation and a làdlàg $\mathbb{F}$-optional process of finite variation;

(viii) a mapping $\widehat{F} = (\widehat{F}^r, \widehat{F}^g)$ where mappings $\widehat{F}^r : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^k$ and $\widehat{F}^g : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^k$ are such that, for any fixed $(y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, the process $(\widehat{F}^r_t(y, z, u))_{t\geq 0}$ belongs to $\mathcal{P}_k(\mathbb{G})$ and the process $(\widehat{F}^g_t(y, z, u))_{t\geq 0}$ belongs to $\mathcal{O}_k(\mathbb{G})$.

We are in a position to introduce a particular class of BSDEs with a random time horizon. For the sake of brevity, they will be called $\mathbb{G}$ BSDEs, as opposed to the associated $\mathbb{F}$ BSDEs, which are introduced in Section 4.1.

Definition 3.1. For a fixed $\widehat{\tau} \in \widehat{\mathcal{T}}$, we say that a triplet $(\widehat{Y}, \widehat{Z}, \widehat{U})$ is a *solution* on $[\![0, \widehat{\tau} \wedge \vartheta]\!]$ to the $\mathbb{G}$ BSDE

$$\begin{aligned}\widehat{Y}_t = \widehat{X}_{\widehat{\tau}\wedge\vartheta} &- \int_{]\!]t,\widehat{\tau}\wedge\vartheta]\!]} \widehat{F}^r_s(\widehat{Y}_s, \widehat{Z}_s, \widehat{U}_s)\, d\widehat{D}^r_s - \int_{[\![t,\widehat{\tau}\wedge\vartheta[\![} \widehat{F}^g_s(\widehat{Y}_s, \widehat{Z}_s, \widehat{U}_s)\, d\widehat{D}^g_{s+} \\ &- \int_{]\!]t,\widehat{\tau}\wedge\vartheta]\!]} \widehat{Z}_s\, d\widetilde{M}^\vartheta_s - \int_{]\!]t,\widehat{\tau}\wedge\vartheta]\!]} \widehat{U}_s\, dm^{\mathbb{G}}_s \end{aligned} \tag{3.2}$$

if $\widehat{Y} \in \mathcal{O}(\mathbb{G})$ is a làglàd process, the processes $\widehat{Z} \in \mathcal{P}_d(\mathbb{G})$ and $\widehat{U} \in \mathcal{O}(\mathbb{F})$ are such that the stochastic integrals in the right-hand side of (3.2) are well defined and equality (3.2) is satisfied on the stochastic interval $[\![0, \widehat{\tau} \wedge \vartheta]\!]$.

The process $\widehat{D}$ from Assumption 3.1 (vii) and the mapping $\widehat{F} = (\widehat{F}^r, \widehat{F}^g)$ from Assumption 3.1 (viii) are henceforth called the *driver* and the *generator* of the $\mathbb{G}$ BSDE, respectively. The processes $m^{\mathbb{G}}$ and $\widetilde{M}^\vartheta$, given by equations (2.2) and (2.4), respectively, are orthogonal $\mathbb{G}$-local martingales stopped at ϑ and they are referred to as *driving martingales*. For explicit integrability conditions, which ensure that the stochastic integral $\widehat{U} \bullet m^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale, see Theorem 2.13 in Choulli et al. [9]. To the best of our knowledge, the issue of well-posedness of the $\mathbb{G}$ BSDE (3.2) is not addressed in the existing comprehensive literature on BSDEs and thus our aim is to contribute to the theory of BSDEs by filling that gap.

In the next definition, we implicitly make the natural postulate of well-posedness of the $\mathbb{G}$ BSDE (3.2) in a suitable space of stochastic processes, which can be left unspecified at this stage.

Definition 3.2. The *nonlinear evaluation* $\widehat{\mathcal{E}}$ is the collection of mappings $\widehat{\mathcal{E}} = \{\widehat{\mathcal{E}}_{\widehat{\sigma},\widehat{\tau}} \,|\, \widehat{\sigma}, \widehat{\tau} \in \widehat{\mathcal{T}},\, \widehat{\sigma} \leq \widehat{\tau}\}$ where for every $\widehat{\sigma}, \widehat{\tau} \in \widehat{\mathcal{T}}$ such that $\widehat{\sigma} \leq \widehat{\tau}$ we have $\widehat{\mathcal{E}}_{\widehat{\sigma},\widehat{\tau}}(\widehat{X}_{\widehat{\tau}}) := \widehat{Y}_{\widehat{\sigma}\wedge\vartheta}$ where the triplet $(\widehat{Y}, \widehat{Z}, \widehat{U})$ is a unique solution to the $\mathbb{G}$ BSDE (3.2) on the interval $[\![0, \widehat{\tau} \wedge \vartheta]\!]$.

Next, under similar assumptions, we introduce the generalized $\mathbb{G}$ RBSDE with a random time horizon ϑ.

Definition 3.3. A quadruplet $(\widehat{Y}, \widehat{Z}, \widehat{U}, \widehat{L})$ is a *solution* on the interval $[\![0, \widehat{\tau} \wedge \vartheta]\!]$ to the $\mathbb{G}$ RBSDE

$$\begin{aligned}\widehat{Y}_t = \widehat{X}_{\widehat{\tau}\wedge\vartheta} &- \int_{]\!]t,\widehat{\tau}\wedge\vartheta]\!]} \widehat{F}^r_s(\widehat{Y}_s, \widehat{Z}_s, \widehat{U}_s)\, d\widehat{D}^r_s - \int_{[\![t,\widehat{\tau}\wedge\vartheta[\![} \widehat{F}^g_s(\widehat{Y}_s, \widehat{Z}_s, \widehat{U}_s)\, d\widehat{D}^g_{s+} \\ &- \int_{]\!]t,\widehat{\tau}\wedge\vartheta]\!]} \widehat{Z}_s\, d\widetilde{M}_s - \int_{]\!]t,\widehat{\tau}\wedge\vartheta]\!]} \widehat{U}_s\, dm^{\mathbb{G}}_s - (\widehat{L}_{\widehat{\tau}\wedge\vartheta} - \widehat{L}_t) \end{aligned} \tag{3.3}$$

if $\widehat{Y} \in \mathcal{O}(\mathbb{G})$ is a làglàd process such that $\widehat{Y} \geq \widehat{X}$, the processes $\widehat{Z} \in \mathcal{P}_d(\mathbb{G})$ and $\widehat{U} \in \mathcal{O}(\mathbb{F})$ are such that the stochastic integrals in the right-hand side of (3.2) are well defined, $\widehat{L} = \widehat{L}^\vartheta$ is a làglàd, increasing, and $\mathbb{G}$-strongly predictable process with $\widehat{L}_0 = 0$ and has the decomposition $\widehat{L} = \widehat{L}^r + \widehat{L}^g$ where the processes $\widehat{L}^r$ and $\widehat{L}^g$ obey the Skorokhod conditions

$$\big(\mathbb{1}_{\{\widehat{Y}_- \neq \widehat{X}_-\}} \bullet \widehat{L}^r\big)^\vartheta_{\widehat{\tau}} = \big(\mathbb{1}_{\{\widehat{Y} \neq \widehat{X}\}} \star \widehat{L}^g_+\big)^\vartheta_{\widehat{\tau}} = 0$$

and equality (3.3) is assumed to hold on $[\![0,\widehat{\tau}\wedge\vartheta]\!]$.

As the arguments used in Subsections 3.1–3.4 can be applied to both the $\mathbb{G}$ BSDE and $\mathbb{G}$ RBSDE, we shall focus on presenting our explanations for the $\mathbb{G}$ BSDE.

3.1 Reduction of the reward process

Let $\mathcal{T}$ denote the class of all finite $\mathbb{F}$-stopping times and, for any fixed $\tau\in\mathcal{T}$, let the *stopped filtration* $\mathbb{F}^{\tau}$ be given by $\mathbb{F}^{\tau}:=(\mathcal{F}_{\tau\wedge t})_{t\geq 0}$. We will now examine the structure of the reward process $\widehat{X}$ specified by (3.1). We claim that, for any $\widehat{\tau}\in\widehat{\mathcal{T}}$, there exists $\tau\in\mathcal{T}$ such that $\widehat{X}_{\widehat{\tau}}=\widehat{X}_{\tau\wedge\vartheta}=\widehat{X}_{\tau}$. First, it is clear from (3.1) that the process $\widehat{X}$ is stopped at ϑ so that $\widehat{X}=\widehat{X}^{\vartheta}$, which immediately implies that $\widehat{X}_{\widehat{\tau}}=\widehat{X}_{\widehat{\tau}\wedge\vartheta}$. Hence, by using also the well known property that for any $\widehat{\tau}\in\widehat{\mathcal{T}}$ there exists $\tau\in\mathcal{T}$ such that $\tau\wedge\vartheta=\widehat{\tau}\wedge\vartheta$, we obtain the following equalities

$$\begin{aligned}\widehat{X}_{\widehat{\tau}}&=\widehat{X}_{\widehat{\tau}\wedge\vartheta}=X_{\widehat{\tau}\wedge\vartheta}\mathbb{1}_{\{\widehat{\tau}\wedge\vartheta<\vartheta\}}+R_{\vartheta}\mathbb{1}_{\{\widehat{\tau}\wedge\vartheta\geq\vartheta\}}=X_{\tau\wedge\vartheta}\mathbb{1}_{\{\tau\wedge\vartheta<\vartheta\}}+R_{\vartheta}\mathbb{1}_{\{\tau\wedge\vartheta\geq\vartheta\}}\\&=X_{\tau}\mathbb{1}_{\{\tau<\vartheta\}}+R_{\vartheta}\mathbb{1}_{\{\tau\geq\vartheta\}}=\widehat{X}_{\tau\wedge\vartheta}=\widehat{X}_{\tau}\end{aligned}$$

so that $\widehat{X}_{\widehat{\tau}}=\widehat{X}_{\tau\wedge\vartheta}=\widehat{X}_{\tau}$ for some stopping time $\tau\in\mathcal{T}$, as was required to show.

Lemma 3.4. *For any $\tau\in\mathcal{T}$, there exists $X(\tau)\in\mathcal{O}(\mathbb{F}^{\tau})$ such that the equality $\widehat{X}_{\tau}=X_{\vartheta}(\tau)$ holds.*

Proof. It suffices to observe that

$$\widehat{X}_{\tau}=R_{\vartheta}\mathbb{1}_{[\![\vartheta,\infty]\!]}(\tau)+X_{\tau}\mathbb{1}_{]\!]0,\vartheta[\![}(\tau)=R_{\vartheta}\mathbb{1}_{[\![0,\tau]\!]}(\vartheta)+X_{\tau}\mathbb{1}_{]\!]\tau,\infty[\![}(\vartheta)=X_{\vartheta}(\tau)\qquad(3.4)$$

where, for any fixed τ, the $\mathbb{F}$-adapted process $X(\tau)$ is given by

$$X(\tau):=R\,\mathbb{1}_{[\![0,\tau]\!]}+X_{\tau}\mathbb{1}_{]\!]\tau,\infty[\![}=R^{\tau}+(X_{\tau}-R_{\tau})\mathbb{1}_{]\!]\tau,\infty[\![}.\qquad(3.5)$$

Since the processes X and R are assumed to be $\mathbb{F}$-optional, by Lemma 3.53 in He et al. [30], the process $X(\tau)$ is $\mathbb{F}^{\tau}$-optional, although it is not a càdlàg process, in general. □

Since $\widehat{X}_{\tau}$ is $\mathcal{G}_{\tau}$-measurable and $X_{\vartheta}(\tau)$ is $\mathcal{G}_{\vartheta}$-measurable, by part (3) of Theorem 3.4 in He et al. [30], the random variable $\widehat{X}_{\tau}=X_{\vartheta}(\tau)$ is $\mathcal{G}_{\tau\wedge\vartheta}$-measurable or, more precisely, it is $\mathcal{F}^{\tau}_{\vartheta}$-measurable and $\mathcal{F}^{\tau}_{\vartheta}\subset\mathcal{G}_{\tau\wedge\vartheta}$. In view of equalities (3.4), we will freely interchange $\widehat{X}_{\widehat{\tau}}$, $\widehat{X}_{\tau}$ and $X_{\vartheta}(\tau)$.

Proposition 3.5. *Given two $\mathbb{G}$-stopping times $\widehat{\sigma},\widehat{\tau}$ such that $\widehat{\sigma}\leq\widehat{\tau}$, there exists $\sigma\leq\tau$ where $\sigma,\tau\in\mathcal{T}$ are such that $\sigma\wedge\vartheta=\widehat{\sigma}\wedge\vartheta$, $\tau\wedge\vartheta=\widehat{\tau}\wedge\vartheta$ and $\widehat{\mathcal{E}}_{\widehat{\sigma},\widehat{\tau}}(\widehat{X}_{\widehat{\tau}\wedge\vartheta})=\widehat{\mathcal{E}}_{\sigma,\tau}(X_{\vartheta}(\tau))$.*

Proof. First, we observe that there exists σ such that $\widehat{\sigma}\wedge\vartheta=\sigma\wedge\vartheta$. Furthermore, if the inequality $\sigma\leq\tau$ fails to hold, then we can take $\sigma'=\sigma\wedge\tau$ and observe that $\sigma'\wedge\vartheta=\sigma\wedge\tau\wedge\vartheta=\widehat{\sigma}\wedge\widehat{\tau}\wedge\vartheta=\widehat{\sigma}\wedge\vartheta$. Using the fact that there exists $\sigma\leq\tau$ such that $\sigma\wedge\vartheta=\widehat{\sigma}\wedge\vartheta$ and $\tau\wedge\vartheta=\widehat{\tau}\wedge\vartheta$, we obtain the following equalities

$$\begin{aligned}&\widehat{\mathcal{E}}_{\widehat{\sigma},\widehat{\tau}}(\widehat{X}_{\widehat{\tau}\wedge\vartheta})=X_{\vartheta}(\tau)-\int_{]\!]\sigma\wedge\vartheta,\tau\wedge\vartheta]\!]}\widehat{F}^{r}_{s}(\widehat{Y}_{s},\widehat{Z}_{s},\widehat{U}_{s})\,d\widehat{D}^{r}_{s}\\&-\int_{[\![\sigma\wedge\vartheta,\tau\wedge\vartheta[\![}\widehat{F}^{g}_{s}(\widehat{Y}_{s},\widehat{Z}_{s},\widehat{U}_{s})\,d\widehat{D}^{g}_{s+}-\int_{]\!]\sigma\wedge\vartheta,\tau\wedge\vartheta]\!]}\widehat{Z}_{s}\,d\widetilde{M}^{\vartheta}_{s}-\int_{]\!]\sigma\wedge\vartheta,\tau\wedge\vartheta]\!]}\widehat{U}_{s}\,dm^{\mathbb{G}}_{s}\end{aligned}$$

and thus we conclude that $\widehat{\mathcal{E}}_{\widehat{\sigma},\widehat{\tau}}(\widehat{X}_{\widehat{\tau}\wedge\vartheta})=\widehat{\mathcal{E}}_{\sigma,\tau}(X_{\vartheta}(\tau))$. □

3.2 Applications to credit risk modeling

This work is largely motivated by the problem of mitigation of financial and insurance risks triggered by an extraneous event. Applications in finance include the pricing of callable American warrants [14, 50], optimal timing of short-selling decisions under recall risk and the modeling of various valuation adjustments, that is, XVAs [51, 11, 12, 37]. For instance, using the terminology from the area of credit risk modeling, the process X may be interpreted as the reward (the promised payoff) to the holder of the contract if the decision to exercise the contract is made before the default time ϑ, while R may represent the closeout payoff (the recovery value), which is received by the holder at time ϑ if the contract was not exercised before ϑ. Furthermore, the process $\widehat{D} = (\widehat{D}^r, \widehat{D}^g)$ represents the cash flows of the contract and the pair $(\widehat{Z}, \widehat{U})$ has the usual interpretation as a hedging portfolio with respect to the default-free assets and a defaultable asset, respectively.

The proposed extensions of previous results to the case of a discontinuous driver, discontinuous hazard process and American style contracts are motivated by practical concerns since our theoretical approach allows for more flexible and realistic models of credit risk where, in principle, both the third party credit risk (e.g., [16, 27]) and the risk of the counterparty's default (e.g., [11, 12]) are comprehensively covered. By introducing jumps in the driver $\widehat{D}$ we are able to model cash flows which are made on a series of discrete dates but, of course, continuous cash flows are also covered by our setup. For a concrete example, we refer to Nie and Rutkowski [40, 41] on the extended Bergman model with collateralization where the continuous compounding assumption is made on the collateral lending and borrowing account (see Assumption 2.4 in [40]). If we assume instead that the collateral funding account B^c is compounded on discrete dates $T_1 < T_2 < \cdots < T_n < T$ with the rate r^c, then the process B^c in [40] obeys the equation $dB^c_t = r^c_t B^c_t \, dD_t$ where $D := \sum_{i=1}^n \mathbb{1}_{[\![T_i,\infty[\![}$.

Consequently, the BSDE from equation (2.9) in [40] becomes a BSDE with a discontinuous driver and a non-linear generator. For similar BSDEs derived in a model with default, we refer for example to equation (16) in Lee and Zhou [37] or equation (4.12) in Bichuch et al. [8]. For further arguments, the reader may also consult Wu [51] where computations of CVA and FVA for derivatives with cash collateral were examined. In particular, it was pointed out in Section 4 of [51], although not explicitly studied there, that in the market practice the cash collateral for interest rate swaps is revised at discrete dates, rather than continuously. Finally, it was also observed in [51] that for application purposes, one needs to generalize their results to American or Bermudan options.

The other important contribution of this work is that neither condition **(C)** nor condition **(A)** are postulated. This means that, firstly, the default-free market modelled using the filtration $\mathbb{F}$ is allowed to have jumps and, secondly, the default event may occur at an $\mathbb{F}$-stopping times with a positive probability. In particular, in the current setup, the hazard process $\widetilde{\Gamma} := \widetilde{G}^{-1} \bullet A^o$ is not necessarily continuous and the size of its discontinuity can be interpreted as the conditional probability that the default event occurs at the time of the jump given that is has not happened up to the moment when jump occurs. These extensions of classical approaches enable one to combine the reduced-form approach and the structural approach to credit risk by allowing to cover both the case of default intensity and the possibility that default can happen concurrently with a family of $\mathbb{F}$-stopping times, which is endogenously specified by the underlying default-free market.

To give a simple illustration of the above arguments, let us consider a market model where the stock process is represented by a semimartingale S. We define $\sigma_1 := \inf\{t \in \mathbb{R}_+ \,|\, \Delta S_t < 0\}$ and $\sigma_2 := \inf\{t \in \mathbb{R}_+ \,|\, S_t < c\}$ for some $c \in \mathbb{R}_+$. Suppose that the default event may occur at time $\sigma := \sigma_1 \wedge \sigma_2$, that is, either at the first time when the stock market experiences a negative shock or when it drops below a constant threshold c. Then we construct the related hazard process $\widetilde{\Gamma}$ by setting, for some constants $\lambda > 0$ and $p \in [0,1]$,

$$\widetilde{\Gamma}_t = \lambda t + (1-p)\mathbb{1}_{\{\sigma \le t\}}$$

and, for simplicity, we set $\vartheta = \inf\{t \in \mathbb{R}_+ \,|\, 1 - \mathcal{E}_t(-\widetilde{\Gamma}_-) > \zeta\}$ where $\mathcal{E}$ denotes the stochastic exponential and ζ is a random variable, which is uniformly distributed on $[0,1]$ and independent of $\mathcal{F}_\infty$. For the construction of a random time with a predetermined hazard process and without the postulate that the hypothesis (H) holds, we refer to Jeanblanc and Li [32]. The parameters $\lambda \in \mathbb{R}_+$ and $p \in [0,1]$ should be estimated using the market data, which is feasible since their interpretation is fairly transparent and thus they are amenable to standard statistical studies

$$\lambda \, dt = \frac{\mathbb{P}(\vartheta \in dt \,|\, \mathcal{F}_t)}{\mathbb{P}(\vartheta \ge t \,|\, \mathcal{F}_t)}, \quad 1 - p = \frac{\mathbb{P}(\vartheta = \sigma \,|\, \mathcal{F}_\sigma)}{\mathbb{P}(\vartheta \ge \sigma \,|\, \mathcal{F}_\sigma)}.$$

We observe that the above construction of default time ϑ hinges on a combination of the reduced-form approach based on the intensity λ with the structural approach based on the $\mathbb{F}$-stopping time σ. In practical applications, one can postulate that the jumps of the hazard process may occur at a predetermined sequence of fixed dates, for instance, the dates of important announcements by a central bank or a regulatory body, which may affect the financial market in an abrupt manner and hence generate negative or positive jumps in share prices.

Finally, the combination of a discontinuous driver with a discontinuous hazard process produces a new interesting feature in the reduced BSDE. Specifically, when the driver process and the hazard process share common jumps or, using the language of finance, when the default may happen at the moment when the discrete compounding of the collateral funding account occurs, then there exist an additional adjustment term in $\widehat{U}$ (see, e.g., (4.13)) related to the jump of the driver given by $\big(F^r_s(R_s) - F^r_s(Y_s)\big)\Delta D^r_s$. This term can be thought of as the adjustment to the hedging portfolio $\widehat{U}$ in order to obtain the correct recovery value. This feature leads to an additional constraint in the reduced BSDE (4.13)–(4.14) and the reduced RBSDE (5.3)–(5.4), which have not appeared in previous works and hence results on these equations could be of independent interest.

3.3 Case of a general Azéma supermartingale

Let us make some comments on the possibility of relaxing Assumption 3.1(ii), that is, allowing the Azéma supermartingale G of ϑ to hit zero. As an example, let us first consider a random time of the form $\vartheta = \vartheta' \wedge T_1$ where the Azéma supermartingale of ϑ' is a strictly positive process and T_1 is an $\mathbb{F}$-stopping time. Then the Azéma supermartingale of ϑ jumps to zero at T_1 and it is not hard to check that all results can be extended to that case by replacing the terminal time τ by $\tau' = \tau \wedge T_1$ and ϑ by ϑ'. Since G is a nonnegative supermartingale, we have that (see, e.g., Theorem 2.62 in He et al. [30])

$$\eta := \inf\{t \in \mathbb{R}_+ \,|\, G_t = 0\} = \inf\{t \in \mathbb{R}_+ \,|\, G_{t-} = 0\} = \lim_{n\to\infty} \eta_n$$

where $\eta_n := \inf\{t \in \mathbb{R}_+ \,|\, G_t \le 1/n\}$. It is known that $G = 0$ on $[\![\eta, \infty[\![$ and, from Lemma 2.14 of [4], we have that $[\![0, \vartheta]\!] \subset \{G_- > 0\}$ and $[\![0, \vartheta[\![\subset \{G > 0\} = [\![0, \eta[\![$ so that $\vartheta \le \eta$.

In order to weaken Assumption 3.1(ii), we introduce the class of random times ϑ such that the Azéma supermartingale G' of $\vartheta' := \vartheta_{\{\vartheta < \eta\}}$ is strictly positive on the interval $[\![0, \eta]\!]$. Specifically, we set (recall that $\widetilde{G}_- = G_-$)

$$\mathcal{K} := \{\vartheta \,|\, \widetilde{G}_\eta > 0\} \tag{3.6}$$

and we will argue that Assumption 3.1(ii) can be replaced by the postulate that ϑ belongs to $\mathcal{K}$. It is clear that the class $\mathcal{K}$ is nonempty and contains not only all random times with a strictly positive Azéma supermartingale, but also their minimum with any $\mathbb{F}$-stopping time. Observe that if a random time ϑ belongs to $\mathcal{K}$ then clearly

$$\widetilde{\eta} := \inf\{t \in \mathbb{R}_+ \,|\, \widetilde{G}_{t-} > \widetilde{G}_t = 0\} = \eta_{\{G_{\eta-} > \widetilde{G}_\eta = 0\}} = \infty$$

and thus the term $\mathbb{1}_{[\![0,\vartheta]\!]} \bullet \big(\Delta M_{\widetilde{\eta}} \mathbb{1}_{[\![\widetilde{\eta},\infty[\![}\big)^p$ in the $\mathbb{G}$-semimartingale decomposition (2.3) of an arbitrary $\mathbb{F}$-local martingale M is in fact null.

Lemma 3.6. *Let ϑ be a random time with the Azéma optional supermartingale $\widetilde{G}$ and let $\vartheta' := \vartheta_{\{\vartheta<\eta\}}$. Then $\vartheta = \vartheta' \wedge \eta$ and the Azéma optional supermartingale $\widetilde{G}'$ of ϑ' is given by $\widetilde{G}' = \widetilde{G}^\eta + \mathbb{1}_{]\!]\eta,\infty[\![} \bullet H$ where $H = {}^o(\mathbb{1}_{\{\vartheta=\eta\}})$. Furthermore, we have that $(A')^o = (A^o)^\eta$ where $A' := \mathbb{1}_{[\![\vartheta',\infty[\![}$.*

Proof. We first observe that $\vartheta' := \vartheta_{\{\vartheta<\eta\}} = \vartheta \mathbb{1}_{\{\vartheta<\eta\}} + \infty \mathbb{1}_{\{\vartheta=\eta\}}$ where in the second equality we have used the inequality $\vartheta \leq \eta$. Using also the equalities $\{\vartheta < \eta\} = \{\vartheta' < \eta\}$ and $\{\vartheta = \eta\} = \{\vartheta' \geq \eta\}$, we obtain

$$\vartheta = \vartheta \mathbb{1}_{\{\vartheta\leq\eta\}} = \vartheta \mathbb{1}_{\{\vartheta<\eta\}} + \vartheta \mathbb{1}_{\{\vartheta=\eta\}} = \vartheta' \mathbb{1}_{\{\vartheta<\eta\}} + \eta \mathbb{1}_{\{\vartheta=\eta\}} = \vartheta' \mathbb{1}_{\{\vartheta'<\eta\}} + \eta \mathbb{1}_{\{\vartheta'\geq\eta\}} = \vartheta' \wedge \eta.$$

Next, to compute the Azéma optional supermartingale of ϑ', we observe that for all $t > 0$,

$$\begin{aligned}\mathbb{P}(\vartheta' < t \,|\, \mathcal{F}_t) &= \mathbb{P}(\vartheta < t, \vartheta < \eta \,|\, \mathcal{F}_t) = \mathbb{P}(\vartheta < t \,|\, \mathcal{F}_t) - \mathbb{P}(\vartheta < t, \vartheta = \eta \,|\, \mathcal{F}_t) \\ &= 1 - \widetilde{G}_t \mathbb{1}_{\{t\leq\eta\}} - \mathbb{P}(\vartheta = \eta \,|\, \mathcal{F}_t) \mathbb{1}_{\{\eta<t\}}\end{aligned}$$

and hence $\widetilde{G}' = \widetilde{G}^\eta + \mathbb{1}_{]\!]\eta,\infty[\![} \bullet H$ where $H = {}^o(\mathbb{1}_{\{\vartheta=\eta\}})$.

The last assertion follows from the uniqueness of the Doob-Meyer-Mertens decomposition of $\widetilde{G}'$. It is worth noting that $(\widetilde{G}')^\eta = (\widetilde{G})^\eta$ and $(G')^\eta = G \mathbb{1}_{[\![0,\eta[\![} + \widetilde{G}_\eta \mathbb{1}_{[\![\eta,\infty[\![}$. □

It is important to observe that if $\vartheta \in \mathcal{K}$, then the supermartingale G' and the optional supermartingale $\widetilde{G}'$ are strictly positive on $[\![0,\tau']\!]$ where $\tau' := \tau \wedge \eta$. Furthermore, the equalities $\widehat{X}_{\widehat{\tau}} = \widehat{X}_{\widehat{\tau}\wedge\vartheta} = \widehat{X}_{\widehat{\tau}\wedge\vartheta'\wedge\eta} = X_{\vartheta'}(\tau')$ hold. We conclude that, on the one hand, all our arguments used to address the case of a strictly positive Azéma supermartingale are still valid when the pair (ϑ,τ) is replaced by (ϑ',τ'). For instance, the BSDE in Proposition 3.5 would not change since $\vartheta \leq \eta$, whereas in Section 4.1 the terminal date can be changed to $\tau' = \tau \wedge \eta$. On the other hand, however, if a random time ϑ is not in $\mathcal{K}$, then technical issues involving either an explosion of integrals or ill-defined terminal condition at time η may arise.

3.4 Extended terminal condition

Let us make some comments on a possible extension of the terminal condition in the $\mathbb{G}$ BSDE. Since the processes X and R are assumed to be $\mathbb{F}$-optional, Lemma 3.4 implies that $X(\tau)$ is an $\mathbb{F}^\tau$-optional process and $X_\vartheta(\tau)$ is $\mathcal{F}^\tau_\vartheta$-measurable. This implies that in our formulation of the $\mathbb{G}$ BSDE (3.2) and the $\mathbb{G}$ RBSDE (3.3) the terminal condition is measurable with respect to $\mathcal{F}^\tau_\vartheta \subset \mathcal{G}_{\tau\wedge\vartheta}$. It is also worth noting that

$$\sigma(V^\tau_\vartheta \,|\, V \in \mathcal{O}(\mathbb{F})) = \mathcal{F}_{\tau\wedge\vartheta} \subset \mathcal{F}^\tau_\vartheta := \sigma(V_\vartheta \,|\, V \in \mathcal{O}(\mathbb{F}^\tau)).$$

Since we do not consider all $\mathcal{G}_{\tau\wedge\vartheta}$-measurable terminal conditions, the multiplicity in the martingale representation property established in Theorem 2.22 of Choulli et al. [9] can be taken to be equal to two, which in fact gives a partial motivation for Definition 3.1 of a solution to the $\mathbb{G}$ BSDE.

In general, the multiplicity in Theorem 2.22 of Choulli et al. [9] is equal to three and thus it would be possible to consider, when a (bounded) terminal condition ζ is $\mathcal{G}_{\tau\wedge\vartheta}$-measurable, a more general $\mathbb{G}$ BSDE driven by $\widetilde{M}^{\vartheta}$, $m^{\mathbb{G}}$ and a pure jump martingale yielding an additional 'correction term' at the terminal time $\tau\wedge\vartheta$. To be more specific, in view of Proposition 3.5, one could study the extended $\mathbb{G}$ BSDE of the form

$$\begin{aligned}\widehat{Y}_t = \zeta &- \int_{\rrbracket t,\tau\wedge\vartheta\rrbracket} \widehat{F}^r_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s,\widehat{J}_s(\tau))\,d\widehat{D}^r_s - \int_{\llbracket t,\tau\wedge\vartheta\llbracket} \widehat{F}^g_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s,\widehat{J}_s(\tau))\,d\widehat{D}^g_{s+} \\ &- \int_{\rrbracket t,\tau\wedge\vartheta\rrbracket} \widehat{Z}_s\,d\widetilde{M}^{\vartheta}_s - \int_{\rrbracket t,\tau\wedge\vartheta\rrbracket} \widehat{U}_s\,dm^{\mathbb{G}}_s - \int_{\rrbracket t,\tau\wedge\vartheta\rrbracket} \widehat{J}_s(\tau)\,dA_s \end{aligned} \tag{3.7}$$

where $\widehat{Z}$ is $\mathbb{F}$-predictable, $\widehat{U}$ is $\mathbb{F}$-optional, $\widehat{J}(\tau)$ is $\mathbb{F}^{\tau}$-progressively measurable and $\mathbb{E}(\widehat{J}_{\vartheta}(\tau)\,|\,\mathcal{F}_{\vartheta})=0$. It is not difficult to check that the last condition implies that $\widehat{J}(\tau)\bullet A$ is a $\mathbb{G}$-local martingale, provided that an appropriate integrability condition is satisfied by $\widehat{J}(\tau)$.

A detailed study of the $\mathbb{G}$ BSDE given by equation (3.7) is beyond the scope of this work since its practical applications are unclear. Let us only point out that if the generator $\widehat{F}$ in (3.7) does not depend on $\widehat{J}$, then one can formally reduce (3.7) to (3.2) and show that a solution to (3.7) can be obtained from a solution to (3.2). To this end, we will need the following auxiliary result.

Lemma 3.7. *Assume that ζ is bounded and $\mathcal{G}_{\tau\wedge\vartheta}$-measurable. Then*
(i) there exists a process $X'(\tau)\in\mathcal{P}r(\mathbb{F}^{\tau})$ such that $\zeta = X'_{\vartheta}(\tau)$;
(ii) there exists a process $X(\tau)\in\mathcal{O}(\mathbb{F}^{\tau})$ such that $X_{\vartheta}(\tau)=\mathbb{E}(X'_{\vartheta}(\tau)\,|\,\mathcal{F}_{\vartheta})$.

Proof. To show the first assertion, we note that since ζ is $\mathcal{G}_{\tau\wedge\vartheta}$-measurable, there exists a process $\widehat{H}\in\mathcal{O}(\mathbb{G})$ such that $\zeta=\widehat{H}_{\tau\wedge\vartheta}$ and thus also, by Proposition 2.11 in Aksamit and Jeanblanc [4], a process $H\in\mathcal{O}(\mathbb{F})$ such that $\widehat{H}\mathbb{1}_{\llbracket 0,\vartheta\llbracket}=H\mathbb{1}_{\llbracket 0,\vartheta\llbracket}$. Furthermore, since $\mathcal{G}_{\tau\wedge\vartheta}\subset\mathcal{G}_{\vartheta}=\mathcal{F}_{\vartheta+}$, there exists a process $H'\in\mathcal{P}r(\mathbb{F})$ such that $\zeta=\widehat{H}_{\tau\wedge\vartheta}=H'_{\vartheta}$ (see Lemma B.1 in Aksamit et al. [1], which is obtained by modifying Proposition 5.3 (b) in Jeulin [33]). We thus have the equalities

$$H'_{\vartheta}\mathbb{1}_{\{\tau<\vartheta\}}=\widehat{H}_{\tau}\mathbb{1}_{\{\tau<\vartheta\}}=H_{\tau}\mathbb{1}_{\{\tau<\vartheta\}}$$

and we can define the $\mathbb{F}^{\tau}$-progressively measurable process

$$X'(\tau):=H'\mathbb{1}_{\llbracket 0,\tau\rrbracket}+H_{\tau}\mathbb{1}_{\rrbracket\tau,\infty\llbracket}, \tag{3.8}$$

which satisfies $X'_{\vartheta}(\tau)=\zeta$. For the second assertion, we note that since $\mathbb{1}_{\llbracket 0,\tau\rrbracket}$ and $H_{\tau}\mathbb{1}_{\rrbracket\tau,\infty\llbracket}$ belong to $\mathcal{O}(\mathbb{F})$, we have

$$\mathbb{E}(X'_{\vartheta}(\tau)\,|\,\mathcal{F}_{\vartheta})=\mathbb{E}(H'_{\vartheta}\,|\,\mathcal{F}_{\vartheta})\mathbb{1}_{\llbracket 0,\tau\rrbracket}(\vartheta)+H_{\tau}\mathbb{1}_{\rrbracket\tau,\infty\llbracket}(\vartheta).$$

By Proposition 2.21 in Choulli et al. [9] there exists an $\mathbb{F}$-optional process X such that $X_{\vartheta}=\mathbb{E}(H'_{\vartheta}\,|\,\mathcal{F}_{\vartheta})$. It now suffices to set $X(\tau):=X\mathbb{1}_{\llbracket 0,\tau\rrbracket}+H_{\tau}\mathbb{1}_{\rrbracket\tau,\infty\llbracket}$ and observe that the process $X(\tau)$ belongs to $\mathcal{O}(\mathbb{F}^{\tau})$. □

By applying Lemma 3.7 to an integrable, $\mathcal{G}_{\tau\wedge\vartheta}$-measurable random variable ζ, we can rewrite $\zeta=X'_{\vartheta}(\tau)=X'_{\vartheta}(\tau)-X_{\vartheta}(\tau)+X_{\vartheta}(\tau)$. Therefore, in view of (3.8), we have $\widehat{J}(\tau):=X'(\tau)-X(\tau)=(H'-X)\mathbb{1}_{\llbracket 0,\tau\rrbracket}$, which shows that $\widehat{J}(\tau)=\widehat{J}(\tau)\mathbb{1}_{\llbracket 0,\tau\rrbracket}$. Then the BSDE (3.7) becomes

$$\begin{aligned}\widehat{Y}_t = X_{\vartheta}(\tau) &- \int_{\rrbracket t,\tau\wedge\vartheta\rrbracket} \widehat{F}^r_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)\,d\widehat{D}^r_s - \int_{\llbracket t,\tau\wedge\vartheta\llbracket} \widehat{F}^g_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)\,d\widehat{D}^g_{s+} \\ &- \int_{\rrbracket t,\tau\wedge\vartheta\rrbracket} \widehat{Z}_s\,d\widetilde{M}^{\vartheta}_s - \int_{\rrbracket t,\tau\wedge\vartheta\rrbracket} \widehat{U}_s\,dm^{\mathbb{G}}_s\end{aligned}$$

where we have used the equalities

$$\int_{\rrbracket t,\tau\rrbracket} \widehat{J}_s(\tau)\, dA_s = \widehat{J}_\vartheta(\tau)\mathbb{1}_{\{t<\vartheta\leq\tau\}} = \widehat{J}_\vartheta(\tau).$$

We conclude that if $(\widehat{Y},\widehat{Z},\widehat{U})$ is a solution to the $\mathbb{G}$ BSDE (3.2), but with $\widehat{\tau}$ replaced by τ, then $(\widehat{Y},\widehat{Z},\widehat{U},\widehat{J}(\tau))$ solves the BSDE (3.7) with a $\mathcal{G}_{\tau\wedge\vartheta}$-measurable terminal condition. Similar arguments can be applied to the case of the RBSDE (3.3). However, in the case of generators depending on $\widehat{J}(\tau)$, the proper form of the adjustment to the terminal condition would be more complicated and its computation would involve the generators $\widehat{F}^r$ and $\widehat{F}^g$.

4 Solution to a generalized BSDE

Our goal is to show that the BSDE (3.2) has a solution, which can be obtained in two steps. In the reduction step, the filtration is shrunk from $\mathbb{G}$ to $\mathbb{F}$ and the BSDE (3.2) is analyzed through an associated reduced BSDE in the filtration $\mathbb{F}$. In the construction step, we show that a solution to the reduced BSDE can be lifted from $\mathbb{F}$ to $\mathbb{G}$ in order to obtain a solution to the BSDE (3.2). Notice that in Sections 4.1 and 4.2 the random times ϑ and $\widehat{\tau}$ are fixed throughout.

4.1 Reduction of a solution to $\mathbb{G}$ BSDE

We first establish some preliminary lemmas related to the concept of shrinkage of filtration. In the main result of this section, Proposition 4.9, we give an explicit representation for the $\mathbb{F}$ BSDE associated with the $\mathbb{G}$ BSDE (3.2). We work here under Assumption 4.1, which will be relaxed in Section 4.2 where an explicit construction of a solution to the $\mathbb{G}$ BSDE is proposed and analyzed.

Assumption 4.1. A solution $(\widehat{Y},\widehat{Z},\widehat{U})$ to the BSDE (3.2) exists on the stochastic interval $\llbracket 0,\widehat{\tau}\wedge\vartheta\rrbracket$ or, equivalently, on the interval $\llbracket 0,\tau\wedge\vartheta\rrbracket$ where $\tau\in\mathcal{T}$ is such that $\tau\wedge\vartheta=\widehat{\tau}\wedge\vartheta$.

Our present goal is to analyze the consequences of Assumption 4.1. We start by recalling that there exist a unique $\mathbb{F}$-optional process Y and a unique $\mathbb{F}$-predictable process Z such that the equalities $\widehat{Y}\mathbb{1}_{\llbracket 0,\vartheta\llbracket} = Y\mathbb{1}_{\llbracket 0,\vartheta\llbracket}$ and $\widehat{Z}\mathbb{1}_{\llbracket 0,\vartheta\rrbracket} = Z\mathbb{1}_{\llbracket 0,\vartheta\rrbracket}$ are valid. Moreover, $Y_\tau = X_\tau$ and the process Y and Z are given by

$$Y = {}^{o}\big(\mathbb{1}_{\llbracket 0,\vartheta\llbracket}\widehat{Y}\big)G^{-1}, \quad Z = {}^{p}\big(\mathbb{1}_{\llbracket 0,\vartheta\rrbracket}\widehat{Z}\big)G_{-}^{-1}. \tag{4.1}$$

Similarly, in view of Assumption 3.1(vii) and Lemma 4.3 below, there exists an right continuous $\mathbb{F}$-adapted process D^r and a left-continuous $\mathbb{F}$-adapted process D^g such that $\widehat{D}^r\mathbb{1}_{\llbracket 0,\vartheta\rrbracket} = D^r\mathbb{1}_{\llbracket 0,\vartheta\rrbracket}$ and $\widehat{D}^g\mathbb{1}_{\llbracket 0,\vartheta\rrbracket} = D^g\mathbb{1}_{\llbracket 0,\vartheta\rrbracket}$. Finally, it is clear that $\widehat{X}\mathbb{1}_{\llbracket 0,\vartheta\llbracket} = X\mathbb{1}_{\llbracket 0,\vartheta\llbracket}$. We shall refer to τ, Y, Z, D^r, D^g and X as the $\mathbb{F}$-*reduction* of $\widehat{\tau}$, $\widehat{Y}$, $\widehat{Z}$, $\widehat{D}^r$, $\widehat{D}^g$ and $\widehat{X}$.

In the following, we slightly abuse the notation and we again denote by Y and Z the stopped processes $Y := Y^\tau$ and $Z := Z^\tau$. Recall that the component $\widehat{U}$ in a solution to (3.2) is assumed to be an $\mathbb{F}$-optional process and thus $\widehat{U}$ is equal to its $\mathbb{F}$-reduction U so that, trivially, $\widehat{U} = U$ and, once again, we will write $U := U^\tau$.

To show more explicitly how the process Y is computed, we observe that

$$\widehat{\mathcal{E}}_{\cdot,\widehat{\tau}}(\widehat{X}_{\widehat{\tau}})\mathbb{1}_{\llbracket 0,\tau\rrbracket}\mathbb{1}_{\llbracket 0,\vartheta\llbracket} = \widehat{\mathcal{E}}_{\cdot,\widehat{\tau}}(\widehat{X}_{\widehat{\tau}})\mathbb{1}_{\llbracket 0,\widehat{\tau}\rrbracket}\mathbb{1}_{\llbracket 0,\vartheta\llbracket}$$

and, in view of Proposition 3.5, there exists $Y\in\mathcal{O}(\mathbb{F})$ such that we have

$$\widehat{\mathcal{E}}_{\cdot,\widehat{\tau}}(\widehat{X}_{\widehat{\tau}})\mathbb{1}_{\llbracket 0,\tau\rrbracket}\mathbb{1}_{\llbracket 0,\vartheta\llbracket} = \widehat{\mathcal{E}}_{\cdot,\tau}(X_\vartheta(\tau))\mathbb{1}_{\llbracket 0,\tau\rrbracket}\mathbb{1}_{\llbracket 0,\vartheta\llbracket} = Y\mathbb{1}_{\llbracket 0,\tau\rrbracket}\mathbb{1}_{\llbracket 0,\vartheta\llbracket}.$$

Therefore, by applying the $\mathbb{F}$-optional projection operator, we obtain

$$ {}^{o}\big(\widehat{\mathcal{E}}_{\cdot,\widehat{\tau}}(\widehat{X}_{\widehat{\tau}})\mathbb{1}_{[\![0,\widehat{\tau}]\!]}\mathbb{1}_{[\![0,\vartheta[\![}\big)\mathbb{1}_{[\![0,\tau]\!]} = {}^{o}\big(\widehat{\mathcal{E}}_{\cdot,\tau}(X_\vartheta(\tau))\mathbb{1}_{[\![0,\tau]\!]}\mathbb{1}_{[\![0,\vartheta[\![}\big)\mathbb{1}_{[\![0,\tau]\!]} = YG\mathbb{1}_{[\![0,\tau]\!]}. $$

A general representation of Y can then be obtained on $[\![0,\tau]\!]$ by noticing that for any $\mathbb{F}$-stopping time σ

$$ Y_\sigma G_\sigma \mathbb{1}_{\{\sigma\le\tau\}} = \mathbb{E}\big(\widehat{\mathcal{E}}_{\sigma,\tau}(X_\vartheta(\tau))\mathbb{1}_{\{\sigma\le\tau\}}\mathbb{1}_{\{\sigma<\vartheta\}}\,|\,\mathcal{F}_\sigma\big)\mathbb{1}_{\{\sigma\le\tau\}}. $$

Our next goal is provide a more explicit computation of the right-hand side in the above equality (see Lemma 4.6).

Remark 4.1. Suppose that Assumption 3.1(ii) is relaxed and we postulate instead that $\vartheta\in\mathcal{K}$ where the class $\mathcal{K}$ is defined by (3.6). Then the modified terminal condition would be $\widehat{X}_{\widehat{\tau}\wedge\vartheta'\wedge\eta} = \widehat{X}_{\tau\wedge\vartheta'\wedge\eta}$ and the reduced terminal condition would become $X_{\tau\wedge\eta}G'_{\tau\wedge\eta}$. Finally, the terminal condition for Y would be $X_{\tau\wedge\eta}$ instead of X_τ. Hence, it would be enough to replace τ with $\tau'=\tau\wedge\eta$ and study the $\mathbb{F}$-BSDE on the interval $[\![0,\tau']\!]$, rather than $[\![0,\tau]\!]$.

The following result can be deduced from Proposition 2.11 in Aksamit and Jeanblanc [4].

Lemma 4.2. *For every $(y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ there exists an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process $F^r(y,z,u)$ such that $\widehat{F}^r_t(y,z,u)\mathbb{1}_{\{\vartheta\ge t\}} = F^r_t(y,z,u)\mathbb{1}_{\{\vartheta\ge t\}}$ for every $t\ge 0$. For every $(y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ there exists an $\mathbb{R}^k$-valued, $\mathbb{F}$-optional process $F^g(y,z,u)$ such that $\widehat{F}^g_t(y,z,u)\mathbb{1}_{\{\vartheta>t\}} = F^g_t(y,z,u)\mathbb{1}_{\{\vartheta>t\}}$ for every $t\ge 0$.*

To reduce the driver $\widehat{D}$ and later the reflection in the $\mathbb{G}$ RBSDE, we prove the following result. Notice that a similar result was established in Jeanblanc et al. [31] in the case where the partition of the space $\Omega\times[0,\infty[$ was independent of time.

Lemma 4.3. *Let $\widehat{D}=\widehat{D}^r+\widehat{D}^g$ be an $\mathbb{G}$-adapted, làglàd, increasing process. Then there exists an $\mathbb{F}$-optional, càdlàg, increasing process D^r and an $\mathbb{F}$-predictable, càglàd, increasing process D^g such that $D^r=\widehat{D}^r$ on $[\![0,\vartheta[\![$ and $D^g=\widehat{D}^g$ on $[\![0,\vartheta]\!]$. If $\widehat{D}$ is a $\mathbb{G}$-strongly predictable increasing process, then D^r can be chosen such that it is an $\mathbb{F}$-predictable, càdlàg, increasing process and $D^r=\widehat{D}^r$ on $[\![0,\vartheta]\!]$.*

Proof. Since $\widehat{D}^r$ belongs to the class $\mathcal{O}(\mathbb{G})$, there exists an $\mathbb{F}$-optional process D^r such that $\widehat{D}^r\mathbb{1}_{[\![0,\vartheta[\![} = D^r\mathbb{1}_{[\![0,\vartheta[\![}$ (see the first equality in (4.1)). Since the optional projection of a càdlàg processes is again a càdlàg process, the process D^r is càdlàg on the set $\{G>0\}=\Omega\times[0,\infty[$ where the last equality is clear since we have assumed that G is strictly positive.

To show that the process D^r is increasing, we observe that, for every $s\le t$,

$$ D^r_t\mathbb{1}_{\{\vartheta>t\}} = \widehat{D}^r_t\mathbb{1}_{\{\vartheta>t\}} \ge \widehat{D}^r_s\mathbb{1}_{\{\vartheta>t\}} = \widehat{D}^r_s\mathbb{1}_{\{\vartheta>s\}}\mathbb{1}_{\{\vartheta>t\}} = D^r_s\mathbb{1}_{\{\vartheta>t\}}. $$

Then, by taking the $\mathcal{F}_t$ conditional expectation of both sides, we deduce that the process D^r is increasing on the set $\{G>0\}=\Omega\times[0,\infty[$.

Furthermore, since the process $\widehat{D}^g$ is càglàd and thus belongs to the class $\mathcal{P}(\mathbb{G})$, there exists an $\mathbb{F}$-predictable process D^g such that $\widehat{D}^g\mathbb{1}_{[\![0,\vartheta]\!]} = D^g\mathbb{1}_{[\![0,\vartheta]\!]}$ (see the second equality in (4.1)). The rest of the proof is similar to the case of D^r except that we now use the properties of the $\mathbb{F}$-predictable projection, rather than the $\mathbb{F}$-optional projection.

Finally, in the case where $\widehat{D}$ is $\mathbb{F}$-strongly predictable, from the decomposition $\widehat{D}=\widehat{D}^r+\widehat{D}^g$ and the fact that $\widehat{D}$ and $\widehat{D}^g$ belong to $\mathcal{P}(\mathbb{G})$, we deduce that $\widehat{D}^r$ belongs to $\mathcal{P}(\mathbb{G})$. Thus there exists an $\mathbb{F}$-predictable process D^r such that $\widehat{D}^r\mathbb{1}_{[\![0,\vartheta]\!]} = D^r\mathbb{1}_{[\![0,\vartheta]\!]}$. This implies that $\widehat{D}^r\mathbb{1}_{[\![0,\vartheta[\![} = D^r\mathbb{1}_{[\![0,\vartheta[\![}$ and, by taking the $\mathbb{F}$-optional projection, we deduce from similar arguments as before, that on the set $\{G>0\}$ the process D^r is increasing and càdlàg. □

Remark 4.4. Clearly if the process $\widehat{D}$ is $\mathbb{F}$-adapted then the equality $\widehat{D}^r = D^r$ holds everywhere and not only on $[\![0,\vartheta[\![$. We remark here that an $\mathbb{F}$-adapted driver $\widehat{D}$ can have certain practical interpretations. For example, one can take $\widehat{D}$ to be the hazard process, that is $\widehat{D} = \widetilde{\Gamma} = \widetilde{G}^{-1} \bullet A^o$, and this can be interpreted as a way to introduce ambiguity in the recovery and the default intensity (see, e.g., Fadina and Schmidt [21]).

The next result is an immediate consequence of Lemma 4.3 and equations (3.1) and (3.2). To alleviate the notation, we will frequently write $\widehat{F}^r_s(\cdot) = \widehat{F}^r_s(\cdot,\widehat{Z}_s,\widehat{U}_s)$, $\widehat{F}^r_g(\cdot) = \widehat{F}^g_s(\cdot,\widehat{Z}_s,\widehat{U}_s), F^r_s(\cdot) = F^r_s(\cdot,Z_s,U_s)$ and $F^g_s(\cdot) = F^g_s(\cdot,Z_s,U_s)$.

Lemma 4.5. *The following equalities are satisfied, for every $t \in \mathbb{R}_+$ on the event $\{t \le \tau\} \cap \{t < \vartheta\}$,*

$$\begin{aligned}\mathbb{E}\Big(\mathbb{1}_{\{\vartheta>t\}}\int_{]\!]t,\tau\wedge\vartheta]\!]}\widehat{F}^r_s(\widehat{Y}_s)\,d\widehat{D}^r_s\,\Big|\,\mathcal{F}_t\Big) &= \mathbb{E}\Big(\int_{]\!]t,\tau]\!]}F^r_s(Y_s)\,dD^r_s\,\Big|\,\mathcal{F}_t\Big)\\ &\quad+\mathbb{E}\Big(\int_{]\!]t,\tau]\!]}\big(F^r_s(R_s)-F^r_s(Y_s)\big)\Delta D^r_s dA^o_s\,\Big|\,\mathcal{F}_t\Big)\end{aligned}\tag{4.2}$$

and, on the event $\{t < \tau\} \cap \{t < \vartheta\}$,

$$\mathbb{E}\Big(\mathbb{1}_{\{\vartheta>t\}}\int_{[\![t,\tau[\![}\mathbb{1}_{\{\vartheta>s\}}\widehat{F}^g_s(\widehat{Y}_s)\,d\widehat{D}^g_{s+}\,\Big|\,\mathcal{F}_t\Big) = \mathbb{E}\Big(\int_{[\![t,\tau[\![}G_sF^g_s(Y_s)\,dD^g_{s+}\,\Big|\,\mathcal{F}_t\Big).$$

Proof. Using Lemma 4.3, the equalities $\widehat{Y}_\vartheta\mathbb{1}_{\{\vartheta\le\tau\}} = R_\vartheta\mathbb{1}_{\{\vartheta\le\tau\}}$ and

$$\mathbb{P}(\vartheta = s\,|\,\mathcal{F}_s) = \widetilde{G}_s - G_s = \Delta A^o_s$$

and noticing that $\{\vartheta \ge s\} \subset \{\vartheta > t\}$ for $s > t$, we obtain

$$\begin{aligned}&\mathbb{E}\Big(\mathbb{1}_{\{\vartheta>t\}}\int_{]\!]t,\tau]\!]}\mathbb{1}_{\{\vartheta\ge s\}}\widehat{F}^r_s(\widehat{Y}_s)\,d\widehat{D}^r_s\,\Big|\,\mathcal{F}_t\Big)\\ &=\mathbb{E}\Big(\int_{]\!]t,\tau]\!]}\mathbb{1}_{\{\vartheta>s\}}F^r_s(Y_s)\,dD^r_s\,\Big|\,\mathcal{F}_t\Big)+\mathbb{E}\Big(\int_{]\!]t,\tau]\!]}\mathbb{1}_{\{\vartheta=s\}}F^r_s(R_s)\,dD^r_s\,\Big|\,\mathcal{F}_t\Big)\\ &=\mathbb{E}\Big(\int_{]\!]t,\tau]\!]}G_sF^r_s(Y_s)\,dD^r_s\,\Big|\,\mathcal{F}_t\Big)+\mathbb{E}\Big(\int_{]\!]t,\tau]\!]}F^r_s(R_s)\Delta A^o_s\,dD^r_s\,\Big|\,\mathcal{F}_t\Big)\\ &=\mathbb{E}\Big(\int_{]\!]t,\tau]\!]}\widetilde{G}_sF^r_s(Y_s)\,dD^r_s\,\Big|\,\mathcal{F}_t\Big)+\mathbb{E}\Big(\int_{]\!]t,\tau]\!]}(F^r_s(Y_s)-F^r_s(R_s))\Delta D^r_s\,dA^o_s\,\Big|\,\mathcal{F}_t\Big).\end{aligned}$$

Similarly, again from Lemma 4.3, we have

$$\mathbb{E}\Big(\mathbb{1}_{\{\vartheta>t\}}\int_{[\![t,\tau[\![}\mathbb{1}_{\{\vartheta>s\}}\widehat{F}^g_s(\widehat{Y}_s)\,d\widehat{D}^g_{s+}\,\Big|\,\mathcal{F}_t\Big) = \mathbb{E}\Big(\int_{[\![t,\tau[\![}G_sF^g_s(Y_s)\,dD^g_{s+}\,\Big|\,\mathcal{F}_t\Big),$$

which gives the required result. □

To simplify further computations we define, for every $(y,z,u) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ and $t \ge 0$,

$$\ddot{F}^r_t(y,z,u) := F^r_t(y,z,u) + \big(F^r_t(R_t,z,u) - F^r_t(y,z,u)\big)G^{-1}_t\Delta A^o_t. \tag{4.3}$$

By combining Lemmas 3.4 and 4.5 with equality (3.2), we obtain the following result.

Lemma 4.6. *The process (Y,Z,U) satisfies on $[\![0,\tau]\!]$*

$$Y_t = G^{-1}_t\,\mathbb{E}\Big(X_\tau G_\tau + \int_{]\!]t,\tau]\!]}R_s\,dA^o_s - \int_{]\!]t,\tau]\!]}\widetilde{G}_s\ddot{F}^r_s\,dD^r_s - \int_{[\![t,\tau[\![}G_sF^g_s\,dD^g_{s+}\,\Big|\,\mathcal{F}_t\Big)$$

where we denote $\ddot{F}^r_s = \ddot{F}^r_s(Y_s,Z_s,U_s)$ and $F^g_s = F^g_s(Y_s,Z_s,U_s)$.

Proof. For any fixed $\mathbb{F}$-stopping time τ, we denote by $K(\tau)$ the $\mathbb{F}$-martingale given by

$$K_t(\tau) := \mathbb{E}\big(X_\tau G_\tau + (R \bullet A^o)_\tau - (\widetilde{G}\ddot{F}^r \bullet D^r)_\tau - (GF^g \star D^g_+)_\tau \,|\, \mathcal{F}_t\big). \tag{4.4}$$

Then the $\mathbb{F}$-optional process Y has the following representation on $[\![0,\tau]\!]$

$$Y_t = G_t^{-1}\left(K_t(\tau) - (R \bullet A^o)_t + (\widetilde{G}\ddot{F}^r \bullet D^r)_t + (GF^g \star D^g_+)_t\right) \tag{4.5}$$

with $Y_\tau = X_\tau$ and thus the asserted equality holds. □

For brevity, we set $C := \widetilde{G}\ddot{F}^r \bullet D^r + GF^g \star D^g_+$ and we note that equality (4.5) can be rewritten as follows

$$Y = G^{-1}\big(K(\tau) - R \bullet A^o + C\big). \tag{4.6}$$

In addition, we define $\widetilde{m} := m - \widetilde{G}^{-1} \bullet [m,m]$ and

$$\widetilde{K}(\tau) := K(\tau) - \widetilde{G}^{-1} \bullet [K(\tau), m].$$

To express the dynamics of the process Y in terms of $\widetilde{m}$ and $\widetilde{K}(\tau)$, we will use the following immediate consequence of Lemma 7.3 from the appendix.

Lemma 4.7. *If $C = C^r + C^g$ is a làglàd process of finite variation and the process Y is given by $Y = G^{-1}(K - R \bullet A^o + C)$ for some $\mathbb{F}$-martingale K, then*

$$Y = Y_0 + \widetilde{G}^{-1} \bullet C^r + G^{-1} \star C^g_+ - (R - Y) \bullet \widetilde{\Gamma} - Y_- G_-^{-1} \bullet \widetilde{m} + G_-^{-1} \bullet \widetilde{K}$$

where $\widetilde{K} := K - \widetilde{G}^{-1} \bullet [K,m]$.

By applying Lemma 4.7 to equality (4.6) and using Lemma 4.3, we obtain the following corollary.

Corollary 4.8. *The process $D = D^r + D^g$ is a làglàd process of finite variation and*

$$Y = Y_0 + \ddot{F}^r \bullet D^r + F^g \star D^g - (R - Y) \bullet \widetilde{\Gamma} - Y_- G_-^{-1} \bullet \widetilde{m} + G_-^{-1} \bullet \widetilde{K}(\tau).$$

Assumption 3.1(vi) yields the existence $\mathbb{F}$-predictable processes $\psi^{Y,Z}$ and ν such that

$$K(\tau) = \psi^{Y,Z} \bullet M, \quad m = \nu \bullet M. \tag{4.7}$$

The next proposition is an immediate consequence of Corollary 4.8 combined with (4.7). As before, we write $F^r_s(\cdot) := F^r_s(\cdot, Z_s, U_s)$ and $F^g_s(\cdot) := F^g_s(\cdot, Z_s, U_s)$ and we give an explicit representation for the $\mathbb{F}$ BSDE associated with the $\mathbb{G}$ BSDE (3.2).

It is worth noting that Proposition 4.9 extends several results from the existing literature where the method of reduction was studied in a particular framework and under additional assumptions, such as the immersion hypothesis or the simplifying conditions **(A)** or **(C)**.

Proposition 4.9. *If the triplet $(\widehat{Y}, \widehat{Z}, \widehat{U})$ is a solution to the $\mathbb{G}$ BSDE* (3.2)*, then the triplet (Y, Z, U) where $U = \widehat{U}$ satisfies on $[\![0,\tau]\!]$*

$$\begin{aligned} Y_t = X_\tau &- \int_{]\!]t,\tau]\!]} F^r_s(Y_s)\, dD^r_s - \int_{[\![t,\tau[\![} F^g_s(Y_s)\, dD^g_{s+} - \int_{]\!]t,\tau]\!]} z_s\, d\widetilde{M}_s \\ &+ \int_{]\!]t,\tau]\!]} \big[R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s))\Delta D^r_s\big]\, d\widetilde{\Gamma}_s \end{aligned}$$

where the process z is given by $z_t := G_{t-}^{-1}\big(\psi_t^{Y,Z} - Y_{t-}\nu_t\big)$.

In the above representation of the $\mathbb{F}$ BSDE associated with the $\mathbb{G}$ BSDE, we can clearly identify the reduced generators F^r and F^g and the form of the adjustment, which is integrated with respect to the hazard process $\widetilde{\Gamma} = \widetilde{G}^{-1} \bullet A^o$ of a random time ϑ.

4.2 Construction of a solution to $\mathbb{G}$ BSDE

In this section, we no longer postulate that a solution to the BSDE (3.2) exists, which means that Assumption 4.1 is relaxed. Our goal is to show that a solution to the $\mathbb{F}$ BSDE (4.13) can be expanded to obtain a solution to the BSDE (3.2) if equation (4.14) has an $\mathbb{F}$-optional solution U. We stress that equations (4.13) and (4.14) are coupled, in the sense that they need to be solved jointly in order to construct a solution to the BSDE (3.2). Obviously, the issues of existence and uniqueness of a solution (Y,Z,U) to equations (4.13) and (4.14) need to be studied under additional assumptions on the generator and all other inputs to the BSDE (3.2). In the next result, we denote by $\widehat{U}$ an arbitrary prescribed $\mathbb{F}$-optional process and we do not use Lemma 4.5.

Lemma 4.10. *For a given process $\widehat{U} \in \mathcal{O}(\mathbb{F})$, let (Y,Z) be an $\mathbb{R} \times \mathbb{R}^d$-valued, $\mathbb{F}$-adapted solution to the BSDE on $[\![0,\tau]\!]$*

$$\begin{aligned} Y_t = X_\tau - \int_{]\!]t,\tau]\!]} F^r_s(Y_s,Z_s,\widehat{U}_s)\,dD^r_s - \int_{[\![t,\tau[\![} F^g_s(Y_s,Z_s,\widehat{U}_s)\,dD^g_{s+} - \int_{]\!]t,\tau]\!]} Z_s\,d\widetilde{M}_s \\ + \int_{]\!]t,\tau]\!]} \big[R_s - Y_s - \big(F^r_s(R_s,Z_s,\widehat{U}_s) - F^r_s(Y_s,Z_s,\widehat{U}_s)\big)\Delta D^r_s\big]\,d\widetilde{\Gamma}_s \end{aligned} \tag{4.8}$$

and let the $\mathbb{G}$-adapted process $\widehat{Y}$ be given by

$$\widehat{Y} := Y_0 + \mathbb{1}_{]\!]0,\vartheta[\![} \bullet Y^r + \mathbb{1}_{]\!]0,\vartheta[\![} \star Y^g + (R_\vartheta - Y_{\vartheta-})\mathbb{1}_{[\![\vartheta,\infty[\![}\mathbb{1}_{\{\tau \geq \vartheta\}}. \tag{4.9}$$

Then $(\widehat{Y},\widehat{Z}) := (\widehat{Y}, Z^\vartheta)$ is a $\mathbb{G}$-adapted solution to the BSDE, on $[\![0,\tau\wedge\vartheta]\!]$,

$$\begin{aligned} \widehat{Y}_t = X_{\tau\wedge\vartheta} &- \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{F}^r_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)\,dD^r_s \\ &- \int_{[\![t,\tau\wedge\vartheta[\![} \widehat{F}^g_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)\,dD^g_{s+} - \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{Z}_s\,d\widetilde{M}_s \\ &- \int_{]\!]t,\tau\wedge\vartheta]\!]} \big[R_s - Y_s - \big(F^r_s(R_s,\widehat{Z}_s,\widehat{U}_s) - F^r_s(Y_s,\widehat{Z}_s,\widehat{U}_s)\big)\Delta D^r_s\big] dm^{\mathbb{G}}_s. \end{aligned} \tag{4.10}$$

Proof. Let us write $C^r := \ddot{F}^r \bullet D^r$ and $C^g := F^g \star D^g_+$. From (4.8) and (4.9), we can deduce that the following equalities hold

$$\widehat{Y}_{\tau\wedge\vartheta} = X_\tau \mathbb{1}_{\{\tau<\vartheta\}} + R_\vartheta \mathbb{1}_{\{\tau\geq\vartheta\}}$$

and

$$\begin{aligned} \widehat{Y} = Y_0 + Z\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{]\!]0,\vartheta[\![} \bullet \widetilde{M}^\vartheta - (R-Y)\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{]\!]0,\vartheta[\![} \bullet \widetilde{\Gamma} \\ + \mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{]\!]0,\vartheta[\![} \bullet C^r + \mathbb{1}_{[\![0,\tau[\![}\mathbb{1}_{[\![0,\vartheta[\![} \star C^g + (R - Y_-)\mathbb{1}_{]\!]0,\tau]\!]} \bullet A. \end{aligned} \tag{4.11}$$

Using again (4.8), we obtain

$$(Y - Y_-)\mathbb{1}_{]\!]0,\tau]\!]} \bullet A = Z\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{[\![\vartheta]\!]} \bullet \widetilde{M}^\vartheta - (R-Y)\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{[\![\vartheta]\!]} \bullet \widetilde{\Gamma} + \mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{[\![\vartheta]\!]} \bullet C^r.$$

Thus, by replacing $(R - Y_-)$ by $(R - Y)$ in the last term of (4.11) and using the equality $m^{\mathbb{G}} = A - \mathbb{1}_{]\!]0,\vartheta]\!]} \bullet \widetilde{\Gamma}$ (see (2.2)), we see that $\widehat{Y}$ is equal to

$$Y_0 + Z\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet \widetilde{M} + (R-y)\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet m^{\mathbb{G}} + \mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet C^r + \mathbb{1}_{[\![0,\tau\wedge\vartheta[\![} \star C^g_+.$$

To establish (4.10), it now remains to show that

$$\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet C^r = \widehat{F}^r(\widehat{Y})\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet D^r - \big(F^r(R) - F^r(Y)\big)\Delta D^r \mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet m^{\mathbb{G}} \tag{4.12}$$

where the variables Z and $\widehat{U}$ are suppressed.

To this end, by using the fact that the equalities $F^r = \widehat{F}^r$ and $Y = \widehat{Y}$ hold on $]\!]0,\vartheta]\!]$ and $\widehat{Y}\mathbb{1}_{[\![\vartheta]\!]} = R\mathbb{1}_{[\![\vartheta]\!]}$, we deduce from (4.3) that

$$\begin{aligned} I_1 &:= \ddot{F}^r(Y,\widehat{U})\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{]\!]0,\vartheta[\![} \bullet D^r = \widehat{F}^r(\widehat{Y})\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet D^r - F^r(R)\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{[\![\vartheta]\!]} \bullet D^r \\ &\quad + \Delta\widetilde{\Gamma}\big(F^r(R) - F^r(Y)\big)\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{]\!]0,\vartheta[\![} \bullet D^r \end{aligned}$$

and

$$I_2 := \ddot{F}^r(Y)\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{[\![\vartheta]\!]} \bullet D^r = \big[F^r(Y) + \Delta\widetilde{\Gamma}\big(F^r(R) - F^r(Y)\big)\big]\mathbb{1}_{]\!]0,\tau]\!]}\mathbb{1}_{[\![\vartheta]\!]} \bullet D^r.$$

Consequently, since $m^{\mathbb{G}} = A - \mathbb{1}_{]\!]0,\vartheta]\!]} \bullet \widetilde{\Gamma}$ is a process of finite variation stopped at ϑ and $\mathbb{1}_{[\![\vartheta]\!]} = \Delta A$, we obtain

$$\begin{aligned} &\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet C^r = I_1 + I_2 \\ &= \widehat{F}^r(\widehat{Y})\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet D^r - \mathbb{1}_{]\!]0,\tau]\!]}\big(F^r(R) - F^r(Y)\big)\big(\Delta A - \Delta\widetilde{\Gamma}\mathbb{1}_{]\!]0,\vartheta]\!]}\big) \bullet D^r \\ &= \widehat{F}^r(\widehat{Y})\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet D^r - \mathbb{1}_{]\!]0,\tau]\!]}\big(F^r(R) - F^r(Y)\big)\Delta D^r \bullet \big(A - \Delta\widetilde{\Gamma}\mathbb{1}_{]\!]0,\vartheta]\!]}\big) \\ &= \widehat{F}^r(\widehat{Y})\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet D^r - \big(F^r(R) - F^r(Y)\big)\Delta D^r\,\mathbb{1}_{]\!]0,\tau\wedge\vartheta]\!]} \bullet m^{\mathbb{G}}, \end{aligned}$$

which shows that equality (4.12) is valid. □

Remark 4.11. If R belongs to the class $\mathcal{P}(\mathbb{F})$, then one can modify the above proof by noticing that $Q := A^o - A^p$ is a finite variation $\mathbb{F}$-martingale and

$$R\widetilde{G}^{-1} \bullet (A^p - A^o) = -R\big(G_-^{-1} \bullet Q + \widetilde{G}^{-1} \bullet Q - G_-^{-1} \bullet Q\big) = -RG_-^{-1} \bullet \widetilde{Q},$$

which, when stopped at ϑ, is a $\mathbb{G}$-martingale. Then, in view of the predictable representation property of M, this term will contribute to the $\mathbb{G}$-martingale term $\widetilde{M}$ in (4.8).

The following proposition is a consequence of Lemma 4.10. It shows that a solution to the $\mathbb{G}$ BSDE can be constructed by first solving the constrained $\mathbb{F}$ BSDE (4.13)–(4.14). Recall that we denote $F_s^r(\cdot) := F_s^r(\cdot, Z_s, U_s)$, $F_s^g(\cdot) := F_s^g(\cdot, Z_s, U_s)$, $\widehat{F}_s^r(\cdot) := \widehat{F}_s^r(\cdot, \widehat{Z}_s, \widehat{U}_s)$ and $\widehat{F}_g^r(\cdot) := \widehat{F}_s^g(\cdot, \widehat{Z}_s, \widehat{U}_s)$.

Proposition 4.12. *Assume that (Y, Z, U) is a solution to the constrained BSDE on $[\![0,\tau]\!]$*

$$\begin{aligned} Y_t = X_\tau - \int_{]\!]t,\tau]\!]} F_s^r(Y_s)\,dD_s^r - \int_{[\![t,\tau[\![} F_s^g(Y_s)\,dD_{s+}^g - \int_{]\!]t,\tau]\!]} Z_s\,d\widetilde{M}_s \\ + \int_{]\!]t,\tau]\!]} \big[R_s - Y_s - \big(F_s^r(R_s) - F_s^r(Y_s)\big)\Delta D_s^r\big]\,d\widetilde{\Gamma}_s \end{aligned} \tag{4.13}$$

where the $\mathbb{F}$-optional process U satisfies the following equality, for all $t \in \mathbb{R}_+$,

$$\int_{]\!]0,t]\!]} U_s\,dm_s^{\mathbb{G}} = \int_{]\!]0,t]\!]} \big[R_s - Y_s - \big(F_s^r(R_s) - F_s^r(Y_s)\big)\Delta D_s^r\big]\,dm_s^{\mathbb{G}}. \tag{4.14}$$

Then the triplet $(\widehat{Y}, \widehat{Z}, \widehat{U}) := (\widehat{Y}, Z^\vartheta, U)$ where the process $\widehat{Y}$ is given by

$$\widehat{Y} := Y_0 + \mathbb{1}_{[\![0,\vartheta[\![} \bullet Y^r + \mathbb{1}_{[\![0,\vartheta[\![} \star Y^g + (R_\vartheta - Y_{\vartheta-})\mathbb{1}_{[\![\vartheta,\infty[\![}\mathbb{1}_{\{\tau\geq\vartheta\}}$$

is a solution to the BSDE (3.2) on $[\![0,\vartheta\wedge\tau]\!]$, that is,

$$\begin{aligned} \widehat{Y}_t = \widehat{X}_{\tau\wedge\vartheta} - \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{F}_s^r(\widehat{Y}_s)\,dD_s^r - \int_{[\![t,\tau\wedge\vartheta[\![} \widehat{F}_s^g(\widehat{Y}_s)\,dD_{s+}^g \\ - \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{Z}_s\,d\widetilde{M}_s - \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{U}_s\,dm_s^{\mathbb{G}}. \end{aligned}$$

In the next step, we will examine the existence of a solution to the constrained BSDE (4.13)–(4.14). Specifically, we seek a triplet (Y, Z, U) of processes that satisfy, for all $t \in [0, \tau]$,

$$\begin{aligned} Y_t = X_\tau - \int_{]\!]t,\tau]\!]} F^r_s(Y_s)\, dD^r_s - \int_{[\![t,\tau[\![} F^g_s(Y_s)\, dD^g_{s+} - \int_{]\!]t,\tau]\!]} Z_s\, dM_s \\ + \int_{]\!]t,\tau]\!]} Z_s\, G_s^{-1}\nu_s\, d[M,M]_s + \int_{]\!]t,\tau]\!]} \big[R_s - Y_s - \big(F^r_s(R_s) - F^r_s(Y_s)\big)\Delta D^r_s\big]\, d\widetilde{\Gamma}_s \end{aligned} \tag{4.15}$$

and, for all $t \in \mathbb{R}_+$ (notice that (4.16) is manifestly stronger than (4.14))

$$U_t = R_t - Y_t - \big(F^r_t(R_t) - F^r_t(Y_t)\big)\Delta D^r_t. \tag{4.16}$$

To examine the existence of a solution to the coupled equations (4.15)–(4.16), we introduce the transformed equations (4.17)–(4.18). Our goal is to remove the quadratic variation term $G^{-1} \bullet [m, M] = G^{-1}\nu \bullet [M, M]$ from (4.15) and place ν inside the generators F^r and F^g, which are assumed to be bounded. In that way, we avoid the need to check the appropriate growth conditions when applying the existing results on the well-posedness of BSDEs.

We define the linear transformation $\bar{Y} := GY$, $\bar{Z} := G_- Z + G^{-1}\bar{Y}\nu$ and $\bar{U} := U$. Then we obtain the transformed generators

$$\bar{F}^r_s(y, z, u) := \widetilde{G}_s F^r_s(G_s^{-1}y, G_{s-}^{-1}(z - G_s^{-1}y\nu_s), u)$$

and

$$\bar{F}^g_s(y, z, u) := G_s F^g_s(G_s^{-1}y, G_{s-}^{-1}(z - G_s^{-1}y\nu_s), u)$$

and we denote $\bar{F}^r_s(\cdot) := \bar{F}^r_s(\cdot, \bar{Z}_s, \bar{U}_s)$ and $\bar{F}^g_s(\cdot) := \bar{F}^g_s(\cdot, \bar{Z}_s, \bar{U}_s)$. Observe that if a solution $(Y, Z, U) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{O}(\mathbb{F})$ to (4.15)–(4.16) exists, then $(\bar{Y}, \bar{Z}, \bar{U}) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{O}(\mathbb{F})$ satisfies the following coupled equations, for all $t \in [0, \tau]$,

$$\begin{aligned} \bar{Y}_t = G_\tau X_\tau - \int_{]\!]t,\tau]\!]} \bar{F}^r_s(\bar{Y}_s)\, dD^r_s - \int_{[\![t,\tau[\![} \bar{F}^g_s(\bar{Y}_s)\, dD^g_{s+} \\ - \int_{]\!]t,\tau]\!]} \bar{Z}_s\, dM_s + \int_{]\!]t,\tau]\!]} \big[\widetilde{G}_s R_s - \Delta\bar{F}^r_s \Delta D^r_s\big]\, d\widetilde{\Gamma}_s \end{aligned} \tag{4.17}$$

and, for all $t \in \mathbb{R}_+$

$$\bar{U}_t = R_t - \bar{Y}_s G_t^{-1} - \widetilde{G}_t^{-1}\Delta\bar{F}^r_t \Delta D^r_t \tag{4.18}$$

where we denote $\Delta\bar{F}^r_s := \bar{F}^r_s(G_s R_s) - \bar{F}^r_s(\bar{Y}_s)$. In the reverse, a solution (Y, Z, U) to the coupled equations (4.15)–(4.16) can be obtained from a solution $(\bar{Y}, \bar{Z}, \bar{U})$ to the coupled equations (4.17)–(4.18) by setting $Y := G^{-1}\bar{Y}$, $Z := G_-^{-1}(\bar{Z} - G^{-1}\bar{Y}\nu)$ and $U := \bar{U}$.

Observe that BSDEs (4.15) and (4.17) have the làglàd driver $D = (D^r, D^g)$, which may share common jumps with the martingale M. To the best of our knowledge, there is a gap in the literature on BSDEs of this form and thus we develop in Section 6.1 a jump-adapted methodology to solve such BSDEs under specific assumptions.

4.3 Relation to the existing literature on BSDEs for credit risk

If a random time ϑ is an $\mathbb{F}$-*pseudo-stopping time* (see Nikeghbali and Yor [42] and Aksamit and Li [5]), then $m = 1$ and thus $\nu = 0$ in (4.7) so that we can deal directly with (4.15). Also, under condition **(C)**, it is possible to eliminate the term $\widetilde{G}^{-1} \bullet [m, M]$ through a change of measure when a random time ϑ is an *invariance time* with respect to $\mathbb{F}$. Recall that the notion of the invariance time was put forward and analyzed in Crépey and Song [13] and its introduction was directly motivated by a study in Crépey and Song

[12] of the reduced BSDE for the CVA (Credit Valuation Adjustment) associated with the counterparty credit risk.

We now take the opportunity to elaborate on the differences between our results and those of Crépey and Song [12]. Technically speaking, in [12] the authors work within a predictable framework, meaning that the random time ϑ exhibits a $\mathbb{G}$-predictable compensator, which is assumed to be absolutely continuous with respect to the Lebesgue measure. Notice that in the context of progressive enlargement, the $\mathbb{G}$-predictable compensator is given by the process $G_-^{-1} \bullet A^p$ stopped at ϑ. In contrast, we work here under an optional setup and make use of the $\mathbb{G}$ martingale $m^{\mathbb{G}}$, and thus the hazard process is given by $\widetilde{\Gamma} = \widetilde{G}^{-1} \bullet A^o$, which is an $\mathbb{F}$-optional process and a $\mathbb{G}$-optional process when stopped at ϑ. Consequently, results obtained in [12] hinge on the classical Doob-Meyer decomposition $G = n - A^p$, rather than the optional decomposition of $G = m - A^o$ (see Definition 2.1).

This key difference manifest itself in two ways. Firstly, as already explained in Section 3.2, we aim to construct models in which financial shocks or jumps in the value of the stock are allowed and the default time ϑ may happen concurrently with these shocks with a positive probability. Therefore, if the recovery process R is $\mathbb{F}$-optional and hence R_ϑ is $\mathcal{G}_\vartheta$-measurable (e.g., a fixed fraction of the pay-off of a standard call option in a jump diffusion model), then in order to make use of the $\mathbb{G}$-predictable compensator, the methodology developed in [12] would require to compute the quantity $\mathbb{E}[R_\vartheta|\mathcal{G}_{\vartheta-}]$ or $\widehat{\xi}$ in the reduced BSDE (4.4) studied in [12], which may prove unfeasible. This should be contrasted with our approach where it is not needed to compute these quantities and all model inputs $(X, R, \widetilde{\Gamma}, m)$ can be explicitly chosen.

Secondly, the $\mathbb{G}$ BSDE (4.3) in [12] is stopped strictly before ϑ (specifically, at $\vartheta-$, see Lemma 2.5) rather than at ϑ and the reduced BSDE (4.4) in [12] was obtained under condition **(C.2)** stating that there exists an equivalent probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ such that any $\mathbb{F}$-local martingales under $\mathbb{Q}$ stopped at $\vartheta-$ are $\mathbb{G}$-local martingales under $\mathbb{Q}$. As a consequence, the change of a probability measure allows one to eliminate the term $G^{-1} \bullet [n, M]$, which is known to coincide with $\widetilde{G}^{-1} \bullet [m, M]$ under condition **(C)**. To be more specific, condition **(C.2)** implies that ϑ is an invariance time or, in other words, there exists an equivalent probability measure $\mathbb{Q}$ such that ϑ is an $\mathbb{F}$-*pseudo-predictable-stopping time* (see Definition 2.1 and Proposition 2.2 in Jeanblanc and Li [32]). Recall that $m = 1$ in the case of an $\mathbb{F}$-pseudo-stopping time, whereas $n = 1$ if ϑ is assumed to be an $\mathbb{F}$-pseudo-predictable-stopping time. For this reason, the term $G^{-1} \bullet [n, M]$ does not appear in the reduced BSDE in [12], which is formulated under an equivalent probability measure $\mathbb{Q}$. For an example of an $\mathbb{F}$-pseudo-predictable-stopping time, which is not an $\mathbb{F}$-pseudo-stopping time, we refer to Example 3.13 in [32].

Finally, we stress that the two approaches are indeed identical if condition **(C)** is postulated. Therefore, in principle, one could attempt to either impose a condition similar to condition **(C.2)** or mimic the approach developed in [13] by studying a new class of random times for which we could stop at ϑ and the drift term $\widetilde{G}^{-1} \bullet [m, M]$ could be removed through a change of a probability measure. Put another way, one could introduce a family of random times for which there exists an equivalent probability measure under which they become $\mathbb{F}$-pseudo-stopping times. However, we believe that the elimination of the term $\widetilde{G}^{-1} \bullet [m, M]$ is not the most important issue (as we show in Example 6.7) since it is not hard to give non-trivial examples where the well-posedness of the reduced BSDE can be obtained without annihilating that term. Therefore, we find it reasonable either to assume that the generator is bounded and apply the previous transformation method or to postulate that ϑ is an $\mathbb{F}$-pseudo-stopping time under $\mathbb{P}$.

4.4 Solution to $\mathbb{G}$ BSDE in the Brownian case

To illustrate our approach from Sections 4.1 and 4.2, we first use results from Essaky et al. [20] to show that in the case of the Brownian filtration, if $F^g = 0$ and U does not appear in the right-hand side of (4.16) or (4.18), then a unique solution $(\bar{Y}, \bar{Z}, \bar{U})$ to (4.17)–(4.18) exists and thus a unique solution (Y, Z, U) to (4.15)–(4.16) exists as well. Specifically, we now take $M = W$ to be a d-dimensional Wiener process in its natural filtration $\mathbb{F}$ and we consider below the constrained $\mathbb{F}$ BSDE (4.15)–(4.16) with $F^g = 0$. Our goal is to demonstrate that, under some natural assumptions, these equations have a unique solution $(Y, Z, U) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{O}(\mathbb{F})$.

To simplify the notation, we write D and F instead of $(D^r, 0)$ and $(F^r, 0)$, respectively, and we consider the situation where $D = (D^1, D^2) = (\langle W \rangle, \widetilde{\Gamma})$ and $F = (F^1, F^2) = (F^1(y, z, u), F^2(y))$ for $(y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$. Then the BSDE (4.17) becomes

$$\begin{aligned} \bar{Y}_t = G_\tau X_\tau - \int_{]\!]t,\tau]\!]} \bar{F}^1_s(\bar{Y}_s)\, d\langle W \rangle_s - \int_{]\!]t,\tau]\!]} \bar{F}^2_s(\bar{Y}_s)\, d\widetilde{\Gamma}_s - \int_{]\!]t,\tau]\!]} \bar{Z}_s \, dW_s \\ + \int_{]\!]t,\tau]\!]} \big[\widetilde{G}_s R_s - \big(\bar{F}^2_s(G_s R_s) - \bar{F}^2_s(\bar{Y}_s)\big)\Delta\widetilde{\Gamma}_s\big]\, d\widetilde{\Gamma}_s \end{aligned} \tag{4.19}$$

where $\bar{F}^1_s(\bar{Y}_s) := \bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)$ and equation (4.18) has an explicit solution given by

$$\bar{U} = (R - \bar{Y} G^{-1}) - (\bar{F}^2(GR) - \bar{F}^2(\bar{Y}))\widetilde{G}^{-1}\Delta\widetilde{\Gamma}. \tag{4.20}$$

We point out that, in the above, our choice of D^2 was somewhat arbitrary and we decided to set $D^2 = \widetilde{\Gamma}$ for simplicity of presentation.

Proposition 4.13. *Assume that:*

(i) for every $t \in \mathbb{R}_+$, the map $F^1_t : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous;

(ii) for every $t \in \mathbb{R}_+$, the map $F^2_t : \mathbb{R} \to \mathbb{R}$ is bounded, Lipschitz continuous and decreasing;

(iii) the dual $\mathbb{F}$-optional projection A^o has a finite number of discontinuities.

Then the BSDE (4.19) *has a solution $(\bar{Y}, \bar{Z})$ and a solution (Y, Z) to* (4.15) *is obtained by setting $Y := G^{-1}\bar{Y}$ and $Z := G^{-1}(\bar{Z} - G^{-1}\bar{Y}\nu)$.*

Proof. To establish the existence of a solution $(\bar{Y}, \bar{Z})$ to (4.19), we will apply Theorem 2.1 in Essaky et al. [20] to the data (P, R, ϑ, F). We observe that the BSDE (4.19) is a special case of equation (2.1) in [20], which has the following form

$$\begin{aligned} \bar{Y}_t = G_\tau X_\tau + \int_{]\!]t,\tau]\!]} f(s, \bar{Y}_s, \bar{Z}_s)\, ds + \int_{]\!]t,\tau]\!]} g(s, \bar{Y}_s)\, d\bar{A}_s \\ + \sum_{t < s \le \tau} h(s, \bar{Y}_{s-}, \bar{Y}_s) - \int_{]\!]t,\tau]\!]} \bar{Z}_s \, dW_s. \end{aligned} \tag{4.21}$$

Indeed, (4.19) can be recovered from (4.21) if we set $\bar{A} := \widetilde{\Gamma}^c$ where $\widetilde{\Gamma}^c$ is the continuous part of $\widetilde{\Gamma}$ and define the mappings f, g and h as follows

$$\begin{aligned} &f(s, \bar{Y}_s, \bar{Z}_s) := -\bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s), \quad g(s, \bar{Y}_s) := \widetilde{G}_s R_s - \bar{F}^2_s(\bar{Y}_s), \\ &h(s, \bar{Y}_{s-}, \bar{Y}_s) := \big(\widetilde{G}_s R_s - F^2_s(\bar{Y}_s)\big)\Delta\widetilde{\Gamma}_s - \big(\bar{F}^2_s(G_s R_s) - \bar{F}^2_s(\bar{Y}_s)\big)(\Delta\widetilde{\Gamma}_s)^2 \end{aligned}$$

where $\bar{U}$ is given by (4.20) and the mapping h is in fact independent of the variable $\bar{Y}_-$.

To apply Theorem 2.1 from Essaky et al. [20], it suffices to check that Assumptions (A.1)–(A.4) on page 2151 of [20] are satisfied. To check Assumption (A.1), we note that the mapping F^1 is assumed to be continuous and, since it is also bounded, there exists a constant $C > 0$ such that $|F^1| \le C \le C(1 + |z|)$, which shows that the linear growth condition is satisfied. Next, from the fact that $|\Delta\widetilde{\Gamma}| \leq 1$, we deduce that $|g|$ is bounded and thus Assumption (A.2) holds as well. Finally, h is bounded and continuous and, since the process $\widetilde{\Gamma}$ is assumed to have a finite number of jumps, it is enough to check condition (c) in Assumption (A.3).

To this purpose, we observe that $0 \leq \Delta\widetilde{\Gamma} \leq 1$ and thus the mapping

$$y \mapsto y + h(s,y) = y - \Delta\widetilde{\Gamma}_s \bar{F}_s^2(y)\big(1 - \Delta\widetilde{\Gamma}_s\big) + \Delta\widetilde{\Gamma}_s\big(\widetilde{G}_s R_s - \Delta\widetilde{\Gamma}_s \bar{F}_s^2(G_s R_s)\big)$$

is nondecreasing and continuous. Finally, the Mokobodski condition postulated in (A.4) is trivially satisfied as we deal here with the BSDE with no reflecting boundaries. Thus, by applying Theorem 2.1 in [20] with T replaced by $\tau \in \mathcal{T}$, we obtain the existence of a maximal solution $(\bar{Y}, \bar{Z})$ to (4.19). □

Example 4.14. Let us consider a special case where ϑ is an $\mathbb{F}$-pseudo-stopping time (see Nikeghbali and Yor [42]) and thus the process ν in (4.7) vanishes. We can suppose that the mapping $z \mapsto F_s^1(y,z,u)$ has a linear (quadratic) growth and thus, since in the case of a Brownian filtration the equality $\widetilde{G} = G_-$ holds, we obtain

$$|\bar{F}_s^1(y,z)| = \widetilde{G}_s\big|F_s^1(G_s^{-1}y, G_{s-}^{-1}(z - G_s^{-1}y\nu_s), u)\big| \leq G_{s-}(1 + G_{s-}^{-1}|z|) \leq (1+|z|),$$

which shows that the boundedness of F^1 postulated in Proposition 4.13 can be relaxed.

5 Solution to a generalized reflected BSDE

Our goal in this section is to study the properties of solutions to the $\mathbb{G}$ RBSDE with a random time horizon ϑ.

5.1 Reduction of a solution to $\mathbb{G}$ RBSDE

As in Section 4.1, in order to show that (3.3) has a solution, we will first reduce a solution to the $\mathbb{G}$ RBSDE (3.3) to a solution of an associated RBSDE in filtration $\mathbb{F}$. Subsequently, we show how a solution to the reduced $\mathbb{F}$ RBSDE, which can be shown to exist under suitable assumptions, can be employed to construct a solution to the $\mathbb{G}$ RBSDE (3.3). Again, we first work under the following temporary postulate, which will be relaxed in Section 5.2.

Assumption 5.1. A solution $(\widehat{Y}, \widehat{Z}, \widehat{U}, \widehat{L})$ to the $\mathbb{G}$ RBSDE (3.3) exists.

Following the approach developed for the non-reflected case, we decompose $\widehat{Y}$ into the pre-default and post-default components

$$\begin{aligned}\widehat{Y}_t\mathbb{1}_{\{\vartheta>t\}} + \widehat{Y}_t\mathbb{1}_{\{\vartheta\leq t\}} &= \widehat{Y}_t\mathbb{1}_{\{\vartheta>t\}} + \widehat{X}_{\vartheta\wedge\tau}\mathbb{1}_{\{\vartheta\leq t\}} = Y_t\mathbb{1}_{\{\vartheta>t\}} + \widehat{X}_\vartheta\mathbb{1}_{\{\vartheta\leq t\}}\\ &= G_t^{-1}\mathbb{E}\big(\widehat{Y}_t\mathbb{1}_{\{\vartheta>t\}}\,\big|\,\mathcal{F}_t\big)\mathbb{1}_{\{\vartheta>t\}} + R_\vartheta\mathbb{1}_{\{\vartheta\leq t\}}.\end{aligned}$$

To compute the component $G_t^{-1}\mathbb{E}\big(\widehat{Y}_t\mathbb{1}_{\{\vartheta>t\}}\,|\,\mathcal{F}_t\big)$ we proceed similarly to the non-reflected case. The new feature here is the use of Lemma 4.3 in order to obtain a reduction of the $\mathbb{G}$-strongly predictable, increasing process $\widehat{L}$. Computations in this section are similar to those in Section 4.1, except for the presence of the reflection process and thus in the following we will focus on new elements. To proceed, similarly to (4.4), we set

$$\begin{aligned}K_t(\tau) := \mathbb{E}\big(&X_\tau G_\tau + (\widetilde{G}\ddot{F}^r \bullet D)_\tau + (GF^g \star D_+^g)_\tau\,|\,\mathcal{F}_t\big)\\ &+ \mathbb{E}\big((R+L)\bullet A^o)_\tau + L_\tau G_\tau\,|\,\mathcal{F}_t\big)\end{aligned} \tag{5.1}$$

where $\ddot{F}^r$ is given by (4.3). From Assumption 3.1, we deduce the existence of $\mathbb{F}$-predictable processes $\psi^{Y,Z,L}$ and ν such that $K(\tau) = \psi^{Y,Z,L}\bullet M$ and $m = \nu\bullet M$.

Proposition 5.1. *The process $Y = {}^o(\widehat{Y}\mathbb{1}_{[\![0,\vartheta[\![})G^{-1}$ satisfies the $\mathbb{F}$ RBSDE, on $[\![0,\tau]\!]$,*

$$\begin{aligned} Y_t = X_\tau - \int_{]\!]t,\tau]\!]} F^r_s(Y_s)\,dD^r_s - \int_{[\![t,\tau[\![} F^g_s(Y_s)\,dD^g_{s+} - \int_{]\!]t,\tau]\!]} z_s\,d\widetilde{M}_s \\ + \int_{]\!]t,\tau]\!]} \big[R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s))\Delta D^r_s\big]\,d\widetilde{\Gamma}_s - (L_\tau - L_t) \end{aligned}$$

where $Y \geq X$, $U = \widehat{U}$, the process Z is the $\mathbb{F}$-reduction of the process $\widehat{Z}$ given by (4.1)*, $z_t := G^{-1}_{t-}\big(\psi^{Y,Z,L}_t - Y_{t-}\nu_t\big)$ is an $\mathbb{F}$-predictable process and L is an $\mathbb{F}$-strongly predictable, increasing process such that $L = \widehat{L}$ on $[\![0,\vartheta]\!]$ and the Skorokhod conditions are satisfied, that is, $(\mathbb{1}_{\{Y_- \neq X_-\}} \bullet L^r)_\tau = (\mathbb{1}_{\{Y \neq X\}} \star L^g_+)_\tau = 0$.*

Proof. In view of Lemmas 3.4, 4.3 and 4.5 the generators $\widehat{F}^r$, $\widehat{F}^g$ and the increasing process $\widehat{L}$ can be reduced to the filtration $\mathbb{F}$ to obtain, on the event $\{\tau \geq t\}$,

$$\begin{aligned} \mathbb{E}(\widehat{Y}_t\mathbb{1}_{\{\vartheta>t\}}\,|\,\mathcal{F}_t) &= \mathbb{E}\big(P_\tau\mathbb{1}_{\{\vartheta>\tau\}} + R_\vartheta\mathbb{1}_{\{t<\vartheta\leq\tau\}} + (\widetilde{G}\ddot{F}^r \bullet D^r)_\tau - (\widetilde{G}\ddot{F}^r \bullet D^r)_t \\ &\quad + (Gf^g \bullet D^g_+)_\tau - (Gf^g \bullet D^g_+)_t + (L_\tau - L_t)\mathbb{1}_{\{\vartheta>\tau\}} + (L_\vartheta - L_t)\mathbb{1}_{\{t<\vartheta\leq\tau\}}\,|\,\mathcal{F}_t\big) \\ &= \mathbb{E}\big(X_\tau G_\tau + (\widetilde{G}\ddot{F}^r \bullet D^r)_\tau - (\widetilde{G}\ddot{F}^r \bullet D^r)_t + (Gf^g \bullet D^g_+)_\tau - (Gf^g \bullet D^g_+)_t \\ &\quad + ((R+L)\bullet A^o)_\tau - ((R+L)\bullet A^o)_t + L_\tau G_\tau - L_t G_t\,|\,\mathcal{F}_t\big) \end{aligned}$$

where the mapping $\ddot{F}^r$ is given in (4.3). Next, an application of the làglàd product rule to LG and the equalities $\widetilde{G} = G_- + \Delta m$ and $L = L_- + \Delta L^r$ yield

$$\begin{aligned} LG &= L_- \bullet m - L_- \bullet A^o + G_- \bullet L^r + G \star L^g_+ + \Delta G \bullet \Delta L^r \\ &= L_- \bullet m - L \bullet A^o + \widetilde{G} \bullet L^r + G \bullet L^g_+. \end{aligned}$$

By combining these computations, we conclude that

$$YG = K(\tau) - \widetilde{G}\ddot{F}^r \bullet D^r - GF^g \star D^g_+ - R \bullet A^o + \widetilde{G} \bullet L^r + G \star L^g_+$$

where $Y := G^{-1}\,{}^o(\widehat{Y}\mathbb{1}_{[\![0,\vartheta[\![})$ and $K(\tau)$ is given by (5.1). The backward dynamics of Y can now be computed from Corollary 4.8

$$\begin{aligned} Y_t = X_\tau - \int_{]\!]t,\tau]\!]} \ddot{F}_s(Y_s)\,dD^r_s - \int_{[\![t,\tau[\![} F^g_s(Y_s)\,dD^g_{s+} - \int_{]\!]t,\tau]\!]} z_s\,dM_s \\ + \int_{]\!]t,\tau]\!]} (R_s - Y_s)\,d\widetilde{\Gamma}_s + \int_{]\!]t,\tau]\!]} \widetilde{G}^{-1}_s z_s\,d[M,n]_s - (L_\tau - L_t) \end{aligned}$$

and thus, after rearranging and using (4.3), we obtain the asserted BSDE.

It remains to check that the appropriate Skorokhod conditions are met by the process L. Recall that the Skorokhod conditions satisfied by $\widehat{L}^r$ and $\widehat{L}^g$ are

$$\big(\mathbb{1}_{\{\widehat{Y}_- \neq X_-\}} \bullet \widehat{L}^r\big)_{\tau\wedge\vartheta} = \big(\mathbb{1}_{\{\widehat{Y} \neq X\}} \star \widehat{L}^g_+\big)_{\tau\wedge\vartheta} = 0. \tag{5.2}$$

By integrating the first equality in (5.2) with respect to G^{-1}_-, we obtain

$$\big(G^{-1}_-\mathbb{1}_{[\![0,\vartheta]\!]}\mathbb{1}_{\{\widehat{Y}_- \neq X_-\}} \bullet \widehat{L}^r\big)_\tau = 0.$$

The equality $\widehat{Y}\mathbb{1}_{[\![0,\vartheta[\![} = Y\mathbb{1}_{[\![0,\vartheta[\![}$ implies that $\widehat{Y}_-\mathbb{1}_{[\![0,\vartheta]\!]} = Y_-\mathbb{1}_{[\![0,\vartheta]\!]}$ and $(\widehat{L}^r)^\vartheta = (L^r)^\vartheta$. Consequently, since $X_- = X_-\mathbb{1}_{[\![0,\vartheta]\!]} + R_\vartheta\mathbb{1}_{]\!]\vartheta,\infty]\!]}$, we get

$$\big(G^{-1}_-\mathbb{1}_{[\![0,\vartheta]\!]}\mathbb{1}_{\{\widehat{Y}_- \neq X_-\}} \bullet \widehat{L}^r\big)_\tau = \big(G^{-1}_-\mathbb{1}_{[\![0,\vartheta]\!]}\mathbb{1}_{\{Y_- \neq X_-\}} \bullet L^r\big)_\tau = 0.$$

Then, by taking the expectation and using the property of the dual $\mathbb{F}$-predictable projection, we obtain

$$\mathbb{E}((G_-^{-1}\mathbb{1}_{[\![0,\vartheta]\!]}\mathbb{1}_{\{Y_-\neq X_-\}}\bullet L^r)_\tau)=\mathbb{E}((\mathbb{1}_{\{Y_-\neq X_-\}}\bullet L^r)_\tau),$$

which implies that L^r obeys the first Skorokhod condition, that is, $(\mathbb{1}_{\{Y_-\neq X_-\}}\bullet L^r)_\tau=0$. Similarly, to check the second Skorokhod condition, we integrate the second equality in (5.2) with respect to G^{-1} and use the equality $\widehat{L}^g_+\mathbb{1}_{[\![0,\vartheta[\![}=L^g_+\mathbb{1}_{[\![0,\vartheta[\![}$ to obtain

$$(G^{-1}\mathbb{1}_{[\![0,\vartheta[\![}\mathbb{1}_{\{\widehat{Y}\neq X\}}\star\widehat{L}^g_+)_\tau=(G^{-1}\mathbb{1}_{[\![0,\vartheta[\![}\mathbb{1}_{\{Y\neq P\}}\star L^g_+)_\tau.$$

By taking the expectation and using the property of the dual $\mathbb{F}$-optional projection, we obtain the equality $\mathbb{E}((\mathbb{1}_{\{Y\neq X\}}\star L^g_+)_\tau)=0$, which in turn implies that $(\mathbb{1}_{\{Y\neq X\}}\star L^g_+)_\tau=0$. □

5.2 Construction of a solution to $\mathbb{G}$ RBSDE

In this section, we relax the postulate that a solution $(\widehat{Y},\widehat{Z},\widehat{U},\widehat{L})$ exists. In the next result, we again denote by $\widehat{U}$ an arbitrary prescribed $\mathbb{F}$-adapted process and we do not use Lemma 4.5.

Lemma 5.2. *Let a process $\widehat{U}\in\mathcal{O}(\mathbb{F})$ be given and let (Y,Z,L) be an $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$-valued, $\mathbb{F}$-adapted solution to the $\mathbb{F}$ RBSDE, on $[\![0,\tau]\!]$,*

$$\begin{aligned}Y_t=X_\tau&-\int_{]\!]t,\tau]\!]}F^r_s(Y_s,Z_s,\widehat{U}_s)\,dD^r_s-\int_{[\![t,\tau[\![}F^g_s(Y_s,Z_s,\widehat{U}_s)\,dD^g_{s+}\\&+\int_{]\!]t,\tau]\!]}\big[R_s-Y_s-\big(F^r_s(R_s,Z_s,\widehat{U}_s)-F^r_s(Y_s,Z_s,\widehat{U}_s)\big)\Delta D^r_s\big]\,d\widetilde{\Gamma}_s\\&-\int_{]\!]t,\tau]\!]}Z_s\,d\widetilde{M}_s-(L_\tau-L_t)\end{aligned}$$

where $Y\geq X$ and L is an $\mathbb{F}$-strongly predictable, increasing process such that the Skorokhod conditions $(\mathbb{1}_{\{Y_-\neq X_-\}}\bullet L^r)_\tau=(\mathbb{1}_{\{Y\neq X\}}\star L^g_+)_\tau=0$ hold and $L_0=0$. Then the triplet $(\widehat{Y},\widehat{Z},\widehat{L}):=(\widehat{Y},Z^\vartheta,L^\vartheta)$ where $\widehat{Y}$ is given by

$$\widehat{Y}:=Y_0+\mathbb{1}_{]\!]0,\vartheta[\![}\bullet Y^r+\mathbb{1}_{]\!]0,\vartheta[\![}\star Y^g_++(R_\vartheta-Y_{\vartheta-})\mathbb{1}_{[\![\vartheta,\infty[\![}\mathbb{1}_{\{\tau\geq\vartheta\}}$$

is a solution to the $\mathbb{G}$ RBSDE, on $[\![0,\tau\wedge\vartheta]\!]$,

$$\begin{aligned}\widehat{Y}_t=X_{\tau\wedge\vartheta}&-\int_{]\!]t,\tau\wedge\vartheta]\!]}\widehat{F}^r_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)\,dD^r_s-\int_{[\![t,\tau\wedge\vartheta[\![}\widehat{F}^g_s(\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)\,dD^g_{s+}\\&-\int_{]\!]t,\tau\wedge\vartheta]\!]}\big[R_s-Y_s-\big(F^r_s(R_s,\widehat{Z}_s,\widehat{U}_s)-F^r_s(Y_s,\widehat{Z}_s,\widehat{U}_s)\big)\Delta D_s\big]\,dm^{\mathbb{G}}_s\\&-\int_{]\!]t,\tau\wedge\vartheta]\!]}\widehat{Z}_s\,d\widetilde{M}_s-(\widehat{L}_{\tau\wedge\vartheta}-\widehat{L}_t)\end{aligned}$$

where $\widehat{Y}\geq X$ and $\widehat{L}=L^\vartheta$ is a $\mathbb{G}$-strongly predictable, increasing process such that the Skorokhod conditions $(\mathbb{1}_{\{\widehat{Y}_-\neq\widehat{X}_-\}}\bullet\widehat{L}^r)_{\vartheta\wedge\tau}=(\mathbb{1}_{\{\widehat{Y}\neq\widehat{X}\}}\star\widehat{L}^g_+)_{\vartheta\wedge\tau}=0$ are valid and $\widehat{L}_0=0$.

Proof. It suffices to set $C^r:=\widetilde{G}^{-1}\ddot{F}^r\bullet D^r+L^r$ and $C^g:=F^g\star D^g_++L^g$ in the proof of Lemma 4.10. The required Skorokhod conditions are also met since

$$\big(\mathbb{1}_{[\![0,\vartheta]\!]}\mathbb{1}_{\{\widehat{Y}_-\neq X^\vartheta_-\}}\bullet L^r\big)_\tau=\big(\mathbb{1}_{[\![0,\vartheta]\!]}\mathbb{1}_{\{Y_-\neq X_-\}}\bullet L^r\big)_\tau=0$$

and

$$\big(\mathbb{1}_{[\![0,\vartheta[\![}\mathbb{1}_{\{\widehat{Y}\neq X^\vartheta\}}\star L^g_+\big)_\tau=\big(\mathbb{1}_{[\![0,\vartheta[\![}\mathbb{1}_{\{Y\neq X\}}\star L^g_+\big)_\tau=0$$

where we have used the following equalities: $\widehat{Y}\mathbb{1}_{[\![0,\vartheta[\![}=Y\mathbb{1}_{[\![0,\vartheta[\![}$, $\widehat{Y}_-\mathbb{1}_{[\![0,\vartheta]\!]}=Y_-\mathbb{1}_{[\![0,\vartheta]\!]}$ and $X_-=X_-\mathbb{1}_{[\![0,\vartheta]\!]}+R_\vartheta\mathbb{1}_{]\!]\vartheta,\infty]\!]}$. □

Proposition 5.3. *Let (Y,Z,U,L) be a solution to the $\mathbb{F}$ RBSDE, on $[\![0,\tau]\!]$,*

$$Y_t = X_\tau - \int_{]\!]t,\tau]\!]} F^r_s(Y_s)\,dD^r_s - \int_{[\![t,\tau[\![} F^g_s(Y_s)\,dD^g_{s+} - \int_{]\!]t,\tau]\!]} Z_s\,d\widetilde{M}_s \\ + \int_{]\!]t,\tau]\!]} \big[R_s - Y_s - \big(F^r_s(R_s) - F^r_s(Y_s)\big)\Delta D^r_s\big]\,d\widetilde{\Gamma}_s - (L_\tau - L_t) \tag{5.3}$$

where $Y \geq X$, L is an $\mathbb{F}$-strongly predictable, increasing process with $L_0 = 0$ and such that the Skorokhod conditions $(\mathbb{1}_{\{Y_- \neq X_-\}} \bullet L^r)_\tau = (\mathbb{1}_{\{Y \neq X\}} \star L^g_+)_\tau = 0$ are obeyed, and the $\mathbb{F}$-optional process U satisfies, for all $t \in \mathbb{R}_+$,

$$\int_{]0,t]} U_s\,dm^{\mathbb{G}}_s = \int_{]0,t]} \big[R_s - Y_s - \big(F^r_s(R_s) - F^r_s(Y_s)\big)\Delta D^r_s\big]\,dm^{\mathbb{G}}_s. \tag{5.4}$$

Then $(\widehat{Y},\widehat{Z},\widehat{U},\widehat{L}) := (\widehat{Y}, Z^\vartheta, U, L^\vartheta)$ where the process $\widehat{Y}$ is given by

$$\widehat{Y} := Y_0 + \mathbb{1}_{[\![0,\vartheta[\![} \bullet Y^r + \mathbb{1}_{[\![0,\vartheta[\![} \star Y^g_+ + (R_\vartheta - Y_{\vartheta-})\mathbb{1}_{[\![\vartheta,\infty[\![}\mathbb{1}_{\{\tau \geq \vartheta\}}$$

is a solution to the $\mathbb{G}$ RBSDE (3.3) on $[\![0,\tau\wedge\vartheta]\!]$, that is,

$$\widehat{Y}_t = X_{\tau\wedge\vartheta} - \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{F}^r_s(\widehat{Y}_s)\,dD^r_s - \int_{[\![t,\tau\wedge\vartheta[\![} \widehat{F}^g_s(\widehat{Y}_s)\,dD^g_{s+} \\ - \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{Z}_s\,d\widetilde{M}_s - \int_{]\!]t,\tau\wedge\vartheta]\!]} \widehat{U}_s\,dm^{\mathbb{G}}_s - (\widehat{L}_{\vartheta\wedge\tau} - \widehat{L}_t) \tag{5.5}$$

where $\widehat{Y} \geq X$ and $\widehat{L} = L^\vartheta$ is a $\mathbb{G}$-strongly predictable, increasing process such that the Skorokhod conditions $(\mathbb{1}_{\{\widehat{Y}_- \neq \widehat{X}_-\}} \bullet \widehat{L}^r)_{\vartheta\wedge\tau} = (\mathbb{1}_{\{\widehat{Y} \neq \widehat{X}\}} \star \widehat{L}^g_+)_{\vartheta\wedge\tau} = 0$ hold and $\widehat{L}_0 = 0$.

Proof. The assertion of the proposition follows from Lemma 5.2 and similar arguments as used in Section 4.2 (see, in particular, the proof of Lemma 4.10). □

We now focus on the existence of a solution to the constrained $\mathbb{F}$ RBSDE (5.3)–(5.4) from Proposition 5.3. As in Section 4.2, we define the linear transformation

$$\bar{Y} := GY,\ \bar{Z} := G_- Z + G^{-1}\bar{Y}\nu,\ \bar{U} := U,\ \bar{L}^r := \widetilde{G} \bullet L^r,\ \bar{L}^g := G \star L^g$$

and the transformed generators $\bar{F}^r$ and $\bar{F}^g$. It is easy to check that if a solution $(Y,Z,U,L) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{O}(\mathbb{F}) \times \overline{\mathcal{P}}(\mathbb{F})$ to the coupled equations (5.3)–(5.4) exists, then $(\bar{Y},\bar{Z},\bar{U},\bar{L}) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F}) \times \mathcal{O}(\mathbb{F}) \times \overline{\mathcal{P}}(\mathbb{F})$ satisfies the following coupled equations (5.6)–(5.7)

$$\bar{Y}_t = G_\tau X_\tau - \int_{]\!]t,\tau]\!]} \bar{F}^r_s(\bar{Y}_s,\bar{Z}_s,\bar{U}_s)\,dD^r_s - \int_{[\![t,\tau[\![} \bar{F}^g_s(\bar{Y}_s,\bar{Z}_s,\bar{U}_s)\,dD^g_{s+} \\ - \int_{]\!]t,\tau]\!]} \bar{Z}_s\,dM_s + \int_{]\!]t,\tau]\!]} \big[\widetilde{G}_s R_s - \Delta F^r_s \Delta D^r_s\big] d\widetilde{\Gamma}_s - (\bar{L}_\tau - \bar{L}_t) \tag{5.6}$$

and

$$\int_{]\!]0,t]\!]} \bar{U}_s\,dm^{\mathbb{G}}_s = \int_{]\!]0,t]\!]} \big[(R_s - \bar{Y}_s G^{-1}_s) - \widetilde{G}^{-1}_s \Delta F^r_s \Delta D^r_s\big]\,dm^{\mathbb{G}}_s \tag{5.7}$$

where

$$\Delta F^r_s := \bar{F}^r_s(G_s R_s, \bar{Z}_s, \bar{U}_s) - \bar{F}^r_s(\bar{Y}_s,\bar{Z}_s,\bar{U}_s)$$

and $\bar{L} = \bar{L}^r + \bar{L}^g$ satisfies

$$(\mathbb{1}_{\{\bar{Y}_- \neq G_- X_-\}} \bullet \bar{L}^r)_\tau = (\mathbb{1}_{\{\bar{Y} \neq GX\}} \star \bar{L}^g)_\tau = 0.$$

In the reverse, a solution (Y,Z,U,L) to equations (5.3)–(5.4) can be obtained from a solution $(\bar{Y},\bar{Z},\bar{U})$ to equations (5.6)–(5.7) by setting $Y := G^{-1}\bar{Y}$, $Z := G^{-1}_-(\bar{Z} - G^{-1}\bar{Y}\nu)$, $U := \bar{U}$, $L^r := \widetilde{G}^{-1} \bullet \bar{L}^r$ and $L^g := G^{-1} \star \bar{L}^g$.

5.3 Solution to $\mathbb{G}$ RBSDE in the Brownian case

We proceed here similarly to Section 4.4. Let $M = W$ be a d-dimensional Wiener process and $\mathbb{F}$ be its natural filtration. We consider the constrained $\mathbb{F}$ RBSDE (5.3)–(5.4) with $F^g = 0$ and a càdlàg lower barrier X. The goal is again to demonstrate that, in some specific settings, the coupled equations (5.6)–(5.7) possess a unique solution (Y, Z, U, L).

As before, we write D and F instead of D^r and F^r, respectively, and we consider the case where $D = (D^1, D^2) = (\langle W\rangle, \widetilde{\Gamma})$ and $F = (F^1, F^2) = (F^1(y, z, u), F^2(y))$ for all $(y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$. Then the BSDE (5.3) becomes

$$\begin{aligned}\bar{Y}_t = G_\tau X_\tau &- \int_{\rrbracket t,\tau\rrbracket} \bar{F}^1_s \, d\langle W\rangle_s - \int_{\rrbracket t,\tau\rrbracket} \bar{F}^2_s(\bar{Y}_s)\, d\widetilde{\Gamma}_s - \int_{\rrbracket t,\tau\rrbracket} \bar{Z}_s \, dW_s \\ &+ \int_{\rrbracket t,\tau\rrbracket} \big[\widetilde{G}_s R_s - \big(\bar{F}^2_s(G_s R_s) - \bar{F}^2_s(\bar{Y}_s)\big)\Delta\widetilde{\Gamma}_s\big] d\widetilde{\Gamma}_s - (\bar{L}_\tau - \bar{L}_t) \end{aligned} \tag{5.8}$$

where $\bar{F}^1_s := \bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)$ and equation (5.4) has a solution $\bar{U}$ given by equality (4.20).

The following result gives sufficient conditions for existence of a solution to $\mathbb{F}$ RBSDEs (5.3) and (5.8) in the case of the Brownian filtration $\mathbb{F}$.

Proposition 5.4. *Assume that:*

(i) for every $t \geq 0$, the map $F^1_t : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous;

(ii) for every $t \geq 0$, the map $F^2_t : \mathbb{R} \to \mathbb{R}$ is bounded, Lipschitz continuous and decreasing;

(iii) the dual $\mathbb{F}$-optional projection A^o has a finite number of discontinuities;

(iv) the process X is càdlàg.

Then the RBSDE (5.8) *has a solution $(\bar{Y}, \bar{Z}, \bar{L})$ and a solution (Y, Z, L) to* (5.3) *can be obtained by setting $Y := G^{-1}\bar{Y}$, $Z := G^{-1}(\bar{Z} - G^{-1}\bar{Y}\nu)$ and $L := \widetilde{G}^{-1} \bullet \bar{L}$.*

Proof. Similarly to the proof of Proposition 4.13, in order to obtain a solution $(\bar{Y}, \bar{Z}, \bar{L})$ to the BSDE (4.19), we apply Theorem 2.1 in [20] to the data (X, R, ϑ, F). We note that (4.19) is a special case of equation (2.1) in [20] of the form

$$\begin{aligned}\bar{Y}_t = G_\tau X_\tau &+ \int_{\rrbracket t,\tau\rrbracket} f(s, \bar{Y}_s, \bar{Z}_s)\, ds + \int_{\rrbracket t,\tau\rrbracket} g(s, \bar{Y}_s)\, d\bar{A}_s - \int_{\rrbracket t,\tau\rrbracket} \bar{Z}_s \, dW_s \\ &+ \sum_{t<s\leq\tau} h(s, \bar{Y}_{s-}, \bar{Y}_s) - (\bar{L}_\tau - \bar{L}_t)\end{aligned}$$

where (5.8) can be recovered if we set $\bar{A} := \widetilde{\Gamma}^c$ (the continuous part $\widetilde{\Gamma}$) and

$$f(s, \bar{Y}_s, \bar{Z}_s) := -\bar{F}^1_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s), \quad g(s, \bar{Y}_s) := \widetilde{G}_s R_s - \bar{F}^2_s(\bar{Y}_s)$$

and

$$h(s, \bar{Y}_{s-}, \bar{Y}_s) := \big(\widetilde{G}_s R_s - F^2_s(\bar{Y}_s)\big)\Delta\widetilde{\Gamma}_s - \big(\bar{F}^2_s(G_s R_s) - \bar{F}^2_s(\bar{Y}_s)\big)(\Delta\widetilde{\Gamma}_s)^2$$

where $\bar{U}$ is given by (4.20). Notice that once again h does not depend on $\bar{Y}_{s-}$. As was explained in the proof of Proposition 4.13, the assumptions in Theorem 2.1 of [20] are satisfied and thus a solution $(\bar{Y}, \bar{Z}, \bar{L})$ exists. Finally, we observe that the $\mathbb{F}$-predictable, increasing process $L := \widetilde{G}^{-1} \bullet \bar{L}$ clearly obeys the Skorokhod conditions since

$$(\mathbb{1}_{\{Y_- \neq X_-\}} \bullet L)_\tau = (\mathbb{1}_{\{\bar{Y}_- \neq G_- X_-\}} \widetilde{G}^{-1} \bullet \bar{L})_\tau = 0$$

and thus the proof is completed. ☐

In Section 6, we deviate from the previous sections and, for given a filtration $\mathbb{F}$, we focus on BSDE (6.1) and RBSDE (6.12) with the feature that the driver is làglàd and may shares jumps with the driving martingale. Even when the driver is càdlàg, there is a gap in the existing literature on BSDEs when the driver shares common jumps with the driving martingale and thus we develop a jump-adapted method to solve BSDEs of this general form.

6 BSDE with a làglàd driver and discontinuous martingales

In the last section, we shall work in a general setting and study solutions to BSDEs and RBSDEs where the driver is làglàd and may share common jumps with the driving d-dimensional martingale M. To the best of our knowledge, such results are not yet available in the literature. More specifically, given a filtration $\mathbb{F}$, we first propose in Section 6.1 a method of solving the $\mathbb{F}$ BSDE

$$v_t = \xi_\tau - \int_{]\!]t,\tau]\!]} f_s^r \, dD_s^r - \int_{[\![t,\tau[\![} f_s^g \, dD_{s+}^g - \int_{]\!]t,\tau]\!]} z_s \, dM_s \tag{6.1}$$

where $f_s^r := f_s^r(v_{s-}, v_s, z_s)$ and $f_s^g := f_s^g(v_s, v_{s+})$. In particular, we observe that in the case where either F^g in (4.15) does not depend on U and Z or U in (4.16) can be solved and does not depend on Z (see, for example, Section 4.4), then the BSDEs (4.15) and (4.17) can be obtained as a special case of the above BSDE (6.1). In Section 6.2, we extend this approach to the case of $\mathbb{F}$ RBSDEs.

6.1 BSDE with a làglàd driver and common jumps

We present below a jump-adapted method of transforming the làglàd BSDE given by (6.1) to a system of more tractable càdlàg BSDEs, which in turn can be further converted into a system of càdlàg BSDEs with a continuous driver. In some special cases, a solution to the latter BSDE can be obtained by utilizing results from the existing literature.

Step 1. From a làglàd to càdlàg driver. For simplicity, in the following we denote $D := D^r + D^g$ so that D is a làglàd process of finite variation. We suppose that the times of right-hand jumps of D (that is, the moments when $\Delta^+ D > 0$) are given by the family $(T_i)_{i=1,2,\dots,p}$ of $\mathbb{F}$-stopping times and we denote $S_0 := 0$, $S_i := T_i \wedge \tau$ and $S_{p+1} := \tau$. We note that at each S_i we have that $v_+ - v = f^g(v, v_+)\Delta D_+^g$, which shows that when the value of v_+ is already known, then the value of v can be obtained as a solution to that equation. This leads to the observation that a solution (v, z) to (6.1) can be constructed by first solving iteratively, for every $i = 0, 1, \dots, p$, the following càdlàg BSDE on each stochastic interval $[\![S_i, S_{i+1}]\!]$

$$v_t^i = \xi^i - \int_{]\!]t,S_{i+1}]\!]} f_s^r(v_{s-}^i, v_s^i, z_s^i) \, dD_s^r - \int_{]\!]t,S_{i+1}]\!]} z_s^i \, dM_s \tag{6.2}$$

where an $\mathcal{F}_{S_{i+1}}$-measurable random variable ξ^i is given by the system of equations, for every $i = 0, 1, \dots, p-1$,

$$v_{S_{i+1}}^{i+1} - \xi^i = f_{S_{i+1}}^g\big(\xi^i, v_{S_{i+1}}^{i+1}\big)\Delta^+ D_{S_{i+1}}$$

with $\xi^p = \xi_\tau$. Then a solution (v, z) to the làglàd BSDE (6.1) is obtained by setting

$$v := \sum_{i=0}^{p} v^i \mathbb{1}_{]\!]S_i,S_{i+1}]\!]}, \quad z := \sum_{i=0}^{p} z^i \mathbb{1}_{]\!]S_i,S_{i+1}]\!]}.$$

Step 2. From a càdlàg to continuous driver. In view of (6.2), in the following we focus on showing that the càdlàg BSDE can be solved, under certain assumptions about the driver and filtration. We now consider the situation where the filtration $\mathbb{F}$ can support discontinuous martingales (e.g., the Brownian-Poisson filtration) and the driver D is possibly discontinuous. More specifically, we study the càdlàg BSDE of the form

$$y_t = \xi_\tau - \int_{]\!]t,\tau]\!]} f_s^r(y_{s-}, y_s, z) \, dD_s^r - \int_{]\!]t,\tau]\!]} z_s \, dM_s \tag{6.3}$$

where a solution $(y, z) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F})$ is such that y is a càdlàg process.

Remark 6.1. Our interest in the BSDE (6.3) is motivated by the need to understand the well-posedness of the pre-default BSDE, which is obtained in a nonlinear reduced-form model without postulating that either condition $(\mathbf{C})$ or $(\mathbf{A})$ holds. The discontinuity in D^r stems from the discontinuity of the hazard process $\widetilde{\Gamma}$ and, in some financial applications, the introduction of the nonlinearity can be interpreted as a way to introduce ambiguity in the recovery and the default intensity (see, e.g., Fadina and Schmidt [21]).

In the following, we suppose that ξ_τ is bounded and $\mathcal{F}_\tau$-measurable and we consider a more general BSDE

$$y_t = \xi_\tau - \int_{\rrbracket t,\tau\rrbracket} f^r_s \, dD^c_s - \sum_{t<s\leq\tau} h(s, y_{s-}, y_s) - \int_{\rrbracket t,\tau\rrbracket} z_s \, dM_s \tag{6.4}$$

where $f^r_s := f^r_s(y_{s-}, y_s, z_s)$ and D^c is the continuous part of the process D^r.

To recover the BSDE (6.3) from (6.4), it suffices to set $h(s, y_{s-}, y_s) := f^r_s(y_{s-}, y_s)\Delta D^r_s$. Consequently, we henceforth suppose that $h = 0$ outside the graph of a finite set of $\mathbb{F}$-predictable stopping times $(T_i)_{i=1,2,\dots,p}$ and we denote $S_0 := 0$, $S_i := T_i \wedge \tau$ and $S_{p+1} := \tau$.

Remark 6.2. Note that a sufficient assumption for the jumps of D^r to be $\mathbb{F}$-predictable stopping times is to postulate that D^r is an $\mathbb{F}$-predictable, increasing process. Furthermore, observe that the condition that $(T_i)_{i=1,2,\dots p}$ are $\mathbb{F}$-predictable stopping times can be relaxed if the mapping h does not depend on y_-.

Remark 6.3. Suppose that $p = 1$ and denote $S = S_1$. Let us assume that a solution to (6.4) on $\rrbracket S, \tau\rrbracket$ has already been found and our goal is to construct its extension to the interval $\llbracket 0, \tau\rrbracket$. We observe that if (y, z) is a solution to (6.4), then

$$y_t = y_0 + \int_{\rrbracket 0,t\rrbracket} f^r_s(y_{s-}, y_s, z_s) \, dD^c_s + \sum_{0<s\leq t} h(s, y_{s-}, y_s) + \int_{\rrbracket 0,t\rrbracket} z_s \, dM_s$$

and hence the jump of the càdlàg process y at time S satisfies

$$\Delta y_S := y_S - y_{S-} = h(S, y_{S-}, y_S) + z_S \, \Delta M_S. \tag{6.5}$$

By taking the conditional expectation of both sides of (6.5) with respect to $\mathcal{F}_{S-}$, we obtain the following equation

$$y_{S-} = \mathbb{E}\big(y_S - h(S, y_{S-}, y_S) \,|\, \mathcal{F}_{S-}\big),$$

which, at least in principle, can be solved for y_{S-} under appropriate additional assumptions. Subsequently, one could compute z_S from equality (6.5). However, if one decides to proceed in that way, then to solve the BSDE (6.4) on $\llbracket 0, S\rrbracket$ one would need to solve (6.4) on $\llbracket 0, S\llbracket$ and thus to study the BSDE driven by the martingale M stopped at $S-$. Since this would be quite cumbersome, we propose in Proposition 6.4 an alternative method where this difficulty is circumvented.

To show the existence of a solution to the BSDE (6.4), we introduce an auxiliary làglàd BSDE

$$\begin{aligned} v_t = \xi_\tau - h(\tau, v_{\tau-}, \xi_\tau) - \int_{\rrbracket t,\tau\rrbracket} f^r_s(v_{s-}, v_s, z_s) \, dD^c_s - \int_{\rrbracket t,\tau\rrbracket} z_s \, dM_s \\ - \sum_{t\leq s<\tau} h(s, v_{s-}, v_{s+}) \end{aligned} \tag{6.6}$$

where a solution $(v, z) \in \mathcal{O}(\mathbb{F}) \times \mathcal{P}_d(\mathbb{F})$ is such that v is a làglàd process.

Proposition 6.4. *Let v be a làglàd process such that (v,z) is a solution to the BSDE* (6.6) *on $[\![0,\tau]\!]$. Then (y,z) where $y := v_+ \mathbb{1}_{[\![0,\tau]\!]} + h(\tau, v_{\tau-}, \xi_\tau)\mathbb{1}_{[\![\tau]\!]}$ is a solution to the càdlàg BSDE* (6.4) *on $[\![0,\tau]\!]$.*

Proof. Suppose that (v,z) is a solution to (6.6). It is clear from (6.6) that the left-hand and right-hand jumps of v are given by $\Delta v = z\Delta M$ and $\Delta^+ v = h(\cdot, v_-, v_+)$, respectively. By the optional sampling theorem, we have that $\mathbb{E}(v_S \,|\, \mathcal{F}_{S-}) = v_{S-}$ for any $\mathbb{F}$-predictable stopping time S. Therefore, if the random variable v_{S+} is known, then the $\mathcal{F}_S$-measurable random variable v_S is a solution to the equation

$$\Delta^+ v_S := v_{S+} - v_S = h\big(S, \mathbb{E}[v_S \,|\, \mathcal{F}_{S-}], v_{S+}\big). \tag{6.7}$$

If we set $y := v_+$ on $[\![0,\tau[\![$, then $y_- = v_-$ and thus

$$\begin{aligned}\Delta y_S &= y_S - y_{S-} = y_{S+} - y_{S-} = v_{S+} - v_{S-} = \Delta^+ v_S + \Delta v_S \\ &= h(\cdot, v_{S-}, v_{S+}) + z_S\,\Delta M_S = h(\cdot, y_{S-}, y_S) + z_S\,\Delta M_S,\end{aligned}$$

which coincides with (6.5). In the next step, we take inspiration from the proof of Theorem 3.1 in Essaky et al. [20] and rewrite (6.4) into

$$v_t = \xi_\tau - h(\tau, v_\tau, \xi_\tau)\Delta D^r_\tau - \int_{]\!]t,\tau]\!]} f^r_s(v_{s-}, v_s, z_s)\, dD^c_s - \int_{]\!]t,\tau]\!]} z_s\, dM_s - \sum_{t\le s<\tau} \Delta^+ v_s.$$

Recall that, by assumption about h, the right-hand jump times of v are given by the family $(T_i)_{i=1,2,\dots,p}$ of $\mathbb{F}$-stopping times and we denote $S_0 := 0$, $S_i := T_i \wedge \tau$ and $S_{p+1} := \tau$. We observe that a solution (v,z) can be obtained by first solving iteratively the following càdlàg BSDE on the stochastic interval $[\![S_i, S_{i+1}]\!]$, for every $i = 0,1,\dots,p$,

$$v^i_t = \xi^i - \int_{]\!]t,S_{i+1}]\!]} f^r_s(v^i_{s-}, v^i_s, z^i_s)\, dD^c_s - \int_{]\!]t,S_{i+1}]\!]} z^i_s\, dM_s \tag{6.8}$$

where ξ^i is an $\mathcal{F}_{S_{i+1}}$-measurable random variable determined by the recursive system of equations, for every $i = 0,1,\dots,p$ (see (6.7))

$$v^{i+1}_{S_{i+1}} - \xi^i = h\big(S_{i+1}, \mathbb{E}[\xi^i \,|\, \mathcal{F}_{S_{i+1}-}], v^{i+1}_{S_{i+1}}\big) \tag{6.9}$$

with the terminal condition $v^{p+1}_{S_{p+1}} = \xi$. In the last step, we aggregate the family of solutions (v^i, z^i) for $i = 0,1,\dots,p$ by setting

$$v := v^0_0 + \sum_{i=0}^p v^i \mathbb{1}_{]\!]S_i, S_{i+1}]\!]}, \quad z := z^0_0 + \sum_{i=0}^p z^i \mathbb{1}_{]\!]S_i, S_{i+1}]\!]}.$$

Then, by an application of the Itô formula, one can check that (v,z) is a solution to the làglàd BSDE (6.6). Furthermore, since $\Delta y_S = v_{S+} - v_{S-}$ and the dynamics of y and v, which are given by (6.4) and (6.6), respectively, are easily seen to coincide on each stochastic interval $]\!]S_i, S_{i+1}[\![$, we conclude that $(y,z) := (v_+, z)$ is a solution to the càdlàg BSDE (6.4) once we made the appropriate adjustment to the last jump of size h at the terminal time τ. □

Remark 6.5. Note that if the recovery process R is $\mathbb{F}$-predictable (so that one can use A^p instead of A^o) and D^r is chosen to be have $\mathbb{F}$-predictable jumps (for instance, if $D^r = (\langle M \rangle, \widetilde{G}^{-1} \bullet A^p)$), then the transformed BSDE (4.17) has the form (6.4) and h vanishes outside the graph of a family of $\mathbb{F}$-predictable stopping times. In that case, assuming that ξ^i can be solved in (6.9), we would be able to consider jumps of a size h depending on v_-.

Example 6.6. Let us show that if appropriate conditions are imposed on the inputs (f^r, D^r, M), then a unique solution (v^i, z^i) to (6.8) can be obtained on each interval $[\![S_i, S_{i+1}]\!]$ for $i = 0, 1, \dots, p$ and hence a solution (y, z) to (6.4) can be constructed as well. In the following, we assume that the process $\langle M \rangle$ is continuous, the function h does not depend on v_- and

$$f^r(v_-, v, z) \bullet D^c = f(v_-, v, z) \bullet \langle M \rangle + g(v) \bullet B$$

where B is an $\mathbb{F}$-adapted, bounded, continuous, increasing process and f and g are some real-valued mappings satisfying appropriate measurability conditions. We note that, as h does not depend on v_-, the assumption that the jump times of D^r (and hence also $(S_i)_{i=1,\dots,p}$) are $\mathbb{F}$-predictable stopping times can be relaxed. Furthermore, the right-hand jumps of the process v are given by $\Delta^+ v_t = h(t, v_{t+})$.

We thus need to analyze the following càdlàg BSDE with a continuous driver, on each stochastic interval $[\![S_i, S_{i+1}]\!]$ for every $i = 0, 1, \dots, p$,

$$\begin{aligned} dv^i_t &= -f_t(v^i_{t-}, v^i_t, z^i_t)\, d\langle M \rangle_t - g(t, v^i_t)\, dB_t - z^i_t\, dM_t, \\ v^i_{S_{i+1}} &= v^{i+1}_{S_{i+1}} - h\big(S_{i+1}, v^{i+1}_{S_{i+1}}\big), \end{aligned}$$

with the terminal condition $v^{p+1}_{S_{p+1}} = \xi_\tau$.

Observe that in the case of a Brownian-Poisson filtration $\mathbb{F}$, the existence and uniqueness of a family of solutions (v^i, z^i) can be deduced from Theorem 53.1 in Pardoux [44] under the postulate that f, g and h are bounded and Lipschitz continuous functions, the process B is bounded, and $M = (W, \widetilde{N})$ where W is a Brownian motion and $\widetilde{N}$ is an independent compensated Poisson process.

Example 6.7. Let the filtration $\mathbb{F}$ be the Brownian-Poisson filtration. We consider below an example given in Gapeev et al. [26] of a supermartingale J valued in $(0, 1]$ which is the solution to the SDE

$$dJ_t = -\lambda J_t\, dt + \frac{b}{\sigma} J_t(1 - J_t)\, dW_t, \quad J_0 = 1.$$

The process J takes a multiplicative form $J_t = Q_t e^{-\lambda t}$ where Q satisfy

$$Q_t = 1 + \int_0^t \frac{b}{\sigma}(1 - J_u) Q_u\, dW_u.$$

For a fixed $p \in (0, 1)$, we consider the supermartingales

$$\widetilde{G}_t = J_t \mathbb{1}_{\{t \le T_1\}} + p J_t \mathbb{1}_{\{T_1 < t\}} = J_t - (1 - p) J_t \mathbb{1}_{\{T_1 < t\}}$$

and $G_t = J_t - (1 - p) J_t \mathbb{1}_{\{T_1 \le t\}}$ and we observe that, by an application of the Itô formula, we have

$$\widetilde{G}_t = 1 + \int_0^t \frac{b}{\sigma} \widetilde{G}_u (1 - J_u)\, dW_u - \int_0^t \lambda \widetilde{G}_u\, du - (1 - p) J_{T_1} \mathbb{1}_{\{T_1 < t\}}.$$

We know from Jeanblanc and Li [32] that it is possible to construct a random time τ such that the Azéma optional supermartingale and the Azéma supermartingale associated with τ are given by $\widetilde{G}$ and G, respectively. In the present example, the equality $\widetilde{G} = G_-$ holds and the martingale m, the dual $\mathbb{F}$-optional projection A^o and the hazard process $\widetilde{\Gamma} = (\widetilde{G}^{-1} \bullet A^o)$ associated with τ are given by the following expressions

$$\begin{aligned} m_t &= 1 + \int_0^t \frac{b}{\sigma} \widetilde{G}_u (1 - J_u)\, dW_u, \\ A^o_t &= \int_0^t \lambda G_{u-}\, du + (1 - p) J_{T_1} \mathbb{1}_{\{T_1 \le t\}}, \end{aligned}$$

and $\widetilde{\Gamma}_t = \lambda t + (1-p)\mathbb{1}_{\{T_1 \le t\}}$ so that $\widetilde{\Gamma}^c_t = \lambda t$ and $\widetilde{\Gamma}^d_t = (1-p)\mathbb{1}_{\{T_1\le t\}}$.

The stopping time T_1 can be viewed as a shock to the underlying financial asset and τ is the timing of a default event. The parameter $p \in (0,1)$ can be regarded as the conditional probability that the default event occurs at T_1 given that the default event has not occurred before T_1. In the following, we denote the compensated Poisson process by $\widetilde{N}$ and for the ease of presentation we set $D^r = \widetilde{\Gamma}/(1-p)$ and $D^g = \widetilde{\Gamma}^d_-/(1-p) = \mathbb{1}_{]\!]T_1,\infty[\![}$. Furthermore, we suppose the generators F^r and F^d does not depend on Y_- and the constrained BSDE (4.13)-(4.14) reduces to a single BSDE given by

$$\begin{aligned}Y_t = X_T &- \int_{]\!]t,T]\!]} \frac{F^r_s(Y_s)}{1-p}\, d\widetilde{\Gamma}_s - \int_{[\![t,T[\![} \frac{F^g_s(Y_s)}{1-p}\, d\widetilde{\Gamma}^d_s + \int_{]\!]t,T]\!]} \frac{b}{\sigma}(1-J_s) Z^1_s\, ds - \int_{]\!]t,T]\!]} Z^1_s\, dW_s \\ &- \int_{]\!]t,T]\!]} Z^2_s\, d\widetilde{N}_s + \int_{]\!]t,T]\!]} \big[R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s))\mathbb{1}_{[\![T_1]\!]}(s)\big] d\widetilde{\Gamma}_s.\end{aligned}$$

On the set $\{T_1 \le T\}$, we observe that the driver of the above BSDE has only one jump at time T_1 and thus on the stochastic interval $]\!]T_1, T]\!]$ we need only to find the solution (y,u) where $u = (u^1,u^2)$ to the BSDE,

$$\begin{aligned}y_t = X_T &- \int_t^T \Big[\frac{\lambda F^r_s(y_s)}{1-p} - \frac{b(1-J_s)u^1_s}{\sigma} - \lambda(R_s - y_s)\Big]\, ds \\ &- \int_t^T u^1_s\, dW_s - \int_t^T u^2_s\, d\widetilde{N}_s. \end{aligned} \tag{6.10}$$

At the jump time T_1, the right jump of Y is given by $\Delta^+ Y_{T_1} = F^g_{T_1}(Y_{T_1})$ and the quantity Y_{T_1} is obtained by solving the equation $y_{T_1} - F^g_{T_1}(Y_{T_1}) = Y_{T_1}$. Assuming that Y_{T_1} can be computed, we see that one is required to solve the càdlàg BSDE, on the stochastic interval $[\![0,T_1]\!]$,

$$\begin{aligned}Y_t = Y_{T_1} &- \int_{]\!]t,T_1]\!]} \frac{F^r_s(Y_s)}{1-p}\, d\widetilde{\Gamma}_s + \int_{]\!]t,T_1]\!]} \frac{b}{\sigma}(1-G_s)\, Z^1_s\, ds - \int_{]\!]t,T_1]\!]} Z^1_s\, dW_s - \int_{]\!]t,T_1]\!]} Z^2_s\, d\widetilde{N}_s \\ &+ \int_{]\!]t,T_1]\!]} \big[R_s - Y_s - (F^r_s(R_s) - F^r_s(Y_s))\mathbb{1}_{[\![T_1]\!]}(s)\big]\, d\widetilde{\Gamma}_s\end{aligned}$$

where the martingale term and the driver may share a common jump at T_1. Again, we observe that the driver jumps at T_1 only and the jump size given by

$$h(T_1, Y_{T_1}) := F^r_{T_1}(Y_{T_1}) - \big[R_{T_1} - Y_{T_1} - (F^r_{T_1}(R_{T_1}) - F^r_{T_1}(Y_{T_1}))\big](1-p).$$

Therefore, the adjusted terminal condition v_{T_1} at T_1 equals

$$v_{T_1} := Y_{T_1} - h(T_1, Y_{T_1}) = R_{T_1} - F^r_{T_1}(R_{T_1}) - p\big[R_{T_1} - Y_{T_1} - (F^r_{T_1}(R_{T_1}) - F^r_{T_1}(Y_{T_1})\big]$$

and we see that we need to solve the following BSDE with a continuous driver, on the stochastic interval $[\![0,T_1]\!]$,

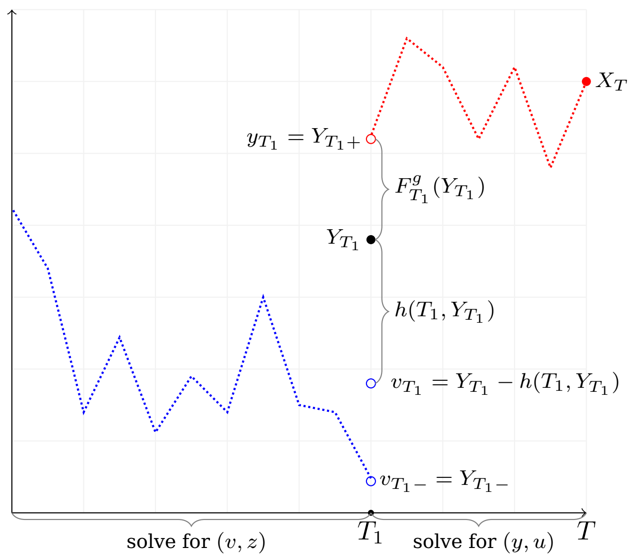
$$\begin{aligned}v_t = v_{T_1} &- \int_{]\!]t,T_1]\!]} \Big[\frac{\lambda F^r_s(v_s)}{1-p} + \frac{b(1-J_s)\, z^1_s}{\sigma} + \lambda(R_s - v_s)\Big]\, ds \\ &- \int_{]\!]t,T_1]\!]} z^1_s\, dW_s - \int_{]\!]t,T_1]\!]} z^2_s\, d\widetilde{N}_s. \end{aligned} \tag{6.11}$$

To this end, let Y_{T_1} be a solution to the equation $y_{T_1} - F^g_{T_1}(Y_{T_1}) = Y_{T_1}$. Then a solution (Y,Z) where $Z = (Z^1, Z^2)$ on the whole interval $[\![0,T]\!]$ can be obtained by setting

$$\begin{aligned}Y &:= v\mathbb{1}_{[\![0,T_1]\!]} + h(T_1, Y_{T_1})\mathbb{1}_{[\![T_1]\!]} + y\mathbb{1}_{]\!]T_1,T]\!]}, \\ Z^i &:= z^i \mathbb{1}_{[\![0,T_1]\!]} + u^i \mathbb{1}_{]\!]T_1,T]\!]}.\end{aligned}$$

Let us now consider the set $\{T_1 > T\}$. Since there are no jumps before T, it suffices to find (v,z) in (6.11) on the whole interval $[\![0,T]\!]$ with the terminal condition $v_T = X_T$.

To showcase the jump-adapted method outlined above, we provide an exhibit



We point out that since $(1-J)$ is bounded by one then, when considering (6.10) and (6.11), we do not need to study the transformed BSDE given in (4.17)-(4.18). This is because, given appropriate assumptions on F^r, the linear growth conditions in z can be easily verified here.

6.2 RBSDE with a làglàd driver and common jumps

Following the structure of Section 6.1, for a given filtration $\mathbb{F}$, we focus on $\mathbb{F}$ RBSDEs of the form

$$v_t = \xi_\tau - \int_{]\!]t,\tau]\!]} f^r_s(v_{s-}, v_s, z_s)\, dD^r_s - \int_{[\![t,\tau[\![} f^g_s(v_s, v_{s+})\, dD^g_{s+} \\ - \int_{]\!]t,\tau]\!]} z_s\, dM_s + l^r_\tau - l^r_t + l^g_\tau - l^g_t \tag{6.12}$$

where l^r and l^g satisfy $(\mathbb{1}_{\{v_- \neq \xi_-\}} \bullet l^r)_\tau = (\mathbb{1}_{\{v \neq \xi\}} \star l^g)_\tau = 0$. We observe that in the case where F^g in (5.3) does not depend on U and Z or that U in (5.4) can be solved and does not depend on Z (for an example, see Section 5.3), then RBSDEs (5.3) and (5.6) can be obtained as a special case of the above RBSDE (6.12). Similar to the non-reflected case, we present below a jump-adapted method to reduce the làglàd RBSDE (6.12) to a system of càdlàg RBSDEs, which can be further reduced to a system of càdlàg RBSDEs with continuous drivers.

Step 1. From a làglàd to càdlàg driver. By examining the right-hand jumps of v, that is, $\Delta^+ v$, and the Skorokhod condition satisfied by l^g, we observe that $\Delta^+ v$ and Δl^g_+ must satisfy the conditions

$$v_+ - v = f^g(v_+, v)\Delta D^g_+ + \Delta l^g_+, \quad (v - \xi)\Delta l^g_+ = 0,$$

which in turn implies that

$$\Delta l^g_+ = \big(\xi - (v_+ - f^g(v, v_+)\Delta D^g_+)\big)^+, \quad v = \xi \vee \big(v_+ - f^g(v, v_+)\Delta D^g_+\big).$$

We thus see that at the jump times of l^g_+ and D^g_+ the quantity v can be computed by solving the second equation (of course, assuming that a solution exists) and Δl^g_+ can be obtained by substitution. In particular, if f^g does not depend on v, then it is clear that we have

$$\Delta l^g_+ = (\xi - (v_+ - f^g(v_+)\Delta D^g_+))^+, \quad v = \xi \vee (v_+ - f^g(v_+)\Delta D^g_+).$$

These arguments lead to the observation that a solution (v,z,l) to (6.12) can be constructed by solving iteratively, for every $i=0,1,\dots,p$, the following càdlàg RBSDE on $[\![S_i,S_{i+1}]\!]$

$$v^i_t=\xi^i-\int_{]\!]t,S_{i+1}]\!]}h^r_s(v^i_{s-},v^i_s,z^i_s)\,dD^r_s-\int_{]\!]t,S_{i+1}]\!]}z^i_s\,dM_s+l^i_{S_{i+1}}-l^i_t \tag{6.13}$$

where the càdlàg increasing process l^i obeys the Skorokhod condition $(\mathbb{1}_{\{\xi_-=v^i_-\}}\bullet l^i)=0$ and $(\xi^i,\Delta l^g_{S_{i+1}+})$ are $\mathcal{F}_{S_{i+1}}$-measurable random variables such that $\Delta l^g_{S_{p+1}+}=0$, $\xi^p=\xi_\tau$ and for $i=0,1,\dots,p-1$,

$$\Delta l^g_{S_{i+1}+}=\Big(\xi_{S_{i+1}}-(v^{i+1}_{S_{i+1}}-h^g_{S_{i+1}}(\xi^i,v^{i+1}_{S_{i+1}})\Delta D^g_{S_{i+1}+})\Big)^+,$$
$$\xi^i=\xi_{S_{i+1}}\vee\Big(v^{i+1}_{S_{i+1}}-h^g_{S_{i+1}}(\xi^i,v^{i+1}_{S_{i+1}})\Delta D^g_{S_{i+1}+}\Big).$$

Then a global solution (v,z,l) where $l=l^r+l^g$ is obtained by setting

$$v=v_0+\sum_{i=0}^p v^i\mathbb{1}_{]\!]S_i,S_{i+1}]\!]},\qquad z=z_0+\sum_{i=0}^p z^i\mathbb{1}_{]\!]S_i,S_{i+1}]\!]},$$
$$l^r=\sum_{i=0}^p(l^{i-1}_{S_i}+l^i)\mathbb{1}_{]\!]S_i,S_{i+1}]\!]},\quad l^g=\sum_{i=1}^p\Delta l^g_{S_i+}\mathbb{1}_{]\!]S_i,\infty[\![},$$

where $l^{-1}_0=0$, $v_0=v^0_0$ and $z_0=z^0_0$.

Step 2. From a càdlàg to continuous driver. In view of the càdlàg RBSDE (6.13), we study the RBSDE of the form

$$y_t=\xi_\tau-\int_{]\!]t,\tau]\!]}f^r_s(y_{s-},y_s,z_s)\,dD^r_s-\int_{]\!]t,\tau]\!]}z_s\,dM_s+l_\tau-l_t \tag{6.14}$$

where a solution $(y,z,l)\in\mathcal{O}(\mathbb{F})\times\mathcal{P}_d(\mathbb{F})\times\mathcal{P}(\mathbb{F})$ is such that y is a càdlàg process and l is a càdlàg, increasing process such that $(\mathbb{1}_{\{y_-\neq\xi_-\}}\bullet l)_\tau=0$ and $l_0=0$. In the following, we consider a more general RBSDE of the form

$$y_t=\xi_\tau-\int_{]\!]t,\tau]\!]}f^r_s(y_{s-},y_s,z_s)\,dD^c_s-\sum_{t<s\le\tau}h(s,y_{s-},y_s)-\int_{]\!]t,\tau]\!]}z_s\,dM_s+l_\tau-l_t \tag{6.15}$$

where a solution $(y,z,l)\in\mathcal{O}(\mathbb{F})\times\mathcal{P}_d(\mathbb{F})\times\mathcal{P}(\mathbb{F})$ is such that y is a càdlàg and l is a càdlàg increasing process such that $(\mathbb{1}_{\{y_-\neq\xi_-\}}\bullet l^r)_\tau=0$ and $l_0=0$.

To recover equation (6.14) from (6.15), it suffices to set $h(s,y_{s-},y_s):=f^r_s(y_{s-},y_s)\Delta D^r_s$. In view of this observation, we further suppose that $h=0$ outside the graph of a finite family of $\mathbb{F}$-predictable stopping times $(T_i)_{i=1,2,\dots,p}$ and we denote $S_0=0$, $S_i=T_i\wedge\tau$ and $S_{p+1}=\tau$. To examine the existence of a solution to the RBSDE (6.15), we introduce an auxiliary RBSDE

$$\begin{aligned}v_t=\xi_\tau-h(\tau,v_{\tau-},\xi_\tau)-\int_{]\!]t,\tau]\!]}f^r_s(v_{s-},v_s,z_s)\,dD^c_s\\ -\sum_{t\le s<\tau}h(s,v_{s-},v_{s+})-\int_{]\!]t,\tau]\!]}z_s\,dM_s+l_\tau-l_t\end{aligned} \tag{6.16}$$

where a solution (v,z,l) is such that v is a làglàd, $\mathbb{F}$-adapted process, the process z is $\mathbb{F}$-predictable and the process l obeys the Skorokhod condition $(\mathbb{1}_{\{v_-\neq\xi_-\}}\bullet l)_\tau=0$ and $l_0=0$.

Proposition 6.8. *Let v be a làglàd process such that (v,z,l) is a solution to the RBSDE* (6.16) *on $[\![0,\tau]\!]$. Then (y,z,l) where $y:=v_+\mathbb{1}_{[\![0,\tau[\![}+h(\tau,v_{\tau-},\xi_\tau)\mathbb{1}_{[\![\tau]\!]}$ solves the càdlàg RBSDE* (6.15) *on $[\![0,\tau]\!]$.*

Proof. Suppose that (v,z,l) is a solution to (6.15). It is clear from (6.15) that the left-hand and right-hand jumps of v are given by $\Delta v = z\Delta M + \Delta l$ and $\Delta^+ v = h(\cdot, v_-, v_+)$, respectively. Note that l must satisfy the reflection condition $(v_{S-} - \xi_{S-})\Delta l_S = 0$ and, by the optional sampling theorem, we have that $\mathbb{E}[v_S \,|\, \mathcal{F}_{S-}] = v_{S-} + \Delta l_S$ for any $\mathbb{F}$-stopping time S. Then, by solving these two equations, we obtain

$$v_{S-} = \xi_{S-} \vee \mathbb{E}[v_S \,|\, \mathcal{F}_{S-}], \quad \Delta l_S = \big(\xi_{S-} - \mathbb{E}[v_S \,|\, \mathcal{F}_{S-}]\big)^+.$$

Therefore, if the random variable v_{S+} is known, then the $\mathcal{F}_S$-measurable random variable v_S is a solution to the equation

$$\Delta^+ v_S := v_{S+} - v_S = h\big(S, \xi_{S-} \vee \mathbb{E}[v_S \,|\, \mathcal{F}_{S-}], v_{S+}\big).$$

Recall that, by assumption about h, the right-hand jump times of v are given by the family $(T_i)_{i=1,2,\dots,p}$ of $\mathbb{F}$-predictable stopping times and we denote $S_0 = 0$, $S_i = T_i \wedge \tau$ and $S_{p+1} = \tau$. We observe that a solution (v,z,l) can be constructed by first solving by iteration, for every $i = 0,1,\dots,p$, the following càdlàg RBSDE on the stochastic interval $[\![S_i, S_{i+1}]\!]$

$$v^i_t = \xi^i - \int_{]\!]t,S_{i+1}]\!]} f^r_s(v^i_{s-}, v^i_s, z^i_s)\, dD^c_s - \int_{]\!]t,S_{i+1}]\!]} z^i_s\, dM_s + l^i_{S_{i+1}} - l^i_t$$

where, for every $i = 0,1,\dots,p$, we have $\mathbb{1}_{\{v^i_- \neq \xi_-\} \cap]\!]S_i,S_{i+1}]\!]} \bullet l^i = 0$ and we denote by ξ^i an $\mathcal{F}_{S_{i+1}}$-measurable random variable satisfying $v^{p+1}_{S_{p+1}} = \xi_\tau$ and

$$v^{i+1}_{S_{i+1}} - \xi^i = h\big(S_{i+1}, \xi_{S_{i+1}-} \vee \mathbb{E}[\xi^i \,|\, \mathcal{F}_{S_{i+1}-}], v^{i+1}_{S_{i+1}}\big).$$

We aggregate the family of solutions (v^i, z^i) for $i = 0,1,\dots,p$ by setting $l^{-1}_0 = 0$ and

$$v = v^0_0 + \sum_{i=0}^p v^i \mathbb{1}_{]\!]S_i,S_{i+1}]\!]}, \quad z = z^0_0 + \sum_{i=0}^p z^i \mathbb{1}_{]\!]S_i,S_{i+1}]\!]}, \quad l = \sum_{i=0}^p (l^{i-1}_{S_i} + l^i)\mathbb{1}_{]\!]S_i,S_{i+1}]\!]}.$$

Using the Itô formula, one can check that (v,z,l) satisfies the làglàd RBSDE (6.15) and l is increasing and satisfies $(\mathbb{1}_{\{v_- \neq \xi_-\}} \bullet l)_\tau = 0$. Furthermore, since $\Delta y_S = v_{S+} - v_{S-}$ and the dynamics of y and v (see (6.15) and (6.16), respectively) are easily seen to coincide on $]\!]S_i, S_{i+1}[\![$, we conclude that $(y,z,l) := (v_+, z, l)$ is a solution to the càdlàg RBSDE (6.15) once we made the appropriate adjustment to the last jump of size h at the terminal time τ, since $v_- = y_-$ and $(\mathbb{1}_{\{v_- \neq \xi_-\}} \bullet l)_\tau = 0$. ☐

Example 6.9. Here we show that if appropriate conditions are imposed on the inputs data (f,D,M), then a unique solution (v^i, z^i) to (6.8) can be obtained on each interval $[\![S_i, S_{i+1}]\!]$ for $i = 0,1,\dots,p$ and thus a solution (y,z,l) to (6.4) can be constructed. In the following, we assume that the process $\langle M\rangle$ is continuous, the function h does not depend on v_- and

$$f^r(v_-, v, z) \bullet D^c = f(v_-, v, z) \bullet \langle M\rangle + g(v) \bullet C$$

where C is an $\mathbb{F}$-adapted, continuous, increasing process. Furthermore, f and g are some real-valued mappings that satisfies appropriate measurability conditions.

We note that since h does not depend on v_-, the assumption that the jumps of D occur at $\mathbb{F}$-predictable stopping times can be relaxed and the right-hand jumps of v are given by $\Delta^+ v_t = h(t, v_{t+})$. Hence one is required to solve the following càdlàg RBSDE with continuous drivers, on each stochastic interval $[\![S_i, S_{i+1}]\!]$ for $i = 0,1,\dots,p$,

$$\begin{aligned} dv^i_t &= -f_t(v^i_t, z^i_t)\, dt - g_t(v^i_t)\, dD^c_t - z^i_t\, dM_t + dl^i_t, \\ v^i_{S_{i+1}} &= v^{i+1}_{S_{i+1}} - h\big(S_{i+1}, v^{i+1}_{S_{i+1}}\big), \end{aligned}$$

where $v^{p+1}_{S_{p+1}} = \xi$ and $l^i \in \mathcal{P}(\mathbb{F})$ is a càdlàg, increasing process with $l^i_0 = 0$ and such that the following equality holds

$$\mathbb{1}_{\{v^i_- \neq \xi_-\} \cap]\!]S_i, S_{i+1}]\!]} \bullet l^i = 0.$$

In the case where $\mathbb{F}$ is a Poisson filtration or, more generally, is generated by the *Teugels martingales* (see Nualart and Schoutens [43] or Schoutens and Teugels [49]), the existence and uniqueness of a solution (v^i, z^i, l^i) can be obtained by an application of Theorem 5 in Ren and El Otmani [48] under the postulate that f, g and h are bounded and Lipschitz continuous functions, the process D is bounded, and M is the compensated Poisson process.

7 Appendix

We assume that the process R is $\mathbb{F}$-optional and we define the làglàd process Q by

$$Q := K - R \bullet A^o + C = K - R \bullet A^o + C^r + C^g$$

where K is an $\mathbb{F}$-local martingale and C is a làdlàg process of finite variation. If Y is a làdlàg process of finite variation or, more generally, an optional semimartingale (which, by definition, is assumed to be a làdlàg process), then Y admits the decomposition $Y = Y^r + Y^g$ where $Y^g_t := \sum_{s<t}(Y_{s+} - Y_s)$ and the càdlàg process Y^r is given by $Y^r := Y - Y^g$.

Lemma 7.1. *Assume that $G > 0$. Then the process G^{-1} satisfies*

$$G^{-1} = G_0 - G_-^{-2} \bullet \widetilde{m} + G^{-1} \bullet \widetilde{\Gamma} \tag{7.1}$$

where $\widetilde{\Gamma} := \widetilde{G}^{-1} \bullet A^o$ and $\widetilde{m} := m - \widetilde{G}^{-1} \bullet [m, m]$. Moreover, for the càdlàg process

$$Q^r := K - R \bullet A^o + C^r$$

we have that

$$[Q^r, G^{-1}] = -G^{-1}G_-^{-1}\big([K, m] - [K, A^o] + [C^r, G]\big) - R\Delta G^{-1} \bullet A^o. \tag{7.2}$$

Proof. For brevity, we write $[G] := [G, G]$ and $[m] = [m, m]$. The Itô formula yields

$$G^{-1} = G_0^{-1} - G_-^{-2} \bullet G + G^{-1}G_-^{-2} \bullet [G] = G_0^{-1} - G_-^{-2} \bullet J \tag{7.3}$$

where $J := G - G^{-1} \bullet [G]$. Since $G = m - A^o$ and thus $\Delta G = \Delta m - \Delta A^o$, we obtain

$$\begin{aligned}[G] &= [m] - [m, A^o] + [A^o, A^o] = [m] - \Delta m \bullet A^o - (\Delta m - \Delta A^o) \bullet A^o \\ &= [m] - \Delta m \bullet A^o - \Delta G \bullet A^o\end{aligned}$$

so that

$$J = G - G^{-1} \bullet [G] = m_- A^o - G^{-1} \bullet [m] + G^{-1}\big(\Delta m \bullet A^o + \Delta G \bullet A^o\big).$$

Using the equalities $\widetilde{m} = m - \widetilde{G}^{-1} \bullet [m]$ and $\Delta m = \widetilde{G} - G_-$, we get

$$J = \widetilde{m} + \widetilde{G}^{-1} \bullet [m] - A^o - G^{-1} \bullet [m] - G^{-1}\big((\widetilde{G} - G_-) \bullet A^o - \Delta G \bullet A^o\big).$$

Since $\widetilde{G} - G = \Delta A^o$, we also have that

$$\begin{aligned}(\widetilde{G}^{-1} - G^{-1}) \bullet [m] &= G^{-1}\widetilde{G}^{-1}(G - \widetilde{G}) \bullet [m] = -G^{-1}\widetilde{G}^{-1}\Delta A^o \bullet [m] \\ &= -G^{-1}\widetilde{G}^{-1}(\Delta m)^2 \bullet A^o = -G^{-1}\widetilde{G}^{-1}(\widetilde{G} - G_-)^2 \bullet A^o.\end{aligned}$$

Consequently,

$$J = \widetilde{m} - A^o + G^{-1}\big(\widetilde{G} - G_- + \Delta G - \widetilde{G}^{-1}(\widetilde{G} - G_-)^2\big) \bullet A^o = \widetilde{m} - G^{-1}G_-^2 \bullet \widetilde{\Gamma},$$

which, when combined with (7.3), shows that (7.1) is valid. To establish (7.2), we first compute

$$\begin{aligned}[Q^r, G^{-1}] &= -G_-^{-2} \bullet [Q^r, G] + G^{-1}G_-^{-2} \bullet [Q^r, [G]] \\ &= -G_-^{-2} \bullet [Q^r, G] + G^{-1}G_-^{-1}\Delta G \bullet [Q^r, G] = -G^{-1}G_-^{-1} \bullet [Q^r, G].\end{aligned}$$

Finally, using the equalities $\Delta G = -GG_-\Delta G^{-1}$ and $G = m - A^o$, we obtain

$$\begin{aligned}[Q^r, G^{-1}] &= -G^{-1}G_-^{-1} \bullet [Q^r, G] = G^{-1}G_-^{-1}R \bullet [A^o, G] - G^{-1}G_-^{-1} \bullet \big([K, G] + [C^r, G]\big) \\ &= -G^{-1}G_-^{-1} \bullet \big([K, G] + [C^r, G]\big) - R\Delta G^{-1} \bullet A^o \\ &= -G^{-1}G_-^{-1} \bullet ([K, m] - [K, A^o] + [C^r, G]) - R\Delta G^{-1} \bullet A^o\end{aligned}$$

and thus equality (7.2) is proven as well. □

We maintain the assumption that $G > 0$ (and thus also $\widetilde{G} > 0$) and we consider the process

$$Y := G^{-1}Q = G^{-1}(K - R \bullet A^o + C). \tag{7.4}$$

Our goal is to derive the dynamics of Y in terms of $\widetilde{\Gamma}, \widetilde{m}$ and

$$\widetilde{K} := K - \widetilde{G}^{-1} \bullet [K, m].$$

In the proof of Lemma 7.3, we will employ the optional integration by parts formula. Recall that, by definition, any *semimartingale* is a càdlàg process but an *optional semimartingale* is not necessarily a càdlàg process although, by definition, it is a làdlàg process.

Let $X = X^r + X^g$ and $Y = Y^r + Y^g$ be làglàd optional semimartingales such as Y is of finite variation. Then the *optional integration by parts formula* reads (see Theorem 8.2 in Gal'čuk [24])

$$XY = X_0Y_0 + X \circ Y + Y \circ X + [X, Y] \tag{7.5}$$

where the *optional stochastic integrals* are given by

$$\begin{aligned}X \circ Y &= X_- \bullet Y^r + X \star Y_+^g, \\ Y \circ X &= Y_- \bullet X^r + Y \star X_+^g,\end{aligned}$$

where X_+^g (respectively, Y_+^g) is the càdlàg version of the càglàd process X^g (respectively, Y^g) and the quadratic covariation $[X, Y]$ equals

$$[X, Y]_t = \sum_{0<s\leq t} \Delta X_s \Delta Y_s + \sum_{0\leq s<t} \Delta^+ X_s \Delta^+ Y_s$$

where we denote $\Delta X_t = X_t - X_{t-}$ and $\Delta^+ X_t = X_{t+} - X_t$.

For the reader's convenience, we formulate a variant of the optional integration by parts formula (7.5), which holds when $X = X^r$ is a (càdlàg) semimartingale and $Y = Y^g$ is a càglàd process of finite variation.

Lemma 7.2. *Let $X = X^r$ be a semimartingale and let $Y = Y^g$ be a càglàd process of finite variation. Then the process XY is làdlàg and satisfies, for every $0 \leq s < t$,*

$$X_tY_t = X_sY_s + \int_{]\!]s,t]\!]} Y_u\, dX_u + \int_{[\![s,t[\![} X_u\, dY_{u+}^g \tag{7.6}$$

where Y_+^g is the càdlàg version of Y^g.

We will use the shorthand notation for (7.6)

$$XY = X_0Y_0 + Y \bullet X + X \star Y^g_+$$

but all equalities in the proof of Lemma 7.3 should be understood in the sense of (7.6), meaning that all integrals with respect to a càdlàg (respectively, càglàd) process should be evaluated on the interval $\rrbracket s,t\rrbracket$ (respectively, on the interval $\llbracket s,t\llbracket$) for arbitrary $0 \le s < t$.

Lemma 7.3. *If the process Y is given by* (7.4) *where C is a làdlàg process of finite variation with the decomposition $C = C^r + C^g$, then*

$$Y_t = Y_0 - \int_{\rrbracket 0,t\rrbracket}(R_s - Y_s)\, d\widetilde{\Gamma}_s - \int_{\rrbracket 0,t\rrbracket} Y_{s-}G^{-1}_{s-}\, d\widetilde{m}_s + \int_{\rrbracket 0,t\rrbracket} G^{-1}_{s-}\, d\widetilde{K}_s + \int_{\rrbracket 0,t\rrbracket} \widetilde{G}^{-1}_s\, dC^r_s + \int_{\llbracket 0,t\llbracket} G^{-1}_s\, dC^g_{s+}. \tag{7.7}$$

Proof. We note that Q satisfies

$$Q = K - R \bullet A^o + C = K - R \bullet A^o + C^r + C^g = Q^r + C^g$$

where

$$Q^r = K - R \bullet A^o + C^r = \widetilde{K} + \widetilde{G}^{-1} \bullet [K,m] - R \bullet A^o + C^r.$$

The integration by parts formulas applied to $Y = G^{-1}Q = G^{-1}Q^r + G^{-1}C^g$ gives

$$\begin{aligned} Y &= G^{-1}Q^r + G^{-1}C^g \\ &= Y_0 + Q_- \bullet G^{-1} + G^{-1}_- \bullet Q^r + [Q^r, G^{-1}] + G^{-1} \star C^g_+ \end{aligned} \tag{7.8}$$

since G^{-1} and Q^r are (càdlàg) semimartingales and thus the Itô integration by parts formula applied to the product $G^{-1}Q^r$ yields

$$G^{-1}Q^r = Q^r_- \bullet G^{-1} + G^{-1}_- \bullet Q^r + [Q^r, G^{-1}]$$

whereas the optional integration by parts formula from Lemma 7.2 gives

$$G^{-1}C^g = C^g \bullet G^{-1} + G^{-1} \star C^g_+.$$

From (7.2) and (7.8), we obtain

$$\begin{aligned} Y &= Y_0 - Q_-\big(G^{-2}_- \bullet \widetilde{m} - G^{-1} \bullet \widetilde{\Gamma}\big) + G^{-1}_- \bullet \big(\widetilde{K} + \widetilde{G}^{-1} \bullet [K,m] - R \bullet A^o + C^r\big) \\ &\quad - G^{-1}_-G^{-1}\big([K,m] - [K,A^o] + [C^r,G]\big) - R\Delta G^{-1} \bullet A^o + G^{-1} \star C^g_+ \\ &= Y_0 - Y_-G^{-1}_- \bullet \widetilde{m} + G^{-1}_- \bullet \widetilde{K} + K + H \end{aligned} \tag{7.9}$$

where

$$\begin{aligned} K &:= G^{-1}_- \bullet C^r + G^{-1} \star C^g_+ - G^{-1}G^{-1}_- \bullet [C^r, G] \\ &= G^{-1}_- \bullet C^r + G^{-1} \star C^g_+ - G^{-1}G^{-1}_-\Delta G \bullet \Delta C^r = G^{-1} \bullet C^r + G^{-1} \star C^g_+ \end{aligned}$$

and

$$\begin{aligned} H &:= G^{-1}G^{-1}_-\Delta K \bullet A^o + G^{-1}_-\big(\widetilde{G}^{-1} - G^{-1}\big) \bullet [K,m] \\ &\quad - RG^{-1} \bullet A^o + Y_-G^{-1}G_-\widetilde{G}^{-1} \bullet A^o = \sum_{i=1}^{4} H^i. \end{aligned}$$

We first recall that $\Delta A^o = \widetilde{G} - G$ and $\Delta m = \widetilde{G} - G_-$. Therefore,

$$\begin{aligned} H^1 + H^2 &= G^{-1}G_-^{-1}\Delta K \bullet A^o + G_-^{-1}\big(\widetilde{G}^{-1} - G^{-1}\big) \bullet [K,m] \\ &= G^{-1}G_-^{-1}\big(\Delta K \bullet A^o - \widetilde{G}^{-1}\Delta A^o \bullet [K,m]\big) = G^{-1}G_-^{-1}\big(\Delta K \bullet A^o - \widetilde{G}^{-1}\Delta K\Delta m \bullet A^o\big) \\ &= G^{-1}G_-^{-1}\big(\Delta K - \widetilde{G}^{-1}(\widetilde{G} - G_-)\Delta K\big) \bullet A^o = G^{-1}\widetilde{G}^{-1}\Delta K \bullet A^o. \end{aligned}$$

Next, we deduce from (7.4) that

$$\Delta K = \Delta(YG) + R\Delta A^o - \Delta C^r = \Delta(YG) + R(\widetilde{G} - G) - \Delta C^r$$

and thus

$$\begin{aligned} H &= G^{-1}\widetilde{G}^{-1}\big(\Delta(YG) + R(\widetilde{G} - G) - \Delta C^r\big) \bullet A^o - RG^{-1} \bullet A^o + Y_- G^{-1}G_-\widetilde{G}^{-1} \bullet A^o \\ &= G^{-1}\widetilde{G}^{-1}\big(YG + R(\widetilde{G} - G) - \Delta C^r\big) \bullet A^o - RG^{-1} \bullet A^o \\ &= \widetilde{G}^{-1}Y \bullet A^o - G^{-1}\widetilde{G}^{-1}\Delta C^r \bullet A^o - R\widetilde{G}^{-1} \bullet A^o \\ &= \widetilde{G}^{-1}Y \bullet A^o - G^{-1}\widetilde{G}^{-1}\Delta A^o \bullet C^r - R\widetilde{G}^{-1} \bullet A^o \\ &= (Y - R) \bullet \widetilde{\Gamma} + (\widetilde{G}^{-1} - G^{-1}) \bullet C^r. \end{aligned}$$

To complete the derivation of (7.7), it suffices to substitute K and H into (7.9). □

In the next lemma, which is a counterpart of Lemma 7.3, we study the dynamics of the process $\widetilde{Y}$ given by

$$\widetilde{Y} := \widetilde{G}^{-1}\big(M - (R \star A^o)\big) \tag{7.10}$$

where the $\mathbb{F}$-martingale M is defined as follows, for every $t \in [0,T]$,

$$M_t := \mathbb{E}_{\mathbb{P}}\big[P_T\widetilde{G}_T + (R \star A^o)_T \,|\, \mathcal{F}_t\big].$$

Notice that the process $\widetilde{Y}$ given by (7.10) is related to the payoff $P_T\mathbb{1}_{\{\vartheta \geq T\}} + R_\vartheta\mathbb{1}_{\{\vartheta < T\}}$, which is encountered in credit risk modeling (see, e.g., Li et al. [38]). We recall that

$$(R \star A^o)_t = \int_{[0,t[} R_s\, dA^o_s$$

and thus $R \star A^o$ is a left-continuous process.

Lemma 7.4. *If $\widetilde{G}_0 = G_0$ then the process $\widetilde{Y}$ given by* (7.10) *satisfies*

$$\widetilde{Y} = \widetilde{Y}_0 + G_-^{-1} \bullet \widetilde{M} - \widetilde{Y}_- G_-^{-1} \bullet \widetilde{m} - (R - \widetilde{Y}_+) \star \widetilde{\Gamma} \tag{7.11}$$

where $\widetilde{M} := M - \widetilde{G}^{-1} \bullet [M,m]$ and $\widetilde{m} := m - \widetilde{G}^{-1} \bullet [m,m]$.

Proof. Note that

$$\widetilde{Y} = \widetilde{G}^{-1}(M - R \star A^o) = \widetilde{G}^{-1}H$$

where we denote $H = M - R \star A^o$. Since

$$\widetilde{G}^{-1} = G_0 - G_-^{-2} \bullet \widetilde{m} + G^{-1} \star \widetilde{\Gamma}$$

the integration by parts formula gives

$$\begin{aligned} \widetilde{Y} &= \widetilde{G}^{-1}\,(M - R \star A^o) = \widetilde{G}^{-1}H \\ &= \widetilde{G}_-^{-1} \bullet H^r + \widetilde{G}^{-1} \star H^g + H_- \bullet (\widetilde{G}^r)^{-1} + H \star (\widetilde{G}^g)^{-1} + [H^r, (\widetilde{G}^r)^{-1}] + [H^g, (\widetilde{G}^g)^{-1}] \\ &= \widetilde{G}_-^{-1} \bullet H^r - R \star \widetilde{\Gamma}^g + H_- \bullet (\widetilde{G}^r)^{-1} + H_+ G^{-1} \star \widetilde{\Gamma}^g + [H^r, (\widetilde{G}^r)^{-1}] \\ &= \widetilde{G}_-^{-1} \bullet H^r + H_- \bullet (\widetilde{G}^r)^{-1} + [H^r, (\widetilde{G}^r)^{-1}] - \big(R - H_+ G^{-1}\big) \star \widetilde{\Gamma}^g \\ &= \widetilde{G}_-^{-1} \bullet M - R \bullet \widetilde{\Gamma}^c - H_- \bullet G_-^{-2} \bullet \widetilde{m} + H_- G^{-1} \bullet \widetilde{\Gamma}^c + [H^r, (\widetilde{G}^r)^{-1}] - \big(R - H_+ G^{-1}\big) \star \widetilde{\Gamma}^g \\ &= \widetilde{G}_-^{-1} \bullet M - H_- G_-^{-2} \bullet \widetilde{m} + G_-^{-2} \bullet [M, \widetilde{m}] - \big(R - H_-(G_-)^{-1}\big) \bullet \widetilde{\Gamma}^c - \big(R - H_+ G^{-1}\big) \star \widetilde{\Gamma}^g. \end{aligned}$$

Using the equalities $\widetilde{Y} = \widetilde{G}^{-1}H, \widetilde{M} = M - \widetilde{G}^{-1} \bullet [M, m]$ and

$$\widetilde{G}_-^{-1} \bullet M = \widetilde{G}_-^{-1} \bullet \widetilde{M} + \widetilde{G}_-^{-1}\widetilde{G}^{-1} \bullet [M, m]$$
$$G_-^{-2} \bullet [M, \widetilde{m}] = G_-^{-2} \bullet [M, m] - G_-^{-2}\widetilde{G}^{-1}\Delta m \bullet [M, m] = -\widetilde{G}G_-^{-1} \bullet [M, m]$$

combined with the property that $\widetilde{Y}$ has at most countable number of jumps, we obtain

$$\begin{aligned}\widetilde{Y} &= G_-^{-1} \bullet \widetilde{M} - \widetilde{Y}_- G_-^{-1} \bullet \widetilde{m} - (R - \widetilde{Y}_-) \bullet \widetilde{\Gamma}^c - (R - \widetilde{Y}_+) \star \widetilde{\Gamma}^g \\ &= G_-^{-1} \bullet \widetilde{M} - \widetilde{Y}_- G_-^{-1} \bullet \widetilde{m} - (R - \widetilde{Y}_+) \star \widetilde{\Gamma}\end{aligned}$$

as was required to show. □

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