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Local limit theorems for a directed random walk on the backbone of a supercritical oriented percolation cluster*

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Abstract

We consider a directed random walk on the backbone of the supercritical oriented percolation cluster in dimensions $d+1$ with $d \geq 3$ being the spatial dimension. For this random walk we prove an annealed local central limit theorem and a quenched local limit theorem. The latter shows that the quenched transition probabilities of the random walk converge to the annealed transition probabilities reweighted by a function of the medium centred at the target site. This function is the density of the unique measure which is invariant for the point of view of the particle, is absolutely continuous with respect to the annealed measure and satisfies certain concentration properties.

Keywords: random walk in dynamical random environment; oriented percolation; supercritical cluster; quenched local limit theorem in random environment; environment viewed from the particle.
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1 Introduction

Random walks in a static or dynamic random environment arise in different models from physical and biological sciences. The investigation of such random walks under different conditions on the environment has been an active research area with a lot of recent progress. In this paper, we analyse a *directed* random walk on the backbone of a supercritical oriented percolation cluster on $\mathbb{Z}^d \times \mathbb{Z}$. This random walk was introduced

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and studied in [4]. There it was shown that the random walk satisfies a law of large numbers and a quenched central limit theorem for all spatial dimensions $d \geq 1$. The main purpose of this work is to extend these results and derive a quenched *local* limit theorem. For this, we will have to restrict ourselves to spatial dimensions $d \geq 3$. Analogous results for a class of ballistic random walks in uniformly elliptic i.i.d. random environments were recently obtained in [2]. This paper has been an inspiration and a guide for the present study.

1.1 The model and background

Consider a discrete space-time field $\omega \coloneqq \{\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ of i.i.d. Bernoulli random variables with parameter $p \in [0,1]$, defined on some (large enough) probability space equipped with a probability measure $\mathbb{P}$. We denote the set of possible values for ω by $\Omega \coloneqq \{0,1\}^{\mathbb{Z}^d \times \mathbb{Z}}$, which we endow with the product topology.

As common in percolation theory, a space-time site $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ is said to be *open* if $\omega(x,n) = 1$ and *closed* if $\omega(x,n) = 0$. A directed *open path* (with respect to ω) from (x,m) to (y,n) for $m \leq n$ is a space-time sequence $(x_m, m), \dots, (x_n, n)$ such that $x_m = x$, $x_n = y$, $\|x_k - x_{k-1}\| \leq 1$ for $k = m+1, \dots, n$ and $\omega(x_k, k) = 1$ for all $k = m, \dots, n$. Here, and in the following $\|\cdot\|$ denotes the sup-norm on $\mathbb{R}^d$. We will write $(x,m) \xrightarrow{\omega} (y,n)$ if such an open path exists and $(x,m) \xrightarrow{\omega} \infty$ if there exists at least one infinite directed open path starting at (x,m), i.e. if for each $n > m$ there is $y \in \mathbb{Z}^d$ so that $(x,m) \xrightarrow{\omega} (y,n)$.

It is well known that there is $p_c = p_c(d) \in (0,1)$ such that $\mathbb{P}\big((0,0) \xrightarrow{\omega} \infty\big) > 0$ if and only if $p > p_c$; see e.g. Theorem 1 in [15]. We consider here only the case of a fixed $p \in (p_c, 1]$. We define by

$$\mathcal{C} \coloneqq \big\{(x,n) \in \mathbb{Z}^d \times \mathbb{Z} : (x,n) \xrightarrow{\omega} \infty\big\} \tag{1.1}$$

the *backbone of the space-time cluster of the oriented percolation*, i.e. the set of all space-time sites which are connected to "time $+\infty$" by a directed open path. Note that $\mathcal{C}$ depends on ω and that we have $\mathbb{P}(|\mathcal{C}| = \infty) = 1$ for $p > p_c$. For future reference we define the process $\xi \coloneqq (\xi_n)_{n \in \mathbb{Z}}$ on $\{0,1\}^{\mathbb{Z}^d}$ by

$$\xi_n(x) = \mathbb{1}_{\mathcal{C}}\big((x,n)\big). \tag{1.2}$$

The process ξ can be interpreted as the time reversal of the stationary discrete time contact process. In particular, for any $n \in \mathbb{Z}$ the random field $\xi_n(\cdot)$ is distributed according to the upper invariant measure of the discrete time contact process, which is non-trivial in the case $p > p_c$. For more details we refer the reader to Section 1 (around equation (1.2)) in [4], see also [5].

Our goal is to study the directed random walk on the cluster $\mathcal{C}$. This random walk was studied in [4] in the case that the initial point of the random walk belongs to the cluster. Here we want to compare the annealed and quenched laws for starting points without checking whether they are on the cluster or not. Thus, we define the random walk slightly differently: It behaves as a simple random walk (which jumps uniformly to one of the sites in the unit ball around the present site) as long as it is not on the cluster and once it hits the cluster it will behave as the random walk from [4]. For a site $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ we define its *neighbourhood* at time $(n+1)$ by

$$U(x,n) \coloneqq \{(y, n+1) : \|x - y\| \leq 1\}. \tag{1.3}$$

Given ω and therefore the random cluster $\mathcal{C}$ and $(y,m) \in \mathbb{Z}^d \times \mathbb{Z}$ we consider random walks $(X_n)_{n \geq m}$ with initial position $X_m = y$ and transition probabilities for $n \geq m$ given by

$$\mathbb{P}(X_{n+1} = z \,|\, X_n = x, \omega) = \begin{cases} |U(x,n) \cap \mathcal{C}|^{-1} & \text{if } (x,n) \in \mathcal{C} \text{ and } (z, n+1) \in \mathcal{C}, \\ |U(x,n)|^{-1} & \text{if } (x,n) \notin \mathcal{C}, \end{cases} \tag{1.4}$$

for $(z, n+1) \in U(x, n)$, and otherwise $\mathbb{P}(X_{n+1} = z \,|\, X_n = x, \omega) = 0$. We write P_ω for the conditional law of $\mathbb{P}$ given ω and E_ω for the corresponding expectation. In particular, for the transition probabilities we have

$$P_\omega(X_{n+1} = z \,|\, X_n = x) = \mathbb{P}(X_{n+1} = z \,|\, X_n = x, \omega). \tag{1.5}$$

For the above random walk starting from position $X_m = y \in \mathbb{Z}^d$ at time $m \in \mathbb{Z}$ we denote by $P_\omega^{(y,m)}$ its *quenched law* and by $E_\omega^{(y,m)}$ the corresponding expectation. The *annealed (or averaged) law* of that random walk is denoted by $\mathbb{P}^{(y,m)}$ and its expectation by $\mathbb{E}^{(y,m)}$. Note that for any $A \in \sigma(X_n : n = m, m+1, \dots)$ we have

$$\mathbb{P}^{(y,m)}(A) = \int P_\omega^{(y,m)}(A)\, d\mathbb{P}(d\omega). \tag{1.6}$$

1.2 Main results: annealed and quenched local limit theorems

In Theorem 1.1 in [4] it is shown that the random walk (X_n) starting in $0 \in \mathbb{Z}^d$ at time 0 satisfies an annealed central limit theorem and the limiting law is a non-trivial centred isotropic d-dimensional normal law. In particular its covariance matrix Σ is of the form $\sigma^2 I_d$ for a positive constant σ^2 and the d-dimensional identity matrix I_d. Recall that in [4] it is assumed that the space-time origin is contained in $\mathcal{C}$ so that the random walk starts and stays on $\mathcal{C}$. This is not a big constraint because the time a random walk needs to hit the cluster $\mathcal{C}$ has exponentially decaying tails; see Lemma B.1 in the appendix.

The annealed CLT from [4] can be strengthened to an annealed *local* CLT. For a proof of the following theorem we refer to Section 3.

Theorem 1.1 (Annealed local CLT)**.** *For $d \geq 1$ and Σ as above we have*

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \Big| \mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Sigma}} \exp\Big(-\frac{1}{2n} x^T \Sigma^{-1} x\Big) \Big| = 0. \tag{1.7}$$

Theorem 3.1 in [4] extends the annealed CLT therein to a quenched version with the same limiting law. Thus, the quenched and annealed laws after N steps are comparable on the level of boxes of side length $N^{1/2}$. This result was later refined in [21, Chapter 3], where a comparison result between the quenched and annealed laws after N steps on the level of boxes of side length $N^{\theta/2}$ for $\theta \in (0,1)$ was obtained. (We recall this result in Theorem 8.1 below.)

The main goal of this paper is to strengthen this further and prove a quenched local limit theorem which is an analogue of Theorem 1.1. In order to state the precise result, we need to introduce some notation. First, for $(y, m) \in \mathbb{Z}^d \times \mathbb{Z}$, we define the *space-time shift operator* σ on Ω by

$$\sigma_{(y,m)}\omega(x, n) := \omega(x + y, n + m) \tag{1.8}$$

and we write $\xi_m(y; \omega)$ for $\xi_m(y)$ read off from a given realization ω as in (1.1) and (1.2). We define the transition kernel for the environment seen from the particle (compare this with (1.4)) by

$$\mathfrak{R} f(\omega) := \sum_{\|y\|\leq 1} g(y; \omega) f(\sigma_{(y,1)}\omega) \tag{1.9}$$

acting on bounded measurable functions $f : \Omega \to \mathbb{R}$, where

$$g(y; \omega) := \mathbb{1}_{\{\sum_{\|z\|\leq 1} \xi_1(z;\omega) > 0,\, \omega(0,0)=1\}} \frac{\xi_1(y; \omega)}{\sum_{\|z\|\leq 1} \xi_1(z; \omega)} + \mathbb{1}_{\{\sum_{\|z\|\leq 1} \xi_1(z;\omega)=0 \text{ or } \omega(0,0)=0\}} \frac{1}{3^d}. \tag{1.10}$$

Definition 1.2. *A measure Q on Ω is called* invariant with respect to the point of view of the particle *if for every bounded continuous function $f : \Omega \to \mathbb{R}$*

$$\int_\Omega \mathfrak{R}f(\omega)\, dQ(\omega) = \int_\Omega f(\omega)\, dQ(\omega). \tag{1.11}$$

Theorem 1.3. *Let $d \geq 3$ and $p \in (p_c, 1]$. Then there exists a unique measure Q on Ω which is invariant with respect to the point of view of the particle satisfying $Q \ll \mathbb{P}$ and the concentration property* (2.9) *below.*

The main result of this paper is a quenched local limit theorem which is an analogue of Theorem 1.11 in [2] in the case of our model.

Theorem 1.4 (Quenched local limit theorem)**.** *Let $d \geq 3$ and $p \in (p_c, 1]$, let Q be the measure from Theorem 1.3 and denote by $\varphi = dQ/d\mathbb{P} \in L_1(\mathbb{P})$ the Radon–Nikodym derivative of Q with respect to $\mathbb{P}$. Then for $\mathbb{P}$ almost every ω we have*

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \left| P_\omega^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)\right| = 0. \tag{1.12}$$

Remark 1.5 (Theorem 1.4 for $d = 1, 2$)**.** While this paper was under review, inspired by work of Tal Peretz [20], we found a way to prove the crucial quenched annealed-comparison result Theorem 8.1, which we quote from [21] for $d \geq 3$, also for $d = 1, 2$. Consequently, all intermediate results can be carried over to $d = 1, 2$ and this allows to prove Theorem 1.4 for $d = 1, 2$ as well. The details will be presented in future work.

Remark 1.6 (Uniqueness of Q)**.** By a general argument (see e.g. [2, Section 7.1]), when it exists the function φ in (1.12) is $\mathbb{P}$ almost surely unique. Furthermore, it will follow from the arguments in the proofs below (cf. also Remark 2.6) that if a measure Q' on Ω is invariant with respect to the point of view of the particle and satisfies $Q' \ll \mathbb{P}$ and (1.12) with $\varphi' = dQ'/d\mathbb{P}$ then this measure Q' satisfies the concentration property (2.9) as well and thus in particular agrees with Q.

Related literature Random walks in static and dynamic random environments is a very active research area. For a review of random walks in random environments and basic concepts and objects we refer the reader to [22]; for a more recent review see [12].

The random walk that we consider here can be seen as a random walk in a dynamic random environment. Its relation to random walks in dynamic random environments in the literature is discussed in some detail in [4, Remark 1.7]. The main differences are that the random environment is not uniformly elliptic and is not i.i.d. In fact the environment that we consider here has even infinite range dependencies, due to the fact that the steps of the random walk are restricted to the backbone of the oriented percolation cluster once it hits the cluster. The environment also does not satisfy mixing conditions such as (conditional) cone-mixing in contrast to the model considered in [16]. In [7] a much weaker mixing assumption than cone-mixing is introduced (literally for a continuous time model) and our model does satisfy their assumption. However, they only prove a LLN for a nearest neighbour random walk in $d = 1$. A comprehensive overview of the recent literature on random walks in dynamic random environments can be found in the introduction of [7]. See also [6, Remark 1.1].

Results on quenched local limit theorems for random walks in random environments are very recent. Our research is inspired by [2] where a quenched local limit theorem was shown (in dimension $d \geq 4$) for the case of an i.i.d. random environment and where the random walk satisfies a ballisticity criterion and has uniformly elliptic transition probabilities. (Note that ballisticity in the "time" direction is trivial in our model. Uniform ellipticity and the i.i.d. property are not satisfied.)

Other results on local limit theorems in random environments that we are aware of are concerned with specific models. In [9] the quenched local CLT is proven for random walks in a time-dependent balanced random environment. In [10] and [11] quenched local limit theorems are obtained for random walks in random environments on a strip. A different class of random walks in random media for which quenched local CLTs have been obtained are the so called random conductance models. For a recent work in this direction and an overview of the literature see [1] and references therein. In a continuous set-up, [13] recently proved a local limit theorem for a diffusion in a Gaussian random medium which is white in time.

Outlook and open questions While we do exhibit a measure Q which is invariant with respect to the point of view of the particle and absolutely continuous with respect to $\mathbb{P}$, we can currently establish uniqueness only in the class of such measures satisfying the additional property (2.9), see Remark 1.6. Furthermore, because of non-ellipticity, Q is not equivalent to $\mathbb{P}$, see the discussion in Remark 2.6 below. We leave open the questions whether property (2.9) is necessary for uniqueness and whether Q is equivalent to $\mathbb{P}$ when restricted to the set $\widetilde{\Omega}$ from (2.11) in Remark 2.6.

We restrict our analysis in this paper to the case $d \geq 3$. This is essentially owed to the fact that Theorem 8.1, which we quote from [21, Thm. 3.24], is presently only available under this assumption. It was proved there using an environment exposure technique from [8], which was also used by [2], and the proof exploited the fact that in dimension at least 3, two independent random walks will almost surely meet only finitely often, irrespective of the number N of steps they take.

As already mentioned in Remark 1.5, since the submission of this paper we were inspired by [20] and realized that we are also able to prove Theorem 1.4 for $d = 1, 2$ since in the proof of Theorem 3.24 in [21] some estimates were more conservative than needed and it turns out that we can extend this theorem to $d = 1, 2$.

We prove in Theorem 1.4 a quenched local limit theorem for a very specific model of a non-elliptic random walk in a non-trivial dynamic random environment, and our proofs do exploit specific properties of this environment, namely the oriented percolation cluster. However, we think that this environment is prototypical for a large class of dynamic environments which can be "mapped" to it by suitable coarse-graining procedures, see [3], Section 3 and the concrete example in Section 4 there. It seems quite possible that given substantial technical effort, our approach to Theorem 1.4 could be extended to the class of environments from [3]. We leave this for future work.

Outline of the paper The proofs of the main results are long and quite technical. Let us describe the main ideas of the proofs and explain how the paper is organised: In Section 2 we first give several auxiliary results which we then use for the proofs of Theorem 1.3 and of Theorem 1.4.

Annealed estimates: In Section 3 we prove several annealed derivative estimates which build on, and extend somewhat, previous work by [21]. These estimates will be used for the proof of the annealed local CLT, Theorem 1.1, also presented in Section 3. Starting with Section 4 the paper is devoted to the proofs of the auxiliary results from Section 2.

Comparison of the quenched and annealed laws: Lemma 2.1, proven in Section 4, provides a comparison between the *quenched* and *annealed* laws on the level of large (but finite) boxes. In particular it shows that the total variation distance between $\mathbb{P}(X_N \in \cdot)$ and $P_\omega(X_N \in \cdot)$ on the level of boxes of side length $M \gg 1$ is small with very high probability as $N \to \infty$ in a suitably quantified way; see equation (2.1). The starting point of the proof of Lemma 2.1 is [21, Theorem 3.24], recalled in Theorem 8.1 below, which gives an analogous result for boxes whose size grows like $N^{\theta/2}$ with $0 < \theta < 1$ as $N \to \infty$, and therefore much slower than the diffusive scale $N^{1/2}$. We augment this with an iteration scheme that is guided by the proof of Theorem 5.1 in [2]. The main argument towards the proof of Lemma 2.1 is formulated as Proposition 4.1 which provides the crucial estimate for the iteration step. The proof of that proposition is long and relies to a large extent on ideas from [2] and is postponed to Section 8. It requires a suitable control of the density of "good" boxes on which an estimate as in equation (2.1) from Lemma 2.1 holds locally uniformly, see Definition 8.2. This deviates from the set-up in [2] because our environment is not i.i.d. and in fact here the boxes are in principle correlated over arbitrary lengths, albeit weakly.

Measure for the point of view of the particle: The function $\varphi = dQ/d\mathbb{P}$ from (1.12) is the density of a measure Q which is invariant with respect to the point of view of the particle and absolutely continuous with respect to $\mathbb{P}$. For the existence of such a measure Q we consider the quenched laws Q_N of the environment seen from the particle after N steps of the walk; see (2.4). The measure Q is constructed as a weak limit of the Cesàro average of the measures Q_N along a subsequence; see (2.6) and (2.7). In Proposition 2.2 and Corollary 2.4 we show that averages of $dQ_N/d\mathbb{P}$ and $dQ/d\mathbb{P}$ over large boxes are close to one with high probability depending on the size of the boxes. It will turn out that the measure Q which we obtain as described above is unique, i.e. it does not depend on the particular subsequence; see Remark 2.6.

Proposition 2.2 and Corollary 2.4 are proven in Section 5. To this end we construct a coupling of Q_N and P_N, the law of the environment viewed relative to the annealed walk (note that $P_N = \mathbb{P}$ for all N). Lemma 2.1 allows for a coupling which puts both walks in the same M-box with very high probability. We strengthen this to a coupling which puts both walks at exactly the same spatial position with uniformly non-vanishing probability; see the proof of Lemma 5.3.

Since we average over the environment in the definition of the annealed law of the random walk in equation (1.6) it is clear that the annealed random walk does not see any specific environment. In contrast to that the quenched random walk knows the exact environment it walks in. So, to compare the annealed and quenched laws of the random walk, the annealed walk needs to see the environment of the quenched random walk. This is done through reweighting by φ. In particular, a consequence of multiplying the annealed law with φ is that this product will be zero for all space-time points $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ in which the contact process ξ is 0 in the environment ω.

In Proposition 2.8 we show that the annealed law of the random walk at time n reweighted with the function φ converges for almost all ω to a probability law on $\mathbb{Z}^d$. It is proven in Section 6.

Hybrid measures: For the proof of Theorem 1.4, instead of comparing the quenched and annealed laws directly, we use the triangle inequality, some "hybrid" measures and space-time convolutions of quenched-annealed measures; see Definition 2.7. In Proposition 2.9, proven in Section 7, we show that the total variation distance of some of these measures converges to 0 as n, the number of steps of the random walk, goes to infinity. An essential tool of the proof of Proposition 2.9 is Lemma 7.1 in which we study the total variation distance of quenched laws of two random walks starting at different positions. The idea is to use couplings with the annealed measures on the level of large (growing) boxes combined with annealed derivative estimates in order to first ensure that the two walks are in the same box with probability bounded away from 0. Using connectivity properties of the oriented percolation clusters (see below) the above described procedure can be iterated to produce a literal coupling where the two walks coincide with high probability after sufficiently many steps. Lemma 7.1 is proven in Section 9.

Oriented percolation results: In the appendix, Section A, we show that two infinite percolation clusters intersect with high probability within a finite time. This result was pointed out in [15], who proved that two infinite clusters do intersect almost surely, but without the quantification of the time of intersection. Finally, in Section B, we show that the probability that a random walk started off the cluster does not hit the cluster within time t decays exponentially with t.

2 Proofs of the main results

In this section we collect several important auxiliary results and present towards the end of this section how to utilise them to prove Theorem 1.3 and Theorem 1.4. The proofs of the auxiliary results are postponed to the subsequent sections.

Our starting point is a lemma which can be seen as an adaptation of Theorem 5.1 in [2] to our setting. Recall between (1.5) and (1.6) the definitions of the quenched measure $P_\omega^{(x,m)}$ and the annealed measure $\mathbb{P}^{(x,m)}$ for the random walk $(X_n)_{n=m,m+1,\dots}$ with $X_m = x$. For any positive real number L we denote by Π_L a partition of $\mathbb{Z}^d$ into boxes of side length $\lfloor L \rfloor$.

Lemma 2.1. *Let $d \geq 3$. For $N, M \in \mathbb{N}$, $c, C > 0$ denote by $K(N) \coloneqq K(N, M, c, C)$ the set of environments $\omega \in \Omega$ such that for every $x \in \mathbb{Z}^d$ satisfying $\|x\| \leq N$*

$$\sum_{\Delta \in \Pi_M} \big| P_\omega^{(x,0)}(X_N \in \Delta) - \mathbb{P}^{(x,0)}(X_N \in \Delta) \big| \leq \frac{C}{M^c} + \frac{C}{N^c}. \tag{2.1}$$

If $c > 0$ is small enough and $C < \infty$ large enough, there are universal positive constants $\tilde{c}$, $\widetilde{C}$, for which we have

$$\mathbb{P}\big(K(N)\big) \geq 1 - \widetilde{C} N^{-\tilde{c} \log N} \quad \textit{for all } N. \tag{2.2}$$

In words, Lemma 2.1 shows that the total variation distance between the annealed measure $\mathbb{P}^{(x,0)}(X_N \in \cdot)$ and the quenched measure $P_\omega^{(x,0)}(X_N \in \cdot)$ on the level of boxes of side length $M \gg 1$ is small with very high probability as $N \to \infty$. The proof of Lemma 2.1 is given in Section 4. It builds on a preliminary result by Steiber [21, Theorem 3.24] which we recall in Theorem 8.1 below. The latter gives an analogous result to Lemma 2.1 for boxes of side length $N^{\theta/2}$ with $0 < \theta < 1$ for large N. In particular, for $N \to \infty$ the side length of these boxes grows much more slowly than the diffusive scale $N^{1/2}$.

Lemma 2.1 allows to construct a coupling of the quenched walk under $P_\omega^{(x,0)}$ and the annealed walk under $\mathbb{P}^{(x,0)}$ which puts both walks in the same M-box with very high probability. We strengthen this coupling to a coupling which puts both walks at exactly the same spatial position with uniformly non-vanishing probability; see Lemma 5.3 below. This, in turn, is essential for the next statement which concerns the difference between the annealed and quenched law of the environment viewed relative to the walk after N steps, which we denote by P_N and Q_N respectively. More precisely, for $N \in \mathbb{N}$, we define Q_N and P_N by

$$P_N(A) \coloneqq \mathbb{E}\Big[\sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_N = x) \mathbb{1}_{\{\sigma_{(x,N)}\omega \in A\}} \Big] \tag{2.3}$$

and

$$Q_N(A) \coloneqq \mathbb{E}\Big[\sum_{x \in \mathbb{Z}^d} P_\omega^{(0,0)}(X_N = x) \mathbb{1}_{\{\sigma_{(x,N)}\omega \in A\}} \Big]. \tag{2.4}$$

Note that, in fact we have $P_N = \mathbb{P}$ for all N; see (5.9).

The following proposition is proven in Section 5.

Proposition 2.2. *For $M \in \mathbb{N}$ let $\Delta_0(M)$ denote a d-dimensional cube of side length M in $\mathbb{Z}^d$ centred at the origin. There exists a universal constant $c > 0$ so that for every $\varepsilon > 0$ there is $M_0 = M_0(\varepsilon) \in \mathbb{N}$ so that for $M \geq M_0$ and all $N \in \mathbb{N}$*

$$\mathbb{P}\Big(\Big|\frac{1}{|\Delta_0(M)|}\sum_{x\in\Delta_0(M)}\frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega) - 1\Big| > \varepsilon\Big) \leq M^{-c\log M}. \tag{2.5}$$

Corollary 2.3. *Let $d \geq 3$ and $p > p_c$. Then, for every $k \in \mathbb{N}$, $\sup_N \mathbb{E}[(\frac{dQ_N}{d\mathbb{P}})^k] < \infty$.*

Proof. For $M \in \mathbb{N}$ large enough, Proposition 2.2 implies

$$\begin{aligned}\mathbb{P}\Big(\frac{dQ_N}{d\mathbb{P}}(\omega) >& 2(2M+1)^d\Big)\\ &\leq \mathbb{P}\Big(\frac{1}{(2M+1)^d}\sum_{x\in\{-M,\dots,M\}^d}\frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,n)}\omega) > 2\Big) \leq M^{-c\log M},\end{aligned}$$

which in turn implies the assertion. ◻

Let us consider the Cesàro sequence

$$\widetilde{Q}_n := \frac{1}{n}\sum_{N=0}^{n-1} Q_N, \quad n = 1, 2, \dots. \tag{2.6}$$

Since $(\widetilde{Q}_n)_n$ are measures on a compact space (recall that Ω carries the product topology), the sequence is tight. In particular, there is a weakly converging subsequence, say $(\widetilde{Q}_{n_k})_k$, and we set

$$Q := \lim_{k\to\infty}\widetilde{Q}_{n_k}. \tag{2.7}$$

Using Corollary 2.3 and the Cauchy–Schwarz inequality for some finite positive constant $\tilde{c}$ we have uniformly for all n

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{N=0}^{n-1}\frac{dQ_N}{d\mathbb{P}}\Big)^2\Big] = \frac{1}{n^2}\sum_{N,N'=0}^{n-1}\mathbb{E}\Big[\frac{dQ_N}{d\mathbb{P}}\frac{dQ_{N'}}{d\mathbb{P}}\Big] \leq \tilde{c}. \tag{2.8}$$

Note that (2.8) implies $Q \ll \mathbb{P}$; see the proof of Theorem 1.3. A standard argument shows that Q is invariant with respect to the point of view of the particle; see Proposition 1.8 in [17] for an abstract argument or the proof of Lemma 1 in [12] for the argument in the case of random walks in random environments.

The proof of the following analogue of Proposition 2.2 for Q instead of Q_n is given in Section 5.

Corollary 2.4. *Recall the notation of Proposition 2.2 and let Q be the measure obtained as a limit in (2.7). There exists a universal constant $c > 0$ so that for every $\varepsilon > 0$ there is $M_0 = M_0(\varepsilon) \in \mathbb{N}$ and for every $M \geq M_0$ we have*

$$\mathbb{P}\Big(\Big|\frac{1}{|\Delta_0(M)|}\sum_{x\in\Delta_0(M)}\frac{dQ}{d\mathbb{P}}(\sigma_{(x,0)}\omega) - 1\Big| > \varepsilon\Big) \leq M^{-c\log M}. \tag{2.9}$$

Proof of Theorem 1.3. By construction and shift invariance of $\mathbb{P}$ we have $Q_N \ll \mathbb{P}$ for every N and therefore $\tilde{Q}_n \ll \mathbb{P}$ for every n. Furthermore, by (2.8) the family of Radon–Nikodym derivatives $(d\tilde{Q}_n/d\mathbb{P})_{n=1,2,\dots}$ is uniformly integrable. These facts together imply that we also have $Q \ll \mathbb{P}$ for any Q obtained as in (2.7). The concentration property is the assertion of Corollary 2.4. For the question of uniqueness of Q see Remark 2.6 below. ◻

Remark 2.5. Using shift-invariance of $\mathbb{P}$, it is easy to see that for Q_N from (2.4) a version of $dQ_N/d\mathbb{P}$ is given by

$$\varphi_N(\omega) = \sum_{x\in\mathbb{Z}^d} P_\omega^{(-N,x)}(X_0 = 0) \tag{2.10}$$

(we have $P^{(0,0)}_{\sigma_{(-x,-N)}\omega}(X_N = x) = P_\omega^{(-N,-x)}(X_0 = 0)$, recall the notation introduced below (1.5)). This formula is the analogue of [8, Proposition 1.2] in our context. In particular, φ_N is a local function of the space-time values of ξ which themselves can be obtained as limits of local functions of ω. By (2.7) we obtain $dQ/d\mathbb{P}$ as a limit of $d\widetilde{Q}_{n_k}/d\mathbb{P}$. Thus, by taking a subsequence of $(n_k)_k$, $dQ/d\mathbb{P}$ can be considered as an almost sure limit of averages of functions of the form in (2.10).

Remark 2.6 (Uniqueness of invariant $Q \ll \mathbb{P}$ with concentration properties of the density)**.** A measure Q obtained as in (2.7) may in principle depend on a particular subsequence. In the proof of Theorem 1.4 we will show that the density $\varphi = dQ/d\mathbb{P}$ of any measure Q satisfying the concentration property (2.9) also satisfies (1.12). As shown in [2, Section 7.1], when it exists such a measure is $\mathbb{P}$ almost surely unique. In particular, in (2.7) we have weak convergence towards the unique Q along any subsequence and therefore we have weak convergence of the Cesàro sequence $(\widetilde{Q}_n)_{n\in\mathbb{N}}$ from (2.6) towards Q. However, we currently do not know whether the sequence $(Q_N)_{N\in\mathbb{N}}$ from (2.4) converges itself.

Using Lemma B.1 and (2.10) from Remark 2.5 one can show that Q is concentrated on

$$\widetilde{\Omega} = \big\{\omega \in \Omega : \omega \text{ contains a doubly infinite directed open path through } (0,0)\big\} \tag{2.11}$$

and thus Q is not equivalent to $\mathbb{P}$ because $0 < \mathbb{P}(\widetilde{\Omega}) < 1$. We note also that Kozlov's classical argument concerning equivalence, see e.g. [12, Thm. 2.12], does not apply because our walks are not elliptic. We do not know whether Q is equivalent to $\mathbb{P}(\,\cdot\,|\widetilde{\Omega})$.

To prove Theorem 1.4 we want to make use of the good control of the difference between the quenched and annealed law on the level of boxes and various properties of the prefactor φ that we have formulated above in Lemma 2.1 and Corollary 2.4. Furthermore, instead of comparing $\mathbb{P}^{(0,0)}(X_N \in \cdot)$ and $P_\omega^{(0,0)}(X_N \in \cdot)$ directly, we compare both of these two measures with auxiliary "hybrid" measures which are introduced in the following definition, analogous to [2], Definition 7.2.

Definition 2.7. *Let Q be the measure on Ω defined in* (2.7)*, which by Theorem 1.3 and its proof is invariant with respect to the point of view of the particle with $Q \ll \mathbb{P}$. Let $\varphi = dQ/d\mathbb{P}$ be the corresponding Radon–Nikodym derivative. For $\omega \in \Omega$ and a given partition Π of $\mathbb{Z}^d$ into boxes of a fixed side length we define the following measures on $\mathbb{Z}^{d+1}$:*

$$\nu_\omega^{\mathrm{ann}\times\mathrm{pre}}(x,n) \coloneqq \frac{1}{Z_{\omega,n}} \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega), \tag{2.12}$$

$$\nu_\omega^{\mathrm{que}}(x,n) \coloneqq P_\omega^{(0,0)}(X_n = x), \tag{2.13}$$

$$\nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}(x,n) \coloneqq P_\omega^{(0,0)}(X_n \in \Delta_x)\frac{\varphi(\sigma_{(x,n)}\omega)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n)}\omega)}. \tag{2.14}$$

Here, $Z_{\omega,n} = \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)$ is the normalizing constant in (2.12) *and Δ_x in* (2.14) *is the unique d-dimensional box that contains x in the partition Π.*

All of the measures introduced in the above definition are different measures of the random walk after n steps: $\nu_\omega^{\mathrm{ann}\times\mathrm{pre}}(\cdot,n)$ is the annealed measure with a prefactor, $\nu_\omega^{\mathrm{que}}(\cdot,n)$ is the quenched measure and $\nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}(\cdot,n)$ is a "hybrid" measure, where the box is chosen according to the quenched measure but then the point inside the box is chosen according to the (annealed) normalised prefactor. Of course the measure $\nu_\omega^{\mathrm{box-que\times pre}}(\cdot, n)$ does depend on the particular partition Π but it will be clear from the context which partition is used.

First we study the behaviour of the normalizing constant in (2.12); see Section 6 for a proof of the following result.

Proposition 2.8. *For $\mathbb{P}$-almost all $\omega \in \Omega$ the normalizing constant $Z_{\omega,n}$ satisfies*

$$\lim_{n\to\infty} Z_{\omega,n} = 1. \tag{2.15}$$

Note that this proposition is an analogous result to Lemma 7.3 in [2].

The following proposition is the key result for the proof of Theorem 1.4. It states that for large n the above introduced measures are close to each other in a suitable norm and is an analogous result to Lemma 7.5 in [2]. To state this precisely, for $\omega \in \Omega$ and any two probability measures ν_ω^1 and ν_ω^2 on $\mathbb{Z}^d \times \mathbb{Z}$ (more precisely these are transition kernels from Ω to $\mathbb{Z}^d \times \mathbb{Z}$) let the *$L^1$ distance of ν_ω^1 and ν_ω^2* at time $n \in \mathbb{Z}$ be defined by

$$\left\|\nu_\omega^1 - \nu_\omega^2\right\|_{1,n} := \sum_{x\in\mathbb{Z}^d} |\nu_\omega^1(x,n) - \nu_\omega^2(x,n)|. \tag{2.16}$$

Furthermore, for $k \le n$ the *space-time convolution of ν_ω^1 and ν_ω^2* is defined by

$$(\nu^1 * \nu^2)_{\omega,k}(x,n) := \sum_{y\in\mathbb{Z}^d} \nu_\omega^1(y, n-k)\nu^2_{\sigma_{(y,n-k)}\omega}(x-y,k). \tag{2.17}$$

We can interpret (2.17) as follows: A random walk takes $n-k$ steps in the random medium ω according to ν_ω^1, then re-centers the medium at its current position in space-time and takes the remaining k steps according to ν_ω^2.

Proposition 2.9. *Fix $0 < 2\delta < \varepsilon < \frac{1}{4}$, and for $n \in \mathbb{N}$ set $k = \lceil n^\varepsilon \rceil$ and $\ell = \lceil n^\delta \rceil$. Let $\Pi = \Pi(\ell)$ be a partition of $\mathbb{Z}^d$ into boxes of side length ℓ. For $\mathbb{P}$-almost every $\omega \in \Omega$ the measures from Definition 2.7 satisfy*

$$\lim_{n\to\infty} \left\|\nu_\omega^{\mathrm{ann\times pre}} - (\nu^{\mathrm{ann\times pre}} * \nu^{\mathrm{que}})_{\omega,k}\right\|_{1,n} = 0, \tag{L1}$$

$$\lim_{n\to\infty} \left\|(\nu^{\mathrm{ann\times pre}} * \nu^{\mathrm{que}})_{\omega,k} - (\nu^{\mathrm{box-que\times pre}} * \nu^{\mathrm{que}})_{\omega,k}\right\|_{1,n} = 0, \tag{L2}$$

$$\lim_{n\to\infty} \left\|(\nu^{\mathrm{box-que\times pre}} * \nu^{\mathrm{que}})_{\omega,k} - (\nu^{\mathrm{que}} * \nu^{\mathrm{que}})_{\omega,k}\right\|_{1,n} = 0. \tag{L3}$$

The proof of the above proposition is given in Section 7. With the results stated in the present section we can give a proof of the quenched local limit theorem.

Proof of Theorem 1.4. Using the triangle inequality we have

$$\sum_{x\in\mathbb{Z}^d} |P_\omega^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)|$$

$$\le \sum_{x\in\mathbb{Z}^d} |P_\omega^{(0,0)}(X_n = x) - (\nu^{\mathrm{box-que\times pre}} * \nu^{\mathrm{que}})_{\omega,k}(x,n)| \tag{2.18}$$

$$+ \sum_{x\in\mathbb{Z}^d} |(\nu^{\mathrm{box-que\times pre}} * \nu^{\mathrm{que}})_{\omega,k}(x,n) - (\nu^{\mathrm{ann\times pre}} * \nu^{\mathrm{que}})_{\omega,k}(x,n)| \tag{2.19}$$

$$+ \sum_{x\in\mathbb{Z}^d} |(\nu^{\mathrm{ann\times pre}} * \nu^{\mathrm{que}})_{\omega,k}(x,n) - \nu_\omega^{\mathrm{ann\times pre}}(x,n)| \tag{2.20}$$

$$+ \sum_{x\in\mathbb{Z}^d} |\nu_\omega^{\mathrm{ann\times pre}}(x,n) - \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)|. \tag{2.21}$$

By Proposition 2.9 the terms in (2.18), (2.19) and (2.20) tend to 0 as n goes to infinity. In order to compare (2.18) with (L3) literally note that we have $P^{(0,0)}_{\omega}(X_n = x) = \nu^{\mathrm{que}} * \nu^{\mathrm{que}})_{\omega,k}(x,n)$ by construction. Finally, by definition of $\nu_{\omega}^{\mathrm{ann}\times\mathrm{pre}}(x,n)$ the term in (2.21) can be written as

$$\Big|\frac{1}{Z_{\omega,n}} - 1\Big| \sum_{x\in\mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) = \Big|\frac{1}{Z_{\omega,n}} - 1\Big| Z_{\omega,n}. \tag{2.22}$$

By Proposition 2.8 it follows that the expression in (2.22) converges to 0 as n tends to infinity. $\square$

3 Annealed estimates and the proof of Theorem 1.1

In this section we collect estimates for the annealed walk that will be needed later in the proofs, and present a proof of Theorem 1.1.

Lemma 3.1 (Annealed derivative estimates)**.** *Let D be a positive constant. For $d \geq 3$, $j = 1, \dots, d$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $x, y \in \mathbb{Z}^d$, such that $\|x - y\| \leq D\sqrt{n}\log^3 n$, denoting by e_j the j-th (canonical) unit vector we have*

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y+e_j,m)}(X_{n+m} = x)| \leq Cn^{-(d+1)/2}, \tag{3.1}$$

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y,m+1)}(X_{n+m} = x)| \leq Cn^{-(d+1)/2}, \tag{3.2}$$

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y,m)}(X_{n+m} = x + e_j)| \leq Cn^{-(d+1)/2}, \tag{3.3}$$

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y,m)}(X_{n-1+m} = x)| \leq Cn^{-(d+1)/2}, \tag{3.4}$$

for some positive constant $C = C(D)$.

Proof. The estimates (3.1) and (3.2) are from [21]; see Lemma 3.9 and its proof in Appendix A.2 there. Note that Lemma 3.9 in [21] the choice of parameters essentially leads to the assumption $\|x - y\| \leq \sqrt{n}\log^3 n$ with y being in a box near the origin. However, the proofs of this lemma show that we can choose an arbitrary constant, that is independent of n and it sufficient to assume $\|x - y\| \leq D\sqrt{n}\log^3 n$. Furthermore, by translation invariance we obtain these upper bound for all starting positions (y, m). By translation invariance we have

$$\mathbb{P}^{(y+e_j,m)}(X_{n+m} = x) = \mathbb{P}^{(y,m)}(X_{n+m} = x - e_j)$$

and

$$\mathbb{P}^{(y,m+1)}(X_{n+m} = x) = \mathbb{P}^{(y,m)}(X_{n-1+m} = x).$$

Thus, the estimates (3.3) and (3.4) follow from (3.1) and (3.2). $\square$

We will also need the following generalization of the annealed derivative estimates in the previous lemma.

Lemma 3.2. *Let $\varepsilon > 0$. For $n \in \mathbb{N}$ large enough and every partition $\Pi_n^{(\varepsilon)}$ of $\mathbb{Z}^d$ into boxes of side length $\lfloor n^{\varepsilon} \rfloor$, we have*

$$\sum_{\Delta\in\Pi_n^{(\varepsilon)}} \sum_{x\in\Delta} \max_{y\in\Delta} \big[\mathbb{P}^{(0,0)}(X_n = y) - \mathbb{P}^{(0,0)}(X_n = x)\big] \leq Cn^{-\frac{1}{2}+3d\varepsilon}. \tag{3.5}$$

Proof. We consider the following set of boxes around the origin of $\mathbb{Z}^d$

$$\widetilde{\Pi}_n^{(\varepsilon)} := \{\Delta \in \Pi_n^{(\varepsilon)} : \Delta \cap [-\sqrt{n}\log^3 n, \sqrt{n}\log^3 n]^d \neq \emptyset\}. \tag{3.6}$$

With this notation we can write the sum on the left hand side of (3.5) as

$$\sum_{\Delta\in\widetilde{\Pi}_n^{(\varepsilon)}}\sum_{x\in\Delta}\max_{y\in\Delta}\big[\mathbb{P}^{(0,0)}(X_n=y)-\mathbb{P}^{(0,0)}(X_n=x)\big] \tag{3.7}$$

$$+\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\widetilde{\Pi}_n^{(\varepsilon)}}\sum_{x\in\Delta}\max_{y\in\Delta}\big[\mathbb{P}^{(0,0)}(X_n=y)-\mathbb{P}^{(0,0)}(X_n=x)\big]. \tag{3.8}$$

So, it is enough to prove suitable upper bounds for these two sums. By Lemma 3.6 from [21] we have

$$\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\widetilde{\Pi}_n^{(\varepsilon)}}\mathbb{P}^{(0,0)}(X_n\in\Delta)\le Cn^{-c\log n} \tag{3.9}$$

for some positive constants C and c. Thus, the double sum (3.8) is bounded from above by

$$\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\widetilde{\Pi}_n^{(\varepsilon)}}\sum_{x\in\Delta}\big[\mathbb{P}^{(0,0)}(X_n\in\Delta)-\mathbb{P}^{(0,0)}(X_n=x)\big]$$
$$=\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\widetilde{\Pi}_n^{(\varepsilon)}}(|\Delta|-1)\mathbb{P}^{(0,0)}(X_n\in\Delta)\le Cn^{d\varepsilon}n^{-c\log n}\le\widetilde{C}n^{-\tilde{c}\log n}$$

for suitably chosen constants $\tilde{c}$ and $\widetilde{C}$. Using annealed derivative estimates from Lemma 3.1 the double sum (3.7) is bounded above by

$$\sum_{\Delta\in\widetilde{\Pi}_n^{(\varepsilon)}}\sum_{x\in\Delta}Cn^{\varepsilon}n^{-\frac{d+1}{2}}\le C(n^{\varepsilon}+\sqrt{n}\log^3 n)^d n^{\varepsilon}n^{-\frac{d+1}{2}}\le Cn^{3d\varepsilon}n^{-1/2}.$$

Combination of the last two displays completes the proof. □

Proof of Theorem 1.1. Let $\varepsilon,\delta>0$ be small (they will later be tuned appropriately). Let $\Pi_n^{(\varepsilon)}$ be a partition of $\mathbb{Z}^d$ in boxes of side length $\lceil\varepsilon\sqrt{n}\,\rceil$. Let $C_\delta>0$ be a constant such that $\mathbb{P}^{(0,0)}(\|X_n\|>C_\delta\sqrt{n})<\delta$; such a constant exists by the annealed central limit theorem, see [4], Theorem 1. Furthermore denote by $\Pi_n^{(\varepsilon,\delta)}$ the subset of boxes in $\Pi_n^{(\varepsilon)}$ intersecting $\{x\in\mathbb{Z}^d:\|x\|\le C_\delta\sqrt{n}\}$. Then

$$\sum_{x\in\mathbb{Z}^d}\Big|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big|$$
$$=\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}\Big|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big| \tag{3.10}$$
$$+\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}\Big|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big|. \tag{3.11}$$

We will show that ε can be chosen so small that the above sum is bounded by 4δ for large enough n. We first find an upper bound for (3.10). By definition of $\Pi_n^{(\varepsilon,\delta)}$ if $\Delta\in\Pi_n^{(\varepsilon)}\setminus\Pi_n^{(\varepsilon,\delta)}$ then we have $\|x\|>C_\delta\sqrt{n}$ for all $x\in\Delta$. Thus, (3.10) is bounded from above by

$$\sum_{\substack{x\in\mathbb{Z}^d\\ \|x\|>C_\delta\sqrt{n}}}\Big(\mathbb{P}^{(0,0)}(X_n=x)+\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big)$$
$$\le\delta+C\exp\Big(-\frac{c}{2}C_\delta^2\Big).$$

By choosing C_δ large enough we can ensure that (3.10) is bounded by 2δ.

Turning to (3.11) we first compare the two terms in $|\cdot|$ with the averages over appropriate boxes. First, let $x \in \mathbb{Z}^d$ be fixed and let $\Delta \in \Pi_n^{(\varepsilon)}$ be the box containing x. Using annealed derivative estimates from Lemma 3.1 we obtain

$$\begin{aligned}
&|\mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\mathbb{P}^{(0,0)}(X_n \in \Delta)| \\
&\qquad = \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\Big|\sum_{y\in\Delta}\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = y)\Big| \\
&\qquad \le \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\sum_{y\in\Delta}\|x-y\|\, n^{-(d+1)/2} \le \lceil \varepsilon\sqrt{n}\,\rceil \cdot n^{-(d+1)/2} \le \frac{\varepsilon}{n^{d/2}}.
\end{aligned}$$

Now consider $\Delta \in \Pi_n^{(\varepsilon,\delta)}$. For every $x \in \Delta$ we have

$$\begin{aligned}
&\Big|\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big) - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big| \\
&\quad = \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big|1 - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \exp\Big(-\frac{1}{2n}(y^T\Sigma^{-1}y - x^T\Sigma^{-1}x)\Big)\,dy\Big| \\
&\quad \le \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d} \\
&\qquad\qquad \times \int_\Delta \Big|1 - \exp\Big(-\frac{1}{2n}((y-x)^T\Sigma^{-1}(y-x) + 2x^T\Sigma^{-1}(y-x))\Big)\Big|\,dy \\
&\quad \le \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \Big|1 - \exp\Big(\frac{1}{2n}(C\varepsilon^2 n + CC_\delta\varepsilon n)\Big)\Big|\,dy \\
&\quad \le \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\cdot C\varepsilon \le C\varepsilon,
\end{aligned}$$

where we have used $\|x-y\| \le \varepsilon\sqrt{n}$, $\|x\| \le C_\delta\sqrt{n}$ and for the fourth line the fact that $|1-\exp(-x)| \le \exp(|x|) - 1$. Using first the triangle inequality and then combining the last two estimates we see that each summand in (3.11) is bounded from above by

$$\begin{aligned}
&\big|\mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\mathbb{P}^{(0,0)}(X_n \in \Delta)\big| \\
&\quad + \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\Big|\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big) - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big| \\
&\quad + \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\Big|\mathbb{P}^{(0,0)}(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big| \qquad (3.12) \\
&\le \frac{C\varepsilon}{n^{d/2}} + \frac{C\varepsilon}{(2\pi n)^{d/2}\sqrt{\det\Sigma}} \\
&\quad + \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\Big|\mathbb{P}^{(0,0)}(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big|.
\end{aligned}$$

The number of vertices summed over all $\Delta \in \Pi_n^{(\varepsilon,\delta)}$ is bounded by $((C_\delta+\varepsilon)\sqrt{n})^d \le C(C_\delta\sqrt{n})^d$. Thus,

$$\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}\Big(\frac{C\varepsilon}{n^{d/2}} + \frac{C\varepsilon}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\Big) \le C\cdot C_\delta^d\varepsilon. \qquad (3.13)$$

Summing the last line in (3.12) with the double sum $\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}$ gives

$$\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\Big|\mathbb{P}^{(0,0)}(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big|. \qquad (3.14)$$

By applying the annealed CLT from [4] (and approximating the indicator $\mathbb{1}_\Delta$ appropriately by continuous and bounded functions) and noting that for fixed ε and δ the set $\Pi_n^{(\varepsilon,\delta)}$ is finite implies that (3.14) goes to zero as n tends to infinity. In particular it is smaller than δ for large enough n.

Combining the estimates above we obtain

$$\sum_{x\in\mathbb{Z}^d}\Big|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big|\le 2\delta+C\cdot C_\delta^d\varepsilon+\delta<4\delta$$

for large enough n and choosing $\varepsilon>0$ so that $C\cdot C_\delta^d\varepsilon<\delta$. This concludes the proof. □

4 Proof of Lemma 2.1

For the proof of of Lemma 2.1 we follow closely the proof of Theorem 5.1 in [2] and adapt their arguments to our model. The general idea is to implement an iteration scheme that carries the annealed-quenched comparison from Theorem 8.1 below along a sequence of more and more slowly growing box scales.

Let us introduce some notation first. Let $\theta>0$ be a (small) constant to be determined in the proof. For $j\in\mathbb{N}$, we set $n_j:=\lfloor N^{2^{-j}}\rfloor$ and $r(N):=\lceil\log_2(\frac{\theta\log N}{\log M})\rceil$. Note that $r(N)$ is the smallest integer satisfying $n_{r(N)}^\theta\le M$. Furthermore we set

$$N_0:=N-\sum_{j=1}^{r(N)}n_j\quad\text{and}\quad N_k:=\sum_{j=1}^{k}n_j+N_0=N_{k-1}+n_k,\ \text{for all }1\le k\le r(N).\qquad(4.1)$$

Finally, for $0\le k\le r(N)$, abusing the notation and suppressing the dependence on θ and n we write for the rest of this section $\Pi_k:=\Pi_{n_k^\theta}$ and define

$$\lambda_k(\omega):=\sum_{\Delta\in\Pi_k}\big|P_\omega^{(0,0)}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta)\big|.\qquad(4.2)$$

Note in particular that $\lambda_{r(N)}$ is twice the total variation distance between the quenched and the annealed measures on boxes of side length $\le M$, which is the term we wish to bound from above to show (2.1). If one wishes to be slightly more precise, then one should replace $N_{r(N)}$ by M, thus obtaining the total variation for boxes of side length M exactly. This, however, does not influence the estimates to follow.

The proof of the following proposition is long and technical and will be given in Section 8.

Proposition 4.1. *There exists constants $C,c,\alpha>0$ and events $G(N),N\in\mathbb{N}$, with $\mathbb{P}(G(N))\ge 1-CN^{-c\log N}$ such that for all $\omega\in G(N)$ we have*

$$\lambda_k\le\lambda_{k-1}+Cn_k^{-\alpha},\quad\forall\,1\le k\le r(N).\qquad(4.3)$$

In particular, $\lambda_{r(N)}\le\lambda_1+C\sum_{k=1}^{r(N)}n_k^{-\alpha}$ for $\omega\in G(N)$.

Proof of Lemma 2.1. The assertion is a consequence of Proposition 4.1 (and Theorem 8.1) and can be proven analogously to the argument in the last part of the proof of Theorem 5.1 in [2], page 2920. □

5 Concentration from coupling: Proofs of Proposition 2.2 and Corollary 2.4

In this section we prove some analogues of the results of Section 6 in [2] and present proofs of Proposition 2.2 and Corollary 2.4.

Lemma 5.1. *There exists a constant $c > 0$ and set of environments $K(N, c)$ satisfying*

$$\mathbb{P}(K(N, c)) \geq 1 - N^{-c \log N} \tag{5.1}$$

such that for every ω there exists a coupling $\Theta_{\omega,N}$ of $\mathbb{P}^{(0,0)}(X_N = \cdot)$ and $P_\omega^{(0,0)}(X_N = \cdot)$ with the property

$$\Theta_{\omega,N}(\Lambda) > c \ \text{ for every } \omega \in K(N, c), \tag{5.2}$$

where $\Lambda := \{(x, x) : x \in \mathbb{Z}^d\}$.

Proof. For $\varepsilon > 0$ and $M \in \mathbb{N}$ denote by $K(N) = K(N, M, \varepsilon)$ the set of environments $\omega \in \Omega$ satisfying

$$\sum_{\Delta \in \Pi_M} |P_\omega^{(0,0)}(X_N \in \Delta) - \mathbb{P}^{(0,0)}(X_N \in \Delta)| < \varepsilon, \tag{5.3}$$

where Π_M is a partition of $\mathbb{Z}^d$ into d-dimensional boxes of side length M. By Lemma 2.1, for every $\varepsilon \in (0, 1)$ there exists a $M \in \mathbb{N}$ such that $\mathbb{P}(K(N)) \geq 1 - N^{-c \log N}$. On the event $K(N)$, the inequality (5.3) tells us that twice the total variation distance between $\mathbb{P}^{(0,0)}(X_N \in \cdot)$ and $P_\omega^{(0,0)}(X_N \in \cdot)$ on Π_M is less than ε and therefore there exists a coupling $\widetilde{\Theta}_{\omega,N,M}$ of both measures on $\Pi_M \times \Pi_M$ such that $\widetilde{\Theta}_{\omega,N,M}(\Lambda_{\Pi_M}) > 1 - \varepsilon$, where $\Lambda_{\Pi_M} = \{(\Delta, \Delta) : \Delta \in \Pi_M\}$.

Using the coupling $\widetilde{\Theta}$ we construct a new coupling of $\mathbb{P}^{(0,0)}(X_N = \cdot)$ and $P_\omega^{(0,0)}(X_N = \cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ which puts positive probability on the diagonal $\Lambda = \{(x, x) : x \in \mathbb{Z}^d\}$. We define $\Theta_{\omega,N}$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ by

$$\begin{aligned}\Theta_{\omega,N}(x, y) := \sum_{\Delta, \Delta' \in \Pi_M} &\widetilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta') \\ &\cdot \mathbb{P}^{(0,0)}(X_N = x | X_{N-M} \in \Delta) \cdot P_\omega^{(0,0)}(X_N = y | X_{N-M} \in \Delta').\end{aligned} \tag{5.4}$$

Since $\widetilde{\Theta}_{\omega,N-M,M}$ is a coupling of $\mathbb{P}^{(0,0)}$ and $P_\omega^{(0,0)}$ on $\Pi_M \times \Pi_M$ one can easily see that by the formula of total probability $\Theta_{\omega,N}$ is indeed a coupling of $\mathbb{P}^{(0,0)}(X_N = \cdot)$ and $P_\omega^{(0,0)}(X_N = \cdot)$.

For $x \in \mathbb{Z}^d$, let Δ_x be the unique cube which contains x in the partition Π_M. Since the side length of each box in the partition Π_M is M it follows that the annealed random walk can reach x from each point in the box Δ_x in less than M steps.

Next we want to show that the coupling gives us a positive chance for the two walks to end up at the same position. In [2] this is done by showing that $\Theta_{\omega,N}(x, x)$ is bounded away from zero for all $x \in \mathbb{Z}^d$. This is not true in our model because we do not have uniform ellipticity for the quenched measure. The idea here is to show that for "typical" ω the measure $\Theta_{\omega,N}(x, x)$ is bounded away from zero for "many" $x \in \mathbb{Z}^d$. To this end for given ω we define the set $\Pi_\omega^x \subset \Pi_M$ as the set of boxes $\Delta \in \Pi_M$ satisfying

$$P_\omega^{(0,0)}(X_N = x | X_{N-M} \in \Delta) > 0. \tag{5.5}$$

Note that if $\Pi_\omega^x = \emptyset$ for x and ω then we have $\Theta_{\omega,N}(x, x) = 0$. Furthermore, by definition of $P_\omega^{(0,0)}(X_N = x | X_{N-1} = y)$ we have

$$P_\omega^{(0,0)}(X_N = x | X_{N-M} \in \Delta) \geq \left(\frac{1}{3^d}\right)^M \tag{5.6}$$

for all $\Delta \in \Pi_\omega^x$. Now using (5.4), (5.6) and uniform ellipticity of the annealed measure we obtain

$$\begin{aligned}\Theta_{\omega,N}(x,x) &\geq \sum_{\Delta\in\Pi_\omega^x} \widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta) \\ &\qquad\qquad \cdot \mathbb{P}^{(0,0)}(X_N = x | X_{N-M}\in\Delta)\cdot P_\omega^{(0,0)}(X_N = x|X_{N-M}\in\Delta) \\ &\geq \sum_{\Delta\in\Pi_\omega^x} \widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta)\eta^M\left(\frac{1}{3^d}\right)^M,\end{aligned}$$

where $\eta\in(0,1)$ is the "uniform ellipticity bound" of the annealed random walk. Now it suffices to show

$$\sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi_\omega^x}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta) \geq \sum_{\Delta\in\Pi_M}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta). \tag{5.7}$$

This follows immediately if we can show that for all $\Delta\in\Pi_M\setminus\cup_{x\in\mathbb{Z}^d}\Pi_\omega^x$ we have

$$\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta) = 0.$$

For that consider a box $\Delta\in\Pi_M\setminus\cup_{x\in\mathbb{Z}^d}\Pi_\omega^x$, i.e. there is no $x\in\mathbb{Z}^d$ with $\Delta\in\Pi_\omega^x$ for the fixed ω. Thus, we have $P_\omega^{(0,0)}(X_N = x|X_{N-M}\in\Delta) = 0$ for all $x\in\mathbb{Z}^d$. It follows that $P_\omega^{(0,0)}(X_{N-M}\in\Delta) = 0$, because there can be no infinitely long open path starting from Δ. We obtain

$$\begin{aligned}\Theta_{\omega,N}(\Lambda) = \sum_{x\in\mathbb{Z}^d}\Theta_{\omega,N}(x,x) &\geq \sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi_\omega^x}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta)\eta^M\left(\frac{1}{3^d}\right)^M \\ &\geq \sum_{\Delta\in\Pi_M}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta)\eta^M\left(\frac{1}{3^d}\right)^M \geq (1-\varepsilon)\eta^M\left(\frac{1}{3^d}\right)^M\end{aligned} \tag{5.8}$$

for every $\omega\in K(N)$. $\square$

Recall the definitions of P_N and Q_N from (2.3) respectively (2.4). Note that for every $N\in\mathbb{N}$ the measure P_N is in fact the measure $\mathbb{P}$ since for every measurable event $A\in\Omega$ we have by translation invariance

$$\begin{aligned}P_N(A) &= \mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{1}_{\{\sigma_{(x,N)}\omega\in A\}}\Big] = \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{E}[\mathbb{1}_{\{\sigma_{(x,N)}\omega\in A\}}] \\ &= \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{P}(\sigma_{(-x,-N)}A) = \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{P}(A) = \mathbb{P}(A).\end{aligned} \tag{5.9}$$

Definition 5.2. *Given two environments $\omega,\omega'\in\Omega$ we define their distance by*

$$\mathrm{dist}(\omega,\omega') = \inf\big\{\|(x,n)\| : \omega' = \sigma_{(x,n)}\omega\big\},$$

where the infimum over an empty set is defined to be infinity.

We denote by Ψ_N the coupling of P_N and Q_N from Lemma 5.1 extended to $\Omega\times\Omega$, that is,

$$\Psi_N(A) = \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\omega,N}(x,y)\mathbb{1}_{\{(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)\in A\}}\Big]. \tag{5.10}$$

The following result is an analogue to Lemma 6.6 in [2].

Lemma 5.3. *For $M, N \in \mathbb{N}$ let $D^{(1)}_{M,N} : \Omega \to [0,\infty]$ and $D^{(2)}_{M,N} : \Omega \to [0,\infty]$ be defined by*

$$D^{(i)}_{M,N}(\omega_i) := \mathbb{E}_{\Psi_N}[\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>M\}} \,|\, \mathfrak{F}_{\omega_i}](\omega_i), \quad i = 1,2,$$

where $\mathfrak{F}_{\omega_1}$, $\mathfrak{F}_{\omega_2}$ are the σ-algebras generated by the first, respectively, second coordinate in $\Omega \times \Omega$ and Ψ_N is defined in (5.10). *For $M \in \mathbb{N}$, there exists an event F_M with the following properties:*

(1) $\mathbb{P}(F_M) \geq 1 - M^{-c \log M}$.

(2) For every $\varepsilon > 0$ one can choose $M = M(\varepsilon)$ large enough

$$\max\Big\{ D^{(1)}_{M,N}(\omega),\ \frac{dQ_N}{d\mathbb{P}}(\omega) D^{(2)}_{M,N}(\omega) \Big\} \leq \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^{\mathrm{c}}}(\omega). \tag{5.11}$$

Proof. Let

$$F_M = \bigcap_{k > M/2} \Big\{ \omega \in \Omega : \forall x \in [-k,k]^d \cap \mathbb{Z}^d, \sum_{\Delta \in \Pi_M} |\mathbb{P}^{(x,0)}(X_k \in \Delta) - P^{(x,0)}_\omega(X_k \in \Delta)| \leq \frac{C_2}{M^{c_1}} + \frac{C_2}{k^{c_1}} \Big\}$$

where Π_M is a partition of $\mathbb{Z}^d$ into boxes of side length M and C_2, c_1 are the (renamed) constants from Lemma 2.1. Thus, $\mathbb{P}(F_M) \geq 1 - M^{-c \log M}$. Fix $\varepsilon > 0$. Then, by the definition of F_M and the coupling $\widetilde{\Theta}_{\omega,k,M}$ constructed in the proof of Lemma 5.1, for every $\omega \in F_M$, every $k > M/2$ and every $x \in [-k,k]^d \cap \mathbb{Z}^d$ we have

$$\widetilde{\Theta}_{\sigma_{(x,k)}\omega,k,M}(\Lambda_{\Pi_M}) > 1 - \frac{2C_2}{M^{c_1}} > 1 - \varepsilon \tag{5.12}$$

for large enough M, where $\Lambda_{\Pi_M} = \{(\Delta,\Delta) : \Delta \in \Pi_M\}$. Note that for $k \leq M/2$ the left hand side of (5.12) is 1 and therefore (5.11) is trivially true for $N \leq M/2$.

Let us now verify the estimates (5.11) for $D^{(1)}_{M,N}$ and $\frac{dQ_N}{d\mathbb{P}} D^{(2)}_{M,N}$ and $N > M/2$. Note that for $\mathbb{P}$-almost every environment $\omega \in \Omega$ we will show that

$$D^{(1)}_{M,N}(\omega) = \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-(x,N)}\omega,N}(x,y) \mathbb{1}_{\{\|x-y\|>M\}} \tag{5.13}$$

and for Q_N-almost every ω

$$D^{(2)}_{M,N}(\omega) = \left(\frac{dQ_N}{d\mathbb{P}}(\omega) \right)^{-1} \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-(y,N)}\omega,N}(x,y) \mathbb{1}_{\{\|x-y\|>M\}}. \tag{5.14}$$

Using (5.10) we have for every measurable event $A \subset \Omega$

$$\begin{aligned}
&E_{\Psi_N}[\mathbb{1}_{\{(\omega_1,\omega_2)\in A\times\Omega\}} \mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>M\}}] \\
&\quad = \Psi_N(A \times \Omega \cap \{(\omega_1,\omega_2) : \mathrm{dist}(\omega_1,\omega_2) > M\}) \\
&\quad = \mathbb{E}\Big[\sum_{x,y \in \mathbb{Z}^d} \Theta_{\omega,N}(x,y) \mathbb{1}_{\{(\sigma_{(x,N)}\omega, \sigma_{(y,N)}\omega) \in A\times\Omega\}} \mathbb{1}_{\{\mathrm{dist}(\sigma_{(x,N)}\omega, \sigma_{(y,N)}\omega) > M\}} \Big] \\
&\quad = \sum_{x,y \in \mathbb{Z}^d} \mathbb{E}\big[\Theta_{\omega,N}(x,y) \mathbb{1}_{\{\sigma_{(x,N)}\omega \in A\}\}} \mathbb{1}_{\{\|x-y\|>M\}}\big] \\
&\quad = \sum_{x,y \in \mathbb{Z}^d} \mathbb{E}\big[\Theta_{\sigma_{-(x,N)}\omega,N}(x,y) \mathbb{1}_{\{\omega \in A\}\}} \mathbb{1}_{\{\|x-y\|>M\}}\big],
\end{aligned}$$

where the last equality follows by translation invariance of $\mathbb{P}$. Since Ψ_N is a coupling of $P_N = \mathbb{P}$ and Q_N the last term equals

$$E_{\Psi_N}\Big[\mathbb{1}_{\{(\omega,\omega')\in A\times\Omega\}}\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}}\Big],$$

which implies (5.13).

For $B_N := \{\omega\colon \frac{dQ_N}{d\mathbb{P}}(\omega) \neq 0\}$ we have $Q_N(B_N^{\complement}) = \Psi_N(\Omega\times B_N^{\complement}) = 0$, and we get similarly

$$\begin{aligned}
&E_{\Psi_N}[\mathbb{1}_{\{\Omega\times A\}}\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>M\}}]\\
&\quad= E_{\Psi_N}[\mathbb{1}_{\{\Omega\times A\cap B_N\}}\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>M\}}]\\
&\quad= \Psi_N(\Omega\times(A\cap B_N)\cap\{(\omega_1,\omega_2):\mathrm{dist}(\omega_1,\omega_2)>M\})\\
&\quad= \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\omega,N}(x,y)\mathbb{1}_{\{(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)\in\Omega\times A\cap B_N\}}\mathbb{1}_{\{\mathrm{dist}(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)>M\}}\Big]\\
&\quad= \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\omega,N}(x,y)\mathbb{1}_{\{\sigma_{(y,N)}\omega\in A\cap B_N\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big]\\
&\quad= \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega,N}(x,y)\mathbb{1}_{\{\omega\in A\cap B_N\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big]\\
&\quad= E_{Q_N}\Big[\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}(\omega)\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega,N}(x,y)\mathbb{1}_{\{\omega\in A\cap B_N\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big]\\
&\quad= E_{\Psi_N}\Big[\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}(\omega_2)\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega_2,N}(x,y)\mathbb{1}_{\{(\omega_1,\omega_2)\in\Omega\times(A\cap B_N)\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big]\\
&\quad= E_{\Psi_N}\Big[\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}(\omega_2)\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega_2,N}(x,y)\mathbb{1}_{\{(\omega_1,\omega_2)\in\Omega\times A\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big],
\end{aligned}$$

which shows (5.14)

If $\Theta_{\sigma_{-(x,N)}\omega,N}(x,y) > 0$ then necessarily $x\in[-N,N]^d\cap\mathbb{Z}^d$ because in N steps the annealed walk can only reach points in this box. It follows that for large enough M, every $\omega\in F_M$ and every $N\geq M$ we have

$$\begin{aligned}
&\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}}\\
&\quad= 1-\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|\leq M\}}\\
&\quad\leq 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(z,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|\leq M\}}\\
&\quad\leq 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\sum_{\Delta\in\Pi_M}\sum_{x,y\in\Delta}\Theta_{\sigma_{-(z,N)}\omega,N}(x,y)\\
&\quad= 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\sum_{\Delta\in\Pi_M}\widetilde{\Theta}_{\sigma_{-(z,N)}\omega,N,M}(\Delta,\Delta)\\
&\quad= 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\widetilde{\Theta}_{\sigma_{-(z,N)}\omega,N,M}(\Lambda_{\Pi_M}) < \varepsilon.
\end{aligned}$$

Thus,

$$D^{(1)}_{M,N}(\omega) = \sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}} \leq \varepsilon\mathbb{1}_{F_M}(\omega)+\mathbb{1}_{F_M^{\complement}}(\omega).$$

For $\omega \in F_M \cap B_N$ we have shown

$$\frac{dQ_N}{d\mathbb{P}}(\omega) D^{(2)}_{M,N}(\omega) = \sum_{x,y\in\mathbb{Z}^d} \Theta_{\sigma_{-(y,N)}\omega,N} \mathbb{1}_{\{\|x-y\|>M\}} \le \varepsilon$$

whereas for $\omega \in F_M \cap B_N^{\complement}$

$$\frac{dQ_N}{d\mathbb{P}}(\omega) D^{(2)}_{M,N}(\omega) = 0$$

and thus

$$\frac{dQ_N}{d\mathbb{P}}(\omega) D^{(2)}_{M,N}(\omega) \le \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^{\complement}}(\omega). \qquad \square$$

Proof of Proposition 2.2. We follow the ideas of the proof of Lemma 6.5 in [2]. To this end, we consider the events

$$B_\varepsilon^- = \{\omega \in \Omega : \frac{1}{|\Delta_0|} \sum_{x\in\Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega) < 1-\varepsilon\}$$
$$B_\varepsilon^+ = \{\omega \in \Omega : \frac{1}{|\Delta_0|} \sum_{x\in\Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega) > 1+\varepsilon\}.$$

First we consider B_ε^-. We decompose this event into two events, first of which has probability $M^{-c\log M}$ and the second is a $\mathbb{P}$ null set. We assume without loss of generality that Δ_0 is centred at the (spatial) origin, set $M_\varepsilon = \frac{\varepsilon}{6d^2}M$, define $\Delta_0^- = \{x \in \mathbb{Z}^d : \|x\| < M - M_\varepsilon\}$ and

$$S_\varepsilon^- = \{\omega \in B_\varepsilon^- : \sigma_{(x,0)}\omega \in F_{M_\varepsilon}, \forall x \in \Delta_0\},$$

where F_{M_ε} is the event from Lemma 5.3. Due to property (1) of F_{M_ε} from Lemma 5.3

$$\begin{aligned}\mathbb{P}(S_\varepsilon^-) &\ge \mathbb{P}(B_\varepsilon^-) - |\Delta_0|\mathbb{P}(F_{M_\varepsilon}^{\complement}) \\ &\ge \mathbb{P}(B_\varepsilon^-) - M^d (M_\varepsilon)^{-c\log M_\varepsilon} \ge \mathbb{P}(B_\varepsilon^-) - M^{-\tilde{c}\log M},\end{aligned}$$

where $\tilde{c}$ is a positive constant. Therefore it is enough to show that $\mathbb{P}(S_\varepsilon^-) = 0$.

We claim that there exists an event $K^- \subset S_\varepsilon^-$ such that

$$\mathbb{P}(K^-) \ge \mathbb{P}(S_\varepsilon^-) \cdot ((4d)^d |\Delta_0|)^{-1} \tag{5.15}$$

and

$$\text{if } \omega, \omega' \in K^-, \ \omega \ne \omega', \text{ then } \operatorname{dist}(\omega,\omega') > 4M. \tag{5.16}$$

For every $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ let $U_{(x,n)}$ be an independent (of everything else defined so far) random variable uniformly distributed on $[0,1]$, and define

$$K^- := \big\{\omega \in S_\varepsilon^- : \forall (x,0) \in 4\Delta_0 \times \{0\} \setminus \{(0,0)\} \text{ if } \sigma_{(x,0)}\omega \in B_\varepsilon^- \text{ then } U_{(x,0)} < U_{(0,0)}\big\}.$$

This means informally, that from each family of environments whose distance is smaller than $4M$ we choose one uniformly. This implies that property (5.16) for K^- holds. Property (5.15) holds because due to translation invariance of $\mathbb{P}$ we have

$$\mathbb{P}(S_\varepsilon^-) \le \mathbb{P}\Big(\bigcup_{x\in 4d\Delta_0} \sigma_{(x,0)}K^-\Big) \le \sum_{x\in 4d\Delta_0} \mathbb{P}\big(\sigma_{(x,0)}K^-\big) = (4d)^d |\Delta_0| \mathbb{P}(K^-).$$

Now, let

$$G = \bigcup_{x \in \Delta_0} \sigma_{(x,0)} K^- \quad \text{and} \quad G^- = \bigcup_{x \in \Delta_0^-} \sigma_{(x,0)} K^-.$$

By property (5.16) of K^- these are in both cases disjoint unions and therefore we have

$$\begin{aligned} \mathbb{P}(G) &= \sum_{x \in \Delta_0} \mathbb{P}(\sigma_{(x,0)} K^-) = |\Delta_0| \mathbb{P}(K^-) \quad \text{and} \\ \mathbb{P}(G^-) &= |\Delta_0^-| \mathbb{P}(K^-) = |\Delta_0| \big(1 - \frac{\varepsilon}{6d^2}\big)^d \mathbb{P}(K^-) > \big(1 - \frac{\varepsilon}{6}\big) \mathbb{P}(G). \end{aligned} \tag{5.17}$$

Going back to the definition of the event B_ε^- and recalling that $K^- \subset S_\varepsilon^- \subset B_\varepsilon^-$ we obtain

$$\begin{aligned} Q_N(G) &= \int_G \frac{dQ_N}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega) = \sum_{x \in \Delta_0} \int_{\sigma_{(x,0)} K^-} \frac{dQ_N}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega) \\ &= \int_{K^-} \sum_{x \in \Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)} \omega)\, d\mathbb{P}(\omega) \\ &\le \int_{K^-} (1-\varepsilon) |\Delta_0|\, d\mathbb{P}(\omega) = (1-\varepsilon)|\Delta_0| \mathbb{P}(K^-) = (1-\varepsilon)\mathbb{P}(G) \end{aligned}$$

Combining this with (5.17), for small enough $\varepsilon > 0$ we obtain

$$\begin{aligned} Q_N(G) \le (1-\varepsilon)\mathbb{P}(G) &= \frac{1-\varepsilon}{1-\varepsilon/6} \left(1 - \frac{\varepsilon}{6}\right) \mathbb{P}(G) \\ &< \frac{1-\varepsilon}{1-\varepsilon/6} \mathbb{P}(G^-) < \left(1 - \frac{\varepsilon}{3}\right) \mathbb{P}(G^-). \end{aligned} \tag{5.18}$$

Let $A^- = \{(\omega, \omega') : \omega \in G^-, \omega' \notin G\}$. Then by (5.17) and (5.18)

$$\begin{aligned} \Psi_N(A^-) &\ge \mathbb{P}(G^-) - Q_N(G) \ge \mathbb{P}(G^-) - \Big(1 - \frac{\varepsilon}{3}\Big)\mathbb{P}(G^-) \\ &\ge \frac{\varepsilon}{3}\mathbb{P}(G^-) > \frac{\varepsilon}{3}\Big(1 - \frac{\varepsilon}{6}\Big)\mathbb{P}(G) > \frac{\varepsilon}{4}\mathbb{P}(G). \end{aligned} \tag{5.19}$$

By construction of K^-, for every $(\omega, \omega') \in A^-$ we have $\operatorname{dist}(\omega, \omega') > M_\varepsilon$ and, therefore,

$$\begin{aligned} \int_G D^{(1)}_{M_\varepsilon,N}\, d\mathbb{P}(\omega) &= \int_{G \times \Omega} D^{(1)}_{M_\varepsilon,N}\, d\Psi_N(\omega, \omega') \ge \int_{G^- \times \Omega} D^{(1)}_{M_\varepsilon,N}\, d\Psi_N(\omega, \omega') \\ &= \int_{\Omega \times \Omega} E_{\Psi_N}[\mathbb{1}_{\{\operatorname{dist}(\omega,\omega') > M_\varepsilon\}} \,|\, \mathfrak{F}_\omega](\omega) \mathbb{1}_{\{G^- \times \Omega\}}(\omega, \omega')\, d\Psi_N(\omega, \omega') \\ &= \int_{\Omega \times \Omega} E_{\Psi_N}[\mathbb{1}_{\{\operatorname{dist}(\omega,\omega') > M_\varepsilon\}} \mathbb{1}_{\{G^- \times \Omega\}}(\omega, \omega') \,|\, \mathfrak{F}_\omega](\omega)\, d\Psi_N(\omega, \omega') \\ &= \int_{\Omega \times \Omega} \mathbb{1}_{\{\operatorname{dist}(\omega,\omega') > M_\varepsilon\}} \mathbb{1}_{\{G^- \times \Omega\}}(\omega, \omega')\, d\Psi_N(\omega, \omega') \\ &\ge \int_{\Omega \times \Omega} \mathbb{1}_{\{\operatorname{dist}(\omega,\omega') > M_\varepsilon\}} \mathbb{1}_{\{A^-\}}(\omega, \omega')\, d\Psi_N(\omega, \omega') \\ &= \int_{\Omega \times \Omega} \mathbb{1}_{A^-}(\omega, \omega')\, d\Psi_N(\omega, \omega') \\ &= \Psi_N(A^-) > \frac{\varepsilon}{4}\mathbb{P}(G). \end{aligned} \tag{5.20}$$

Since $G \subset F_{M_\varepsilon}$ by definition, using Lemma 5.3 with M_ε and $\varepsilon/5$ instead of M and ε we obtain

$$\int_G D^{(1)}_{M_\varepsilon,N}(\omega)\, d\mathbb{P}(\omega) \le \int_G \frac{\varepsilon}{5} \mathbb{1}_{F_{M_\varepsilon}}(\omega) + \mathbb{1}_{F^{\complement}_{M_\varepsilon}}(\omega)\, d\mathbb{P}(\omega) = \int_G \frac{\varepsilon}{5}\, d\mathbb{P}(\omega) = \frac{\varepsilon}{5}\mathbb{P}(G). \tag{5.21}$$

Combining (5.20) and (5.21) we conclude that $\mathbb{P}(G) = 0$ and, therefore $\mathbb{P}(K^-) = 0$. By property (5.15) of K^- this implies that $\mathbb{P}(S_\varepsilon^-) = 0$ and finally $\mathbb{P}(B_\varepsilon^-) \le M^{-c\log M}$.

Next we turn to the event B_ε^+. As before we set $M_\varepsilon = \frac{\varepsilon}{6d^2}M$ and assume that Δ_0 is centred at the origin. Define $\Delta_0^+ \coloneqq \{x \in \mathbb{Z}^d : \|x\| < M + M_\varepsilon\}$ and let

$$S_\varepsilon^+ = \big\{\omega \in B_\varepsilon^+ : \sigma_{(x,0)}\omega \in F_{M_\varepsilon}, \forall x \in \Delta_0^+\big\},$$

where F_{M_ε} is, as before, the event from Lemma 5.3. Due to property (1) of F_{M_ε}

$$\begin{aligned}\mathbb{P}(S_\varepsilon^+) &\ge \mathbb{P}(B_\varepsilon^+) - |\Delta_0^+|\mathbb{P}(F_{M_\varepsilon}^{\mathsf{C}}) \ge \mathbb{P}(B_\varepsilon^+) - (1+\frac{\varepsilon}{6d^2})^d M^d (M_\varepsilon)^{-c\log M_\varepsilon}\\ &\ge \mathbb{P}(B_\varepsilon^+) - M^{-\tilde{c}\log M}\end{aligned}$$

and again it is enough to show that $\mathbb{P}(S_\varepsilon^+) = 0$. As for S_ε^- we claim that there exists an event $K^+ \subset S_\varepsilon^+$ such that

$$\mathbb{P}(K^+) \ge \mathbb{P}(S_\varepsilon^+)\cdot((4d)^d|\Delta_0^+|)^{-1} \tag{5.22}$$

and

$$\text{if } \omega,\omega' \in K^+ \text{ with } \omega \ne \omega', \text{ then } \operatorname{dist}(\omega,\omega') > 4(M+M_\varepsilon). \tag{5.23}$$

We define K^+ similar to K^-, that is

$$K^+ \coloneqq \big\{\omega \in S_\varepsilon^+ : \forall (x,0) \in 4\Delta_0^+ \times \{0\} \setminus \{(0,0)\} \text{ if } \sigma_{(x,0)}\omega \in B_\varepsilon^+ \text{ then } U_{(x,0)} < U_{(0,0)}\big\}.$$

Let

$$H = \bigcup_{x\in\Delta_0} \sigma_{(x,0)}K^+ \quad \text{and} \quad H^+ = \bigcup_{x\in\Delta_0^+} \sigma_{(x,0)}K^+.$$

Both are, by property (5.23) of K^+ disjoint unions. Therefore we have for $\varepsilon > 0$ small enough

$$\begin{aligned}\mathbb{P}(H) &= |\Delta_0|\mathbb{P}(K^+) \quad \text{and}\\ \mathbb{P}(H^+) &= |\Delta_0^+|\mathbb{P}(K^+) = \Big(1+\frac{\varepsilon}{6d^2}\Big)^d |\Delta_0|\mathbb{P}(K^+) < \Big(1+\frac{\varepsilon}{5}\Big)\mathbb{P}(H).\end{aligned} \tag{5.24}$$

From $K^+ \subset S_\varepsilon^+ \subset B_\varepsilon^+$ we obtain

$$\begin{aligned}Q_N(H) &= \int_H \frac{dQ_N}{d\mathbb{P}}(\omega)\,d\mathbb{P}(\omega) = \sum_{x\in\Delta_0}\int_{\sigma_{(x,0)}K^+} \frac{dQ_N}{d\mathbb{P}}(\omega)\,d\mathbb{P}(\omega)\\ &= \int_{K^+}\sum_{x\in\Delta_0}\frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega)\,d\mathbb{P}(\omega)\\ &> \int_{K^+}|\Delta_0|(1+\varepsilon)\,d\mathbb{P}(\omega) = (1+\varepsilon)|\Delta_0|\mathbb{P}(K^+) = (1+\varepsilon)\mathbb{P}(H).\end{aligned} \tag{5.25}$$

Combination of this with (5.24), for small enough $\varepsilon > 0$ then yields

$$\begin{aligned}Q_N(H) &> (1+\varepsilon)\mathbb{P}(H) = \frac{1+\varepsilon}{1+\varepsilon/5}\Big(1+\frac{\varepsilon}{5}\Big)\mathbb{P}(H)\\ &> \frac{1+\varepsilon}{1+\varepsilon/5}\mathbb{P}(H^+) > \Big(1+\frac{\varepsilon}{3}\Big)\mathbb{P}(H^+).\end{aligned} \tag{5.26}$$

Let $A^+ := \{(\omega,\omega') : \omega \notin H^+, \omega' \in H\}$. Then by (5.26)

$$\begin{aligned}\Psi_N(A^+) &\geq Q_N(H) - \mathbb{P}(H^+)\\ &> Q_N(H) - \frac{1}{1+\varepsilon/3}Q_N(H) = \frac{\varepsilon/3}{1+\varepsilon/3}Q_N(H) \geq \frac{\varepsilon}{4}Q_N(H).\end{aligned} \tag{5.27}$$

By the construction of K^+, for every $(\omega,\omega') \in A^+$ we have $\operatorname{dist}(\omega,\omega') > M_\varepsilon$ and, therefore,

$$\begin{aligned}\int_H D^{(2)}_{M_\varepsilon,N}(\omega)\,dQ_N(\omega) &= \int_{\Omega\times H} D^{(2)}_{M_\varepsilon,N}(\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} D^{(2)}_{M_\varepsilon,N}(\omega')\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} E_{\Psi_N}[\mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\,|\,\mathfrak{F}_{\omega'}](\omega')\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} E_{\Psi_N}[\mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,|\,\mathfrak{F}_{\omega'}](\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} \mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &\geq \int_{\Omega\times\Omega} \mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{A^+}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} \mathbb{1}_{A^+}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \Psi_N(A^+) \geq \frac{\varepsilon}{4}Q_N(H).\end{aligned} \tag{5.28}$$

Since $H \subset F_{M_\varepsilon}$ by definition, $\mathbb{P}(H) \leq Q_N(H)$ by (5.25), and using Lemma 5.3 with M_ε and $\frac{\varepsilon}{5}$ instead of M and ε we obtain

$$\begin{aligned}\int_H D^{(2)}_{M_\varepsilon,N}\,dQ_N(\omega) &\leq \int_{H\cap B_N}\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}\Big[\frac{\varepsilon}{5}\mathbb{1}_{F_{M_\varepsilon}\cap B_N} + \mathbb{1}_{(F_{M_\varepsilon}\cap B_N)^\complement}\Big]\,dQ_N(\omega)\\ &= \int_{H\cap B_N}\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}\Big[\frac{\varepsilon}{5}\mathbb{1}_{F_{M_\varepsilon}\cap B_N} + \mathbb{1}_{(F_{M_\varepsilon}\cap B_N)^\complement}\Big]\,dQ_N(\omega)\\ &= \int_{H\cap B_N}\Big[\frac{\varepsilon}{5}\mathbb{1}_{F_{M_\varepsilon}\cap B_N} + \mathbb{1}_{(F_{M_\varepsilon}\cap B_N)^\complement}\Big]\,d\mathbb{P}(\omega)\\ &= \int_{H\cap B_N}\frac{\varepsilon}{5}\,d\mathbb{P}(\omega)\\ &= \frac{\varepsilon}{5}\mathbb{P}(H\cap B_N) \leq \frac{\varepsilon}{5}\mathbb{P}(H) \leq \frac{\varepsilon}{5}Q_N(H),\end{aligned} \tag{5.29}$$

where we recall from Lemma 5.3 that $B_N = \{\omega : \frac{dQ_N}{d\mathbb{P}}(\omega) \neq 0\}$ and note that $B_N^\complement$ is a Q_N null set. Combining (5.28) and (5.29), we conclude that $Q_N(H) = 0$ and, therefore, by (5.25) we have $\mathbb{P}(H) = 0$. It follows that $\mathbb{P}(K^+) = 0$, which by property (5.22) of K^+ implies that $\mathbb{P}(S_\varepsilon^+) = 0$ and finally that (2.5) holds. □

Proof of Corollary 2.4. To show that Proposition 2.2 holds for Q as well we define Ψ as the weak limit of $\{\frac{1}{n}\sum_{N=0}^{n-1}\Psi_N\}_{n=1}^\infty$ along any converging sub-sequence $\{n_k\}_{k\geq 1}$ (tightness of Ψ_N follows similarly to the discussion below Corollary 2.3). Note that Ψ is a coupling of $\mathbb{P}$ and Q on $\Omega\times\Omega$. Furthermore let

$$D^{(i)}_M(\omega_i) := E_\Psi[\mathbb{1}_{\operatorname{dist}(\omega_1,\omega_2)>dM}\,|\,\mathcal{F}_{\omega_i}](\omega_i), \quad i = 1,2.$$

Now we want to prove inequality (5.11) from Lemma 5.3 for $D_M^{(1)}$ and $D_M^{(2)}$. It is enough to show that along some sub-sequence $\{n_\ell\}_{l\geq 1}$ of $\{n_k\}_{k\geq 1}$

$$D_M^{(1)}(\omega) = \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} D_{M,N}^{(1)}(\omega) \quad \mathbb{P}\text{-a.s.} \tag{5.30}$$

and

$$D_M^{(2)}(\omega) = \Big(\frac{dQ}{d\mathbb{P}}(\omega)\Big)^{-1} \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \frac{dQ_N}{d\mathbb{P}}(\omega) D_{M,N}^{(2)}(\omega) \quad Q\text{-a.s.} \tag{5.31}$$

In fact, if the above equalities hold, then for $\mathbb{P}$-almost every ω we have

$$\begin{aligned}
D_M^{(1)}(\omega) &= \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} D_{M,N}^{(1)}(\omega) \\
&= \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[\sum_{N=0}^{M-1} D_{M,N}^{(1)}(\omega) + \sum_{N=M}^{n_\ell-1} D_{M,N}^{(1)}(\omega) \Big] \\
&\leq \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[M + \sum_{N=M}^{n_\ell-1} D_{M,N}^{(1)}(\omega) \Big] \\
&\leq \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[M + \sum_{N=M}^{n_\ell-1} (\varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega)) \Big] \\
&= \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega).
\end{aligned}$$

In addition for $D_M^{(2)}$ we have for Q almost all ω

$$\begin{aligned}
\frac{dQ}{d\mathbb{P}}(\omega) D_M^{(2)}(\omega) &= \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \frac{dQ_N}{d\mathbb{P}}(\omega) D_{M,N}^{(2)}(\omega) \\
&\leq \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[\sum_{N=0}^{M-1} \frac{dQ_N}{d\mathbb{P}}(\omega) + \sum_{N=M}^{n_\ell-1} \frac{dQ_N}{d\mathbb{P}}(\omega) D_{M,N}^{(2)}(\omega) \Big] \\
&\leq \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[\sum_{N=0}^{M-1} \frac{dQ_N}{d\mathbb{P}}(\omega) + \sum_{N=M}^{n_\ell-1} (\varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega)) \Big] \\
&\leq \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega).
\end{aligned}$$

Let us now prove (5.30) and (5.31). Starting with (5.30) let $A \subset \Omega$ be a measurable event. We have

$$\begin{aligned}
&\mathbb{E}[D_M^{(1)}(\omega_1) \mathbb{1}_A(\omega_1)] \\
&\quad = E_\Psi[\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2) > dM\}} \mathbb{1}_{A\times\Omega}(\omega_1,\omega_2)] \\
&\quad = \Psi(\{(\omega_1,\omega_2) \in \Omega\times\Omega : \mathrm{dist}(\omega_1,\omega_2) > dM\} \cap A\times\Omega) \\
&\quad = \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \Psi_N(\{(\omega_1,\omega_2) \in \Omega\times\Omega : \mathrm{dist}(\omega_1,\omega_2) > dM\} \cap A\times\Omega) \\
&\quad = \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} E_{\Psi_N}[\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2) > dM\}} \mathbb{1}_{A\times\Omega}(\omega_1,\omega_2)] \\
&\quad = \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \mathbb{E}[D_{M,N}^{(1)}(\omega_1) \mathbb{1}_A(\omega_1)]
\end{aligned}$$

$$= \lim_{\ell\to\infty} \mathbb{E}\Big[\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1} D^{(1)}_{M,N}(\omega_1)\mathbb{1}_A(\omega_1)\Big]$$

where we used the definitions of Ψ and of $D^{(1)}_{M,N}$ as the conditional expectation. This implies convergence of $\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1} D^{(1)}_{M,N}$ to $D^{(1)}_{M}$ in $L^1(\mathbb{P})$. Thus, by standard arguments we can choose a subsequence that converges $\mathbb{P}$-almost surely. For $D^{(2)}_M$ we obtain in a similar way that Q-almost surely we have

$$\begin{aligned}
&E_Q[D^{(2)}_M(\omega)\mathbb{1}_A(\omega_2)]\\
&\quad= E_\Psi[\mathbb{1}_{\mathrm{dist}(\omega_1,\omega_2)>dM}\mathbb{1}_{\Omega\times A}(\omega_1,\omega_2)]\\
&\quad= \Psi(\{\mathrm{dist}(\omega_1,\omega_2)>dM\}\cap\Omega\times A)\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}\Psi_N(\{\mathrm{dist}(\omega_1,\omega_2)>dM\}\cap\Omega\times A)\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_{\Psi_N}[\mathbb{1}_{\mathrm{dist}(\omega_1,\omega_2)>dM}\mathbb{1}_{\Omega\times A}(\omega_1,\omega_2)]\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_{\Psi_N}[D^{(2)}_{M,N}(\omega_2)\mathbb{1}_{\Omega\times A}(\omega_1,\omega_2)]\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_{Q_N}[D^{(2)}_{M,N}(\omega_2)\mathbb{1}_A(\omega_2)]\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_Q\Big[\big(\frac{dQ}{d\mathbb{P}}(\omega_2)\big)^{-1}\cdot\frac{dQ_N}{d\mathbb{P}}(\omega_2)\cdot D^{(2)}_{M,N}(\omega_2)\mathbb{1}_A(\omega_2)\Big]\\
&\quad= \lim_{\ell\to\infty}E_Q\Big[\big(\frac{dQ}{d\mathbb{P}}(\omega_2)\big)^{-1}\cdot\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}\frac{dQ_N}{d\mathbb{P}}(\omega_2)\cdot D^{(2)}_{M,N}(\omega_2)\mathbb{1}_A(\omega_2)\Big].
\end{aligned}$$

Thus, Lemma 5.3 holds for $D^{(1)}_M$ and $D^{(2)}_M$ instead of $D^{(1)}_{M,N}$ and $D^{(2)}_{M,N}$ respectively.

Since the only results we need for the proof of Proposition 2.2 are Lemma 2.1 and Lemma 5.3, we can walk through the proof of Proposition 2.2 and repeat the same steps for $\frac{dQ}{d\mathbb{P}}$ to show Corollary 2.4. ◻

The following proposition is an analogue to Proposition 7.1 from [2]. Note that the assertion is not model-specific as it expresses a general property of the density of a measure which is invariant for the point of view of the particle in the setting of a random walk in random environment. Recall that $\varphi = dQ/d\mathbb{P}$ is the Radon-Nikodym derivative of Q with respect to $\mathbb{P}$ from Definition 2.7.

Proposition 5.4. *For $\mathbb{P}$-almost every ω, every $n\in\mathbb{N}_0$, every $x\in\mathbb{Z}^d$ and all $k\le n$*

$$\varphi(\sigma_{(x,n)}\omega) = \sum_{y\in\mathbb{Z}^d} P_\omega^{(x+y,n-k)}(X_n = x)\varphi(\sigma_{(x+y,n-k)}\omega).$$

Proof. Let $n\in\mathbb{N}$. First we consider the case $k=1$. For every bounded measurable function $h:\Omega\to\mathbb{R}$ we have (recall the notation in (1.9) and (1.10))

$$\begin{aligned}
\int_\Omega h(\omega)\varphi(\sigma_{(x,n)}\omega)\,d\mathbb{P}(\omega) &= \int_\Omega h(\sigma_{(-x,-n)}\omega)\varphi(\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega h(\sigma_{(-x,-n)}\omega)\,dQ(\omega)
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Re h(\sigma_{(-x,-n)}\omega)\, dQ(\omega) \\
&= \int_\Omega (\Re h(\sigma_{(-x,-n)}\omega))\varphi(\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \sum_{\|y\|\le 1} g(\omega,y) h(\sigma_{(-x+y,1-n)}\omega)\varphi(\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \sum_{\|y\|\le 1} g(\sigma_{(x-y,n-1)}\omega,y) h(\omega)\varphi(\sigma_{(x-y,n-1)}\omega)\, d\mathbb{P}(\omega).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\varphi(\sigma_{(x,n)}\omega) &= \sum_{\|y\|\le 1} g(\sigma_{(x-y,n-1)}\omega)\varphi(\sigma_{(x-y,n-1)}\omega) \\
&= \sum_{\|y\|\le 1} P^{(0,0)}_{\sigma_{(x-y,n-1)}\omega}(X_1=y)\varphi(\sigma_{(x-y,n-1)}\omega) \\
&= \sum_{\|y\|\le 1} P^{(x-y,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x-y,n-1)}\omega) \\
&= \sum_{y\in\mathbb{Z}^d} P^{(x+y,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x+y,n-1)}\omega).
\end{aligned}
$$

By applying the operator $\Re$ a second time we see that

$$
\begin{aligned}
&\int_\Omega h(\omega)\varphi(\sigma_{(x,n)}\omega)\, d\mathbb{P} \\
&= \int_\Omega h(\omega) \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x+y_1,n-1)}\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega h(\sigma_{(-x-y_1,-n+1)}\omega) \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\sigma_{(-x-y_1,-n+1)}\omega}(X_1=x)\varphi(\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \Big[\Big(\Re(h(\sigma_{(-x-y_1,-n+1)}\omega) \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\sigma_{(-x-y_1,-n+1)}\omega}(X_1=x))\Big)\Big]\varphi(\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \sum_{\|y_2\|\le 1} g(\omega,y_2) h(\sigma_{(-x-y_1+y_2,-n+2)}\omega) \\
&\qquad\qquad \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\sigma_{(-x-y_1+y_2,-n+2)}\omega}(X_1=x)\varphi(\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \sum_{\|y_2\|\le 1} g(\sigma_{(x+y_1-y_2,n-2)}\omega,y_2) h(\omega) \\
&\qquad\qquad \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x+y_1-y_2,n-2)}\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \sum_{\|y_2\|\le 1} P^{(x+y_1+y_2,n-2)}_{\omega}(X_1=x+y_1) \\
&\qquad\qquad \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\omega}(X_1=x) h(\omega)\varphi(\sigma_{(x+y_1+y_2,n-2)}\omega)\, d\mathbb{P}(\omega).
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\varphi(\sigma_{(x,n)}\omega)\\
&\quad= \sum_{\|y_1\|\le 1}\sum_{\|y_2\|\le 1} P_\omega^{(x+y_1+y_2,n-2)}(X_1 = x+y_1) P_\omega^{(x+y_1,n-1)}(X_1 = x)\varphi(\sigma_{(x+y_1+y_2,n-2)}\omega)\\
&\quad= \sum_{y\in\mathbb{Z}^d} P_\omega^{x+y,n-2}(X_2 = x)\varphi(\sigma_{(x+y,n-2)}\omega).
\end{aligned}$$

Inductively we obtain

$$\varphi(\sigma_{(x,n)}\omega) = \sum_{y\in\mathbb{Z}^d} P_\omega^{(x+y,n-k)}(X_k = x)\varphi(\sigma_{(x+y,n-k)}\omega)$$

for all $k \le n$. □

6 Proof of Proposition 2.8

Let Π be a partition of $\mathbb{Z}^d$ into boxes of side length $\lfloor n^\delta \rfloor$ with $0 < \delta < \frac{1}{6d}$. Since $\mathbb{P}^{(0,0)}(X_n = x) = 0$ for $\|x\| > n$ only boxes in $\Pi_n := \{\Delta \in \Pi : \Delta \cap [-n,n]^d \neq \emptyset\}$ have to be considered. We have

$$\begin{aligned}
|Z_{\omega,n} - 1| &= \Big|\sum_{x\in\mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big|\\
&= \Big|\sum_{\Delta\in\Pi_n}\sum_{x\in\Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big|.
\end{aligned}\tag{6.1}$$

By the annealed CLT from [4] for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\mathbb{P}^{(0,0)}(\|X_n\| \ge C_\varepsilon\sqrt{n}) < \varepsilon$$

We want to use this fact below and separate the sum in the last line of (6.1) into boxes in $\widehat{\Pi}_n = \{\Delta \in \Pi_n : \Delta \cap \{x \in \mathbb{Z}^d : \|x\| \le C_\varepsilon\sqrt{n}\} \neq \emptyset\}$ and in $\Pi_n \setminus \widehat{\Pi}_n$. Using the triangle inequality we obtain

$$\begin{aligned}
&|Z_{\omega,n} - 1|\\
&\quad\le \Big|\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\sum_{x\in\Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big| &(6.2)\\
&\qquad+ \Big|\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\Big(\frac{1}{|\Delta|}\sum_{y\in\Delta}[\mathbb{P}^{(0,0)}(X_n = y) - \mathbb{P}^{(0,0)}(X_n = x)]\Big)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big| &(6.3)\\
&\qquad+ \Big|\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\frac{1}{|\Delta|}\sum_{y\in\Delta} \mathbb{P}^{(0,0)}(X_n = y)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big|. &(6.4)
\end{aligned}$$

We start with an upper bound of (6.2). By Corollary 2.4 there exists a constant C, such that, due to translation invariance of $\mathbb{P}$, with $\mathbb{P}$ probability of a least $1 - Cn^{-c\log n}$ for every $\Delta \in \Pi_n$ we have $\sum_{y\in\Delta}[\varphi(\sigma_{(y,n)}\omega)+1] \le C|\Delta|$. Under this event we can bound (6.2) from above by

$$\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\sum_{x\in\Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) + 1] \le C\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\max_{x\in\Delta}\mathbb{P}^{(0,0)}(X_n = x)|\Delta|.$$

Using Lemma 3.2 with $\delta > 0$ replacing ε there we see that (6.2) is bounded from above by

$$C\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\sum_{y\in\Delta}\Big[\max_{x\in\Delta}\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = y)\Big] + C\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\sum_{y\in\Delta}\mathbb{P}^{(0,0)}(X_n = y)$$

$$\begin{aligned}&\le C\varepsilon + C\sum_{\Delta\in\Pi_n}\sum_{y\in\Delta}\Big[\max_{x\in\Delta}\mathbb{P}^{(0,0)}(X_n=x)-\mathbb{P}^{(0,0)}(X_n=y)\Big]\\&\le C\varepsilon + Cn^{-\frac{1}{2}+3d\delta}.\end{aligned}$$

Since $\delta<\frac{1}{6d}$ it follows by the Borel–Cantelli lemma that

$$\limsup_{n\to\infty}\Big|\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\sum_{x\in\Delta}\mathbb{P}^{(0,0)}(X_n=x)[\varphi(\sigma_{(x,n)}\omega)-1]\Big|\le C\varepsilon,\qquad \mathbb{P}\text{-a.s.}\tag{6.5}$$

Next we turn to (6.3). First note that by the annealed derivative estimates from Lemma 3.1 we have for $x,y\in\Delta$, $\Delta\in\widehat{\Pi}_n$

$$|\mathbb{P}^{(0,0)}(X_n=x)-\mathbb{P}^{(0,0)}(X_n=y)|\le C\,\|x-y\|\,n^{-\frac{d+1}{2}}\le Cn^{-\frac{d+1}{2}+\delta}.\tag{6.6}$$

By triangle inequality, (6.6) and again, as above, using Corollary 2.4 for the bound $\sum_{y\in\Delta}[\varphi(\sigma_{(y,n)}\omega)+1]\le C|\Delta|$ the expression (6.3) is bounded from above by

$$\begin{aligned}&\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\frac{1}{|\Delta|}\sum_{y\in\Delta}|\mathbb{P}^{(0,0)}(X_n=y)-\mathbb{P}^{(0,0)}(X_n=x)||\varphi(\sigma_{(x,n)}\omega)-1|\\&\qquad\le Cn^{-\frac{d+1}{2}+\delta}\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\frac{1}{|\Delta|}\sum_{y\in\Delta}\big[\varphi(\sigma_{(x,n)}\omega)+1\big]\\&\qquad\le Cn^{-\frac{d+1}{2}+\delta}\sum_{\Delta\in\widehat{\Pi}_n}\sum_{y\in\Delta}C\\&\qquad\le \widetilde{C}(C_\varepsilon\sqrt{n})^d n^{-\frac{d+1}{2}+\delta}\le \widehat{C}_\varepsilon n^{-\frac{1}{2}+\delta}.\end{aligned}$$

with probability at least $1-Cn^{-c\log n}$. Thus, as $n\to\infty$, by the Borel–Cantelli lemma the expression (6.3) tends to 0 $\mathbb{P}$-almost surely.

Finally we consider (6.4). By triangle inequality and $\mathbb{P}^{(0,0)}(X_n=y)\le Cn^{-d/2}$ for all y we have

$$\begin{aligned}&\Big|\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\frac{1}{|\Delta|}\sum_{y\in\Delta}\mathbb{P}^{(0,0)}(X_n=y)[\varphi(\sigma_{(x,n)}\omega)-1]\Big|\\&\qquad\le\sum_{\Delta\in\widehat{\Pi}_n}\frac{1}{|\Delta|}\sum_{y\in\Delta}\mathbb{P}^{(0,0)}(X_n=y)\Big|\sum_{x\in\Delta}[\varphi(\sigma_{(x,n)}\omega)-1]\Big|\\&\qquad\le Cn^{-d/2}\sum_{\Delta\in\widehat{\Pi}_n}\Big|\sum_{x\in\Delta}[\varphi(\sigma_{(x,n)}\omega)-1]\Big|\\&\qquad\le Cn^{-d(1/2-\delta)}\sum_{\Delta\in\widehat{\Pi}_n}\frac{1}{|\Delta|}\Big|\sum_{x\in\Delta}[\varphi(\sigma_{(x,n)}\omega)-1]\Big|.\end{aligned}$$

Using Corollary 2.4 we obtain

$$\begin{aligned}&\mathbb{P}\Big(Cn^{-d(1/2-\delta)}\sum_{\Delta\in\widehat{\Pi}_n}\frac{1}{|\Delta|}\Big|\sum_{x\in\Delta}[\varphi(\sigma_{(x,n)}\omega)-1]\Big|>\varepsilon\Big)\\&\qquad\le\mathbb{P}\Big(\exists\Delta\in\widehat{\Pi}_n:\frac{1}{|\Delta|}\Big|\sum_{x\in\Delta}[\varphi(\sigma_{(x,n)}\omega)-1]\Big|>\frac{\varepsilon}{CC_\varepsilon^d}\Big)\\&\qquad\le CC_\varepsilon^d n^{d(1/2-\delta)}\mathbb{P}\Big(\frac{1}{|\Delta_0|}\Big|\sum_{x\in\Delta_0}[\varphi(\sigma_{(x,n)}\omega)-1]\Big|>\frac{\varepsilon}{CC_\varepsilon^d}\Big)\\&\qquad\le CC_\varepsilon^d n^{d(1/2-\delta)}n^{-c\delta^2\log n}\le\tilde{C}n^{-\tilde{c}\log n},\end{aligned}$$

where $\Delta_0 \in \widehat{\Pi}_n$ is an arbitrarily fixed box. Thus, for $\varepsilon > 0$ as $n \to \infty$ the $\limsup$ of (6.4) is bounded from above by ε $\mathbb{P}$-almost surely. Combining all three bounds of (6.2)–(6.4), we see that there is a constant $\widehat{C}$ so that for all $\varepsilon > 0$

$$\limsup_{n\to\infty} |Z_{\omega,n} - 1| \le \widehat{C}\varepsilon, \quad \mathbb{P}\text{-almost surely,}$$

which concludes the proof. □

7 Proof of Proposition 2.9

The following result is an essential tool to prove Proposition 2.9 and will be proven in Section 9.

Lemma 7.1. *Let $0 < \theta < 1/2$ and $b > 0$. Define the set*

$$D(n) := \bigcap_{\substack{x,y\in\mathbb{Z}^d:\\ \|x\|,\|y\|\le n^b,\\ \|x-y\|\le n^\theta}} \left\{ \left\| P_\omega^{(x,0)}(X_n \in \cdot) - P_\omega^{(y,0)}(X_n \in \cdot) \right\|_{\mathrm{TV}} \le \mathrm{e}^{-c\frac{\log n}{\log\log n}} \right\}. \tag{7.1}$$

Then there are constants $C, c > 0$ so that $\mathbb{P}(D(n)) \ge 1 - Cn^{-c\log n}$.

Note that the restriction $\|x\|, \|y\| \le n^b$ in the definition of $D(n)$ in (7.1) is necessary because with probability 1 we have an environment where there exist (somewhere far out in space) two neighbouring points $x, y \in \mathbb{Z}^d$ so that the sites $(x,0)$ and $(y,0)$ are both connected to infinity but the respective clusters do not intersect for the first n time steps.

Remark 7.2. The above lemma is the analogue of Lemma 7.7 from [2] in our setting. Note that the bound stated in Lemma 7.7 from [2] is too optimistic to hold in general. However, its assertion can be weakened and one obtains a bound which is still strong enough to prove Lemma 7.5 in [2] by going a similar route as in the proof of Lemma 7.1 here.

Proof of Proposition 2.9, (L1). For this part we make use of the fact that, due to the annealed derivative estimates from Lemma 3.1 for $|x-y| \le k$, $|\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_{n-k} = y)| \le Ck/(n-k)^{(d+1)/2} \approx n^{-(d+1)/2+\varepsilon}$, since $k = \lceil n^\varepsilon \rceil \ll n$. Furthermore we use the fact that by definition as a density of the invariant measure of the environment with respect to the point of view of the particle, the prefactor can be "transported" along the quenched transition probabilities; see Proposition 5.4. Finally we use the concentration property of Corollary 2.4; see equation (2.9).

We have to show

$$\begin{aligned} \lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \Big| & \frac{1}{Z_{\omega,n}} \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) \\ & - \frac{1}{Z_{\omega,n-k}} \sum_{y\in\mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_{n-k} = y)\varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x-y) \Big| = 0. \end{aligned} \tag{7.2}$$

Note that the by the triangle inequality the sum on the left hand side is bounded from above by

$$\begin{aligned} \sum_{x\in\mathbb{Z}^d} \Big| \frac{1}{Z_{\omega,n}} - \frac{1}{Z_{\omega,n-k}} \Big| & \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) \\ & + \frac{1}{Z_{\omega,n-k}} \sum_{x\in\mathbb{Z}^d} \Big| \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) \end{aligned}$$

$$- \sum_{y \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_{n-k} = y)\varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x - y)\Big|.$$

By definition of $Z_{\omega,n}$, recall from Definition 2.7, the first sum in the above display equals to

$$\Big|\frac{1}{Z_{\omega,n}} - \frac{1}{Z_{\omega,n-k}}\Big| Z_{\omega,n},$$

which by Proposition 2.8 almost surely goes to 0 as n and $n-k$ both tend to ∞. Thus, taking also into account the trivial deterministic bound on the speed of the random walk, for (7.2) it suffices to show

$$\begin{aligned} \lim_{n\to\infty} \sum_{x \in \mathbb{Z}^d \cap [-n,n]^d} \Big| & \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) \\ & - \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \mathbb{P}^{(0,0)}(X_{n-k} = y)\varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x - y)\Big| = 0. \end{aligned} \tag{7.3}$$

Denoting by $B_n = \{x \in \mathbb{Z}^d : \|x\| \le \sqrt{n}\log^3 n\}$ and using the triangle inequality an upper bound of the sum in (7.3) is given by

$$\begin{aligned} & \sum_{x \in B_n} \Big| \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \big[\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_{n-k} = y)\big] \\ & \qquad \times \varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x - y)\Big| \end{aligned} \tag{7.4}$$

$$\begin{aligned} & + \sum_{x \in B_n} \mathbb{P}^{(0,0)}(X_n = x) \\ & \qquad \times \Big|\varphi(\sigma_{(x,n)}\omega) - \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x - y)\Big| \end{aligned} \tag{7.5}$$

$$\begin{aligned} & + \sum_{x \in \mathbb{Z}^d \cap [-n,n]^d \setminus B_n} \Big| \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) \\ & \qquad - \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \mathbb{P}^{(0,0)}(X_{n-k} = y)\varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x - y)\Big|. \end{aligned} \tag{7.6}$$

By the annealed derivative estimates (see Lemma 3.1) the term in (7.4) is bounded from above by

$$\begin{aligned} & \sum_{x \in B_n} \Big| \sum_{\substack{y \in \mathbb{Z}^d \\ \|x-y\| \le k}} \big[\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_{n-k} = y)\big] \\ & \qquad \times \varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x - y)\Big| \\ & \le \frac{2Ck}{(n-k)^{(d+1)/2}} \sum_{x \in B_n} \sum_{\substack{y \in \mathbb{Z}^d \\ \|x-y\| \le k}} \varphi(\sigma_{(y,n-k)}\omega) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x - y) \\ & \le \frac{2Ck(\sqrt{n}\log^3 n + k)^d}{(n-k)^{(d+1)/2}} \frac{1}{(\sqrt{n}\log^3 n + k)^d} \sum_{\substack{y \in \mathbb{Z}^d \\ \mathrm{dist}(y,B_n) \le k}} \varphi(\sigma_{(y,n-k)}\omega). \end{aligned}$$

Now using Corollary 2.4 and the fact that $k = \lceil n^\varepsilon \rceil < n^{1/4}$ for $\mathbb{P}$-almost every ω the last term tends to zero as n tend to infinity.

Next we deal with (7.5). Recall that by Proposition 5.4 we have

$$\varphi(\sigma_{(x,n)}\omega) = \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d} \varphi(\sigma_{(x,n-k)}\omega)P_\omega^{(y,n-k)}(X_k = x)$$

for every $x \in \mathbb{Z}^d$ such that $x + [-k,k]^d \cap \mathbb{Z}^d \subset [-n,n]^d \cap \mathbb{Z}^d$. This holds for every $x \in B_n$ and therefore the expression (7.5) equals 0.

Finally, for (7.6), using Lemma 3.6 from [21], we have $\mathbb{P}^{(0,0)}(X_n \notin B_n) \le Cn^{-c\log n}$. Recall that $k = \lceil n^\varepsilon \rceil$ and note that if $P_\omega^{(y,n-k)}(X_k = x) > 0$ then $\|x - y\| \le k$. Thus, for $x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n$ and large enough n

$$\|y\| \ge \|x\| - \|x - y\| \ge \sqrt{n}\log^3 n - k \ge \frac{1}{2}\sqrt{n}\log^3 n.$$

This implies, again due to Lemma 3.6 from [21] that $\mathbb{P}^{(0,0)}(X_{n-k} = y) \le Cn^{-c\log n}$. Therefore, the expression (7.6) is bounded from above by

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n} \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) \\
&\qquad + \sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n}\ \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d} \mathbb{P}^{(0,0)}(X_{n-k} = y)\varphi(\sigma_{(y,n-k)}\omega)P_\omega^{(y,n-k)}(X_k = x) \\
&\le Cn^{-c\log n} \sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n} \varphi(\sigma_{(x,n)}\omega) \\
&\qquad + Cn^{-c\log n} \sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n}\ \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d} \varphi(\sigma_{(y,n-k)}\omega)P_\omega^{(y,n-k)}(X_k = x) \\
&\le Cn^{-c\log n} \sum_{x\in\mathbb{Z}^d\cap[-n,n]^d} \varphi(\sigma_{(x,n)}\omega) + Cn^{-c\log n} \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d} \varphi(\sigma_{(y,n-k)}\omega).
\end{aligned}$$

By Corollary 2.4 we have

$$\mathbb{P}\Big(\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d} \varphi(\sigma_{(x,n)}\omega) \le (2n+1)^d\Big) > 1 - n^{-c\log n},$$

as well as

$$\mathbb{P}\Big(\sum_{y\in\mathbb{Z}^d\cap[-n,n]^d} \varphi(\sigma_{(y,n-k)}\omega) \le (2n+1)^d\Big) > 1 - Cn^{-c\log n}.$$

Thus, the probability of the event that (7.6) is bounded above by $4Cn^{-c\log n}n^d$ converges to 1 super-algebraically fast. Hence the expression (7.6) converges to 0 $\mathbb{P}$-almost surely. $\square$

Proof of Proposition 2.9, (L2). First note that, it is enough to show that

$$\left\|\nu_\omega^{\mathrm{ann}\times\mathrm{pre}} - \nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}\right\|_{1,n-k} \xrightarrow{n\to\infty} 0,$$

since the last k steps are according to the quenched law for both hybrid measures. Then, as the measure $\nu^{\mathrm{box-que}\times\mathrm{pre}}$ suggests, we make use of the comparison between the quenched and the annealed laws on the level of boxes we derived from Lemma 2.1. We also use the concentration properties of φ from Corollary 2.4.

Let $k \in \{0,\ldots,n\}$ be fixed. Note that we have

$$\left\|(\nu^{\mathrm{ann}\times\mathrm{pre}} * \nu^{\mathrm{que}})_{\omega,k} - (\nu^{\mathrm{box-que}\times\mathrm{pre}} * \nu^{\mathrm{que}})_{\omega,k}\right\|_{1,n} \le \left\|\nu_\omega^{\mathrm{ann}\times\mathrm{pre}} - \nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}\right\|_{1,n-k}$$

$$= \sum_{x\in\mathbb{Z}^d} \varphi(\sigma_{(x,n-k)}\omega)\Big|\frac{\mathbb{P}^{(0,0)}(X_{n-k}=x)}{Z_{\omega,n-k}} - \frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big|,$$

where $Z_{\omega,n-k}$ is the normalizing constant from Definition 2.7. By using Proposition 2.8 it is enough to show that $\mathbb{P}$-almost surely

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d} \varphi(\sigma_{(x,n-k)}\omega)\Big|\mathbb{P}^{(0,0)}(X_{n-k}=x) - \frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big| = 0. \tag{7.7}$$

Let $A_n = \{x\in\mathbb{Z}^d : \|x\|\le C(\varepsilon')\sqrt{n}\}$, with $C(\varepsilon')$ chosen so that $\mathbb{P}^{(0,0)}(\|X_{n-k}\| > \frac{C(\varepsilon')}{2}\sqrt{n-k}) < \varepsilon'$ for n large enough. Note that ε' can be chosen independently of ε and δ from Proposition 2.9. Using the triangle inequality the sum in (7.7) is bounded by

$$\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus A_n} \varphi(\sigma_{(x,n-k)}\omega)\Big|\mathbb{P}^{(0,0)}(X_{n-k}=x) - \frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big| \tag{7.8}$$

$$+ \sum_{x\in A_n} \varphi(\sigma_{(x,n-k)}\omega)\Big|\mathbb{P}^{(0,0)}(X_{n-k}=x) - \frac{\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)}{|\Delta_x|}\Big| \tag{7.9}$$

$$+ \sum_{x\in A_n} \varphi(\sigma_{(x,n-k)}\omega)\Big|\frac{\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)}{|\Delta_x|} - \frac{\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big| \tag{7.10}$$

$$+ \sum_{x\in A_n} \varphi(\sigma_{(x,n-k)}\omega)\Big|\frac{\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)} - \frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big|. \tag{7.11}$$

Now we deal with the four terms separately. Expression (7.8) is bounded from above by

$$\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus A_n} \mathbb{P}^{(0,0)}(X_{n-k}=x)\varphi(\sigma_{(x,n-k)}\omega) + P_\omega^{(0,0)}(\|X_{n-k}\| > C(\varepsilon')\sqrt{n}).$$

We obtain $\limsup_{n\to\infty}\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus A_n}\mathbb{P}^{(0,0)}(X_{n-k}=x)\varphi(\sigma_{(x,n-k)}\omega)\le C\varepsilon'$ by the same arguments used to bound (6.2) in the proof of Proposition 2.8. For the second term we can argue as in the proof of Claim 2.15 from [2], to obtain that for a set of environments, with $\mathbb{P}$ probability $> 1-\sqrt{\varepsilon'}$, for large enough n

$$P_\omega^{(0,0)}(\|X_{n-k}\| > C(\varepsilon')\sqrt{n}) \le P_\omega^{(0,0)}\Big(\|X_n\| > \frac{C(\varepsilon')}{2}\sqrt{n}\Big) \le \sqrt{\varepsilon'}.$$

Since $\varepsilon' > 0$ was arbitrary, this proves that (7.8) goes to zero as n goes to infinity.

Next we turn to (7.9). The annealed derivative estimates (recall Lemma 3.1) yield that it is bounded from above by

$$\begin{aligned}
&\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x}|\mathbb{P}^{(0,0)}(X_{n-k}=x) - \mathbb{P}^{(0,0)}(X_{n-k}=y)| \\
&\qquad \le C\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x}\frac{1}{(n-k)^{(d+1)/2}}\|x-y\| \\
&\qquad \le Cdn^\delta\frac{1}{(n-k)^{(d+1)/2}}\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega) \\
&\qquad = \frac{Cn^{\delta+d/2}}{(n-k)^{(d+1)/2}}\Big(\frac{1}{n^{d/2}}\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\Big) \xrightarrow{n\to\infty} 0, \quad \mathbb{P}\text{-a.s.,}
\end{aligned}$$

where for the limit we use Corollary 2.4, the fact that $k = \lceil n^\varepsilon \rceil$ and $\delta < \varepsilon < \frac{1}{4}$.

Next we deal with (7.10). Writing $\widehat{\Pi}_n = \{\Delta \in \Pi : \Delta \cap A_n \neq \emptyset\}$, using annealed derivative estimates and Corollary 2.4 we see that (7.10) is bound by

$$\begin{aligned}
&\sum_{x \in A_n} \varphi(\sigma_{(x,n-k)}\omega) \frac{1}{|\Delta_x|} \mathbb{P}^{(0,0)}(X_{n-k} \in \Delta_x) \Big| 1 - \frac{1}{\frac{1}{|\Delta_x|}\sum_{y \in \Delta_x} \varphi(\sigma_{(y,n-k)}\omega)} \Big| \\
&\quad \le \frac{C}{(n-k)^{d/2}} \sum_{x \in A_n} \varphi(\sigma_{(x,n-k)}\omega) \Big| 1 - \frac{1}{\frac{1}{|\Delta_x|}\sum_{y \in \Delta_x} \varphi(\sigma_{(y,n-k)}\omega)} \Big| \\
&\quad \le C \Big(\frac{n-k}{n}\Big)^{-d/2} \frac{1}{n^{d/2}} \sum_{\Delta \in \widehat{\Pi}_n} \sum_{x \in \Delta} \varphi(\sigma_{(x,n-k)}\omega) \Big| 1 - \frac{1}{\frac{1}{|\Delta_x|}\sum_{y \in \Delta_x} \varphi(\sigma_{(y,n-k)}\omega)} \Big| \\
&\quad = C \Big(1 - \frac{k}{n}\Big)^{-d/2} \frac{1}{n^{d/2}} \sum_{\Delta \in \widehat{\Pi}_n} \sum_{x \in \Delta} \frac{\varphi(\sigma_{(x,n-k)}\omega)|\Delta_x|}{\sum_{y \in \Delta_x} \varphi(\sigma_{(y,n-k)}\omega)} \Big| \frac{1}{|\Delta_x|} \sum_{y \in \Delta_x} \varphi(\sigma_{(y,n-k)}\omega) - 1 \Big| \\
&\quad = C \Big(1 - \frac{k}{n}\Big)^{-d/2} \frac{1}{n^{(d/2)(1-2\delta)}} \sum_{\Delta \in \widehat{\Pi}_n} \Big| \frac{1}{|\Delta|} \sum_{x \in \Delta} \varphi(\sigma_{(x,n-k)}\omega) - 1 \Big|.
\end{aligned}$$

Using the same argument that was used for (6.4), we get that by the Borel–Cantelli lemma the last term goes to zero $\mathbb{P}$-a.s.

Finally, we estimate (7.11). It is bounded from above by

$$\begin{aligned}
&\sum_{x \in A_n} \frac{\varphi(\sigma_{(x,n-k)}\omega)}{\sum_{y \in \Delta_x} \varphi(\sigma_{(y,n-k)}\omega)} |\mathbb{P}^{(0,0)}(X_{n-k} \in \Delta_x) - P_\omega^{(0,0)}(X_{n-k} \in \Delta_x)| \\
&\qquad\qquad\qquad = \sum_{\Delta \in \widehat{\Pi}_n} |\mathbb{P}^{(0,0)}(X_{n-k} \in \Delta) - P_\omega^{(0,0)}(X_{n-k} \in \Delta)|.
\end{aligned}$$

For the last term we can use Theorem 8.1 which implies that it is bounded by $Cn^{-\frac{1}{3}\delta}$ for $\mathbb{P}$-almost every ω and large enough n. Therefore $\mathbb{P}$ almost surely it converges to zero as n tends to infinity. □

Proof of Proposition 2.9, (L3). Note that the first measure chooses, at time $n-k$, a box according to the quenched law and a point in that box weighted by the prefactor, whereas the second measure chooses a box and a point in that box according to the quenched law at time $n-k$. These points are then the starting points for the quenched random walks for the remaining k steps. We use the fact that, given enough time (much more than the square of the starting distance), the total variation distance for two quenched random walks starting from any pair of sites in a box with side length $\lceil n^\ell \rceil$ is, given enough time, i.e. much more than the square of the side length of the box, is small with high probability, see Lemma 7.1.

The proof follows along the same lines as in [2]. We will highlight the point in the proof where we deviate. We have

$$\begin{aligned}
&\big\| (\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k} - (\nu^{\text{que}} * \nu^{\text{que}})_{\omega,k} \big\|_{1,n} \\
&= \sum_{x \in \mathbb{Z}^d} \big| (\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}(x,n) - (\nu^{\text{que}} * \nu^{\text{que}})_{\omega,k}(x,n) \big| \\
&= \sum_{x \in \mathbb{Z}^d} \Big| \sum_{y \in \mathbb{Z}^d} P_\omega^{(0,0)}(X_{n-k} \in \Delta_y) \frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z \in \Delta_y} \varphi(\sigma_{(z,n-k)}\omega)} P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x-y) \\
&\qquad\qquad - \sum_{y \in \mathbb{Z}^d} P_\omega^{(0,0)}(X_{n-k} = y) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x-y) \Big|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in\mathbb{Z}^d}\Big|\sum_{\Delta\in\Pi}\sum_{y\in\Delta} P_\omega^{(y,n-k)}(X_k=x)P_\omega^{(0,0)}(X_{n-k}\in\Delta)\\
&\qquad\qquad \cdot\Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big)\Big|\\
&\le \sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi}\Big|\sum_{y\in\Delta} P_\omega^{(y,n-k)}(X_k=x)P_\omega^{(0,0)}(X_{n-k}\in\Delta)\\
&\qquad\qquad \cdot\Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big)\Big|.
\end{aligned}
\tag{7.12}
$$

Since for every $\Delta\in\Pi$ and $x\in\mathbb{Z}^d$ we have

$$
\sum_{y\in\Delta}\frac{1}{|\Delta|}\sum_{v\in\Delta} P_\omega^{(v,n-k)}(X_k=x)\Big[\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big] = 0
$$

it follows that (7.12) equals

$$
\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\Big|\sum_{y\in\Delta}\Big[P_\omega^{(y,n-k)}(X_k=x) - \Big(\frac{1}{|\Delta|}\sum_{w\in\Delta}P_\omega^{(w,n-k)}(X_k=x)\Big)\Big]\\
&\qquad\qquad \cdot\Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big)\Big|\\
&= \sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\Big|\frac{1}{|\Delta|}\sum_{y\in\Delta}\sum_{w\in\Delta}\Big[P_\omega^{(y,n-k)}(X_k=x) - P_\omega^{(w,n-k)}(X_k=x)\Big]\\
&\qquad\qquad \cdot\Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big)\Big|\\
&\le \sum_{\Delta\in\Pi}\sum_{x\in\mathbb{Z}^d} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\sum_{y\in\Delta}\frac{1}{|\Delta|}\sum_{w\in\Delta}\Big|P_\omega^{(y,n-k)}(X_k=x) - P_\omega^{(w,n-k)}(X_k=x)\Big|\\
&\qquad\qquad \cdot\Big|\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big|
\end{aligned}
\tag{7.13}
$$

Until this point the steps are basically the same as in [2]. Here we deviate from their proof. Note that $P_\omega^{(0,0)}(X_{n-k}\in\Delta)=0$ if $\Delta\cap[-n+k,n-k]^d=\emptyset$. For $\Delta\cap[-n+k,n-k]^d\neq\emptyset$ we have $y,w\in\Delta$ implies that $\|y\|,\|w\|\le n=k^{1/\varepsilon}$ and $\|y-w\|\le n^\delta=k^{\delta/\varepsilon}$.

Using Lemma 7.1 we see that (7.13) is bounded from above by

$$
\begin{aligned}
&\sum_{\Delta\in\Pi} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\sum_{y\in\Delta}\Big|\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big|\\
&\qquad\qquad \cdot\frac{1}{|\Delta|}\sum_{w\in\Delta}\sum_{x\in\mathbb{Z}^d}\Big|P_\omega^{(y,n-k)}(X_k=x) - P_\omega^{(w,n-k)}(X_k=x)\Big|\\
&\le \mathrm{e}^{-c\frac{\log k}{\log\log k}}\sum_{\Delta\in\Pi} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\\
&\qquad\qquad \cdot\sum_{y\in\Delta}\Big|\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y\,|\,X_{n-k}\in\Delta)\Big|\\
&\le 2\mathrm{e}^{-c\frac{\log k}{\log\log k}}\sum_{\Delta\in\Pi} P_\omega^{(0,0)}(X_{n-k}\in\Delta) = 2\mathrm{e}^{-c\frac{\log k}{\log\log k}} \le C\mathrm{e}^{-\tilde c\frac{\log n}{\log\log n}}
\end{aligned}
$$

since $k=\lceil n^\varepsilon\rceil$. The right hand side goes to 0 for $n\to\infty$. □

8 Proof of Proposition 4.1

The starting point is a result from [21]. Define

$$\mathcal{P}(N) := \Big(\Big[-\frac{1}{24}\sqrt{N}\log^3 N, \frac{1}{24}\sqrt{N}\log^3 N\Big]^d \times \Big[0, \frac{1}{3}N\Big]\Big) \cap (\mathbb{Z}^d \times \mathbb{Z}). \tag{8.1}$$

For $\theta \in (0,1)$ and $(x,m) \in \mathcal{P}(N)$ let $G'((x,m),N)$ denote the event that for every box $\Delta \subset \mathbb{Z}^d$ of side length $N^{\theta/2}$ we have

$$\big|P_\omega^{(x,m)}(X_{m+N} \in \Delta) - \mathbb{P}^{(x,m)}(X_{m+N} \in \Delta)\big| \le CN^{-d(1-\theta)/2-\frac{1}{6}\theta}. \tag{8.2}$$

Furthermore set

$$G'(N) := \bigcap_{(x,m)\in\mathcal{P}(N)} \big(G'((x,m),N) \cup \{\xi_m(x) = 0\}\big). \tag{8.3}$$

Theorem 8.1 (Theorem 3.24 in [21])**.** *Let $d \ge 3$. There exist positive constants c and C, such that for all $(x,m) \in \mathcal{P}(N)$ we have*

$$\mathbb{P}^{(x,m)}\big(G'((x,m),N)\big) \ge 1 - CN^{-c\log N} \tag{8.4}$$

and

$$\mathbb{P}\big(G'(N)\big) \ge 1 - CN^{-c\log N}. \tag{8.5}$$

The following notion of *good sites* and *good boxes* will be needed in the proof of Proposition 4.1. On such boxes the annealed and quenched laws are "close" to each other. Recall the process $\xi = (\xi_n)_{n\in\mathbb{Z}}$ from (1.2) and the definition of n_k from the beginning of Section 4. Recall also that Π_k is a partition of $\mathbb{Z}^d$ into the boxes of side length $\lfloor n_k^\theta \rfloor$.

Definition 8.2. *For a given realisation $\omega \in \Omega$, we say that $(x,m) \in \mathbb{Z}^d \times \mathbb{Z}$ is $(k-1,\theta,\varepsilon)$-*good *if either $\xi_m(x;\omega) = 0$ or $\xi_m(x;\omega) = 1$ and the following two conditions are satisfied*

$$\sup_{\Delta'\in\Pi_k} \big|P_\omega^{(x,m)}(X_{m+n_k} \in \Delta') - \mathbb{P}^{(x,m)}(X_{m+n_k} \in \Delta')\big| \le n_k^{\theta d-\frac{d}{2}-\varepsilon}, \tag{8.6}$$

$$P_\omega^{(x,m)}\Big(\max_{s\le n_k} \|X_{m+s} - x\| > \sqrt{n_k}\log^3 n_k\Big) \le Cn_k^{-c\log n_k}. \tag{8.7}$$

*Otherwise the site is said to be $(k-1,\theta,\varepsilon)$-*bad*. We say that for $\Delta \in \Pi_{k-1}$ and $m \in \mathbb{Z}$ the box $\Delta \times \{m\}$ is $(k-1,\theta,\varepsilon)$-*good *if each $(x,m) \in \Delta \times \{m\}$ is $(k-1,\theta,\varepsilon)$-*good*. Otherwise we say that $\Delta \times \{m\}$ is $(k-1,\theta,\varepsilon)$-*bad*.*

The following lemma is a direct consequence of Theorem 8.1.

Lemma 8.3. *For all $\Delta \in \Pi_{k-1}$ there are positive constants C and c so that*

$$\mathbb{P}\left(\Delta \textit{ is } (k-1,\theta,\varepsilon)\textit{-good}\right) \ge 1 - Cn_k^{-c\log n_k}. \tag{8.8}$$

The assertion of Proposition 4.1 is the analogue of the inequality (5.1) in [2]. The strategy of the proof there is as follows. First, using the triangle inequality and the Markov property an upper bound of λ_k is obtained which is given by a sum of four terms (5.2) – (5.5) in [2]. Second, for each of these four terms an upper bound is shown. Three of these upper bounds, the ones for (5.2), (5.4) and (5.5), are not difficult and can be proven in the same way as in [2]. We will omit their proofs here and refer to Appendix C. For (5.3), [2] use a notion of "good" boxes and the fact that for their model they are independent at a large but finite distance. The definition of those good boxes translates to our Definition 8.2, where it is clear that the dependence on ξ prevents us from directly using any argument hinging on independence at a finite distance. We circumvent this problem by defining a new type of boxes for which we are able to work with independence, see the ideas below Proposition 8.4. Using those boxes as an approximation for the good boxes we prove a lower bound on the probability of hitting a good box in Proposition 8.4.

Proof of Proposition 4.1. To prove Proposition 4.1 we need to show inequality (4.3) which we recall here

$$\lambda_k \leq \lambda_{k-1} + Cn_k^{-\alpha}, \quad \forall\, 1 \leq k \leq r(N).$$

for some positive constants α and C on the event $G(N)$ from (8.20).

Fix $\omega \in G(N)$. Recall the definition

$$\lambda_k = \sum_{\Delta \in \Pi_k} \left| P_\omega^{(0,0)}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta) \right|$$

from equation (4.2). Note that, by the triangle inequality, we obtain

$$\lambda_k \leq (8.9) + (8.10) + (8.11) + (8.12),$$

where

$$\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \Bigg| \sum_{u \in \Delta'} P_\omega^{(u,N_{k-1})}(X_{N_k} \in \Delta) \times \left[P_\omega^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') \right] \Bigg|, \tag{8.9}$$

$$\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \Bigg| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') \times [P_\omega^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta)] \Bigg|, \tag{8.10}$$

$$\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \Bigg| \sum_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \times [\mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u)], \tag{8.11}$$

$$\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \Bigg| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, X_{N_{k-1}} \in \Delta') \Bigg|. \tag{8.12}$$

The following estimates and their proofs are analogous to the estimates of the terms in (5.2), (5.4) and (5.5) in [2]

$$(8.9) \leq \lambda_k, \quad (8.11) \leq C\frac{(\log n_k)^{3d}}{n_k^{1/2-2\theta}} + Cn_k^{-c\log n_k}, \quad (8.12) \leq Cn_k^{-c}.$$

We provide the proofs adapted to our notation and setting in the Appendix C.

Using these estimates combined with the estimate $(8.10) \leq C'' n_{k-1}^{-\varepsilon/4}$ proven below, for each of the summands respectively we obtain

$$\lambda_k \leq \lambda_{k-1} + C'' n_{k-1}^{-\epsilon/4} + C\frac{(\log n_k)^{3d}}{n_k^{1/2-2\theta}} + Cn_k^{-c\log n_k} + Cn_k^{-c} \leq \lambda_{k-1} + \tilde{C} n_k^{-\alpha}$$

for appropriate choices of $\alpha > 0$ and $\tilde{C} > 0$. The fact that $\mathbb{P}(G_N) \geq 1 - CN^{-c\log N}$ is proved in Proposition 8.4 below. □

Proof of the analogue of an upper bound of (5.3) in [2]. First, by the triangle inequality, (8.10) is bounded from above by

$$\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta') \sum_{\Delta\in\Pi_k}\big|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)\big|. \tag{8.13}$$

Next we define Π^1_{k-1} as the set of boxes $\Delta'\in\Pi_{k-1}$ with the property

$$\Delta'\cap\{x\in\mathbb{Z}^d:\|x\|\le\sqrt{N_{k-1}}\log^3 N_{k-1}\}\neq\emptyset.$$

By Lemma 3.6 in [21] it follows

$$\sum_{\Delta'\notin\Pi^1_{k-1}}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')\le CN_{k-1}^{-c\log N_{k-1}} \tag{8.14}$$

and consequently (8.10) is bounded from above by

$$CN_{k-1}^{-c\log N_{k-1}}+\sum_{\Delta'\in\Pi^1_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta') \sum_{\Delta\in\Pi_k}\big|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)\big|. \tag{8.15}$$

Recall Definition 8.2. We will write “good” for $(k-1,\theta,\varepsilon)$-good to simplify the notation. By Lemma 8.3 we have $\mathbb{P}\big(\Delta\text{ is good}\big)\ge 1-Cn_k^{-c\log n_k}$. For $u\in\mathbb{Z}^d$ define by $\Pi_k^{(1,u)}$ the set of boxes $\Delta\in\Pi_k$ satisfying (note that $\mathbb{E}^{(u,0)}[X_{n_k}]=u$)

$$\Delta\cap\big\{x\in\mathbb{Z}^d:\big\|x-u\big\|\le\sqrt{n_k}\log^3 n_k\big\}\neq\emptyset. \tag{8.16}$$

If a box $\Delta'\in\Pi^1_{k-1}$ is good, then for $u\in\Delta'$

$$\begin{aligned}
&\sum_{\Delta\in\Pi_k}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&=\sum_{\Delta\in\Pi_k^{(1,u)}}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&\quad+\sum_{\Delta\in\Pi_k\setminus\Pi_k^{(1,u)}}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&\le\sum_{\Delta\in\Pi_k^{(1,u)}}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|+Cn_k^{-c\log n_k}\\
&\le|\Pi_k^{(1,u)}|Cn_k^{\theta d-\frac{d}{2}-\varepsilon}+Cn_k^{-c\log n_k}\\
&\le Cn_k^{\frac{d}{2}-\theta d+\theta d-\frac{d}{2}-\varepsilon}(\log n_k)^{3d}+Cn_k^{-c\log n_k}\\
&\le C(n_k^{-\varepsilon}(\log n_k)^{3d}+n_k^{-c\log n_k})\le Cn_k^{-\varepsilon/2},
\end{aligned} \tag{8.17}$$

where we used in the first inequality that by Lemma 3.6 from [21]

$$\mathbb{P}^{(0,0)}\big(\|X_n\|>\sqrt{n}\log^3 n\big)\le Cn^{-c\log n}$$

and that $|\Pi_k^{(1,u)}|\le Cn_k^{d/2-\theta d}(\log n_k)^{3d}$.

It follows that (8.10) is bounded from above by

$$\begin{aligned}
&CN_{k-1}^{-c\log N_{k-1}} + \sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is good}}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P^{(0,0)}_\omega(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta')Cn_k^{-\varepsilon/2}\\
&\quad+\sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is bad}}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P^{(0,0)}_\omega(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta')\\
&\qquad\qquad\times\sum_{\Delta\in\Pi_k}|P^{(u,N_{k-1})}_\omega(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&\le CN_{k-1}^{-c\log N_{k-1}}+Cn_k^{-\varepsilon/2}+C\sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is bad}}}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta').
\end{aligned}\tag{8.18}$$

Now we want to find an estimate for the probability of hitting a bad box. For some $\beta>0$, to be chosen later, we consider the following event

$$G_{N,n_{k-1}} := \Big\{\sum_{\Delta\in\Pi_{k-1}}\mathbb{1}_{\{\Delta\text{ is good}\}}\mathbb{P}^{(0,0)}\big(X_{N_{k-1}}\in\Delta\big)\ge 1-C'n_k^{-\beta}\Big\}\tag{8.19}$$

and define

$$G_N := \bigcap_{k=1}^{r(N)} G_{N,n_k}.\tag{8.20}$$

We want to mimic the proof in [2] and for that we need to define a new type of boxes to approximate the density of bad boxes. The problem with following the proof in [2] arises from the fact that our environment is, due to the dependence on infinitely long open paths, not i.i.d. To overcome that problem the idea is to exchange the environment ξ with a process that only has finite range dependencies. We will use this idea to show in Proposition 8.4 below that

$$\mathbb{P}(G_N)\ge 1-CN^{-c\log(N)}.\tag{8.21}$$

Note that $n_{k-1}=n_k^2$. Thus, on G_N the expression (8.10) is bounded from above by

$$\begin{aligned}
&CN_{k-1}^{-c\log N_{k-1}}+Cn_k^{-\varepsilon/2}+C\sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is bad}}}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')\\
&\qquad\le CN_{k-1}^{-c\log N_{k-1}}+Cn_k^{-\varepsilon/2}+C'n_{k-1}^{-\beta}\le C''n_{k-1}^{-\varepsilon/4}.
\end{aligned}\tag{8.22}$$

As can be seen in the proof of Proposition 8.4 we can choose $\beta\ge\varepsilon/4$ to obtain the last inequality in (8.22). ☐

Proposition 8.4. *For the events G_N from* (8.20) *there exists $N_0\in\mathbb{N}$ such that, for all $N\ge N_0$ we have that*

$$\mathbb{P}(G_N)\ge 1-CN^{-c\log N}.\tag{8.23}$$

Let $\beta>0$ and put $f(n_k)=\log^2 n_k$. First we need another notion of *good* sites. Given a realization ω we define for all $(x,\ell)\in\mathbb{Z}^d\times\mathbb{Z}$ the set $C_m(x,\ell)$ as the set of sites at time $\ell+m\in\mathbb{Z}$ which can be reached from (x,ℓ) via an open path w.r.t. ω. We start by defining for $k=1,2,\dots$ a field $\tilde\xi^k := (\tilde\xi^k_t(x))_{t\in\mathbb{Z}^d}$ as follows

(i) $\tilde\xi^k_t(x)=\xi_t(x)$ for all $(x,t)\in\mathbb{Z}^d\times\{n_k+f(n_k),n_k+f(n_k)+1,\dots\}$

(ii) For all $(x,t) \in \mathbb{Z}^d \times \{\dots, n_k + f(n_k) - 2, n_k + f(n_k) - 1\}$ we set $\tilde{\xi}_t^k(x) = 1$ if $C_{n_k+f(n_k)-t}(x,t) \neq \emptyset$. Otherwise we set $\tilde{\xi}_t^k(x) = 0$.

Note that $\xi \leq \tilde{\xi}^k$ since for (x,t) with $t < n_k + f(n_k)$ we set $\tilde{\xi}_t(x) = 1$ if (x,t) has an open path of length at least $n_k + f(n_k) - t$ instead of requiring an infinite open path. For $\xi_t(x) \neq \tilde{\xi}_t^k(x)$ we necessarily must have $t < n_k + f(n_k)$ and there must exist an open path started at (x,t) whose length is at least $n_k + f(n_k) - t$ but the contact process started at (x,t) has to eventually die out, i.e. there is no infinite open path starting in (x,t).

The following lemma gives us an upper bound on that probability. The result is well known in the oriented percolation and contact process world. For a proof see for instance Lemma A.1. in [4].

Lemma 8.5. *For $p > p_c$ there exist $C, c > 0$ such that for all $(x,t) \in \mathbb{Z}^d \times \mathbb{Z}$*

$$\mathbb{P}\Big((x,t) \to^{\omega} \mathbb{Z}^d \times \{t+n\} \text{ and } (x,t) \nrightarrow^{\omega} \mathbb{Z}^d \times \{\infty\}\Big) \leq C\mathrm{e}^{-cn}, \quad n \in \mathbb{N}.$$

As a direct consequence we get the following corollary.

Corollary 8.6. *For $x \in \mathbb{Z}^d$ define*

$$D_{n_k}(x) \coloneqq \big(x + [-n_{k-1}^{\theta} - n_k, n_{k-1}^{\theta} + n_k]^d \times [0, n_k]\big) \cap (\mathbb{Z}^d \times \mathbb{Z}).$$

For $p > p_c$ there exist constants $C, c > 0$ such that

$$\mathbb{P}\Big(\tilde{\xi}_t^k(y) = \xi_t(y) \text{ for all } (y,t) \in D_{n_k}(x)\Big) \geq 1 - C\mathrm{e}^{-c\log^2 n_k}. \tag{8.24}$$

Proof. Note that $\theta > 0$ is a small constant and can be chosen such that we have $n_{k-1}^{\theta} = n_k^{2\theta} \leq n_k$ and thus $|D_{n_k}(x)| \leq 2^d n_k^{d+1}$. By definition of $\tilde{\xi}^k$ $\tilde{\xi}_t^k(y) \neq \xi_t(y)$ implies that there is at least one open but finite path whose length is larger that $f(n_k)$. Using Lemma 8.5 the assertion (8.24) follows by the choice of $f(n_k) = \log^2 n_k$. (Here one can see that other choices of $f(n_k)$ are possible as well.) □

Let $(\tilde{X})$ be a random walk in the environment $\tilde{\xi}^k$ with transition probabilities given by

$$P_{\omega,\tilde{\xi}^k}(\tilde{X}_{n+1} = y \,|\, \tilde{X}_n = x) = \begin{cases} |U(x,n) \cap \tilde{\mathcal{C}}^k|^{-1} & \text{if } (x,n) \in \tilde{\mathcal{C}}^k \text{ and } (y,n+1) \in \tilde{\mathcal{C}}^k, \\ |U(x,n)|^{-1} & \text{if } (x,n) \notin \tilde{\mathcal{C}}^k \end{cases} \tag{8.25}$$

for $(y, n+1) \in U(x,n)$ and 0 otherwise and where $\tilde{\mathcal{C}}^k \coloneqq \{(x,n) \in \mathbb{Z}^d \times \mathbb{Z} : \tilde{\xi}_n^k(x) = 1\}$.

Given a realisation ω, we say that (x,m) is $(k-1, \theta, \varepsilon, \tilde{\xi}^k)$-good if it satisfies the conditions from Definition 8.2 with ξ replaced by $\tilde{\xi}^k$ and X replaced by $\tilde{X}$ in the quenched probabilities.

Lemma 8.7. *For all $(x,t) \in \mathbb{Z}^d \times \mathbb{Z}$ we have that*

$$\mathbb{P}((x,t) \text{ is } (k-1, \theta, \varepsilon, \tilde{\xi}^k)\text{-good}) \geq 1 - Cn_k^{-c\log n_k}. \tag{8.26}$$

Proof. Due to Lemma 8.3 it suffices to show that with probability at least $1 - Cn_k^{-c\log n_k}$ we have $\tilde{\xi}_t^k(y) = \xi_t(y)$ for all $(y,t) \in D_{n_k}(x)$. This is exactly the assertion of Corollary 8.6. On that event (x,t) is $(k-1, \theta, \varepsilon)$-good if and only if (x,t) is $(k-1, \theta, \varepsilon, \tilde{\xi}^k)$-good. □

Proof of Proposition 8.4. Recall the definition of $G_{N,n_{k-1}}$ from (8.19). To estimate the probability of hitting a bad box we can now mimic the proof in [2] since we get a lower bound by estimating the probability for the $(k-1, \theta, \varepsilon, \tilde{\xi}^k)$-good boxes. By construction those boxes are independent of each other at distance $> 5n_k$. Define

$$\Pi_{k-1}^{(0)} = \Big\{\Delta' \in \Pi_{k-1}^1 : \mathrm{dist}(\Delta', \underline{0}) \leq \Big\lfloor \sqrt{N_{k-1}} \Big\rfloor\Big\} \tag{8.27}$$

and for $r \geq 1$ let

$$\Pi_{k-1}^{(r)} = \Big\{ \Delta' \in \Pi_{k-1}^1 \colon \Big\lfloor 2^{r-1}\sqrt{N_{k-1}} \Big\rfloor < \mathrm{dist}(\Delta', \underline{0}) \leq \Big\lfloor 2^{r}\sqrt{N_{k-1}} \Big\rfloor \Big\}. \tag{8.28}$$

Note that $(\Pi_{k-1}^{(r)})_{r\geq 0}$ is a partition of Π_{k-1}^1 into disjoint subsets according to the distance of the boxes from the origin which allows us to estimate the hitting probabilities of the bad boxes. Using the annealed local CLT (Theorem 1.1), we have

$$\begin{aligned} &\sum_{\substack{\Delta' \in \Pi_{k-1}^1 \\ \text{is bad}}} \bar{\mathbb{P}}^{(0,0)}(X_{N_{k-1}} \in \Delta') \\ &\qquad \leq \sum_{r=0}^{\lceil \log_2(\log N_{k-1})^3 \rceil} |\Pi_{k-1}^{(r)} \cap \{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| C n_{k-1}^{d\theta} N_{k-1}^{-d/2} \mathrm{e}^{-cr^2} \end{aligned}$$

holds for some constants $C, c > 0$ and $\bar{\mathbb{P}}$ is the measure for the changed environments $\tilde{\xi}^k$.

In order to estimate the number of bad boxes in each $\Pi_{k-1}^{(r)}$ we define the event $\widetilde{G}_N = \widetilde{G}_N(C)$ by

$$\widetilde{G}_N := \bigcap_{k=1}^{r(N)} \bigcap_{r=0}^{\lceil \log_2(\log N_{k-1})^3 \rceil} \Big\{ |\Pi_{k-1}^{(r)} \cap \{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| \leq C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} \Big\},$$

where $\beta > 0$ is a constant to be tuned later. Let $\tilde{p}_{k-1}$ be the probability for a box $\Delta' \in \Pi_{k-1}$ to be $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad. Note that $\tilde{p}_k \in \mathcal{O}(n_k^{-c\log n_k})$ and on the event $\widetilde{G}_N$

$$\begin{aligned} \sum_{\substack{\Delta' \in \Pi_{k-1}^1 \\ \text{is bad}}} \bar{\mathbb{P}}^{(0,0)}(X_{N_{k-1}} \in \Delta') &\leq \sum_{r=0}^{\lceil \log_2(\log N_{k-1})^3 \rceil} C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} n_{k-1}^{d\theta} N_{k-1}^{-d/2} \mathrm{e}^{-cr^2} \\ &\leq \sum_{r=0}^{\lceil \log_2(\log N_{k-1})^3 \rceil} C 2^{dr} (\sqrt{N_{k-1}}/n_{k-1}^{\theta})^d n_{k-1}^{d\theta} N_{k-1}^{-d/2} \mathrm{e}^{-cr^2} n_{k-1}^{-\beta} \leq C n_{k-1}^{-\beta}. \end{aligned}$$

Now it suffices to show that $\mathbb{P}(\widetilde{G}_N(C)) \geq 1 - CN^{-c\log(N)}$ for some constant $C > 0$. To do so, fix $k \geq 1$ and note that boxes $\Delta' \in \Pi_{k-1}$ at distance $5n_k$ are, by construction of $\tilde{\xi}^k$, good or bad independently of each other. To see this note that $2(n_{k-1}^{\theta} + n_k + f(n_k)) < 5n_k$ and recall that $\tilde{\xi}_t^k(y) = 1$ if there exists an open path connecting (y,t) to $\mathbb{Z}^d \times \{n_k + f(n_k)\}$ and $\tilde{\xi}_t^k(y) = 0$ otherwise. Let $(\Pi_{k-1}^{r,j})_j$ be a partition of $\Pi_{k-1}^{(r)}$ into at most $(5n_k)^d$ subsets of boxes so that the distance between each pair of boxes in $\Pi_{k-1}^{r,j}$ is bigger than $5n_k$, for every j, and the number of boxes in $\Pi_{k-1}^{r,j}$ is between $|\Pi_{k-1}^{(r)}|/(2(5n_k)^d)$ and $2|\Pi_{k-1}^{(r)}|/(5n_k)^d$.

If the number of $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad boxes in $\Pi_{k-1}^{(r)}$ is bigger than $C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}$, then there exists at least one j so that the number of bad boxes in $\Pi_{k-1}^{r,j}$ is larger than $C|\Pi_{k-1}^{r,j}|n_{k-1}^{-\beta}$. Since the boxes in $\Pi_{k-1}^{r,j}$ are good or bad independently of each other, their number is bounded and they are bad with probability $\tilde{p}_{k-1}$, it follows by Hoeffding's inequality that

$$\begin{aligned} &\bar{\mathbb{P}}\big(|\Pi_{k-1}^{(r)} \cap \{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\big) \\ &\qquad \leq (5n_k)^d \bar{\mathbb{P}}\big(|\Pi_{k-1}^{r,1} \cap \{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| \geq \lceil C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}/(5n_k)^d \rceil\big) \\ &\qquad \leq (5n_k)^d \exp\big(-(Cn_{k-1}^{-\beta} - 2\tilde{p}_{k-1})^2 |\Pi_{k-1}^{(r)}|/(5n_k)^{3d}\big) \end{aligned}$$

$$\begin{aligned}&\le \tilde{C}(5n_k)^d \exp\bigl(-Cn_{k-1}^{-2\beta}|\Pi_{k-1}^{(r)}|/(5n_k)^{3d}\bigr)\\&\le \tilde{C}(5n_k)^d \exp\bigl(-C2^{rd}N^{\frac{-2\beta}{2^{k-1}}+\frac{d}{2}-\frac{d\theta}{2^{k-1}}-\frac{3d}{2^k}}\bigr)\\&= \tilde{C}(5n_k)^d \exp\bigl(-C2^{rd}N^{\frac{d}{2}-(\frac{4\beta+2d\theta+3d}{2^k})}\bigr),\end{aligned} \tag{8.29}$$

where the right hand side decays stretched exponentially in N for $k \ge 4$ if β is small enough, e.g. $\beta = 1$ (which is still sufficient for the proof of (8.10)). For $1 \le k \le 3$ notice that

$$\begin{aligned}\bar{\mathbb{P}}\bigl(|\Pi_{k-1}^{(r)}\cap&\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\bigr)\\&\le \bar{\mathbb{P}}\bigl(\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\} \neq \emptyset\bigr)\\&\le |\Pi_{k-1}^{(r)}|\tilde{p}_{k-1}\\&\le (\sqrt{N}\log^3(N))^d\tilde{p}_{k-1} \le (\sqrt{N}\log^3(N))^d N^{-c\log(N)} \le CN^{-c\log(N)}.\end{aligned} \tag{8.30}$$

Using the estimates above together with the definition of $\widetilde{G}_N$ shows that

$$\begin{aligned}&\bar{\mathbb{P}}(\widetilde{G}_N^{\complement})\\&= \bar{\mathbb{P}}\Biggl(\bigcup_{k=1}^{r(N)}\bigcup_{r=0}^{\lceil\log_2(\log N_{k-1})^3\rceil}\Bigl\{|\Pi_{k-1}^{(r)}\cap\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\Bigr\}\Biggr)\\&\le \sum_{k=1}^{r(N)}\sum_{r=0}^{\lceil\log_2(\log N_{k-1})^3\rceil}\bar{\mathbb{P}}\Bigl(|\Pi_{k-1}^{(r)}\cap\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\Bigr)\\&\le r(N)\lceil\log_2(\log N_{k-1})^3\rceil CN^{-c\log(N)} \le C\log\log(N)\cdot\log(N)^{5/6}N^{-c\log(N)}\\&\le N^{-\tilde{c}\log(N)}.\end{aligned}$$

Next we show that the number of $(k-1,\theta,\varepsilon)$-bad boxes in ξ is on the same order as the number of $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad boxes in $\tilde{\xi}^k$ with high probability. First we define, in a slight abuse of notation, the sets

$$\begin{aligned}D_{n_k}(\Delta) &:= \{(x,t)\in\mathbb{Z}^d\times\mathbb{Z} : \operatorname{dist}(x,\Delta)\le n_k, t\in[0,n_k]\},\\A_{k,\Delta} &:= \{\omega\in\Omega : \xi_t(x) = \tilde{\xi}_t^k(x) \text{ for all } (x,t)\in D_{n_k}(\Delta)\}\end{aligned}$$

for all $\Delta \in \Pi_{k-1}^{(r)}$. Note that $D_{n_k}(\Delta)$ is the same box as $D_{n_k}(x)$ if x is the center of Δ. Using the above defined partitions $(\Pi_{k-1}^{r,j})_j$ we see that for every choice of $\Delta,\Delta' \in \Pi_{k-1}^{r,j}$ the events $A_{k,\Delta}$ and $A_{k,\Delta'}$ are independent, since $\operatorname{dist}(\Delta,\Delta') > 5n_k$. Since $\xi \le \tilde{\xi}^k$ the number of $(k-1,\theta,\varepsilon)$-good boxes in ξ is less or equal to the number of $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad boxes in $\tilde{\xi}^k$.

To shorten the notation we say for a box $\Delta \in \Pi_{k-1}^{(r)}$ that it is good in ξ if it is $(k-1,\theta,\varepsilon)$-good and good in $\tilde{\xi}^k$ if it is $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-good. A box can only be bad in ξ and good in $\tilde{\xi}^k$ for $\omega \in A_{k,\Delta}^{\complement}$. Using Corollary 8.6 we get $\mathbb{P}(A_{k,\Delta}^{\complement}) \le Cn_k^{-c\log n_k}$, and thus, again by Hoeffding's inequality,

$$\begin{aligned}&\mathbb{P}\Bigl(|\Pi_{k-1}^{(r)}\cap\{\text{bad in }\xi\}| - |\Pi_{k-1}^{(r)}\cap\{\text{bad in }\tilde{\xi}^k\}| \ge C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\Bigr)\\&\quad\le \mathbb{P}\Bigl(\exists j \text{ s.t. } |\Pi_{k-1}^{r,j}\cap\{\text{bad in }\xi\}| - |\Pi_{k-1}^{r,j}\cap\{\text{bad in }\tilde{\xi}^k\}| \ge C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\frac{1}{(5n_k)^d}\Bigr)\\&\quad\le (5n_k)^d\mathbb{P}\Bigl(|\Pi_{k-1}^{r,j}\cap\{\text{bad in }\xi\}| - |\Pi_{k-1}^{r,j}\cap\{\text{bad in }\tilde{\xi}^k\}| \ge C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\frac{1}{(5n_k)^d}\Bigr)\\&\quad\le (5n_k)^d\mathbb{P}\Bigl(\sum_{\Delta\in\Pi_{k-1}^{r,j}}\mathbb{1}_{A_{k,\Delta}^{\complement}} \ge C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\frac{1}{(5n_k)^d}\Bigr)\\&\quad\le \tilde{C}(5n_k)^d\exp\Bigl(-C2^{rd}N^{\frac{d}{2}-(\frac{4\beta+2d\theta+d}{2^k})}\Bigr).\end{aligned}$$

Again the right hand side decays stretched exponentially in N for $k \geq 4$ for $\beta > 0$ small enough. For $k \leq 3$ we can repeat the ideas of (8.30). The reason we can prove an upper bound in the same way as in (8.29) and (8.30) is that the probability for a box to be bad in $\tilde{\xi}^k$ is of the same order as $\mathbb{P}(A_{k,\Delta}^{\complement})$, namely $n_k^{-c \log n_k}$. Define

$$A_N \coloneqq \bigcap_{k=1}^{r(N)} \bigcap_{r=0}^{\lceil \log_2 (\log N_{k-1})^3 \rceil} \left\{ |\Pi_{k-1}^{(r)} \cap \{\text{bad in } \xi\}| - |\Pi_{k-1}^{(r)} \cap \{\text{bad in } \tilde{\xi}^k\}| \geq C |\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} \right\}$$

then by the same arguments as above we also get

$$\mathbb{P}(A_N^{\complement}) \leq N^{-c \log N}.$$

Since $\widetilde{G}_N \cap A_N \subset G_N$ the claim follows. □

9 Mixing properties of the quenched law: proof of Lemma 7.1

Definition 9.1. *Let Π_M be a partition of $\mathbb{Z}^d$ into boxes of side lengths M, let $C > 0$ and let ω be a realisation of the environment. We call a box $\Delta \in \Pi_M$* social with respect to ω *at time $N \in \mathbb{N}$, if for any pair of points $x, y \in \Delta$ there exists $z \in \mathbb{Z}^d$ such that*

$$P_\omega^{(x,N)}(X_{N+\lceil CM \rceil} = z) > 0, \quad \textit{and} \quad P_\omega^{(y,N)}(X_{N+\lceil CM \rceil} = z) > 0.$$

Note that if $P_\omega^{(x,N)}(X_{N+\lceil CM \rceil} = z) > 0$, then by construction $P_\omega^{(x,N)}(X_{N+\lceil CM \rceil} = z) \geq (3^{-d})^{CM}$.

The next result shows that the density of social boxes is suitably high.

Lemma 9.2. *For every $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$ and constants $c, C > 0$ such that for all $M \geq M_0$ there exists a set of environments S_M satisfying*

$$\sum_{\substack{\Delta \in \Pi_M \\ \Delta \text{ is not social}}} \mathbb{P}^{(x,0)}(X_n \in \Delta) < \varepsilon \quad \textit{for all } \omega \in S_M$$

and $\mathbb{P}(S_M) \geq 1 - C\mathrm{e}^{-c \log n}$. (Recall that the property of Δ being social depends on ω.)

Corollary 9.3. *For every $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$ so that for all $M > M_0$ there are environments $\bar{S}_M$ such that*

$$\sum_{\substack{\Delta \in \Pi_M \\ \Delta \text{ is not social}}} P_\omega^{(x,0)}(X_n \in \Delta) < 2\varepsilon$$

for all $\omega \in \bar{S}_M$ and $\mathbb{P}(\bar{S}_M) \geq 1 - Cn^{-c \log n}$.

Proof. Combine Lemma 9.2 and Lemma 2.1. □

Proof of Lemma 9.2. The proof idea is similar to the one we have used to prove the high density of good boxes; see the proof of Proposition 8.4. We set

$$p_M \coloneqq \mathbb{P}(\Delta \text{ is not social}).$$

As a direct consequence of Lemma A.1 for every $\Delta \in \Pi_M$ we have that $p_M \leq C\mathrm{e}^{-cM}$ for some positive constants C, c. We define

$$S_M \coloneqq \bigcap_{r=0}^{\log_2 \log^3 n} \left\{ |\Pi_M^{(r)} \cap \{\text{not social boxes}\}| < C |\Pi_M^{(r)}| p_M \right\}, \tag{9.1}$$

where

$$\Pi_M^{(0)} = \{\Delta \in \Pi_M : \mathrm{dist}(\Delta, 0) \le \sqrt{n}\},$$
$$\Pi_M^{(r)} = \{\Delta \in \Pi_M : 2^{r-1}\sqrt{n} < \mathrm{dist}(\Delta, 0) \le 2^r\sqrt{n}\} \quad \text{for } r \ge 1.$$

By Lemma 3.6 from [21] we have $\mathbb{P}^{(0,0)}(\|X_n\| \ge \sqrt{n}\log^3 n) \le Cn^{-c\log n}$ and so for $\omega \in S_M$ (note that being social depends on ω)

$$\begin{aligned}\sum_{\substack{\Delta\in\Pi_M\\ \Delta \text{ is not social}}} \mathbb{P}^{(0,0)}(X_n \in \Delta) &\le Cn^{-c\log n} + \sum_{r=0}^{\log_2\log^3 n} \sum_{\substack{\Delta\in\Pi_M^{(r)}\\ \Delta \text{ is not social}}} \mathbb{P}^{(0,0)}(X_n \in \Delta)\\
&\le \sum_{r=0}^{\log_2\log^3 n} C|\Pi_M^{(r)}|p_M \frac{1}{n^{d/2}} \exp\left(-\frac{1}{2n}(2^{r-1}\sqrt{n})^2\right)\\
&\le C\sum_{r=0}^{\log_2\log^3 n} \left(\frac{2^r\sqrt{n}}{M}\right)^d \frac{1}{n^{d/2}} \exp(-cr^2)p_M\\
&\le Cp_M \sum_{r=0}^{\log_2\log^3 n} \frac{1}{M^d} \exp(-cr^2 + rd\log 2)\\
&\le C'p_M\end{aligned}$$

where we used the annealed local CLT in the second inequality. It remains to show that $\mathbb{P}^{(0,0)}(S_M) \ge 1 - C\mathrm{e}^{-c\log n}$. We have

$$\begin{aligned}\mathbb{P}^{(0,0)}(S_M^c) &= \mathbb{P}^{(0,0)}\big(\exists r \le \log_2\log^3 n : |\Pi_M^{(r)} \cap \{\text{not social boxes}\}| > C|\Pi_M^{(r)}|p_M\big)\\
&\le \sum_{r=0}^{\log_2\log^3 n} \mathbb{P}^{(0,0)}\big(|\Pi_M^{(r)} \cap \{\text{not social boxes}\}| > C|\Pi_M^{(r)}|p_M\big).\end{aligned}$$

Next, let $(\Pi_M^{r,j})_{j\in J}$ be a further partition of $\Pi_M^{(r)}$ so that for each $j \in J$ the distance between any pair of distinct boxes in $\Pi_M^{r,j}$ is bigger than $3CM$ and

$$\frac{|\Pi_M^{(r)}|}{2(3CM)^d} \le |\Pi_M^{r,j}| \le \frac{2|\Pi_M^{(r)}|}{(3CM)^d}.$$

Note that the index set $J = J(M,r)$ is finite (in fact we have $|J| \le 2(3CM)^d$) and that by construction the boxes in $\Pi_M^{r,j}$ are social or not social independently of each other. If $|\Pi_M^r \cap \{\text{not social boxes}\}| > C|\Pi_M^{(r)}|p_M$ then there exists a j such that $|\Pi_M^{r,j} \cap \{\text{not social boxes}\}| > C|\Pi_M^{(r)}|p_M/(3CM)^d$. Using Hoeffding's inequality for $r \ge 1$ we obtain

$$\begin{aligned}&\mathbb{P}^{(0,0)}\big(|\Pi_M^{(r)} \cap \{\text{not social boxes}\}| > C|\Pi_M^{(r)}|p_M\big)\\
&\quad\le \sum_{j\in J} \mathbb{P}^{(0,0)}\left(|\Pi_M^{r,j} \cap \{\text{not social boxes}\}| > \frac{C|\Pi_M^{(r)}|p_M}{(3CM)^d}\right)\\
&\quad= \sum_{j\in J} \mathbb{P}^{(0,0)}\left(|\Pi_M^{r,j} \cap \{\text{not social boxes}\}| - |\Pi_M^{r,j}|p_M > \Big(\frac{C|\Pi_M^{(r)}|}{(3CM)^d} - |\Pi_M^{r,j}|\Big)p_M\right)\\
&\quad\le \sum_{j\in J} \exp\Big(-2p_M^2\Big(C\frac{|\Pi_M^{(r)}|}{(3CM)^d} - |\Pi_M^{r,j}|\Big)^2\Big)\end{aligned}$$

$$\leq \sum_{j\in J} \exp\Big(-2p_M^2(C-2)\frac{|\Pi_M^{(r)}|^2}{(3CM)^{2d}}\Big)$$
$$\leq 2(3CM)^d \exp\Big(-Cp_M^2\frac{(2^{r-1}\sqrt{n})^{2d}}{(3CM)^{2d}}\Big).$$

Similarly for $r=0$ we have

$$\mathbb{P}^{(0,0)}\big(|\Pi_M^{(0)} \cap \{\text{not social boxes}\}| > C|\Pi_M^{(0)}|p_M\big) \leq 2(3CM)^d \exp\Big(-Cp_M^2\frac{\sqrt{n}^{2d}}{(3CM)^{2d}}\Big).$$

Using the above estimates we obtain

$$\mathbb{P}^{(0,0)}(S_M^c)$$
$$\leq 2(3CM)^d \exp\Big(-Cp_M^2\frac{\sqrt{n}^{2d}}{(3CM)^{2d}}\Big) + \sum_{r=1}^{\log_2 \log^3 n} 2(3CM)^d \exp\Big(-Cp_M^2\frac{(2^{r-1}\sqrt{n})^{2d}}{(3CM)^{2d}}\Big)$$
$$\leq \log_2 \log^3(n)\cdot \exp\Big(-Cp_M^2\frac{\sqrt{n}^{2d}}{(3CM)^{2d}}\Big) \leq Cn^{-c\log n}.$$

□

Proof of Lemma 7.1. The proof relies on a construction of a suitable coupling of $P_\omega^{(x,0)}(X_n \in \cdot)$ and $P_\omega^{(y,0)}(X_n \in \cdot)$. First we show that there is a coupling on the level of boxes with side length M, where M is a constant. Let Π_M be a partition of $\mathbb{Z}^d$ in boxes of side length M and fix x and y. Set

$$F_{n^\theta} := \bigcap_{k\geq n^\theta} \Big\{\omega : \forall z \in [-k,k]^d \cap \mathbb{Z}^d,$$
$$\sum_{\Delta\in\Pi_M} |\mathbb{P}^{(z,0)}(X_k\in\Delta) - P_\omega^{(z,0)}(X_k\in\Delta)| \leq \frac{C_1}{k^{c_2}} + \frac{C_1}{M^{c_2}}\Big\},$$

and

$$F(x,y) := \bigcap_{\substack{(\tilde{x},m)\in\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{x}-x\|\leq n\\ m\leq n}} \sigma_{(\tilde{x},m)}F_{n^\theta} \cap \bigcap_{\substack{(\tilde{y},m)\in\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{y}-y\|\leq n\\ m\leq n}} \sigma_{(\tilde{y},m)}F_{n^\theta}$$

By Lemma 2.1 we have $\mathbb{P}(F_{n^\theta}) \geq 1 - n^{-c\log n}$ and thus $\mathbb{P}(F(x,y)) \geq 1 - Cn^{-c\log n}$. In the following we assume that the indices of the random walks are integers, otherwise we take the integer part. Now choosing M and n large enough for $\|x-y\| \leq n^\theta$ on the event $F(x,y)$ we obtain

$$\sum_{\Delta\in\Pi_M} |P_\omega^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta) - P_\omega^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta)|$$
$$\leq \sum_{\Delta\in\Pi_M} |P_\omega^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta) - \mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta)|$$
$$+ \sum_{\Delta\in\Pi_M} |P_\omega^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta)|$$
$$+ \sum_{\Delta\in\Pi_M} |\mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta)|$$
$$\leq \frac{1}{8} + \frac{1}{8} + \sum_{\Delta\in\Pi_M} |\mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta}\in\Delta)|$$

$$
\begin{aligned}
&\le \frac{1}{4} + Cn^{-c\log n} \\
&\qquad + \sum_{\Delta\in\Pi_M^{x,y}(n^{2\theta}\log^{8d} n^{\theta})} |\mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^{\theta}} \in \Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^{\theta}} \in \Delta)| \\
&\le \frac{1}{4} + Cn^{-c\log n} + |\Pi_M^{x,y}(n^{2\theta}\log^{8d} n^{\theta})| dn^{\theta} C(n^{2\theta}\log^{8d} n^{\theta})^{-\frac{d+1}{2}} \\
&\le \frac{1}{4} + Cn^{-c\log n} + 2\Big(n^{\theta}\log^{4d}(n^{\theta})\log^{3}(n^{2\theta}\log^{8d} n^{\theta})\Big)^{d} dn^{\theta} C(n^{2\theta}\log^{8d} n^{\theta})^{-\frac{d+1}{2}} \\
&= \frac{1}{4} + Cn^{-c\log n} + C\Big(\log(n^{2\theta}\log^{8d} n^{\theta})\Big)^{3d}\log^{-4d}(n^{\theta}) \\
&< \frac{1}{2},
\end{aligned}
$$

for n large enough, where

$$
\Pi_M^{x,y}(m) := \Big\{\Delta\in\Pi_M : \Delta\cap\{z\in\mathbb{Z}^d : \min(\|x-z\|, \|y-z\|) \le \sqrt{m}\log^3 m\} \neq \emptyset\Big\}
$$

and we used Lemma 3.6 from [21] and the annealed derivative estimates; see Lemma 3.1. The number of steps we chose might seem a bit strange at first. The choice becomes more clear by looking at the last inequality above. There we see that, with the methods we use, we need a bit more steps than the square of the current distance. One can calculate that any additional factor $\log^m(n^{\theta})$ with $m > 6d$ is enough to get the estimate. So there exists a coupling $\Xi^{x,y}_{\omega,n^{2\theta}\log^{8d} n^{\theta}}$ of $P_\omega^{(x,0)}(X_{n^{2\theta}\log^{8d} n^{\theta}} \in \cdot)$ and $P_\omega^{(y,0)}(X_{n^{2\theta}\log^{8d} n^{\theta}} \in \cdot)$ on $\Pi_M\times\Pi_M$ such that for $\omega\in F(x,y)$

$$
\Xi^{x,y}_{\omega,n^{2\theta}\log^{8d} n^{\theta}}\big(\{(\Delta,\Delta):\Delta\in\Pi_M\}\big) > \frac{1}{2}.
$$

Recall $\bar{S}_M$ from Corollary 9.3. We have for

$$
\omega\in H(x,y) := F(x,y)\cap \bigcap_{\substack{(\tilde{x},m)\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{x}-x\|\le n\\ m\le n}} \sigma_{(\tilde{x},m)}\bar{S}_M \cap \bigcap_{\substack{(\tilde{y},m)\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{y}-y\|\le n\\ m\le n}} \sigma_{(\tilde{y},m)}\bar{S}_M
$$

that

$$
\sum_{\substack{\Delta\in\Pi_M\\ \Delta\text{ is social}}} \Xi^{x,y}_{\omega,n^{2\theta}\log^{8d} n^{\theta}}(\Delta,\Delta) > \frac{1}{2} - \varepsilon(M) > \frac{1}{4}.
$$

By Corollary 9.3 we obtain $\mathbb{P}(H(x,y)) \ge 1 - Cn^{-c\log n}$. Thus, by the definition of social boxes (Definition 9.1), we can construct a coupling $\widetilde{\Xi}^{x,y}_{\omega,n^{\theta}}$ of $P_\omega^{(x,0)}(X_{n^{2\theta}\log^{8d} n^{\theta}+CM} \in \cdot)$ and $P_\omega^{(y,0)}(X_{n^{2\theta}\log^{8d} n^{\theta}+CM} \in \cdot)$ satisfying $\widetilde{\Xi}^{x,y}_{\omega,n^{\theta}}(\{(z,z) : z\in\mathbb{Z}^d\}) > \frac{1}{4}(\frac{1}{3^d})^{2CM}$. If this coupling is successful, we let the random walks go along the same path until time n. In case it isn't, we try to couple from their current position. Note that $\omega\in H(x,y)$ ensures that we can repeat the coupling attempt at the new positions.

For the rest of the proof let $n_k := n^{\theta}\log^{k(4d+3)} n$, $k\in\mathbb{N}_0$ and $s_k := n_k^2\log^{8d} n_k + CM$. The n_k will represent the distance between the walkers at the start of an attempt at coupling and s_k will be the number of steps necessary for the attempt. Furthermore let $S_k := \sum_{i=0}^k s_i$.

By Lemma 3.6 from [21], we know that with probability of at least $1 - Cn^{-c\log n}$ the distance between the random walks will only be

$$
\big(n^{2\theta}\log^{8d} n^{\theta}\big)^{1/2}\log^3\big(n^{2\theta}\log^{8d} n^{\theta}\big) \le n^{\theta}\log^{4d}(n^{\theta})\log^3(n) \le n^{\theta}\log^{4d+3} n = n_1,
$$

as long as $8d \le (1-2\theta)\frac{\log n}{\log\log n^{\theta}}$. This condition is not a restriction, since we will let $n\to\infty$.

Let us now iterate the coupling procedure. If the coupling in step $k-1$ is not successful, i.e. if the walks are not at the same point, we try to couple again starting from the current positions. This leads to an iterative coupling $\widehat{\Xi}$ of the following form: $\widehat{\Xi}^{x,y}_{\omega,0} = \widetilde{\Xi}^{x,y}_{\omega,n_0} = \widetilde{\Xi}^{x,y}_{\omega,n^\theta}$ and for $k \geq 1$

$$\begin{aligned}\widehat{\Xi}^{x,y}_{\omega,k}(z_1,z_2) = \sum_{a,b\in\mathbb{Z}^d} & \widehat{\Xi}^{x,y}_{\omega,k-1}(a,b) \\ & \cdot \Big[\mathbb{1}_{\{a=b\}}\mathbb{1}_{\{z_1=z_2\}}P^{(a,S_{k-1})}_\omega(X_{S_k}=z_1) + \mathbb{1}_{\{0<\|a-b\|\leq n_k\}}\widetilde{\Xi}^{a,b}_{\omega,n_k}(z_1,z_2) \\ & \quad + \mathbb{1}_{\{\|a-b\|>n_k\}}P^{(a,S_{k-1})}_\omega(X_{S_k}=z_1)P^{(b,S_{k-1})}_\omega(X_{S_k}=z_2)\Big],\end{aligned}$$

where $\widetilde{\Xi}^{a,b}_{\omega,n_k}$ is a coupling of $P^{(a,S_{k-1})}_\omega(X_{S_k}\in\cdot)$ and $P^{(b,S_{k-1})}_\omega(X_{S_k}\in\cdot)$. The idea is that the random walks will stay together once they are at the same site. We try to couple them via $\widetilde{\Xi}^{a,b}_{\omega,n_k}$ if their distance is not too large and we let them evolve independently otherwise.

Since at distance n_k for the next coupling we walk s_k steps and with high probability have at most a distance of $s_k^{1/2}\log^3 s_k$, the above coupling will work as long as $k \leq \frac{(1-2\theta)\log n}{(8d+6)\log\log n} - \frac{8d}{8d+6}$ holds, which we show below. We obtain

$$s_k^{1/2}\log^3 s_k = \Big(n_k^2\log^{8d} n_k + CM\Big)^{1/2}\log^3\Big(n_k^2\log^{8d} n_k + CM\Big).$$

Now for $k \leq \frac{(1-2\theta)\log n}{(4d+3)\log\log n}$ and n large enough

$$n_k^2\log^{8d} n_k + CM \leq n_k^2 \log^{8d} n$$

and

$$\log^{4d} n_k = \log^{4d}\Big(n^\theta \log^{k(4d+3)}(n)\Big) \leq \log^{4d} n.$$

Thus, we have

$$s_k^{1/2}\log^3 s_k \leq n_k \log^{4d}(n)\log^3\Big(n_k^2\log^{8d} n\Big).$$

Furthermore, if $k \leq \frac{(1-2\theta)\log n}{(8d+6)\log\log n} - \frac{8d}{8d+6}$ then

$$\begin{aligned}2\log n_k + 8d\log\log n &= 2\log\big(n^\theta\log^{k(4d+3)} n\big) + 8d\log\log n \\ &= 2\theta\log n + k(8d+6)\log\log n + 8d\log\log n \leq \log n\end{aligned}$$

It follows that

$$s_k^{1/2}\log^3 s_k \leq 2n_k\log^{4d} n\log^3 n = 2n^\theta\log^{(k+1)(4d+3)}(n) = n_{k+1}.$$

So after we try the k-th coupling we are, with high probability, at distance n_{k+1}. The probability for each try to be successful is bounded from below by $\frac{1}{4}(\frac{1}{3^d})^{2CM}$ and we have $\frac{(1-2\theta)\log n}{(8d+6)\log\log n} - 1$ attempts. So the time we need for those attempts is

$$\begin{aligned}\sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} s_k &= \sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} n_k^2\log^{8d} n_k + CM \\ &\leq \sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} n^{2\theta}\log^{k(8d+6)}(n)\log^{8d}(n) + CM \\ &= \frac{(1-2\theta)\log n}{(8d+6)\log\log n}CM + n^{2\theta}\log^{8d}(n)\sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1}\Big(\log^{(8d+6)}(n)\Big)^k.\end{aligned} \tag{9.2}$$

Note that

$$(\log n)^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}(8d+6)} = \exp\big((1-2\theta)\log n\big) = n^{1-2\theta}$$

and therefore the right hand side of (9.2) is bounded from above by

$$\begin{aligned}\frac{(1-2\theta)\log n}{(8d+6)\log\log n}CM + n^{2\theta}\log^{8d}(n)\frac{n^{1-2\theta}-1}{\log^{(8d+6)}(n)-1}&\\ \le \frac{(1-2\theta)\log n}{(8d+6)\log\log n}CM + \frac{n}{\log^5(n)}&\\ = n\left(\frac{(1-2\theta)\log n}{n(8d+6)\log\log n}CM + \frac{1}{\log^5 n}\right) < n,&\end{aligned}$$

for n large enough. And the probability for the above coupling to fail is smaller than

$$(1-p^*)^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} \le \mathrm{e}^{-c\frac{\log n}{\log\log n}}$$

where $p^* = \frac{1}{4}(\frac{1}{3^d})^{2CM}$ and $c > 0$ is a constant. So for a fixed pair of points x, y with $\|x-y\| \le n^\theta$ we have

$$\left\|P_\omega^{(x,0)}(X_n \in \cdot) - P_\omega^{(y,0)}(X_n \in \cdot)\right\|_{\mathrm{TV}} \le \mathrm{e}^{-c\frac{\log n}{\log\log n}}$$

with probability at least $1 - n^{-c\log n}$. Thus we get for every $b > 0$

$$\begin{aligned}\mathbb{P}(D(n)) &= \mathbb{P}\Big(\bigcap_{\substack{x,y\in\mathbb{Z}^d:\\ \|x\|,\|y\|\le n^b,\\ \|x-y\|\le n^\theta}} \Big\{\left\|P_\omega^{(x,0)}(X_n \in \cdot) - P_\omega^{(y,0)}(X_n \in \cdot)\right\|_{\mathrm{TV}} \le \mathrm{e}^{-c\frac{\log n}{\log\log n}}\Big\}\Big)\\ &\ge 1 - \sum_{\substack{x,y\in\mathbb{Z}^d:\\ \|x\|,\|y\|\le n^b,\\ \|x-y\|\le n^\theta}} \mathbb{P}\Big(\Big\{\left\|P_\omega^{(x,0)}(X_n \in \cdot) - P_\omega^{(y,0)}(X_n \in \cdot)\right\|_{\mathrm{TV}} > \mathrm{e}^{-c\frac{\log n}{\log\log n}}\Big\}\Big)\\ &\ge 1 - n^{d(b+\theta)}n^{-c\log n} \ge 1 - Cn^{-c'\log n}.\end{aligned}$$

Note that $b > 0$ can be chosen arbitrarily large, but the constants C and c' will have to adjusted accordingly. □

A Intersection of clusters of points connected to infinity

The following lemma is a quantification of Theorem 2 from [15] which was pointed out there without a proof. We give a proof using a key result from [14].

Lemma A.1. *Let $d \ge 2$, $p > p_c$. Then there are positive constants M and C and c such that for all $x, y \in \mathbb{Z}^d$ with $\|x-y\| \le M$*

$$\mathbb{P}\big(B(x,y;M,C)|(x,0)\to\infty,(y,0)\to\infty\big) \ge 1 - \exp(-cM), \tag{A.1}$$

where $B(x,y;M,C)$ is the set of all $\omega \in \Omega$ for which there is $z \in \mathbb{Z}^d$ satisfying

$$(x,0) \xrightarrow{\omega} (z,CM), \quad (y,0) \xrightarrow{\omega} (z,CM) \quad \textit{and} \quad (z,CM) \xrightarrow{\omega} \infty.$$

Proof. For $A \subset \mathbb{Z}^d$ we put $\eta_t^A(x) = \mathbb{1}_{\{(y,0)\to(x,t)\text{ for some }y\in A\}}$ (this is the discrete time contact process starting from all sites in A infected at time 0). Write

$$B(x,t) := \big\{\exists\, z : \|x-z\| \le c_1 t \text{ and } \eta_t^{\{x\}}(z) \ne \eta_t^{\mathbb{Z}^d}(z)\big\}$$

for the "bad" event that coupling in a ball around x has not occurred at time t. We obtain from [14, Thm. 1, Formula (3)] that

$$\mathbb{P}\big(B(x,t) \cap \{(x,0) \to \infty\}\big) \le C\mathrm{e}^{-ct} \tag{A.2}$$

for certain constants $c_1, C, c \in (0,\infty)$ (which depend on d and on $p > p_c$). Literally, the result in [14] is proved for the continuous time version of the contact process, but we believe that the same holds in discrete time.

Now consider $x, y \in \mathbb{Z}^d$ with $\|x-y\| \le M$. Pick C_2 so large that

$$J := \{z : \|z-x\| \le C_2M \text{ and } \|z-y\| \le C_2M\}$$

satisfies $\#J \ge M^d$. Applying (A.2) with $t = C_2M$ for x and for y gives

$$\begin{aligned}\mathbb{P}\Big(&\big(B(x,C_2M) \cup B(y,C_2M)\big) \cap \{(x,0)\to\infty, (y,0)\to\infty\}\Big) \\ &\le \mathbb{P}\big(B(x,C_2M) \cap \{(x,0)\to\infty\}\big) + \mathbb{P}\big(B(y,C_2M) \cap \{(y,0)\to\infty\}\big) \le 2C\mathrm{e}^{-cCM_2}\end{aligned}$$

hence

$$\mathbb{P}\Big(\eta^{\{x\}}_{C_2M}(z) = \eta^{\mathbb{Z}^d}_{C_2M}(z) = \eta^{\{y\}}_{C_2M}(z)\,\forall z \in J \;\Big|\; (x,0)\to\infty, (y,0)\to\infty\Big) \ge 1 - C'\mathrm{e}^{-cC_2M}.$$

Furthermore

$$\begin{aligned}\mathbb{P}\Big(\exists\, z \in J\ :&\eta^{\mathbb{Z}^d}_{C_2M}(z) = 1 \text{ and } (z, C_2M) \to \infty \;\Big|\; (x,0)\to\infty, (y,0)\to\infty\Big) \\ &\ge \mathbb{P}\Big(\exists\, z \in J\ :\ \eta^{\mathbb{Z}^d}_{C_2M}(z) = 1 \text{ and } (z, C_2M)\to\infty\Big) \ge 1 - C\mathrm{e}^{-cM^d}\end{aligned}$$

where we used the FKG inequality in the first inequality. For the second inequality we use the fact that extinction starting from A is exponentially unlikely in $\#A$ (see Theorem 2.30 (b) in [18]) and the fact that $\eta^{\mathbb{Z}^d}_{C_2M}$ dominates the upper invariant measure which itself dominates a product measure on $\{0,1\}^{\mathbb{Z}^d}$ with some density $\rho > 0$ (see Corollary 4.1 in [19]).

Combining, we find that for

$$A(x,y,C_2,M) := \{\exists\, z \in \mathbb{Z}^d\ :\ (x,0) \to (z,C_2M), (y,0)\to(z,C_2M), (z,C_2M)\to\infty\}$$

we have

$$\mathbb{P}\big(A(x,y,C_2,M) \;\big|\; (x,0)\to\infty, (y,0)\to\infty\big) \ge 1 - C'\mathrm{e}^{-cC_2M} - C\mathrm{e}^{-cM^d}. \qquad \square$$

B Quenched random walk finds the cluster fast

Since we allow the quenched random walk to start outside the cluster we need some kind of control on the time it needs to hit the cluster. The following lemma will yield exactly that.

Lemma B.1. *Let $d \ge 1$ and define the set $A_n = A_n(C', c') := \{\omega \in \Omega : P^{(0,0)}_\omega(\xi_i(X_i) = 0, i = 1,\dots,n) \le C'\mathrm{e}^{-c'n}\}$. There exist constants $C, c > 0$, so that for every $p > p_c(d)$ and small enough C' and c' we have*

$$\mathbb{P}(A_n^{\mathsf{C}}) \le C\mathrm{e}^{-cn} \quad \text{for all } n = 1,2,\dots.$$

Proof. Note that by our definition of the quenched law, see equation (1.4), the quenched random walk performs a simple random walk until it hits the cluster $\mathcal{C}$. Thus, on the event that the random walk doesn't hit the cluster, we can switch the random walk with a simple random walk $(Y_n)_n$ that is independent of the environment. Using Lemma 2.11 from [3] it follows

$$\begin{aligned}
&\mathbb{P}^{(0,0)}\big(\xi_0(X_0) = \cdots = \xi_n(X_n) = 0\big)\\
&\qquad = \sum_{x_1,\ldots,x_n} \mathbb{P}^{(0,0)}\big((X_1,\ldots,X_n) = (x_1,\ldots,x_n), \xi_0(0) = \cdots = \xi_n(x_n) = 0\big)\\
&\qquad = \sum_{x_1,\ldots,x_n} \mathbb{P}^{(0,0)}\big((Y_1,\ldots,Y_n) = (x_1,\ldots,x_n), \xi_0(0) = \cdots = \xi_n(x_n) = 0\big)\\
&\qquad = \sum_{x_1,\ldots,x_n} \mathbb{P}^{(0,0)}\big((Y_1,\ldots,Y_n) = (x_1,\ldots,x_n)\big)\mathbb{P}\big(\xi_0(0) = \cdots = \xi_n(x_n) = 0\big)\\
&\qquad \le \tilde{C}\mathrm{e}^{-\tilde{c}n},
\end{aligned}$$

where $\tilde{C}$ and $\tilde{c}$ are certain constants depending only on p and d.

Using the definition of the annealed law we get

$$\begin{aligned}
&\mathbb{P}^{(0,0)}\big(\xi_0(X_0) = \cdots = \xi_n(X_n) = 0\big)\\
&\qquad = \int_{A_n} P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1,\ldots,n)\, d\mathbb{P}(\omega)\\
&\qquad\qquad + \int_{A_n^\complement} P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1,\ldots,n)\, d\mathbb{P}(\omega)\\
&\qquad \ge \int_{A_n^\complement} P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1,\ldots,n)\, d\mathbb{P}(\omega)\\
&\qquad > \mathbb{P}(A_n^\complement) C' \mathrm{e}^{-c'n}
\end{aligned}$$

and since

$$\int_{A_n^\complement} P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1\ldots,n)\, d\mathbb{P}(\omega) \le \tilde{C}\mathrm{e}^{-\tilde{c}n}$$

we obtain that $\mathbb{P}(A_n^\complement) \le C\mathrm{e}^{-cn}$ with $c = \tilde{c} - c' > 0$ by choosing $c' < \tilde{c}$. □

C Remaining upper bounds for the proof of Proposition 4.1

In this section we prove the three remaining upper bounds for the proof of Proposition 4.1.

Proof of the upper bound of (8.9). Consider

$$\begin{aligned}
&\sum_{\Delta\in\Pi_k} \sum_{\Delta'\in\Pi_{k-1}} \Big| \sum_{u\in\Delta'} P_\omega^{(u,N_{k-1})}(X_{N_k} \in \Delta)\\
&\qquad \times \big[P_\omega^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta')P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta')\big]\Big|.
\end{aligned} \tag{C.1}$$

To get an upper bound for (C.1) the arguments in [2] do not require any specific properties of the model and apply to our model as well. The steps are as follows: by the triangle inequality followed by elementary computations (C.1) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta\in\Pi_k} \sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} P_\omega^{(u,N_{k-1})}(X_{N_k} \in \Delta)\\
&\qquad \times \big|P_\omega^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta')P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta')\big|\\
&= \sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} |P_\omega^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta')P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta')|
\end{aligned}$$

$$= \sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') |P_\omega^{(0,0)}(X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta')|$$
$$= \sum_{\Delta'\in\Pi_{k-1}} |P_\omega^{(0,0)}(X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta')| = \lambda_{k-1}.$$
□

Proof of the upper bound of (8.11). Consider

$$\sum_{\Delta\in\Pi_k} \sum_{\Delta'\in\Pi_{k-1}} \Big| \sum_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \times [\mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u)]. \tag{C.2}$$

For any two probability measures μ and $\tilde{\mu}$ on $\mathbb{Z}^d$ we have

$$\sum_{u\in\Delta'} f(u)\mu(u) - \sum_{u\in\Delta'} f(u)\tilde{\mu}(u) \le \max_{u\in\Delta'} f(u) - \min_{u\in\Delta'} f(u).$$

Thus, the expression (C.2) can be bounded from above by

$$\begin{aligned}
&\sum_{\Delta\in\Pi_k} \sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \Big| \max_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \min_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \Big| \\
&\quad \le \sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \\
&\qquad \times \sum_{\Delta\in\Pi_k^{(1,u)}} \Big| \max_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \min_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \Big| + C n_k^{-c\log n_k},
\end{aligned} \tag{C.3}$$

where for $\Pi_k^{(1,u)}$ is the set defined in (8.16).

Using $\mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) = \sum_{v\in\Delta} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} = v)$ we have

$$\begin{aligned}
&\max_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \min_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \\
&\qquad \le \sum_{v\in\Delta} \max_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} = v) - \min_{u\in\Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} = v) \\
&\qquad \le \sum_{v\in\Delta} \mathrm{diam}(\Delta') \frac{C}{n_k^{(d+1)/2}} \\
&\qquad \le (n_k^\theta)^d n_{k-1}^\theta \frac{C}{n_k^{(d+1)/2}},
\end{aligned}$$

where the second to last inequality follows by the annealed derivative estimates from Lemma 3.1. Altogether the expression (C.2) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \sum_{\Delta\in\Pi_k^{(1,u)}} (n_k^\theta)^d n_{k-1}^\theta \frac{C}{n_k^{(d+1)/2}} + C n_k^{-c\log n_k} \\
&\le C\Big(\sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \Big(\frac{\sqrt{n_k}(\log n_k)^3}{n_k^\theta} \Big)^d \frac{(n_k^\theta)^d n_{k-1}^\theta}{n_k^{(d+1)/2}} + n_k^{-c\log n_k} \Big) \\
&\le C\Big((\log n_k)^{3d} \frac{n_{k-1}^\theta}{n_k^{1/2}} + n_k^{-c\log n_k} \Big) \le C\Big(\frac{(\log n_k)^{3d}}{n_k^{1/2-2\theta}} + n_k^{-c\log n_k} \Big).
\end{aligned}$$
□

Proof of the upper bound of (8.12). Consider

$$\sum_{\Delta\in\Pi_k} \sum_{\Delta'\in\Pi_{k-1}} \Big| \sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, X_{N_{k-1}} \in \Delta') \Big|. \tag{C.4}$$

Recall the regeneration times introduced in [4]. There they are defined for a random walk on the backbone of the oriented percolation cluster, whereas we allow the random walk to start outside the cluster. As noted in [4, Remark 2.3], the local construction, which they use to obtain the regeneration times, can be extended to starting points outside the cluster. Let $B_{m,\tilde{m}}$ be the event that the first regeneration time greater than m will happen before $m+\tilde{m}^\beta$, for some small constant $\beta > 0$ to be tuned appropriately later. By Lemma 2.5 from [4] the distribution of the regeneration increments has exponential tail bounds, and thus $\mathbb{P}(B_{m,\tilde{m}}) \le C\mathrm{e}^{-c\tilde{m}^\beta}$. First, note that by the theorem of total probability and the triangle inequality (C.4) is bounded from above by

$$\begin{aligned} &\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta|X_{N_{k-1}}=u)| \\ &\qquad \le \sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \\ &\qquad\qquad \times\sum_{\Delta\in\Pi_k}\big(\big|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big| \\ &\qquad\qquad\qquad +\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big) \end{aligned} \tag{C.5}$$

First note that

$$\begin{aligned} &\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{N_{k-1},n_k}|X_{N_{k-1}}=u) \\ &\qquad\qquad\qquad\qquad = \mathbb{P}(B^{\mathsf{C}}_{N_{k-1},n_k})\le C\mathrm{e}^{-cn_k^\beta}. \end{aligned}$$

The remaining part of the right hand side of (C.5) is bounded from above by

$$\begin{aligned} &\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}\big(\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{0,n_k}) \\ &\qquad +\big|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B_{0,n_k})-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big|\big). \end{aligned}$$

Using the same arguments as above we obtain

$$\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{0,n_k})=\mathbb{P}(B^{\mathsf{C}}_{N_{k-1},n_k})\le C\mathrm{e}^{-cn_k^\beta}$$

and thus it remains to find a suitable upper bound for

$$\begin{aligned} &\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \\ &\qquad \sum_{\Delta\in\Pi_k}\big|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B_{0,n_k})-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big| \end{aligned}$$

Let $\tilde{\tau}_{N_{k-1}}$ denote the first regeneration time greater than N_{k-1}. By splitting the probabilities above into the sum over the possible times at which the regeneration can occur and the possible sites at which the random walk can be at the time of the regeneration we see that the term in the above display equals to

$$\begin{aligned} &\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \\ &\quad\cdot\sum_{\Delta\in\Pi_k}\Big|\sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}}\mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(u,N_{k-1})}(\tilde{\tau}_{N_{k-1}}=t,X_{\tilde{\tau}_{N_{k-1}}}=v) \\ &\quad-\sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}}\mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(0,0)}(\tilde{\tau}_{N_{k-1}}=t,X_{\tilde{\tau}_{N_{k-1}}}=v|X_{N_{k-1}}=u)\Big|. \end{aligned} \tag{C.6}$$

The modulus in the last two lines of the above display is bounded from above by

$$\begin{aligned}
&\Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) \sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(u,N_{k-1})}(\tilde\tau_{N_{k-1}}=t, X_{\tilde\tau_{N_{k-1}}}=v)\\
&\quad - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\\
&\qquad\qquad\cdot \sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}}=t, X_{\tilde\tau_{N_{k-1}}}=v|X_{N_{k-1}}=u)\Big|\\
&\le \Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\Big|\\
&\quad + \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(u,N_{k-1})}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta)\\
&\quad + \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta|X_{N_{k-1}}=u)
\end{aligned}$$

Plugging that into the sums in (C.6) we obtain that an upper bound of (C.4) is given by

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\\
&\qquad \sum_{\Delta\in\Pi_k}\Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\Big|\\
&+ \sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\\
&\qquad \sum_{\Delta\in\Pi_k} \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(u,N_{k-1})}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta)\\
&+ \sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\sum_{\Delta\in\Pi_k} \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\\
&\qquad\qquad\qquad \mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta, X_{N_{k-1}}=u)\\
&+ C\mathrm{e}^{-cn_k^\beta}.
\end{aligned}$$

Recall the definition of $\Pi_k^{(1,u)}$ from (8.16).

Now define $\Pi_k^{1,u,\beta}$ as the set boxes $\Delta\in\Pi_k$ for which

$$\Delta\cap\Big(\bigcup_{v:\ \|v-u\|\le n_k^\beta}\{x\in\mathbb{Z}^d:\|x-v\|\le\sqrt{n_k}\log^3 n_k\}\Big)\ne\emptyset. \tag{C.7}$$

Using Lemma 3.6 from [21] we obtain

$$\sum_{\Delta\notin\Pi_k^{1,u,\beta}} \mathbb{P}^{(v,0)}(X_{N_k-t}\in\Delta)\le \mathbb{P}^{(v,0)}\big(|X_{N_k-t}-v|>\sqrt{N_k-t}\log^3 N_k-t\big)\le Cn_k^{-c\log n_k}$$

for all $v \in \mathbb{Z}^d$ with $\|v-u\| \le n_k^\beta$ and all $t \in [N_{k-1}, N_{k-1}+n_k^\beta]$. Using this it follows

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \\
&\qquad \sum_{\Delta\in\Pi_k} \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:|u-v|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(u,N_{k-1})}(\tilde\tau_{N_{k-1}} > N_{k-1}+n_k^\beta) \\
&\le |\Pi_k^{1,u,\beta}|\mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}} > N_{k-1}+n_k^\beta) + Cn_k^{-c\log n_k} \\
&\le n_k^{\beta d} n_k^{d/2(1-2\theta)}(\log n_k)^{3d} C\mathrm{e}^{-cn_k^\beta} + Cn_k^{-c\log n_k} \le Cn_k^{-c\log n_k},
\end{aligned}$$

where we have used the fact that, by the definition of $\Pi_k^{(1,u)}$ in (8.16), $|\Pi_k^{1,u,\beta}| \le n_k^{\beta d}|\Pi_k^{(1,u)}|$. Similarly

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} \sum_{\Delta\in\Pi_k} \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:|u-v|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) \\
&\qquad\qquad \mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}} > N_{k-1}+n_k^\beta, X_{N_{k-1}}=u) \qquad\qquad \text{(C.8)} \\
&\le |\Pi_k^{1,u,\beta}|\mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}} > N_{k-1}+n_k^\beta) + Cn_k^{-c\log n_k} \\
&\le n_k^{\beta d} n_k^{d/2(1-2\theta)}(\log n_k)^{3d} C\mathrm{e}^{-cn_k^\beta} + Cn_k^{-c\log n_k} \le Cn_k^{-c\log n_k}.
\end{aligned}$$

Altogether it follows that (C.4) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \\
&\times \sum_{\Delta\in\Pi_k^{1,u,\beta}} \Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\Big| \\
&\qquad\qquad\qquad + Cn_k^{-c\log n_k} + C\mathrm{e}^{-cn_k^\beta}
\end{aligned}$$

Using the annealed derivative estimates from Lemma 3.1 we obtain

$$\begin{aligned}
&\Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\Big| \\
&\le |\Delta|\Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta \\ x\in\Delta}} \mathbb{P}^{(v,t)}(X_{N_k}=x) - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta] \\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta \\ y\in\Delta}} \mathbb{P}^{(v,t)}(X_{N_k}=y)\Big| \\
&\le |\Delta| C(n_k^\beta + n_k^\theta) n_k^{-\frac{d+1}{2}} \\
&\le n_k^{d\theta} C(n_k^\beta + n_k^\theta) n_k^{-\frac{d+1}{2}}.
\end{aligned}$$

Now if we choose $\beta=\theta$ and θ small enough, we get that the above expression is smaller than $Cn_k^{-\frac{2d+1}{4}}$. Putting everything together we get the upper bound

$$\begin{aligned}
&C\mathrm{e}^{-cn_k^\theta} + Cn_k^{-c\log n_k} + \sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \sum_{\Delta\in\Pi_k^{1,u,\beta}} n_k^{-\frac{d}{2}-\frac{1}{4}} \\
&\le C\mathrm{e}^{-cn_k^\theta} + Cn_k^{-c\log n_k} + \sum_{\Delta'\in\Pi_{k-1}} \sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}\in u)|\Pi_k^{1,u,\beta}| n_k^{-\frac{d}{2}-\frac{1}{4}} \\
&\le C\mathrm{e}^{-cn_k^\theta} + Cn_k^{-c\log n_k} + \sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta') Cn_k^{\beta d} n_k^{d(1-\theta)}(\log n_k)^{3d} n_k^{-\frac{d}{2}-\frac{1}{4}}
\end{aligned}$$

$$\leq C\mathrm{e}^{-cn_k^\theta} + Cn_k^{-c\log n_k} + Cn_k^{d/2}(\log n_k)^{3d} n_k^{-\frac{d}{2}-\frac{1}{4}}$$
$$= C\mathrm{e}^{-cn_k^\theta} + Cn_k^{-c\log n_k} + C(\log n_k)^{3d} n_k^{-1/4},$$

where we used the fact that $|\Pi_k^{1,u,\beta}| \leq n_k^{\beta d}|\Pi_k^{(1,u)}| \leq Cn_k^{\beta d} n_k^{d(1-\theta)} \log^{3d} n_k$ and that we choose $\beta = \theta$. Thus, recalling equation (C.4), we obtain

$$\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \Big| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u)\mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, X_{N_{k-1}} \in \Delta') \Big| \leq Cn_k^{-c} \quad \text{(C.9)}$$

for some constants $C, c > 0$. □

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