

Stability estimates for singular SDEs and applications*

Lucio Galeati[†]

Chengcheng Ling[‡]

Abstract

We prove several stability estimates, comparing solutions driven by different (b^i, σ^i) , both for Itô and Stratonovich SDEs, possibly depending on negative Sobolev norms of the difference $b^1 - b^2$. We then discuss several applications of these results to McKean–Vlasov SDEs, criteria for strong compactness of solutions and Wong–Zakai type theorems.

Keywords: SDEs with singular coefficients; stability; Krylov–Röckner condition; distributional drifts; McKean–Vlasov equations; strong compactness of solutions; Wong–Zakai theorem.

MSC2020 subject classifications: 60H10; 60H50; 60F15; 60J60.

Submitted to EJP on August 7, 2022, final version accepted on February 1, 2023.

1 Introduction

In this paper we consider multidimensional singular SDEs of the form

$$dX_t = b_t(X_t) dt + \sigma_t(X_t) dW_t, \quad X|_{t=0} = X_0 \quad (1.1)$$

where $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and W is a d -dimensional Brownian motion. The interval $[0, T]$ is finite, although arbitrarily large, and the initial condition X_0 is possibly random, with finite m -moment for suitable $m \in [1, \infty)$, and independent of W . The drift b is integrable, possibly unbounded; b and σ will be always assumed to satisfy relatively standard assumptions under which strong well-posedness is known to hold. To properly formulate them, let us first define for $\beta \in [0, 1]$ the sets

$$\mathcal{J}_\beta := \left\{ (p, q) : p, q \in \left(\frac{2}{2-\beta}, \infty \right), \quad \frac{2}{q} + \frac{d}{p} < 2 - \beta \right\};$$

we will be mostly interested in considering \mathcal{J}_0 and \mathcal{J}_1 . In the next statement, given $p, q \in [1, \infty]$, we will use the short notation $L_t^q L_x^p$ to denote the Lebesgue–Bochner space $L^q([0, T]; L^p(\mathbb{R}^d))$; see the end of the introduction for a summary of the main notations used throughout this paper.

*LG was funded by the DFG under Germany’s Excellence Strategy – GZ 2047/1, project-id 390685813 during the completion of this work. CL acknowledges financial support by the DFG via Research Unit FOR 2402.

[†]École Polytechnique Fédérale de Lausanne, Bâtiment MA, 1015 Lausanne, Switzerland.

E-mail: lucio.galeati@epfl.ch

[‡]Technische Universität Wien, Institute of Analysis and Scientific Computing, 1040 Wien, Austria.

E-mail: chngcheng.ling@asc.tuwien.ac.at

Assumption 1.1. *The following hold.*

(\mathcal{H}^b) *There exist $(p_1, q_1) \in \mathcal{J}_1$ such that $b \in L_t^{q_1} L_x^{p_1}$.*

(\mathcal{H}_1^σ) *There exists a constant K such that σ is uniformly bounded and nondegenerate, namely*

$$K^{-1}|\xi|^2 \leq |\sigma^*(t, x)\xi|^2 \leq K|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, (t, x) \in [0, T] \times \mathbb{R}^d.$$

(\mathcal{H}_2^σ) *σ is uniformly continuous in space, uniformly in time, with modulus of continuity h , in the sense that*

$$|\sigma(t, x) - \sigma(t, y)| \leq h(|x - y|) \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^{2d}.$$

(\mathcal{H}_3^σ) *There exist $(p_2, q_2) \in \mathcal{J}_1$ such that $\nabla\sigma \in L_t^{q_2} L_x^{p_2}$.*

It is well established that under Assumption 1.1, the SDE (1.1) is strongly well-posed for any initial X_0 independent of W , see e.g. [37] and the basic recap in Section 2.1. In this work we are instead interested in deriving *stability estimates*, in the sense of comparing two distinct solutions X^1, X^2 driven by the same W but with respect to different initial conditions, drifts and diffusions. The next statement exemplifies some of our main findings in this regard.

Theorem 1.2. *Let $(b^1, \sigma^1), (b^2, \sigma^2)$ satisfy Assumption 1.1 for the same parameters $(p_1, q_1), (p_2, q_2), K, h$ and denote by X^1, X^2 the associated solutions to (1.1). Then for any $m \in [1, \infty), T > 0$ and any $(\tilde{p}_1, \tilde{q}_1) \in \mathcal{J}_0, (\tilde{p}_2, \tilde{q}_2) \in \mathcal{J}_1$ there exists a constant C (depending on all the aforementioned parameters as well as $d, \|b^i\|_{L_t^{\tilde{q}_1} L_x^{\tilde{p}_1}}$ and $\|\nabla\sigma^i\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}}$) such that*

$$\left\| \sup_{t \in [0, T]} |X_t^1 - X_t^2| \right\|_{L_\omega^m} \leq C \left[\|X_0^1 - X_0^2\|_{L_\omega^m} + \|b^1 - b^2\|_{L_t^{\tilde{q}_1} L_x^{\tilde{p}_1}} + \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}} \right]. \quad (1.2)$$

If moreover $q_2 = \infty$ and $4/q_1 + d/p_1 < 1$, we have

$$\left\| \sup_{t \in [0, T]} |X_t^1 - X_t^2| \right\|_{L_\omega^m} \leq \tilde{C} \left[\|X_0^1 - X_0^2\|_{L_\omega^m} + \|b^1 - b^2\|_{L_t^{\tilde{q}_1} W_x^{-1, p_1}} + \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}} \right] \quad (1.3)$$

where \tilde{C} is another positive constant with similar dependence on the parameters.

To the best of our knowledge, the first paper to discuss the importance of quantitative stability estimates is [40]. Therein, in the case of common diffusion $\sigma^1 = \sigma^2$, an estimate for $X^1 - X^2$ in function of $\|b^1 - b^2\|_{L_t^{q_1} L_x^{p_1}}$ is presented in Theorem 1.1-(E); instead another estimate, in the case $b^1 = b^2 = 0$ and $\sigma^1 \neq \sigma^2$ is provided in [40, Lemma 5.3]. The results is however subject to several restrictions, most importantly the assumption $q = p > d + 2$ needed to show well-posedness of the corresponding Kolmogorov equation, and the focus on $m = 2$. Later, again for $\sigma^1 = \sigma^2$, the stability result was generalized in [27, Theorem 3.3], for any values $(p, q) \in \mathcal{J}_1$, but again the resulting estimate is only in function of $\|b^1 - b^2\|_{L_t^q L_x^p}$ and holds for $m = 2$. A quite general stability estimate, true for any $m \in [1, \infty)$, is presented in [38, Theorem 3.10], where the SDE is also allowed to have a multiplicative Lévy noise; however, the condition $\nabla b^1 \in L_t^{q'} L_x^{p'}$ for suitable $p', q' \in (1, \infty)$ must be assumed to obtain the stability result. To the best of our knowledge, the first paper to consider stability estimates in negative Sobolev norms (namely in the style of (1.3)) is [11], which considers the more general framework of SDEs driven by fBm; however in the Brownian setting $H = 1/2$, it only allows for additive noise $\sigma = \mathbb{I}$ and drifts b of spatial regularity at least C^α for some $\alpha > 0$. The same work also shows how such estimates can be used to solve McKean-Vlasov SDEs. Theorem 1.2

can be considered a great generalization of all of the aforementioned results, allowing for any value $m \in [1, \infty)$, different integrability coefficients $(p_1, q_1) \neq (p_2, q_2)$ for b and σ in Assumption 1.1 as well as parameters $(\tilde{q}_1, \tilde{p}_1), (\tilde{q}_2, \tilde{p}_2)$ different from them in the estimates (1.2) and (1.3). Let us also mention that this work is hugely inspired by the previous work [25] by one of the authors, where numerical schemes for singular SDEs are considered; although technically it doesn't contain stability estimates, it is clear that the quantity ϖ_n appearing in [25, Theorem 1.1] is closely related to the error committed by comparing two SDEs driven respectively by drift b^n and b respectively, in the case of common diffusion σ . We refer to Section 3.4.1 for a deeper discussion on this point. Finally, strong moments estimates of the form (1.2)-(1.3) are not the only form of stability one might be interested in exploring; in a different direction, based on estimating the distance of the transition probabilities of the two processes X^1 and X^2 , it is worth mentioning the works [3, 21, 2].

The proof of Theorem 1.2 will be presented in Section 2.3 and is in fact consequence of a more abstract and much more general stability result, see Proposition 2.12; the same strategy of proof allows for numerous variants, see Remark 2.14, as well as similar results for Stratonovich SDEs, cf. Corollary 2.13.

In order to motivate the importance of results in the style of Theorem 1.2, we will present in Section 3 several applications, ranging from existence and uniqueness results for distribution dependent (or McKean-Vlasov) SDEs, criteria for strong compactness of solutions to singular SDEs and Wong-Zakai type theorems. We will also shortly discuss in Section 3.4 other applications of such results which have already appeared in the literature, although without it being stated explicitly; indeed, we believe one of nice contributions of this work is also to highlight the importance of stability estimates for SDEs in a wide range of problems, which have so far been treated without recognizing their key role in them.

Remark 1.3. For simplicity in this paper we only considered drifts with global bounds in $L_t^q L_x^p$. Up to technical details, we expect the same strategy to work in many other cases, including:

- a) $b, \nabla \sigma \in L_t^q \tilde{L}_x^p$ where \tilde{L}_x^p is the so called localized L_x^p considered in [37]; one of the advantages of this case is that bounded b is covered as well.
- b) The case of b of the form $b = b^1 + \dots + b^n$ with $b^i \in L_t^{q_i} L_x^{p_i}$ with $(q_i, p_i) \in \mathcal{J}_1$.
- c) The presence of an additional Lipschitz term, i.e. $b = b^1 + b^2$ with $b^1 \in L_t^q L_x^p$ and $b^2 \in L_t^\infty C_x^1$ as in [39].
- d) Coefficients belonging to mixed-normed spaces, i.e. $b \in L_t^q L_{x_1}^{p_1} \dots L_{x_d}^{p_d}$ with $2/q + \sum_i 1/p_i < 1$, as considered in [28].

Notations and conventions

We always work on a finite time interval $[0, T]$. We write $a \lesssim b$ to mean that there exists a positive constant C such that $a \leq Cb$; we use the index $a \lesssim_\lambda b$ to highlight the dependence $C = C(\lambda)$. For vectors $x, y \in \mathbb{R}^d$, we write $x \cdot y := \sum_{i=1}^d x_i y_i$ and $|x| = \sqrt{x \cdot x}$; for matrices $A, B \in \mathbb{R}^{d \times d}$, we use the convention $A : B := \sum_{1 \leq i, j \leq d} A_{ij} B_{ij}$. Given a real number z , we denote by $\lfloor z \rfloor$ and $\{z\}$ respectively its integer and fractional parts.

For any $N \in \mathbb{N}$ and $p \in [1, \infty]$, we denote by $L^p(\mathbb{R}^d; \mathbb{R}^N)$ the standard Lebesgue space; when there is no risk of confusion in the parameter N , we will simply write L_x^p for short and denote by $\|\cdot\|_{L_x^p}$ the corresponding norm. Similarly for the Bessel potential spaces $W_x^{\beta, p} = W^{\beta, p}(\mathbb{R}^d; \mathbb{R}^N)$, which are defined for $\beta \in \mathbb{R}$, with corresponding norm

$$\|f\|_{W_x^{\beta, p}} := \|(\mathbb{I} - \Delta)^{\beta/2} f\|_{L_x^p};$$

Stability estimates

For $\alpha \in [0, \infty)$, $C_x^\alpha = C^\alpha(\mathbb{R}^d; \mathbb{R}^N)$ stands for the usual Hölder continuous function space, made of continuous bounded functions with continuous and bounded derivatives up to order $[\alpha] \in \mathbb{N}$ and with globally $\{\alpha\}$ -Hölder continuous derivatives of order $[\alpha]$.

We denote by $C_T = C([0, T]; \mathbb{R}^d)$ the path space of continuous functions on $[0, T]$, endowed with the supremum norm $\|\varphi\|_{C_T} = \sup_{t \in [0, T]} |\varphi_t|$.

Given a Banach space E and a parameter $q \in [1, \infty]$, we denote by $L_t^q E = L^q(0, T; E)$ the space of measurable functions $f : [0, T] \rightarrow E$ such that

$$\|f\|_{L_t^q E} := \left(\int_0^T \|f_t\|_E^q dt \right)^{\frac{1}{q}} < \infty$$

with the usual convention of the essential supremum norm in the case $q = \infty$. Similarly, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $m \in [1, \infty)$, we denote by $L_\omega^m E = L^m(\Omega, \mathcal{F}, \mathbb{P}; E)$ the space of E -valued \mathcal{F} -measurable random variables X such that

$$\|X\|_{L_\omega^m E} := \mathbb{E}[\|X\|_E^m]^{\frac{1}{m}} < \infty$$

where \mathbb{E} denotes expectation w.r.t. \mathbb{P} . The above definitions can be concatenated by choosing at each step a different E , so that one can define $L_\omega^m C_T$, $L_t^q L_x^p$ and so on. Whenever $q = p$, we might write for simplicity $L_{t,x}^p$ in place of $L_t^p L_x^p$.

2 Stability estimates

2.1 Preliminaries

We shortly recall here several facts related to the SDE (1.1) which will be needed in the sequel, including Girsanov theorem, Zvonkin transform and Khasminskii type estimates. They are by now very classical tools in the study of singular SDEs, first introduced in the pioneering works by Zvonkin [43] and Veretennikov [35]; see for instance [37] for a nice self-contained presentation (in a slightly different setting).

We start by recalling that any solution X can be regarded after a change of measure as a solution to a driftless SDE, thanks to Girsanov theorem.

Lemma 2.1. *Let (b, σ) satisfy Assumption 1.1 and let X be the unique solution to (1.1). Then there exists another measure \mathbb{Q} , equivalent to \mathbb{P} , and a \mathbb{Q} -Brownian motion \tilde{W} such that X solves the SDE $dX_t = \sigma_t(X_t) d\tilde{W}_t$. Furthermore it holds*

$$\mathbb{E}_{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^n \right] < \infty \quad \forall n \in \mathbb{Z}.$$

Proof. An application of [37, Lemma 4.1] for the choice $f = |b|^2$, $p = 2p_1$, $q = 2q_1$ allows to deduce that the solution X satisfies

$$\mathbb{E} \left[\exp \left(\lambda \int_0^T |b_r|^2(X_r) dr \right) \right] < \infty \quad \forall \lambda > 0.$$

From this, the finiteness of all (positive and negative) moments of the density $d\mathbb{Q}/d\mathbb{P}$ associated to Girsanov's theorem is a standard argument, see for instance the proof of [7, Proposition 2.4]. \square

The next classical ingredient we need are suitable estimates for parabolic PDEs of the form (2.1), strictly related to the so called *Zvonkin transform*. In the next statement, we are adopting the notations $a = \sigma\sigma^*$ and $a : D^2 f = \sum_{i,j} a_{ij} \partial_{ij}^2 f$, $b \cdot \nabla f = \sum_i b_i \partial_i f$.

Lemma 2.2. *Let (b, σ) satisfy Assumption 1.1. Then there exists $\lambda_0 \geq 1$, depending on T, d, K, h, p_1, q_1 and $\|b\|_{L_t^{q_1} L_x^{p_1}}$, such that for all $\lambda \geq \lambda_0$ there is a unique solution $u \in L_t^{q_1} W_x^{2,p_1} \cap W_t^{1,q_1} L_x^{p_1}$ to the equation*

$$\partial_t u + \frac{1}{2} a : D^2 u + b \cdot \nabla u - \lambda u = -b, \quad u|_{t=T} = 0. \tag{2.1}$$

Furthermore there exists $\epsilon > 0$, depending on d, p_1, q_1 , such that

$$\lambda^\epsilon \|u\|_{L_t^\infty C_x^1} + \|\partial_t u\|_{L_t^{q_1} L_x^{p_1}} + \|u\|_{L_t^{q_1} W_x^{2,p_1}} \lesssim \|b\|_{L_t^{q_1} L_x^{p_1}}. \tag{2.2}$$

Proof. The result is a special case of [37, Theorem 3.2]; indeed, by time reversal we can reduce ourselves to the case of a forward parabolic equation with $u|_{t=0} = 0$ as therein. Let $\epsilon > 0$ be a sufficiently small parameter satisfying $2/q_1 + d/p_1 > 1 - 4\epsilon$ and choose $p' \geq p$ large enough so that $d/p' = 2\epsilon$; setting $q' = \infty$ and $\alpha = 1 + 4\epsilon$, we can apply the result from [37, Theorem 3.2] to get an estimate for $\lambda^\epsilon \|u\|_{L_t^\infty W_x^{\alpha,p'}}$. The embedding $W_x^{\alpha,p'} \hookrightarrow C_x^{\alpha-d/p'} \hookrightarrow C_x^1$ finally allows to conclude. \square

The next lemma, which allows to control integrals of the form $\int_0^T f_r(X_r)$ for f merely satisfying $L_t^q L_x^p$ -integrability, is also classical; its proof comes from a combination of so called *Krylov's estimates* and *Khasminskii's lemma*.

Lemma 2.3. *Let (b, σ) satisfy Assumption 1.1, X be a solution to (1.1) and $(\tilde{p}, \tilde{q}) \in \mathcal{J}_0$. Then there exists a constant $C > 0$ (depending on $T, d, K, h, p_1, q_1, \|b\|_{L_t^{q_1} L_x^{p_1}}, \tilde{p}, \tilde{q}$) such that*

$$\mathbb{E} \left[\exp \left(\lambda \int_0^T f_r(X_r) \, dr \right) \right] \leq C \exp \left(C \lambda^{\tilde{q}} \|f\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}}^{\tilde{q}} \right) \quad \forall \lambda > 0, \quad f \in L_t^{\tilde{q}} L_x^{\tilde{p}}. \tag{2.3}$$

Proof. By [37, Theorem 1.1 (A)], for any $s < t$ it holds

$$\mathbb{E} \left[\int_s^t f(X_r) \, dr \middle| \mathcal{F}_s \right] \lesssim \|f\|_{L_t^{\tilde{q}}(s,t;L_x^{\tilde{p}})} =: w(s,t)^{1/\tilde{q}},$$

where w is a continuous control; estimate (2.3) then follows from an application of the quantitative version of Khasminskii's lemma from [25, Lemma 3.5], for the choice $\gamma = 1/\tilde{q}$. \square

Our next ingredient will be an alternative estimate for additive functionals $\int_0^t f_r(X_r) \, dr$ in the case where f only enjoys negative regularity, see Lemma 2.5 below; this result is less classical and improves on the one from [25, Proposition 6.6]. To this end, we first need to recall a result on parabolic PDEs with distributional forcing, first established in [25].

Lemma 2.4. *Let σ satisfy Assumption 1.1 with $q_2 = \infty$, $f \in L_t^{\tilde{q}} W_x^{-1,\tilde{p}}$ with $\tilde{p}, \tilde{q} \in (1, \infty)$ and $\frac{1}{p_2} + \frac{1}{\tilde{p}} < 1$. Then there exists a unique solution¹ $v \in L_t^{\tilde{q}} W_x^{1,\tilde{p}} \cap W_t^{1,\tilde{q}} W_x^{-1,\tilde{p}}$ to the PDE*

$$\partial_t v + \frac{1}{2} a : D^2 v = f, \quad v|_{t=T} = 0. \tag{2.4}$$

Furthermore, there exists a constant $C > 0$ (depending on $T, d, K, p_2, \|\nabla \sigma\|_{L_t^\infty L_x^{p_2}}, \tilde{p}, \tilde{q}$) such that

$$\|v\|_{L_t^{\tilde{q}} W_x^{1,\tilde{p}}} + \|\partial_t v\|_{L_t^{\tilde{q}} W_x^{-1,\tilde{p}}} \leq C \|f\|_{L_t^{\tilde{q}} W_x^{-1,\tilde{p}}}. \tag{2.5}$$

¹Here we mean the weak solution in the sense of [25, Definition A.1]; in particular, it can be shown that $a : D^2 v$ is a well-defined distribution belonging to $L_t^{\tilde{q}} W_x^{-\tilde{p}}$, so that identity (2.4) holds in the sense of distributions. See [25] for more details.

Proof. In the case $T = 1$, the statement is a special case of [25, Theorem A.4]; the case of general intervals $[0, T]$ follows by a simple rescaling argument. \square

Lemma 2.5. *Let b, σ satisfy Assumption 1.1 with $q_2 = \infty$, X be a solution to (1.1) and $(\tilde{p}, \tilde{q}) \in \mathcal{J}_1$. Then there exists a constant $C > 0$ (depending on $T, d, K, h, p_1, q_1, p_2, \|b\|_{L_t^{q_1} L_x^{p_1}}, \|\nabla \sigma\|_{L_t^\infty L_x^{p_2}}, \tilde{p}, \tilde{q}$) such that for all smooth function f it holds*

$$\mathbb{E} \left[\exp \left(\lambda \sup_{t \in [0, T]} \left| \int_0^t f_r(X_r) dr \right| \right) \right] \leq C \exp \left(C \lambda^{\tilde{q}} \|f\|_{L_t^{\tilde{q}} W_x^{-1, \tilde{p}}}^{\tilde{q}} \right) \quad \forall \lambda > 0. \quad (2.6)$$

By linearity and density, we can extend the definition of $\int_0^t f_r(X_r) dr$ as a continuous process to any $f \in L_t^{\tilde{q}} W_x^{-1, \tilde{p}}$, in which case estimate (2.6) still holds.

Proof. First observe that, thanks to Lemma 2.1, we can assume without loss of generality $b \equiv 0$; we can also assume f to be \mathbb{R} -valued, otherwise we can argue componentwise, and by homogeneity we can take $\lambda = 1$.

Given a smooth f , denote by v the solution to the parabolic PDE (2.4). Since $(\tilde{p}, \tilde{q}) \in \mathcal{J}_1, (p_2, \infty) \in \mathcal{J}_1$, it also holds $\tilde{p} \wedge p_2 > 2$ and thus $1/\tilde{p} + 1/p_2 < 1$; in particular, the assumptions of Lemma 2.4 are satisfied and thus estimate (2.5) holds. Furthermore, using the assumption $(\tilde{p}, \tilde{q}) \in \mathcal{J}_1$ and interpolation estimates, one can find $\theta = 1/\tilde{q} + \epsilon$ with $\epsilon > 0$ small enough such that

$$\begin{aligned} \|v\|_{L_{t,x}^\infty} &\lesssim \|v\|_{L_t^\infty C_x^{2\epsilon}} \lesssim \|v\|_{L_t^\infty W^{2(1-\theta)-1, \tilde{p}}} \\ &\lesssim \|\partial_t v\|_{L_t^{\tilde{q}} W_x^{-1, \tilde{p}}}^\theta \|v\|_{L_t^{\tilde{q}} W_x^{1, \tilde{p}}}^{1-\theta} \lesssim \|f\|_{L_t^{\tilde{q}} W_x^{-1, \tilde{p}}}. \end{aligned}$$

On the other hand, since $dX_t = \sigma_t(X_t) dW_t$, by Itô's formula we get

$$\int_0^t f_r(X_r) dr = v_t(X_t) - v_0(x) - \int_0^t (\sigma^* \nabla v)_r(X_r) \cdot dW_r.$$

Therefore we have the estimate

$$\begin{aligned} &\mathbb{E} \left[\exp \left(\sup_{t \in [0, T]} \left| \int_0^t f_r(X_r) dr \right| \right) \right] \\ &\leq e^{2\|v\|_{L_{t,x}^\infty}} \mathbb{E} \left[\exp \left(\sup_{t \in [0, T]} \left| \int_0^t (\sigma^* \nabla v)_r(X_r) \cdot dW_r \right| \right) \right]; \end{aligned}$$

using Doob's inequality for the submartingale $M_t := \exp \left(\int_0^t (\sigma^* \nabla v)_r(X_r) \cdot dW_r \right)$, as well as the basic inequality $e^{|x|} \leq e^x + e^{-x}$, we can find a constant $\kappa > 0$ such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |M_t| \right] \leq I^+ + I^-, \quad I^\pm := \mathbb{E} \left[\exp \left(\pm \kappa \int_0^T (\sigma^* \nabla v)_r(X_r) \cdot dW_r \right) \right].$$

Let us show how to estimate I^+ , the other term being treated similarly. By estimate (2.5), $\nabla v \in L_t^{\tilde{q}} L_x^{\tilde{p}}$, thus $|\sigma^* \nabla v|^2 \in L_t^{\tilde{q}/2} L_x^{\tilde{p}/2}$ where by our assumptions $(\tilde{q}/2, \tilde{p}/2) \in \mathcal{J}_0$. In particular, Lemma 2.3 applies to $f = |\sigma^* \nabla v|^2$; but then by Girsanov's theorem and Novikov's condition, this implies that

$$N_t := \exp \left(2\kappa \int_0^t (\sigma^* \nabla v)_r(X_r) \cdot dW_r - 2\kappa^2 \int_0^t |(\sigma^* \nabla v)_r(X_r)|^2 dt \right)$$

is a martingale, which together with Hölder's inequality yields

$$\begin{aligned} I^+ &\leq \mathbb{E}[N_t]^{1/2} \mathbb{E} \left[\exp \left(2\kappa^2 \int_0^T |(\sigma^* \nabla v)_r(X_r)|^2 dr \right) \right]^{1/2} \\ &\lesssim C \exp \left(C \|\sigma^* \nabla v\|_{L_t^{\tilde{q}/2} L_x^{\tilde{p}/2}}^2 \right). \end{aligned}$$

Combining all of the above with the estimate

$$\|\sigma^* \nabla v\|^2 \Big|_{L_t^{\tilde{q}/2} L_x^{\tilde{p}/2}} \lesssim \|\nabla v\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}} \lesssim \|f\|_{L_t^{\tilde{q}} W_x^{-1, \tilde{p}}},$$

where the second passage comes from (2.5), and relabelling the constant C to include κ , we finally obtain the desired (2.6). \square

Remark 2.6. The realisation that suitable diffusion processes can be integrated along distributions goes back at least to the works of Bass and Chen [1], where the case of measures in a suitable Kato class is considered. Among more recent developments let us mention [9] and [41, Theorem 5.1], where autonomous $f \in W_x^{-\alpha, p}$ are allowed for $\alpha \leq 1/2, p > d/(1 - \alpha)$. As already mentioned, Lemma 2.5 is an improvement over [25], as it allows $\nu = 1$ without further assumptions of f and reaches exponential integrability.

Among other applications of Lemma 2.5 of independent interest, let us mention that for the choice $f = \div b \in L_t^{\tilde{q}_1} W_x^{-1, \tilde{p}_1}$ it yields uniform L_ω^m estimates for the Jacobian $J\phi_t(x) = \det D_x \phi_t(x)$ of the flow ϕ associated to the SDE (1.1), which is formally given by the formula

$$J\phi_t(x) = \exp \left(\int_0^t \operatorname{div} b_r(X_r^x) \, dr \right).$$

Compare this fact with [8, Section 3], where the additional assumption $\div b \in L^p, p > 2$ is imposed in order to analyse such term and derive regularity estimates.

Remark 2.7. Interpolating between $L_t^{\tilde{q}_0} L_x^{\tilde{p}_0}$ and $L_t^{\tilde{q}_1} W_x^{-1, \tilde{p}_1}$ for $(\tilde{q}_0, \tilde{p}_0) \in \mathcal{J}_0, (\tilde{q}_1, \tilde{p}_1) \in \mathcal{J}_1$ and applying Lemmas 2.3-2.5, one can in fact prove a more general statement: for any $\beta \in [0, 1], (\tilde{q}_\beta, \tilde{p}_\beta) \in \mathcal{J}_\beta$ and any $f \in L_t^{\tilde{q}_\beta} W_x^{-\beta, \tilde{p}_\beta}$, the process $\int_0^\cdot f_r(X_r) \, dr$ is well defined and

$$\mathbb{E} \left[\exp \left(\lambda \sup_{t \in [0, T]} \left| \int_0^t f_r(X_r) \, dr \right| \right) \right] \leq C \exp \left(C \lambda^{\tilde{q}_\beta} \|f\|_{L_t^{\tilde{q}_\beta} W_x^{-\beta, \tilde{p}_\beta}}^{\tilde{q}_\beta} \right) \quad \forall \lambda > 0.$$

Remark 2.8. Estimates (2.3)-(2.6) immediately imply moment bounds of the form

$$\mathbb{E} \left[\left(\int_0^T f_r(X_r) \, dr \right)^m \right] \lesssim \|f\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}}^m, \quad \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t f_r(X_r) \, dr \right|^m \right] \lesssim \|f\|_{L_t^{\tilde{q}} W_x^{-1, \tilde{p}}}^m, \quad (2.7)$$

for all $m \in [1, \infty)$. For instance, to deduce the first estimate in (2.7), it suffices to apply (2.3) for the choice $\lambda = 1, \tilde{f} = f/\|f\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}}$, so that

$$\|f\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}}^{-m} \mathbb{E} \left[\left(\int_0^T f_r(X_r) \, dr \right)^m \right] \lesssim_m \mathbb{E} \left[\exp \left(\int_0^T \tilde{f}_r(X_r) \, dr \right) \right] \lesssim 1.$$

As a final ingredient, we need to recall an inequality involving maximal functions; the standard version, sometimes referred to as Hajlasz inequality, is classical [34], but the one-sided one presented here seems to be of much recent derivation, cf. [4].

Lemma 2.9. *Let f be a measurable function such that $\nabla f = \psi^1 + \psi^2$, where $\psi^1 \in L_t^{\tilde{q}_1} L_x^{\tilde{p}_1}$ for some parameters $(\tilde{q}_1, \tilde{p}_1), (\tilde{q}_2, \tilde{p}_2)$ satisfying $\tilde{q}_i \in [1, \infty)$ and $\tilde{p}_i \in (d, \infty)$; then there exists a negligible Lebesgue set $N \subset [0, T] \times \mathbb{R}^d$ and another measurable function g such that*

$$|f_t(x) - f_t(y)| \leq [g_t^1(x) + g_t^2(x)] |x - y| \quad \forall y \in \mathbb{R}^d, \forall (t, x) \in ([0, T] \times \mathbb{R}^d) \setminus N; \quad (2.8)$$

moreover there exists a constant $C = C(d, \tilde{p}_1, \tilde{p}_2)$ such that g can be chosen of the form $g = g^1 + g^2$ satisfying $\|g^i\|_{L_t^{\tilde{q}_i} L_x^{\tilde{p}_i}} \leq C \|\psi^i\|_{L_t^{\tilde{q}_i} L_x^{\tilde{p}_i}}$ for $i = 1, 2$.

Proof. In the case of autonomous functions f , the statement is a simple consequence of [4, Lemma 5.1]; more precisely, by fixing another $p' \in (d, \tilde{p}_1 \wedge \tilde{p}_2)$ and applying the result therein, inequality (2.8) holds for $g(x) \sim [\mathcal{M}(|\nabla f|^{p'})(x)]^{1/p'}$, where \mathcal{M} denotes the Hardy–Littlewood maximal function operator. By standard properties of \mathcal{M} , $g \lesssim g^1 + g^2$ for $g^i(x) := \mathcal{M}(|\psi^i|^{p'})(x)^{1/p'}$; by construction, $|\psi^i|^{p'} \in L_x^{\tilde{p}_i/p'}$ with $\tilde{p}_i/p' > 1$, therefore again by properties of \mathcal{M} it holds $\|\mathcal{M}(|\psi^i|^{p'})\|_{L^{p/p'}} \lesssim \|\psi^i\|_{L_x^{\tilde{p}_i}}^{p'}$, from which we can deduce that $\|g^i\|_{L_x^{\tilde{p}_i}} \lesssim \|\psi^i\|_{L_x^{\tilde{p}_i}}$.

In the case of time-dependent functions f , one can either: i) observe that for a.e. $t \in [0, T]$, $\nabla f_t \in L_x^{\tilde{p}_1} + L_x^{\tilde{p}_2}$ and apply the result at such fixed t ; or ii) consider a sequence f^n of smooth functions such that $f^n \rightarrow f$ and they admits decompositions $\nabla f^n = \psi^{n,1} + \psi^{n,2}$ with $\psi^{n,i} \rightarrow \psi^i$ in $L_t^{\tilde{q}_i} L_x^{\tilde{p}_i}$; then to any f^n there is an associated $g^n = g^{n,1} + g^{n,2} = \mathcal{M}(|\psi^{n,1}|^{p'})^{1/p'} + \mathcal{M}(|\psi^{n,2}|^{p'})^{1/p'}$ such that (2.8) holds and clearly $g^{n,i} \rightarrow g^i$ in $L_t^{\tilde{q}_i} L_x^{\tilde{p}_i}$, so that inequality (2.8) still holds after the limit (up to a Lebesgue negligible set). \square

2.2 A general abstract estimate

We present here a fundamental estimate which is at the heart of the proof of Theorem 1.2, as well as of several applications. It focuses on comparing the solution X to the SDE (1.1) associated to (X_0, b, σ) w.r.t. to any semimartingale perturbation of it; more precisely, we will consider a process Y solving

$$Y_t = Y_0 + \int_0^t [b_s(Y_s) + R_s^1] ds + \int_0^t [\sigma_s(Y_s) + R_s^2] dW_s \tag{2.9}$$

where R^1, R^2 are predictable processes, taking values respectively in \mathbb{R}^d and $\mathbb{R}^{d \times d}$. In the following, we will enforce the following conditions on Y .

Assumption 2.10. *The following hold for any $T > 0$:*

- i) $\int_0^T [|b_s(Y_s)| + |R_s^1| + |\sigma_s(Y_s)|^2 + |R_s^2|^2] ds < \infty$ \mathbb{P} -a.s.
- ii) For any $f \in L_t^{\tilde{q}} L_x^{\tilde{p}}$ with $(\tilde{q}, \tilde{p}) \in \mathcal{J}_0$, it holds $\int_0^T |f_r(Y_r)| dr < \infty$ \mathbb{P} -a.s. Moreover for any sequence of smooth, bounded functions f^ϵ such that $f^\epsilon \rightarrow f$ in $L_t^{\tilde{q}} L_x^{\tilde{p}}$ as $\epsilon \rightarrow 0$, it holds

$$\lim_{\epsilon \rightarrow 0} \int_0^T |f^\epsilon(Y_s) - f(Y_s)| ds = 0 \quad \mathbb{P}\text{-a.s.}$$

- iii) The processes Y, R^1, R^2 are adapted to the filtration generated by (Y_0, W, \tilde{W}) , where \tilde{W} is another \mathbb{R}^m -valued process, for some $m \in \mathbb{N}$, and Y_0, W, \tilde{W} are independent.

Remark 2.11. While Condition i) is natural in order to give meaning to (2.9), the others are less classical. Point ii) will be needed in order to make sense of an Itô-type formula for functions with only integrable second order derivatives. Instead iii) simplifies the proof of Proposition 2.12 below by allowing to disintegrate w.r.t. the initial conditions (X_0, Y_0) ; depending on the specific structure of the processes R^1, R^2 it might not be even needed. In the simplest cases, one can just take $\tilde{W} \equiv 0$, but there are also applications of interest where a nontrivial choice of \tilde{W} must be considered, as we will see in Section 3.4.2 in the context of interacting particle systems.

Before presenting the statement, we need a few preparations. Given (b, σ) satisfying Assumption 1.1, by Lemma 2.2 we can choose λ large enough so that the solution u to (2.1) has Lipschitz constant smaller than $1/2$ for all $t \in [0, T]$; from now on, we will always work with such a fixed λ and denote by u the associated solution, so that the map

$\phi_t(x) := x + u_t(x)$ is 2-Lipschitz with 2-Lipschitz inverse for all $t \in [0, T]$. For such u , we define an associated process

$$V_t := \int_0^t \left(\frac{1}{2} [(R^2 + \sigma)(R^2 + \sigma)^* - \sigma\sigma^*] : D^2u + R^1 \cdot (I + \nabla u) \right)_r (Y_r) dr + \int_0^t [R^2(I + \nabla u)]_r(Y_r) dW_r \tag{2.10}$$

which is well defined thanks to Assumption 2.10.

The proof of the next result shares some clear analogies with those of stochastic Grönwall lemmas, cf. [25, Lemma 3.8], as well as [12] for a general overview; however, in order not to impose any restriction on the m -th moment we want to estimate, we will exploit crucially the structure of the SDE which allows us to “recenter” it at each step, a property which doesn’t hold in the more general setting of those results.

Proposition 2.12. *Let X be a solution to (1.1) associated to (X_0, b, σ) satisfying Assumption 1.1 and let Y be a solution to (2.9) satisfying Assumption 2.10. Then for any $m \in [1, \infty)$, any $\gamma > 1$ and any $T > 0$ there exists a constant C (depending on $m, \gamma, T, d, (q_i, p_i), K, h, \|b\|_{L_t^{q_1} L_x^{p_1}}, \|\nabla\sigma\|_{L_t^{q_2} L_x^{p_2}}$), such that*

$$\left\| \sup_{t \in [0, T]} |X_t - Y_t| \right\|_{L_\omega^m} \leq C \left(\|X_0 - Y_0\|_{L_\omega^m} + \left\| \sup_{t \in [0, T]} |V_t| \right\|_{L_\omega^{m\gamma}} \right) \tag{2.11}$$

where the process V is defined as in (2.10).

Proof. It suffices to consider the case where the initial data X_0 and Y_0 are deterministic; indeed in the general case, as they are random variables independent of the driving (W, \tilde{W}) , we can first condition on the knowledge of (X_0, Y_0) , obtain the estimate, and then take expectation again to conclude. From now on we will fix $m \in [1, \infty)$.

We start by applying Zvonkin transform to both X and Y ; by classical computations based on generalized Itô’s formula (see e.g. [22, Theorem 3.7] or [37, Lemma 4.1]), setting $\tilde{\sigma} := \sigma(I + \nabla u)$, it holds

$$d(\phi_t(X_t)) = \lambda u_t(X_t) dt + \tilde{\sigma}_t(X_t) dW_t;$$

the computation for Y is slightly more involved, but it is not difficult to check that, thanks to our definition of V , in the end one arrives at

$$d(\phi_t(Y_t)) = \lambda u_t(Y_t) dt + \tilde{\sigma}_t(Y_t) dW_t + dV_t.$$

Indeed, the computation is rigorously justified thanks to Assumption 2.10, as we can first apply Itô formula to ϕ^ϵ , associated to some smooth approximation b^ϵ of b , and then pass to the limit as $\epsilon \rightarrow 0$. As a consequence, if we integrate both equations over any interval $[s, t]$ and take the modulus, we find the estimate

$$|\phi_t(X_t) - \phi_t(Y_t)| \leq |\phi_s(X_s) - \phi_s(Y_s)| + \lambda \int_s^t |u_r(X_r) - u_r(Y_r)| dr + \left| \int_s^t [\tilde{\sigma}_r^1(X_r) - \tilde{\sigma}_r^1(Y_r)] dW_r \right| + \bar{V},$$

where we set $\bar{V} := 2 \sup_{t \in [0, T]} |V_t|$; using the properties of ϕ, u , we arrive at

$$\sup_{t \in [s, s+h]} |X_t - Y_t| \lesssim |X_s - Y_s| + \lambda h \sup_{t \in [s, s+h]} |X_t - Y_t| + \sup_{t \in [s, s+h]} \left| \int_s^t [\tilde{\sigma}_r^1(X_r) - \tilde{\sigma}_r^1(Y_r)] dW_r \right| + \bar{V}. \tag{2.12}$$

Observe that this is a pathwise inequality that holds for any choice of s, h , which we may and will take to be random as well.

Moreover, since $\nabla\sigma \in L_t^{q_2}L_x^{p_2}$ and $u \in L_t^\infty C_x^1 \cap L_t^{q_1}W_x^{2,p_1}$, the new diffusion term satisfies $\nabla\tilde{\sigma} \in L_t^{q_1}L_x^{p_1} + L_t^{q_2}L_x^{p_2}$ as well²; therefore we can apply Lemma 2.9 to find functions $g^i \in L_t^{q_i}L_x^{p_i}$, $i = 1, 2$, for which (2.8) holds for $f = \tilde{\sigma}$. Let us define the process

$$A_t := \int_0^t [\lambda^2 T + |g_r^1|^2(X_r) + |g_r^2|^2(X_r)] dr$$

for a constant $\tilde{K} > 0$ to be chosen later; we also set an increasing sequence of stopping times by $\tau^0 = 0$ and

$$\tau^{n+1} = \inf \{ t \geq \tau^n : t \leq T, A_t - A_{\tau^n} \geq (4\tilde{K})^{-\frac{2}{m}} \},$$

with the convention that $\tau_{n+1} = T$ if there is no $t \in [\tau_n, T]$ such that $A_t - A_{\tau^n} = (4\tilde{K})^{-2/m}$. Now let us take $s = \tau^n$ in (2.12) and elevate both sides to the m -th power, so to obtain

$$\sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \lesssim_m [|Z_{\tau^n}|^m + \lambda^m h^m \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m + \sup_{t \in [0, h]} |M_t^n|^m + \bar{V}^m],$$

where we set $Z := X - Y$ and

$$M_t^n := \int_{\tau^n}^{\tau^n+t} [\tilde{\sigma}_r(X_r) - \tilde{\sigma}_r(Y_r)] dW_r.$$

By Doob's optimal stopping theorem $(M_t^n)_{t \in [0, T]}$ is a continuous martingale w.r.t. to the filtration $(\mathcal{G}_t)_{t \in [0, T]} = (\mathcal{F}_{\tau^n+t})_{t \in [0, T]}$; by the pathwise BDG inequality from [33, Theorem 3], there exists another continuous $(\mathcal{G}_t)_{t \in [0, T]}$ -martingale $(\tilde{M}_t^n)_{t \in [0, T]}$ such that

$$\sup_{t \in [0, h]} |M_t^n|^m \leq [M_h^n]^{m/2} + \tilde{M}_h^n$$

where $([M^n]_t)_{t \in [0, T]}$ denotes the quadratic variation of $(M^n)_{t \in [0, T]}$. We claim that P-a.s. it holds

$$[M^n]_t = \int_{\tau^n}^{\tau^n+t} |\tilde{\sigma}_r(X_r) - \tilde{\sigma}_r(Y_r)|^2 dr \leq \int_{\tau^n}^{\tau^n+t} [|g_r^1|^2(X_r) + |g_r^2|^2(X_r)] |Z_r|^2 dr. \quad (2.13)$$

To see this, let $N \subset [0, T] \times \mathbb{R}^d$ the negligible Lebesgue set associated to $\tilde{\sigma}$, whose existence is guaranteed by Lemma 2.9; setting $f(t, x) := \mathbb{1}_N(t, x)$, it holds $\|f\|_{L_t^q L_x^p} = 0$ for all $(q, p) \in [1, \infty)$ and an application of Lemma 2.3 implies that

$$\int_0^T \mathbb{1}_N(t, X_t) dt = 0 \quad \text{P-a.s.}$$

In view of the above, an application of Lemma 2.9 yields the desired P-a.s. inequality (2.13), as the presence of the exceptional set N doesn't play any role.

Combining everything and taking $h = \tau_{n+1} - \tau_n$, we arrive at a P-a.s. inequality of the form

$$\begin{aligned} \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m &\leq C \left[|Z_{\tau^n}|^m + \tilde{M}_{\tau_{n+1}-\tau_n}^n + |\bar{V}|^m \right] \\ &\quad + \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \left[\lambda^m T^{m/2} (\tau^{n+1} - \tau^n)^{m/2} \right. \\ &\quad \left. + \left(\int_{\tau^n}^{\tau^{n+1}} (|g_1|^2 + |g_2|^2)(X_r) dr \right)^{m/2} \right] \\ &\leq C \left[|Z_{\tau^n}|^m + \tilde{M}_{\tau_{n+1}-\tau_n}^n + |\bar{V}|^m \right] + 2C (A_{\tau_{n+1}} - A_{\tau^n})^{m/2} \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m. \end{aligned}$$

²In the sense that $\nabla\tilde{\sigma} = \psi_1 + \psi_2$ with $\psi^i \in L_t^{q_i}L_x^{p_i}$, cf. the statement of Lemma 2.9.

Taking $C = \tilde{K}$ as the desired constant in the definition of τ^n , we find

$$\sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \leq 2\tilde{K} \left[|Z_{\tau^n}|^m + \tilde{M}_{\tau^{n+1} - \tau^n}^n + |\bar{V}|^m \right];$$

using the fact that $(\tilde{M}_t^n)_{t \in [0, T]}$ is a $(\mathcal{G}_t)_{t \in [0, T]}$ -martingale, taking expectation we are left with

$$\mathbb{E} \left[\sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \right] \leq 2\tilde{K} \mathbb{E} \left[\sup_{t \in [\tau^{n-1}, \tau^n]} |Z_t|^m \right] + 2\tilde{K} \mathbb{E}[|\bar{V}|^m],$$

which iteratively yields (for an appropriately chosen constant $\kappa > 0$ in function of \tilde{K})

$$\mathbb{E} \left[e^{-\kappa n} \sup_{t \in [\tau^{n-1}, \tau^n]} |Z_t|^m \right] \leq |x_0 - y_0|^m + \mathbb{E}[|\bar{V}|^m]. \tag{2.14}$$

Now set $\Delta := (4\tilde{K})^{-2/m}$, $C := 2\kappa/\Delta$, then it holds

$$\begin{aligned} \mathbb{E} \left[e^{-CA_T} \sup_{t \in [0, T]} |Z_t|^m \right] &= \sum_{n \in \mathbb{N}} \mathbb{E} \left[e^{-CA_T} \sup_{t \in [0, T]} |Z_t|^m \mathbb{1}_{A_T \in [\Delta n, \Delta(n+1)]} \right] \\ &\leq \sum_{n \in \mathbb{N}} \mathbb{E} \left[e^{-2\kappa n} \sup_{j=1, \dots, n+1} \sup_{t \in [\tau_{j-1}, \tau_j]} |Z_t|^m \mathbb{1}_{A_T \in [\Delta n, \Delta(n+1)]} \right] \\ &\leq \sum_{n \in \mathbb{N}} e^{-\kappa n} \sum_{j=1}^{n+1} \mathbb{E} \left[e^{-\kappa n} \sup_{t \in [\tau_{j-1}, \tau_j]} |Z_t|^m \right] \\ &\lesssim |x_0 - y_0|^m + \mathbb{E}[|\bar{V}|^m], \end{aligned}$$

where in the last passage we applied (2.14). Observe that since $g^i \in L_t^{q_i} L_x^{p_i}$, $|g^i|^2 \in L_t^{q_i/2} L_x^{p_i/2}$ with $(q_i/2, p_i/2) \in \mathcal{J}_0$, thus by Lemma 2.3 the process A satisfies $\mathbb{E}[\exp(\lambda A_T)] < \infty$ for all $\lambda \in \mathbb{R}$; therefore for any $\delta \in (0, 1)$ it holds

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |Z_t|^{\delta m} \right] &\leq \mathbb{E} \left[e^{-CA_T} \sup_{t \in [0, T]} |Z_t|^m \right]^\delta \mathbb{E} \left[e^{C \frac{\delta}{1-\delta} A_T} \right]^{1-\delta} \\ &\lesssim_\delta |x_0 - y_0|^{\delta m} + \mathbb{E}[|\bar{V}|^m]^\delta. \end{aligned}$$

Taking the $(\delta m)^{-1}$ -power on both sides and relabelling $\tilde{m} = m\delta$, $\gamma\tilde{m} = m$, overall yields the desired result. \square

2.3 Comparison of SDEs with different coefficients

With Proposition 2.12 at hand, we are now ready to give the

Proof of Theorem 1.2. Let us set $X^1 = X$, $X^2 = Y$, $b^1 = b$, $\sigma^1 = \sigma$, $R_t^1 = (b^2 - b^1)_t (X_t^2)$ and $R_t^2 = (\sigma^2 - \sigma^1)_t (X_t^2)$ in the setting of Section 2.2; it is then clear by Section 2.1 that Assumption 2.10 holds and so by Proposition 2.12 our task is reduced to provide estimates for $\|\bar{V}\|_{L^{m\gamma}}$. The process V , as defined in (2.10), in this case can be decomposed as $V = I^1 + I^2 + I^3$ for

$$\begin{aligned} I_t^1 &:= \frac{1}{2} \int_0^t [(a^2 - a^1) : D^2 u]_r (X_r^2) dr, \\ I_t^2 &:= \int_0^t [(\sigma^2 - \sigma^1)(I + \nabla u)]_r (X_r^2) dW_r, \\ I_t^3 &:= \int_0^t [(b^2 - b^1) \cdot (I + \nabla u)]_r (X_r^2) dr, \end{aligned}$$

where $a^i = \sigma^i(\sigma^i)^*$ for $i = 1, 2$; we treat each term separately. From now on for simplicity we will write m in place of $m\gamma$, as it doesn't affect the computation.

Recall that, under Assumption 1.1, by Lemma 2.2 it holds $u \in L_t^{q_1} W_x^{2,p_1} \cap L_t^\infty C_x^1$; moreover $(p_1, q_1), (\tilde{p}_2, \tilde{q}_2) \in \mathcal{J}_1$, thus if we define $1/p' = 1/p_1 + 1/\tilde{p}_2$ and $1/q' = 1/q_1 + 1/\tilde{q}_2$, it holds $(p', q') \in \mathcal{J}_0$. By Hölder's inequality and estimate (2.7), we can then control I^1 by

$$\begin{aligned} \left\| \sup_{t \in [0, T]} |I_t^1| \right\|_{L_\omega^m} &\lesssim \|(a^2 - a^1) : D^2 u\|_{L_t^{q'} L_x^{p'}} \\ &\lesssim \|a^2 - a^1\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}} \|D^2 u\|_{L_t^{q_1} L_x^{p_1}} \\ &\lesssim \|\sigma^2 - \sigma^1\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}}, \end{aligned}$$

where in the final passage we also exploited the bound $\|\sigma^i\|_{L_{t,x}^\infty} \leq K$. A similar idea applies to I^2 , combined with BDG's inequality:

$$\begin{aligned} \left\| \sup_{t \in [0, T]} |I_t^2| \right\|_{L_\omega^m} &\lesssim \left\| \int_0^T |(\sigma_r^2 - \sigma_r^1)(I + \nabla u_r)|^2 (X_r^2) dr \right\|_{L_\omega^{m/2}}^{1/2} \\ &\lesssim \|\sigma^2 - \sigma^1\|_{L_t^{\tilde{q}_2/2} L_x^{\tilde{p}_2/2}}^{1/2} \|I + \nabla u\|_{L_t^\infty L_x^\infty} \\ &\lesssim \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}} \end{aligned} \tag{2.15}$$

where this time we applied (2.7) for $(\tilde{p}_2/2, \tilde{q}_2/2) \in \mathcal{J}_0$. Similarly, since $(\tilde{p}_1, \tilde{q}_1) \in \mathcal{J}_0$, it holds

$$\begin{aligned} \left\| \sup_{t \in [0, T]} |I_t^3| \right\|_{L_\omega^m} &\lesssim \|(b^2 - b^1) \cdot (I + \nabla u)\|_{L_t^{\tilde{p}_1} L_x^{\tilde{q}_1}} \\ &\lesssim \|b^2 - b^1\|_{L_t^{\tilde{p}_1} L_x^{\tilde{q}_1}} \|I + \nabla u\|_{L_t^\infty L_x^\infty} \\ &\lesssim \|b^2 - b^1\|_{L_t^{\tilde{p}_1} L_x^{\tilde{q}_1}}. \end{aligned} \tag{2.16}$$

In the case $q_2 = \infty$ and $4/q_1 + d/p_1 < 1$, we can give an alternative estimate for I^3 ; to this end, let us recall that by the product rules for distributions (see e.g. [25, Lemma A.2 (ii)]), for $p \geq d \vee 2$ it holds

$$\|fg\|_{W_x^{-1,p}} \lesssim \|f\|_{W_x^{-1,p}} \|g\|_{W_x^{1,p}} \quad \forall f \in W_x^{-1,p}, g \in W_x^{1,p}. \tag{2.17}$$

By Hölder's inequality and the second estimate in (2.7) (for $\tilde{q} = q_1/2, \tilde{p} = p_1$, which under our assumptions satisfy $(\tilde{q}, \tilde{p}) \in \mathcal{J}_1$), we then find

$$\begin{aligned} \left\| \sup_{t \in [0, T]} |I_t^3| \right\|_{L_\omega^m} &\lesssim \|(b^2 - b^1) \cdot (I + \nabla u)\|_{L_t^{q_1/2} W_x^{-1,p_1}} \\ &\lesssim \|b^2 - b^1\|_{L_t^{q_1} W_x^{-1,p_1}} \|u\|_{L_t^{q_1} W_x^{2,p_1}}. \end{aligned} \tag{2.18}$$

Overall, estimates (2.15), (2.15), combined with (2.16) (respectively (2.18)) and Proposition 2.12 yield (1.2) (respectively (1.3)). \square

We can also obtain stability estimates for Stratonovich SDEs.

Corollary 2.13. *Let Assumption 1.1 hold and consider X^i solutions to the Stratonovich SDEs*

$$dX_t^i = b_t^i(X_t^i) dt + \sigma_t^i(X_t^i) \circ dW_t, \quad X^i|_{t=0} = X_0^i.$$

Then for any $m \in [1, \infty)$ and any $(\tilde{p}_1, \tilde{q}_1), (\tilde{p}_2, \tilde{q}_2) \in \mathcal{J}_0$ and $(\tilde{p}_3, \tilde{q}_3) \in \mathcal{J}_1$, it holds

$$\begin{aligned} \left\| \sup_{t \in [0, T]} |X_t^1 - X_t^2| \right\|_{L_\omega^m} &\lesssim \|X_0^1 - X_0^2\|_{L_\omega^m} + \|b^1 - b^2\|_{L_t^{\tilde{q}_1} L_x^{\tilde{p}_1}} \\ &\quad + \|\nabla(\sigma^1 - \sigma^2)\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}} + \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_3} L_x^{\tilde{p}_3}}. \end{aligned} \tag{2.19}$$

If additionally $q_2 = \infty$ and $4/q_1 + d/p_1 < 1$, we have

$$\left\| \sup_{t \in [0, T]} |X_t^1 - X_t^2| \right\|_{L_\omega^m} \lesssim \|X_0^1 - X_0^2\|_{L_\omega^m} + \|b^1 - b^2\|_{L_t^{q_1} W_x^{-1, p_1}} + \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_3} L_x^{\tilde{p}_3}}. \quad (2.20)$$

Proof. The proof is very similar to that of Theorem 1.2, so we mostly sketch it. We can rewrite the Stratonovich SDEs into the corresponding Itô ones, with corresponding new drift terms \tilde{b}^i in place of b^i given by

$$\tilde{b}_t^i := b_t^i + \frac{1}{2}(\sigma^i \cdot \nabla)\sigma^i, \quad \text{where } [(\sigma \cdot \nabla)\sigma]_j = \sum_{k, l} \partial_l \sigma_{jk} \sigma_{lk}.$$

Estimates (2.19)-(2.20) then follow respectively from (1.2)-(1.3), once we estimate the difference $\tilde{b}^1 - \tilde{b}^2$ in appropriate norms. In order to obtain (2.19), we can use

$$(\sigma^1 \cdot \nabla)\sigma^1 - (\sigma^2 \cdot \nabla)\sigma^2 = (\sigma^1 \cdot \nabla)(\sigma^1 - \sigma^2) + [(\sigma^1 - \sigma^2) \cdot \nabla]\sigma^2$$

and then

$$\begin{aligned} \|(\sigma^1 \cdot \nabla)(\sigma^1 - \sigma^2)\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}} &\leq \|\sigma^1\|_{L_{t,x}^\infty} \|\nabla\sigma^1 - \nabla\sigma^2\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}}, \\ \|[(\sigma^1 - \sigma^2) \cdot \nabla]\sigma^2\|_{L_t^{q'} L_x^{p'}} &\leq \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_3} L_x^{\tilde{p}_3}} \|\nabla\sigma^2\|_{L_t^{q_2} L_x^{p_2}}, \end{aligned}$$

where we set $1/p' = 1/\tilde{p}_3 + 1/p_2$, $1/q' = 1/\tilde{q}_3 + 1/q_2$ so that $(p', q') \in \mathcal{J}_0$. Instead for obtaining (2.20), in the case $q_2 = \infty$ we can estimate the first term differently. To this end, let us recall that by [25, Lemma A.2 (iii)], whenever $p > d$ and $1/\bar{p} + 1/p < 1$, it holds

$$\|fg\|_{W_x^{-1, \bar{p}}} \lesssim (\|g\|_{L_x^\infty} + \|\nabla g\|_{L_x^p}) \|f\|_{W_x^{-1, \bar{p}}} \quad \forall f \in W_x^{-1, p}, g \in W_x^{1, p}. \quad (2.21)$$

Applying estimate (2.21) for the choice $p = p_2$ and $\bar{p} = \tilde{p}_3$, we obtain

$$\begin{aligned} \|\sigma^1 \cdot \nabla(\sigma^1 - \sigma^2)\|_{L_t^{\tilde{q}_3} W_x^{-1, \tilde{p}_3}} &\lesssim [\|\sigma^1\|_{L_{t,x}^\infty} + \|\nabla\sigma^1\|_{L_t^\infty L_x^p}] \|\nabla(\sigma^1 - \sigma^2)\|_{L_t^{\tilde{q}_3} W_x^{-1, \tilde{p}_3}} \\ &\lesssim \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_3} L_x^{\tilde{p}_3}}. \end{aligned}$$

Combined with Proposition 2.12 and estimates (2.15), (2.15), (2.16) and (2.18), this yields the conclusion. \square

Remark 2.14. Theorem 1.2 and Corollary 2.13 admit several other variants, in the sense of employing other quantities of the form $\|b^1 - b^2\|_{L_t^{\tilde{q}_1} W_x^{-\beta, \tilde{p}_1}}$ in the r.h.s. of (1.3) and (2.20). Indeed going through the proof, the only relevant difference comes from estimating the term I^3 related to $\int_0^\cdot [(b^2 - b^2) \cdot (I + \nabla u)]_r(X_r^2) dr$, which can be accomplished as in Remark 2.7 under the assumption $q_2 = \infty$. There are two different regimes, related to the parameters (β, q_1, p_1) :

- $\beta < 1 - 2/q_1$. In this case, since $(p_1, q_1) \in \mathcal{J}_1$, we can assume β to satisfy $\beta > d/p_1$ as well (this proves a stronger estimate, as for $\tilde{\beta} \leq \beta$ one can use $W_x^{-\tilde{\beta}, p} \hookrightarrow W_x^{-\beta, p}$). By Lemma 2.2 and standard interpolation techniques, it is easy to show that $\nabla u \in L_t^\infty W_x^{\beta, p_1}$ and one can then apply [25, Lemma A.2 (ii)] to obtain

$$\|(b^2 - b^1) \cdot (I + \nabla u)\|_{L_t^{\tilde{q}_1} W_x^{-\beta, \tilde{p}_1}} \lesssim \|b^1 - b^2\|_{L_t^{\tilde{q}_1} W_x^{-\beta, \tilde{p}_1}} (1 + \|\nabla u\|_{L_t^\infty W_x^{\beta, p_1}}),$$

eventually leading to

$$\left\| \sup_{t \in [0, T]} |X_t^1 - X_t^2| \right\|_{L_\omega^m} \lesssim \|X_0^1 - X_0^2\|_{L_\omega^m} + \|b^1 - b^2\|_{L_t^{\tilde{q}_1} W_x^{-\beta, \tilde{p}_1}} + \|\sigma^1 - \sigma^2\|_{L_t^{\tilde{q}_2} L_x^{\tilde{p}_2}} \quad (2.22)$$

for any choice of parameters $(\tilde{p}_1, \tilde{q}_1) \in \mathcal{J}_\beta$. This is in agreement with (1.2), which is recovered for $\beta = 0$.

2. $\beta > 1 - 2/q_1$. In this case we only know that $\nabla u \in L_t^{\bar{q}} W^{\beta, p_1}$ for $1/\bar{q} = 1/q - (1 - \beta)/2$; combined again with [25, Lemma A.2 (ii)], this implies $(b^2 - b^1) \cdot (I + \nabla u) \in L_t^{q'} W_x^{-\beta, \tilde{p}_1}$ for $1/q' = 1/\bar{q} + 1/\tilde{q}_1$ and the relevant condition becomes $(\tilde{p}_1, q') \in \mathcal{J}_\beta$. Overall we still obtain a stability estimate where $\|b^1 - b^2\|_{L_t^{\tilde{q}_1} W_x^{-\beta, \tilde{p}_1}}$ appears, but now only under the more restrictive condition

$$\frac{2}{q_1} + \frac{2}{\tilde{q}_1} + \frac{d}{\tilde{p}_1} < 3 - 2\beta, \quad \frac{d}{p_1} < \beta. \tag{2.23}$$

Condition (2.23) agrees with $4/q_1 + d/p_1 < 1$ needed to derive (1.3), which corresponds to $\beta = 1$, $\tilde{p}_1 = p_1$ and $\tilde{q}_1 = q_1$.

3 Applications

3.1 McKean-Vlasov equations

Throughout this section, for any $t \in [0, T]$ and any X , we will denote by X^t the stopped process $X_s^t := X_{t \wedge s}$, which we ask the reader not to confuse with the pointwise evaluation X_t . We consider McKean-Vlasov (or distribution dependent) SDEs of the form

$$dX_t = B_t(X_t, \mu_t) dt + \Sigma_t(X_t, \mu_t) dW_t, \quad \mu_t = \mathcal{L}(X^t) \tag{3.1}$$

for suitable coefficients B, Σ , see Assumption 3.1 below.

In the case of regular (e.g. Lipschitz) continuous functions B, Σ , equation (3.1) can be solved classically; however in the presence of a singular drift B , several important results in the literature guaranteeing existence and uniqueness for (3.1) require the diffusion term Σ not depend on μ , see e.g. [29, Section 3.2], or anyway more restrictive assumption on Σ , cf. [31, Section 4]; the reason lies in the use of transport-cost inequalities and Girsanov transform, see also [24]. On the other hand there is another class of results, including [16, 17], possibly involving the use of mixed Wasserstein-total variation distances, which naturally allows Σ to depend on μ ; the shortcoming is a more restrictive assumption on B and Σ , e.g. they must be Lipschitz in μ uniformly in x , which rules out basic cases of interest like $B(x, \mu) = b * \mu$ for unbounded b . Let us stress that the literature on the topic is quite vast and there are many other works not properly fitting in this dichotomy, see for instance [15, 30, 19]; see also the papers [26, 23] where, under certain assumptions on the drifts, the authors are additionally able to establish convergence of the particle system, as well as develop numerical schemes with explicit rates of convergence. Finally, let us mention the remarkable work [42] for existence and uniqueness results, in the presence of μ -dependent Σ , under yet another set of assumptions.

Here we do not follow any of the aforementioned approaches, instead we derive existence and uniqueness results for (3.1) by readapting the strategy from [11], based on the use of stability estimates in negative Sobolev spaces and Wasserstein distances. Our assumptions cover the basic convolutional case $B(x, \mu) = b * \mu$ (cf. Corollary 3.5), but also allow for Σ to depend on μ , as well as a more general abstract class of coefficients B and Σ with nonlinear dependence on μ (cf. Assumption 3.1).

In the following, given a separable Banach space E , we denote by $\mathcal{P}(E)$ the set of probability measures on E and by W_m the associated m -Wasserstein distance, namely

$$W_m(\mu, \nu) = \inf \left\{ \left(\int \|x - y\|_E^m \pi(dx, dy) \right)^{1/m} : \pi \in \Pi(\mu, \nu) \right\}$$

where $\Pi(\mu, \nu) \subset \mathcal{P}(E \times E)$ denotes the set of couplings of (μ, ν) . We refer to [36] for a complete survey on the properties of W_m ; let us only recall that $W_m(\mu, \nu) \leq W_{\tilde{m}}(\mu, \nu)$

whenever $m \leq \tilde{m}$ and that for any $m \in [1, \infty)$, $\mu, \nu \in \mathcal{P}(E)$ there exists an optimal coupling for $W_m(\mu, \nu)$, which can be realised as a pair of E -valued random variables (Y, Z) such that $\mathcal{L}(Y) = \mu$, $\mathcal{L}(Z) = \nu$ and $W_m(\mu, \nu) = \mathbb{E}[\|Y - Z\|_E^m]^{1/m}$. We will mainly consider E as the path space $C_T := C([0, T]; \mathbb{R}^d)$ or $E = \mathbb{R}^d$.

We are now ready to introduce our assumptions on the coefficients B, Σ .

Assumption 3.1. *There exist $p_1, p_2 \in (d, \infty)$, $m \in [1, \infty)$ and $C > 0$ such that the maps $B : [0, T] \times \mathbb{R}^d \times \mathcal{P}(C_T) \rightarrow \mathbb{R}^d$, $\Sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}(C_T) \rightarrow \mathbb{R}^{d \times d}$ satisfy the following:*

- i) $\|B_t(\cdot, \mu)\|_{L_x^{p_1}} \leq C$, $\|B_t(\cdot, \mu) - B_t(\cdot, \nu)\|_{W_x^{-1, p_1}} \leq C W_m(\mu, \nu)$ for all $t \in [0, T]$, $\mu, \nu \in \mathcal{P}(C_T)$.
- ii) $\|\nabla \Sigma_t(\cdot, \mu)\|_{L_x^{p_2}} \leq C$, $\|\Sigma_t(\cdot, \mu) - \Sigma_t(\cdot, \nu)\|_{L_x^{p_2}} \leq C W_m(\mu, \nu)$ for all $t \in [0, T]$, $\mu, \nu \in \mathcal{P}(C_T)$.
- iii) $C^{-1}|\xi|^2 \leq |\Sigma(t, x, \mu)^* \xi|^2 \leq C|\xi|^2$ for all $\xi \in \mathbb{R}^d$, $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}(C_T)$.

Under Assumption 3.1, one can give meaning to equation (3.1) as follows: given a continuous adapted process Y , we can define $\mu_t = \mathcal{L}(Y^t)$ and then

$$b_t^Y(x) := B_t(x, \mu_t), \quad \sigma_t^Y(x) := \Sigma_t(x, \mu_t); \tag{3.2}$$

it can be readily checked that (b^Y, σ^Y) satisfy Assumption 1.1³, thus the associated SDE is wellposed. We then define X to be a solution to (3.1) if and only if it is a solution to the SDE associated to (b^X, σ^X) ; the concepts of strong existence, pathwise uniqueness and uniqueness in law for (3.1) can then be readapted similarly. Let us finally recall that, as usual for McKean–Vlasov SDEs, we will consider the initial datum X_0 to be random and independent of W ; depending on how one wants to formulate it, the data of the problem can then be considered equivalently (X_0, B, Σ) or (μ_0, B, Σ) , where $\mu_0 = \mathcal{L}(X_0)$.

Theorem 3.2. *Let B, Σ satisfy Assumption 3.1 for some $m \in [1, \infty)$. Then for any $\mu_0 \in \mathcal{P}_m(\mathbb{R}^d)$ strong existence, pathwise uniqueness and uniqueness in law holds for the SDE (3.1). A similar statement holds for the Stratonovich version of (3.1), i.e. with $\Sigma_t(X_t, \mu_t) dW_t$ replaced by $\Sigma_t(X_t, \mu_t) \circ dW_t$.*

Proof. We give the proof in the Itô case, the Stratonovich one being identical (up to applying Corollary 2.13 in place of Theorem 1.2). Standard arguments involving the structure of eq. (3.1) (see for instance [10, Propositions 3.5-3.7]) allow to show that any weak solution is a strong one and that pathwise uniqueness implies uniqueness in law; thus our task is reduced to show existence and uniqueness among strong solutions. Consider the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$ generated by (X_0, W) and let E be the space of continuous, \mathcal{F} -adapted paths $Y \in L_\omega^m C_T$, endowed with the metric

$$d_E(Y, Z) := \sup_{t \in [0, T]} e^{-\lambda t} \mathbb{E} \left[\sup_{s \leq t} |Y_s - Z_s|^m \right]^{1/m} = \sup_{t \in [0, T]} e^{-\lambda t} \|Y^t - Z^t\|_{L_\omega^m C_T}$$

for a suitable $\lambda > 0$ to be chosen later; define a map \mathcal{I} from E to itself, which to a given process Y associates the unique strong solution $\mathcal{I}(Y)$ to the SDE associated to (X_0, b^Y, σ^Y) . It is then clear that X is a solution to (3.1) if and only if $X = \mathcal{I}(X)$ and that our task is reduced to showing that \mathcal{I} is a contraction on (E, d) , for suitable choice of λ . To this end, let us fix a parameter $q \in (2, \infty)$ sufficiently large so that $4/q + d/p_1 < 1$,

³Since $\nabla \sigma^Y \in L_x^p$ for some $p > d$, Morrey's inequality implies that it is α -Hölder in space for $\alpha = 1 - d/p$, uniformly in time; thus conditions concerning the modulus of continuity of σ^X are automatically satisfied.

Stability estimates

$2/q + d/p_2 < 1$; then by estimate (1.3) for the choice $(\tilde{p}_2, \tilde{q}_2) = (p_2, q)$ and Assumption 3.1, for any $t \in [0, T]$ it holds

$$\begin{aligned} \|\mathcal{I}(Y)^t - \mathcal{I}(Z)^t\|_{L^\infty_{\omega} C_T}^q &\lesssim \|b^Y - b^Z\|_{L^q_t W_x^{-1, p_1}}^q + \|\sigma^Y - \sigma^Z\|_{L^q_t L^{p_2}}^q \\ &\lesssim \int_0^t \mathbb{W}_m(\mathcal{L}(Y^s), \mathcal{L}(Z^s))^q ds \\ &\lesssim \int_0^t e^{\lambda qs} d_E(Y, Z)^q ds \lesssim \lambda^{-1} e^{\lambda qt} d_E(Y, Z)^q; \end{aligned}$$

overall, this implies the existence of another constant $\tilde{C} > 0$ such that

$$d_E(\mathcal{I}(Y), \mathcal{I}(Z)) \leq \tilde{C} \lambda^{-1/q} d_E(Y, Z),$$

which implies contractivity of \mathcal{I} once we choose λ large enough. \square

We now pass to provide sufficient conditions for Assumption 3.1 in a practical case of interest, given by *convolutional drifts*; in particular here we take $\mu_t = \mathcal{L}(X_t) \in \mathcal{P}(\mathbb{R}^d)$ and consider maps B, Σ with linear dependence on μ by means of a convolution.

Lemma 3.3. *For any $p \in [1, \infty]$, $f \in L^p_x$ and g such that $\nabla g \in L^p_x$, it holds*

$$\|f * (\mu - \nu)\|_{W_x^{-1, p}} \lesssim \|f\|_{L^p_x} \mathbb{W}_1(\mu, \nu), \quad \|g * (\mu - \nu)\|_{L^p_x} \lesssim \|\nabla g\|_{L^p_x} \mathbb{W}_1(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{P}(\mathbb{R}^d).$$

Proof. Let us first prove the claim for g . Recall the basic fact that, for any $h_i \in \mathbb{R}^d$, it holds

$$\|g(\cdot + h_1) - g(\cdot + h_2)\|_{L^p_x} \leq \|\nabla g\|_{L^p_x} |h_1 - h_2|.$$

Let (Y, Z) be an optimal coupling for (μ, ν) , then by definition of convolution it holds

$$\begin{aligned} \|g * (\mu - \nu)\|_{L^p_x} &= \|\mathbb{E}[g(\cdot - Y) - g(\cdot - Z)]\|_{L^p_x} \leq \mathbb{E}[\|g(\cdot - Y) - g(\cdot - Z)\|_{L^p_x}] \\ &\leq \|\nabla g\|_{L^p_x} \mathbb{E}[|Y - Z|] = \|\nabla g\|_{L^p_x} \mathbb{W}_1(\mu, \nu). \end{aligned}$$

The case of f then follows from a duality argument. Observe that $\langle f * (\mu - \nu), g \rangle = \langle f, g * (\tilde{\mu} - \tilde{\nu}) \rangle$, where $\tilde{\mu}(A) = \mu(-A)$ is the reflection of μ ; it is clear that $\mathbb{W}_1(\tilde{\mu}, \tilde{\nu}) = \mathbb{W}_1(\mu, \nu)$, therefore

$$|\langle f * (\mu - \nu), g \rangle| \leq \|f\|_{L^p_x} \|g * (\tilde{\mu} - \tilde{\nu})\|_{L^{p'}} \leq \|f\|_{L^p_x} \|g\|_{W_x^{1, p'}} \mathbb{W}_1(\mu, \nu),$$

which yields the conclusion by taking the supremum over all $g \in W_x^{1, p'}$ with $\|g\|_{W_x^{1, p'}} = 1$. \square

Lemma 3.4. *Let σ satisfy conditions (\mathcal{H}_1^σ) - (\mathcal{H}_3^σ) from Assumption 1.1 with $q_2 = \infty$, $p_2 > d$ and constant K and let $\pi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ satisfy*

$$\nabla \pi \in L_t^\infty L_x^{p_2}, \quad \sup_{t, x} |\pi_t(x)| \leq (1 - \delta) K^{-1/2}$$

*for some $\delta \in (0, 1)$. Then $\Sigma_t(x, \mu) := \sigma_t(x) - (\pi_t * \mu)(x)$ satisfies conditions ii)-iii) from Assumption 3.1.*

Proof. By standard properties of convolution, for any $\mu \in \mathcal{P}(\mathbb{R}^d)$ it holds

$$\|\nabla(\pi_t * \mu)\|_{L_x^{p_2}} \leq \|\nabla \pi_t\|_{L_x^{p_2}} \leq \|\nabla \pi\|_{L_t^\infty L_x^{p_2}},$$

which immediately implies a uniform bound on $\|\nabla \Sigma_t(\cdot, \mu)\|_{L_x^{p_2}}$; a similar argument shows uniform boundedness of $\Sigma_t(x, \mu)$. By our assumptions and triangular inequality, it holds

$$|\Sigma_t(x, \mu)\xi| \geq |\sigma_t(x)\xi| - \|\pi * \mu\|_{L_{t,x}^\infty} |\xi| \geq |\sigma_t(x)\xi| - \|\pi\|_{L_{t,x}^\infty} |\xi| \geq \delta K^{-1/2} |\xi|$$

which shows uniform ellipticity. \square

Combining Lemma 3.3 and Lemma 3.4 immediately implies the following

Corollary 3.5. *Let $b \in L_t^\infty L_x^{p_1}$ for some $p_1 \in (d, \infty)$ and let (σ, π) satisfy the hypothesis of Lemma 3.4; consider*

$$B_t(X_t, \mathcal{L}(X_t)) := (b_t * \mathcal{L}(X_t))(X_t), \quad \Sigma_t(X_t, \mathcal{L}(X_t)) := \sigma_t(X_t) - (\pi_t * \mathcal{L}(X_t))(X_t).$$

Then strong wellposedness holds for the associated McKean–Vlasov SDE (3.1).

Remark 3.6. It is clear that other variants of Theorem 3.2 are allowed; for instance Assumption 3.1-i) can be modified by requiring the existence of parameters (q_1, p_1) satisfying $4/q_1 + d/p_1 < 1$ and a function $\ell \in L_t^{q_1}$ such that

$$\|B_t(\cdot, \mu)\|_{L_x^{p_1}} \leq \ell_t, \quad \|B_t(\cdot, \mu) - B_t(\cdot, \nu)\|_{W_x^{-1, p_1}} \leq \ell_t \mathbb{W}_m(\mu, \nu).$$

In the style of Remark 2.14, one could also formulate a condition on B involving $W_x^{-\nu, p_1}$ -norms for other values $\nu \in [0, 1]$, although the choice $\nu = 1$ is natural in view of Lemma 3.3.

Remark 3.7. Unfortunately, our result in general does not cover the choice $\Sigma_t(x, \mu) := (\sigma_t * \mu)(x)$; the problem is that, even if σ is uniformly elliptic, the same doesn't need to hold for $\sigma * \mu$, as cancellations might happen inside the convolution. This difficulty can be overcome under more information on σ ; for instance if $\sigma_t(x) = \gamma_t(x)A$, where $\gamma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and A is a fixed, uniformly elliptic $\mathbb{R}^{d \times d}$ matrix, then the continuity and uniform ellipticity requirements on σ imply that $\gamma_t(x) \geq \delta > 0$ (or $\gamma_t(x) \leq -\delta < 0$, which doesn't change the analysis). Due to the positivity of μ , we then find

$$|(\sigma_t * \mu)^*(x)\xi| = |\gamma_t * \mu(x)| |A^*\xi| \geq \delta |A^*\xi|,$$

showing again uniform ellipticity of $\Sigma_t(x, \mu) = (\sigma_t * \mu)(x)$.

Remark 3.8. Other examples of B and Σ , satisfying conditions similar to Assumption 3.1, are discussed in [11, Section 2.1]. Among them let us mention the case of dependence on statistics of μ , e.g. $B_t(x, \mu) = b_t(x - \langle \phi, \mu \rangle)$ where $\langle \phi, \mu \rangle := \int_{C_T} \phi(\omega) \mu(\delta\omega)$ and $\phi : C_T \rightarrow \mathbb{R}^d$ is a uniformly Lipschitz map. Indeed, by properties of translations in Sobolev spaces and Kantorovich–Rubinstein duality, it holds

$$\|b_t(\cdot - \langle \phi, \mu \rangle) - b_t(\cdot - \langle \phi, \nu \rangle)\|_{W_x^{-1, p}} \lesssim \|b_t\|_{L_x^p} |\langle \phi, \mu - \nu \rangle| \lesssim \|b_t\|_{L_x^p} \|\phi\|_{Lip} \mathbb{W}_1(\mu, \nu).$$

3.2 Strong compactness of solutions to singular SDEs

We show here that, under quite minimal assumptions, the solutions to SDEs of the form (1.1), driven by the same reference Brownian motion W , form a compact set in the strong topology of $L_\omega^m C_T$. This is much stronger than standard tightness results for $\mathcal{L}(X)$.

Throughout this section we will assume for simplicity $X_0 = x \in \mathbb{R}^d$ to be deterministic and fixed, so the dependence of the solution on it will not appear in the sequel. Given a pair (b, σ) satisfying Assumption 1.1, we denote by $X^{b, \sigma}$ the associated solution to the Itô SDE (1.1) with initial condition x ; similarly, $\tilde{X}^{b, \sigma}$ denotes the solution to the corresponding Stratonovich SDE.

We start by dealing with autonomous drift and diffusion; thus we fix some parameters $p_1, p_2 \in (d, \infty)$ corresponding to $b \in L_x^{p_1}, \nabla \sigma \in L_x^{p_2}$.⁴

Theorem 3.9. *Consider a family of coefficients $\mathcal{A} \subset \{(b, \sigma) : b \in L_x^{p_1}, \sigma \in L_x^\infty, \nabla \sigma \in L_x^{p_2}\}$ satisfying Assumption 1.1 in a uniform way; namely, there exist constants $M, K > 0$ such*

⁴As in Section 3.1, $\nabla \sigma \in L_x^{p_2}$ implies σ being globally Hölder, so that we can drop assumption (\mathcal{H}_2^c) and the dependence on a modulus of continuity h .

that for all $(b, \sigma) \in \mathcal{A}$ it holds $\|b\|_{L_x^{p_1}} + \|\sigma\|_{L_x^\infty} + \|\nabla\sigma\|_{L_x^{p_2}} \leq M$ and $|\sigma^*\xi| \geq K^{-1}|\xi|^2$ for all $\xi \in \mathbb{R}^d$. Then the families $\{X^{b,\sigma} : (b, \sigma) \in \mathcal{A}\}$ and $\{\tilde{X}^{b,\sigma} : (b, \sigma) \in \mathcal{A}\}$ are precompact in $L_\omega^m C_T$, for all $m \in [1, \infty)$.

Proof. We focus on the Itô case, the Stratonovich one being almost identical up to applying Corollary 2.13 instead of Theorem 1.2. By well-known results concerning the SDE (1.1) and our assumptions, the solutions $X^{b,\sigma}$ satisfy uniform moment estimates, namely for all $m \in [1, \infty)$ it holds $\sup_{(b,\sigma) \in \mathcal{A}} \|X^{b,\sigma}\|_{L_\Omega^m C_T} < \infty$; thus it suffices to show precompactness w.r.t. convergence in probability.

Given any sequence $(b^n, \sigma^n)_n \subset \mathcal{A}$, by the assumptions and weak compactness we can extract a (not relabelled) subsequence such that b^n converge weakly in $L_x^{p_1}$ to some b and σ^n converge weakly-* in L_x^∞ to some σ , with the additional property that $\nabla\sigma^n$ converge weakly to $\nabla\sigma$ in $L_x^{p_2}$. We claim that X^{b^n, σ^n} converge in probability to $X^{b,\sigma}$, which yields the conclusion in the Itô case.

From now on we will adopt the shorter notation $X^n = X^{b^n, \sigma^n}$, $X = X^{b,\sigma}$. To show the claim, let us fix some $\delta > 0$ and choose $R > 0$ large enough so that $\mathbb{P}(\|X^n\|_{C_T} > R) < \delta$, $\mathbb{P}(\|X\|_{C_T} > R) < \delta$, which is possible by the uniform moment estimates; next, let us choose $\psi \in C_c^\infty$ such that $\psi \equiv 1$ on B_R and $\psi \equiv 0$ on the complement of B_{R+1} , where $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$. Denote by \bar{X}^n the solutions associated to $(\psi b^n, \psi \sigma^n)$, similarly \bar{X} . Clearly ψb^n converge weakly to ψb , which together with the compact embedding $L_x^{p_1} \hookrightarrow W_{x,loc}^{-1,p_1}$ and the presence of the cutoff ψ implies $\|\psi b^n - \psi b\|_{W_x^{-1,p_1}} \rightarrow 0$ as $n \rightarrow \infty$; similarly $\|\psi \sigma^n - \psi \sigma\|_{L_x^{p_2}} \rightarrow 0$, as the sequence $\{\psi \sigma^n\}_n$ is compact in $L_x^{p_2}$ by Rellich-Kondrachov theorem and it is weakly converging to $\psi \sigma$. By Theorem 1.2 we then deduce that $\bar{X}^n \rightarrow \bar{X}$ in $L_\omega^m C_T$. On the other hand, by construction \bar{X}^n and X^n coincide as long as X^n do not get outside B_R , similarly for X , thus we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(\|X^n - X\| > \epsilon) &\leq \lim_{n \rightarrow \infty} \mathbb{P}(\|X^n\| > R) + \mathbb{P}(\|X\| > R) + \mathbb{P}(\|\bar{X}^n - \bar{X}\| > \epsilon) \\ &\leq 2\delta + \lim_{n \rightarrow \infty} \mathbb{P}(\|\bar{X}^n - \bar{X}\| > \epsilon) = 2\delta. \end{aligned}$$

By the arbitrariness of $\delta > 0$, the conclusion follows. □

In the case of time-dependent coefficients, we cannot go through the same proof, since local compactness in weak topologies is not obvious anymore; however, under mild additional time regularity assumptions, we can invoke the partial compactness results from [32, Section 9]. In the next statement, we fix $(p_1, q_1) \in \mathcal{J}_1$ and consider $q_2 = \infty$, $p_2 \in (d, \infty)$.

Corollary 3.10. Consider a family $\mathcal{A} \subset \{(b, \sigma) : b \in L_t^{q_1} L_x^{p_1}, \sigma \in L_{t,x}^\infty, \nabla\sigma \in L_t^\infty L_x^{p_2}\}$ with the following property: there exist some parameters $s > 0$ and $N \in \mathbb{N}$, as well as constants $M, K > 0$ such that for all $(b, \sigma) \in \mathcal{A}$ it holds ⁵

$$\|b\|_{L_t^{q_1} L_x^{p_1}} + \|b\|_{W_t^{s,1} W_x^{-N,p_1}} + \|\sigma\|_{L_{t,x}^\infty} + \|\nabla\sigma\|_{L_t^\infty L_x^{p_2}} + \|\sigma\|_{W_t^{s,1} W_x^{-N,p_2}} \leq M$$

and $|\sigma^*\xi| \geq K^{-1}|\xi|^2$ for all $\xi \in \mathbb{R}^d$. Then the families $\{X^{b,\sigma} : (b, \sigma) \in \mathcal{A}\}$ and $\{\tilde{X}^{b,\sigma} : (b, \sigma) \in \mathcal{A}\}$ are precompact in $L_\omega^m C_T$, for all $m \in [1, \infty)$.

Proof. Introducing the cutoff ψ as in the proof of Theorem 3.9, we can always assume (b, σ) to be compactly supported on B_R ; thus the only relevant issue is to show the

⁵Here we follow the notation from [32]; for a Banach space E , the space $W_t^{s,1} E = W^{s,1}(0, T; E)$ is defined as the set of integrable functions $f : [0, T] \rightarrow E$ such that

$$\|f\|_{W_t^{s,1} E} := \|f\|_{L_t^1 E} + \llbracket f \rrbracket_{W_t^{s,1} E} < \infty, \quad \text{where} \quad \llbracket f \rrbracket_{W_t^{s,1} E} := \int_{[0,T]^2} \frac{\|f_r - f_u\|_E}{|r-u|^{1+s}} dr du.$$

existence of a subsequence (b^n, σ^n) converging to (b, σ) in a suitable weak topology, which still has to be strong enough in order for our stability estimates to apply. For any $\beta > 0$, we can apply [32, Corollary 7] for the choice $X = L_x^{p_1}$, $B = W_x^{-\beta, p_2}$ and $Y = W_x^{-\beta-N, p_2}$ to deduce the existence of a subsequence such that $b^n \rightarrow b$ in $L_t^{\tilde{q}_1} W_x^{-\beta, p_1}$ for any $\tilde{q}_1 < q_1$, where $b \in L_t^{q_1} L_x^{p_1}$; similarly $\sigma^n \rightarrow \sigma$ in $L_t^{\tilde{q}_2} L_x^{p_2}$ for any $\tilde{q}_2 < \infty$, where σ still satisfies the same bounds as the elements from \mathcal{A} . In particular, the SDE associated to (b, σ) is well defined, with solution $X^{b, \sigma}$. Since $(p_1, q_1) \in \mathcal{J}_1$ and $p_2 > d$, we can always find \tilde{q}_1, \tilde{q}_2 and β satisfying the above and such that additionally $\beta < 1 - 2/q_1$, $(\tilde{q}_1, p_1) \in \mathcal{J}_\beta$ and $(\tilde{q}_2, p_2) \in \mathcal{J}_1$, which by the stability estimate (2.22) implies $X^{b^n, \sigma^n} \rightarrow X^{b, \sigma}$ and thus the conclusion in the Itô case.

The reasoning for Stratonovich SDEs is almost identical, but we give some additional details since we haven't precisely stated a version of (2.22) in this case. By rewriting the SDE in the corresponding Itô form, it suffices to verify that $\sigma^n \cdot \nabla \sigma^n$ converge strongly to $\sigma \cdot \nabla \sigma$ in suitable topologies; by [32, Corollary 7], we can show $\sigma^n \rightarrow \sigma$ in $L_t^{\tilde{q}_2} W_x^{\beta, p_2}$ for any $\beta < 1$, thus $\nabla \sigma^n \rightarrow \nabla \sigma$ in $L_t^{\tilde{q}_2} W_x^{\beta-1, p_2}$ and by (2.17) we can deduce that $\sigma^n \cdot \nabla \sigma^n \rightarrow \sigma \cdot \nabla \sigma$ in $L_t^{\tilde{q}_2/2} W_x^{\beta-1, p_2}$. As the argument holds for any $\tilde{q}_2 < \infty$ and any $\beta < 1$ and by assumption $p_2 > d$, we can always choose such parameters in a way that $(\tilde{q}_2/2, p_2) \in \mathcal{J}_{1-\beta}$, yielding the conclusion again by an application of (2.22). \square

Remark 3.11. We conjecture that the time regularity condition, in the form of a uniform bound in $W_t^{s,1} W_x^{-N,p}$, is not needed in order to derive the strong compactness result of Corollary 3.10. In order to avoid it, one would need a stability estimate where the difference of the drifts is measured in a negative norm of both time and space, e.g. $\|b^1 - b^2\|_{W_t^{-s, q_1} W_x^{-\beta, p_1}}$. In this direction, let us mention that it is possible to obtain estimates for integrals of the form $\int_t^r f_r(X_r) dr$, by means of *nonlinear Young integrals* (cf. [5, 14]), for instance when $f \in C_t^{-\gamma} C_x^1$ for $\gamma < 1/2$; combining this result with Lemma 2.5, giving instead estimates when f has positive regularity in time but negative in space, and interpolation techniques, there is some hope to achieve estimates for f enjoying (possibly very small) negative regularity in both variables. We leave this problem for future investigations.

3.3 Improved Wong-Zakai theorem

In this section we apply our stability results in the context of Wong-Zakai approximation of singular SDEs. The main difficulty lies in the fact that, due to the singularity of the drift b , the SDE associated to a smooth approximation W^n in place of W does not need to be wellposed, making even the correct formulation of the result unclear. To overcome this difficulty, we follow the strategy introduced in [27], based on first replacing the coefficients (b, σ) by some smooth approximations of them and then applying the Wong-Zakai result therein; one thus needs to control the overall error committed in this two-step approximation. Compared to [27], we improve the result from by not only allowing b to be singular, but also considering σ only Sobolev differentiable and not C_x^2 .

We start by introduction some basic notations and conventions for this section.

For simplicity, here we will only consider autonomous coefficients (b, σ) satisfying Assumption 1.1, as well as deterministic initial data $x_0 \in \mathbb{R}^d$. We always work on the Wiener space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, that is to say, $\Omega := \{\omega \in \mathcal{C}([0, \infty), \mathbb{R}^d) : \omega(0) = 0\}$, \mathbb{P} is the Wiener measure defined on the Borel σ -algebra \mathcal{F} of Ω , $W_t(\omega) := \omega(t)$, $t \geq 0$ is a Wiener process and $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by $(W_t)_{t \geq 0}$.

We shall consider the following class of approximations:

Definition 3.12. [18, VI. Definition 7.1] *By an approximation of the Wiener process $(W_t)_{t \in [0, T]}$, we mean a family $\{W^n\}_{n \geq 0}$ of d -dimensional continuous processes defined on the Wiener space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ such that*

Stability estimates

- (1) for every $\omega \in \Omega$, $t \mapsto W_t^n(\omega)$ is continuous and piecewise continuously differentiable;
- (2) $W_0^n(\omega)$ is $\mathcal{F}_{1/n}$ -measurable and $\mathbb{E}W_0^n = 0$;
- (3) $W_{t+k/n}^n(\omega) = W_t^n(\theta_{k/n}\omega) + \omega(k/n)$, for every $k \in \mathbb{N}$, $t \geq 0$ and $\omega \in \Omega$, where θ_t is the shift operator defined by $(\theta_t\omega)(s) = \omega(t+s) - \omega(t)$;
- (4) there exists a positive constant C such that

$$\mathbb{E}[|W_0^n|^6] \leq Cn^{-3}, \quad \mathbb{E}\left(\int_0^{1/n} |\dot{W}_s^n|^6 ds\right)^6 \leq Cn^{-3}.$$

From [18, VI. Section 7] we know that for such a approximation sequence W^n we have

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[\sup_{0 \leq t \leq T} |W_t - W_t^n|^2\right] = 0$$

for every $T > 0$. We further introduce the following notations for $t > 0$ and $i, j \in \{1, \dots, d\}$:

$$s_{ij}^n(t) := s_{ij}(t, n) := \frac{1}{2t} \mathbb{E}\left[\int_0^t (W_{s,i}^n \dot{W}_{s,j}^n - W_{s,i}^n \dot{W}_{s,i}^n) ds\right] \quad (3.3)$$

and

$$c_{ij}^n(t) := c_{ij}(t, n) := \frac{1}{t} \mathbb{E}\left[\int_0^t \dot{W}_{s,i}^n (W_{t,j}^n - W_{s,j}^n) ds\right]. \quad (3.4)$$

Notice that $s(t, n)$ is a skew-symmetric $d \times d$ -matrix for each t and n , i.e. $s_{ij}(t, n) = -s_{ji}(t, n)$. In order to control the convergence rate of the solutions X^n driven by W^n to X driven by W , we will impose the following:

Assumption 3.13. *There exists a skew-symmetric $d \times d$ -matrix s and a rate function f_n such that*

$$|s_{ij}^n(n^{-1}) - s_{ij}| \leq f_n, \quad \lim_{n \rightarrow \infty} f_n = 0, \quad i, j = 1, \dots, d.$$

Under Assumption 3.13, we define

$$c_{ij} := s_{ij} + \frac{1}{2}\delta_{ij}, \quad i, j = 1, \dots, d. \quad (3.5)$$

Correspondingly, for any pair of functions $\varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ sufficiently regular, we define a vector field $c : \varphi \cdot \nabla \psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ componentwise by

$$(c : \varphi \cdot \nabla \psi)_k(x) := \sum_{i,j,l=1}^d c_{ij} \varphi_{il}(x) \partial_l \psi_{jk}(x) \quad \forall k = 1, \dots, d.$$

The next result gives us the dependence between the convergence rate of X^n to X and the coefficients b, σ ; it is just a useful rewriting of [18, VI. Theorem 7.2], making explicitly the dependence on the coefficients (b, σ) and the rate of convergence. For more details on its derivation, we refer the interested reader to Appendix A.

⁶ $\dot{W}_s^n := \frac{dW_s^n}{ds}$

Theorem 3.14 (Wong–Zakai). *Let $b \in C_x^1$, $\sigma \in C_x^2$ and $(W^n)_{n \geq 1}$ be an approximation of W , in the sense of Definition 3.12, satisfying Assumption 3.13. Consider the SDEs*

$$X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s + \int_0^t (c : \sigma \cdot \nabla \sigma)(X_s) ds,$$

and

$$X_t^n = x_0 + \int_0^t b(X_s^n) ds + \int_0^t \sigma(X_s^n) dW_s^n.$$

Then for each $T > 0$ there exists a constant C such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - X_t^n|^2 \right] \leq \exp \left(C(1 + \|b\|_{C_x^1}^2 + \|\sigma\|_{C_x^1}^4 + \|\sigma\|_{C_x^0}^2 \|\sigma\|_{C_x^2}^2) \right) (f_n^2 + n^{-\frac{1}{5}}), \quad (3.6)$$

where f_n is from Assumption 3.13 and C depends on T and $\max_{i,j} |c_{ij}|$, but not on b , σ and n .⁷

In order to control the error we commit by applying Theorem 3.14 at the level of smooth approximations of our coefficients, we introduce the following classes for b and σ .

Definition 3.15. *Let $b \in L_x^{p_1}$ for some $p_1 \in (d, \infty)$, $p_1 \geq 2$, and $r \in [0, \infty) \rightarrow [0, \infty)$ be an increasing function. For a sequence $(b^n)_{n \geq 1}$ of continuous functions $b^n : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we write $(b^n)_{n \geq 1} \in \mathcal{A}(b, p_1, r)$ if:*

- (i) $b^n \in C_x^1$,
- (ii) $\|b - b^n\|_{L_x^{p_1}} \rightarrow 0$ as $n \rightarrow \infty$,
- (iii) $\|b^n\|_{C_x^1} \leq r(n)\|b\|_{L_x^{p_1}}$.

Definition 3.16. *Let $p_2 \in (d, \infty)$, $p_2 \geq 2$, and let σ satisfy condition (\mathcal{H}_1^σ) from Assumption 1.1 with constant K , $\nabla \sigma \in L_x^{p_2}$; let $r \in [0, \infty) \rightarrow [0, \infty)$ be an increasing function. For a sequence $(\sigma^n)_{n \geq 1}$ of regular functions $\sigma^n : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^d$, we write $(\sigma^n)_{n \geq 1} \in \mathcal{A}(\sigma, p_2, r)$ if:*

- (i) σ^n satisfy condition (\mathcal{H}_1^σ) with the same constant K , uniformly in n ,
- (ii) $\sigma^n \in C_x^2$,
- (iii) $\|\sigma - \sigma^n\|_{L_x^{p_2}} \rightarrow 0$ as $n \rightarrow \infty$,
- (iv) $\sup_{n \in \mathbb{N}} \|\nabla \sigma^n\|_{L_x^{p_2}} < \infty$ and $\|\sigma^n\|_{C_x^1}^2 + K\|\sigma^n\|_{C_x^2} \leq r(n)\|\sigma\|_{L_x^{p_2}}$.

We are now ready to state our result.

Theorem 3.17. *Let $(p_1, p_2) \in (d, \infty)^2$, $p_i \geq 2$, and let $b \in L_x^{p_1}$, σ satisfy condition (\mathcal{H}_1^σ) in Assumption 1.1 and $\nabla \sigma \in L_x^{p_2}$. Assume further that $(W^n)_{n \geq 1}$ is an approximation of W , in the sense of Definition 3.12, satisfying Assumption 3.13. Let X be the solution to*

$$X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s + \int_0^t (c : \sigma \cdot \nabla \sigma)(X_s) ds \quad (3.7)$$

for c as defined in (3.5). Let $(b^n)_{n \geq 1} \in \mathcal{A}(b, p_1, r)$, $(\sigma^n)_{n \geq 1} \in \mathcal{A}(\sigma, p_2, r)$ be approximation sequences for b and σ such that

$$\lim_{n \rightarrow \infty} \exp \left(Cr(n)^2 (\|b\|_{L_x^{p_1}}^2 + \|\sigma\|_{L_x^{p_2}}^2) \right) (f_n^2 + n^{-1/5}) = 0, \quad (3.8)$$

⁷It is claimed in [27, Theorem 3.9] that the resulting error in (3.6) is $f_n^2 + n^{-1+\epsilon}$, where ϵ can be chosen arbitrarily small; unfortunately this is not the case, as there are other error terms of the form $n^{-\epsilon}$ which become too slow if ϵ is chosen too small. For more details, see Appendix A.

where C is the constant appearing in (3.6) and f_n is given in Assumption 3.13. Then we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t - X_t^n|^2 \right] = 0, \tag{3.9}$$

where $(X_t^n)_{t \geq 0}$ is the solution to the equation

$$X_t^n = x_0 + \int_0^t b^n(X_s^n) ds + \int_0^t \sigma^n(X_s^n) \dot{W}_s^n ds.$$

Proof. Observe that, under our assumptions, $c : \sigma \cdot \nabla \sigma \in L_x^{p_2}$, therefore the SDE (3.7) is well-posed (either by [37] or the facts recalled in Section 2.1). Let $X_{m,t}$ be the solution to

$$X_{m,t} = x_0 + \int_0^t b^m(X_{m,s}) ds + \int_0^t \sigma^m(X_{m,s}) dW_s + \int_0^t (c : \sigma^m \cdot \nabla \sigma^m)(X_{m,s}) ds.$$

By Theorem 3.14, we know that for any $m \geq 1$ it holds

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_{m,t} - X_{m,t}^n|^2 \right] \\ & \lesssim \exp \left(C (\|b^m\|_{C_x^1}^2 + \|\sigma\|_{C_x^1}^4 + \|\sigma\|_{C_x^0}^2 \|\sigma^m\|_{C_x^2}^2) (f_n^2 + n^{-1/5}) \right) \\ & \lesssim \exp \left(C r(m)^2 (\|b\|_{L_x^p}^2 + \|\sigma\|_{L_x^p}^2) (f_n^2 + n^{-1/5}) \right), \end{aligned}$$

where C is independent of m and n and $X_{m,t}^n$ is the solution to

$$X_{m,t}^n = x_0 + \int_0^t b^m(X_{m,s}^n) ds + \int_0^t \sigma^m(X_{m,s}^n) \dot{W}_s^n ds.$$

By choosing $m = n$ and using the condition (3.8), we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_{n,t} - X_{n,t}^n|^2 \right] = 0.$$

On the other hand, arguing similarly to the proof of Corollary 2.13, using the estimates

$$\begin{aligned} \|c : (\sigma^m - \sigma) \cdot \nabla \sigma^m\|_{L_x^{p_2/2}} & \lesssim \|\sigma^m - \sigma\|_{L_x^{p_2}} \|\sigma^m\|_{L_x^{p_2}} \lesssim \|\sigma^m - \sigma\|_{L_x^{p_2}}, \\ \|c : \sigma \cdot \nabla (\sigma^m - \sigma)\|_{W_x^{-1,p_2}} & \lesssim \|\nabla (\sigma^m - \sigma)\|_{W_x^{-1,p_2}} (\|\sigma\|_{L_x^\infty} + \|\nabla \sigma\|_{L_x^{p_2}}) \lesssim \|\sigma^m - \sigma\|_{L_x^{p_2}}, \end{aligned}$$

it is not difficult to show that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t - X_{n,t}|^2 \right] \lesssim \|b - b^n\|_{L_x^{p_1}}^2 + \|\sigma - \sigma^n\|_{L_x^{p_2}}^2.$$

Overall we obtain

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t - X_t^n|^2 \right] \lesssim \|b - b^n\|_{L_x^{p_1}}^2 + \|\sigma - \sigma^n\|_{L_x^{p_2}}^2 + \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_{n,t} - X_{n,t}^n|^2 \right]$$

which implies (3.9) by letting $n \rightarrow \infty$. □

3.4 Other applications of stability estimates in the literature

We briefly discuss here two further examples of implicit applications of the stability estimates from Proposition 2.12 in the literature. In this case we do not provide any new results, yet we believe that highlighting this step makes the arguments from the papers [25] and [13] clearer.

3.4.1 Euler-Maruyama schemes for singular SDEs

Similarly to Section 3.3, when dealing with singular drifts one cannot try to simulate the solution to the SDE directly, as the unboundedness of b can be a major source of errors. Again, one can circumvent this problem by introducing a two step approximation, based on first mollifying b , and then applying an Euler–Maruyama scheme to the SDE with approximate drift b^n . This idea has seen precursors in [20], in the study of weak error rates for integrable drifts, and in [6], in the one-dimensional setting with b only enjoying some negative Hölder regularity. Here we follow the recent work [25], which studied the strong error in the general multidimensional setting, with (b, σ) satisfying Assumption 1.1; therein this two-step approximation was directly unified in a unique way and thus stability estimates were only used in a very implicit way.

Let (b, σ) satisfy Assumption 1.1 and X be the solution to (1.1); there are two different (yet similar) ways one can proceed.

On one hand, given a sequence of smooth bounded approximations $(b^n)_n$ of b , one can introduce the solution \tilde{X}^n associated to (b^n, σ) and then apply the Euler–Maruyama scheme of step n to (b^n, σ) , yielding a numerical approximation $\tilde{X}^{n,n}$; the overall error splits into $e = e^1 + e^2$, associated respectively to $X^n - \tilde{X}^n$ and $\tilde{X}^n - \tilde{X}^{n,n}$. The first error e^1 can be estimated by means of either Theorem 1.2 or the more refined estimates from Section 2.3 and yields a quantity in the style of ϖ_n as defined in [25]. The error e_2 instead is the purely numerical one, which has to be treated by different techniques and will result in a polynomial decay rate $n^{-\alpha}$. Alternatively, one can proceed more directly as in [25] and, given the approximations $(b^n)_n$, compare X to the solutions X^n to

$$dX_t^n = b_t^n(X_{k_n(t)}^n) dt + \sigma_t(X_{k_n(t)}^n) dW_t, \quad X^n|_{t=0} = X_0^n, \quad (3.10)$$

where $k_n(t) = \lfloor tn \rfloor / n$, $\lfloor \cdot \rfloor$ denoting the integer part. For fixed $n \in \mathbb{N}$, we are in the setting of Proposition 2.12 by considering $Y_t := X_t^n$, corresponding to (2.9) for the choice

$$R_t^1 := b_t^n(X_{k_n(t)}^n) - b_t(X_t^n), \quad R_t^2 := \sigma_t(X_{k_n(t)}^n) - \sigma_t(X_t^n).$$

Assuming for the moment that (Y, R^1, R^2) satisfy Assumption 2.10, an application of Proposition 2.12 then yields

$$\left\| \sup_{t \in [0, T]} |X_t - X_t^n| \right\|_{L_\omega^m} \lesssim \|X_0 - X_0^n\|_{L_\omega^m} + \left\| \sup_{t \in [0, T]} |V_t| \right\|_{L_\omega^{m\gamma}},$$

for V as defined in (2.10). By our assumptions, the properties of u and martingale inequalities, it is easy to deduce that

$$\begin{aligned} \left\| \sup_{t \in [0, T]} |V_t| \right\|_{L_\omega^m} &\lesssim \left\| \int_0^T |(b^n - b)_t(X_t^n)| dt \right\|_{L_\omega^{m\gamma}} + \left\| \int_0^T |(b_t^n(X_t^n) - b_t^n(X_{k_n(t)}^n))| dt \right\|_{L_\omega^{m\gamma}} \\ &\quad + \left\| \int_0^T |(\sigma_t(X_t^n) - \sigma_t(X_{k_n(t)}^n))| |D^2 u_t(X_t^n)| dt \right\|_{L_\omega^{m\gamma}} \\ &\quad + \left\| \left(\int_0^T |(\sigma_t(X_t^n) - \sigma_t(X_{k_n(t)}^n))|^2 dt \right)^{1/2} \right\|_{L_\omega^{m\gamma}}. \end{aligned} \quad (3.11)$$

Again, the first term gives rise to the approximation error e^1 related to ϖ_n , while all the others are numerical errors producing a decay $n^{-\alpha}$. This is the point where the fine analysis on the properties of the process X^n conducted in [25, Section 5], also based on novel tools like stochastic sewing techniques, must kick in. The same kind of results in fact verify that (Y, R^1, R^2) verify Assumption 2.10, see for instance Theorem 5.1 and Lemma 5.14 from [25].

Nonetheless, we find it important to stress how reaching an estimate of the form (3.11) can be performed in the framework of well established SDE tools and do not need any substantially new techniques.

3.4.2 Interacting particle systems

We now set ourselves in the setting of [13], which studied the propagation of chaos for interacting particle systems with singular kernel of Krylov-Röckner type. Therein again Proposition 2.12 was applied, although implicitly; we make this point clear as follows.

For any $N \in \mathbb{N}$, consider the interacting particle system

$$dX_t^{N,i} = B_t(X_t^{N,i}, \mu_t^N) dt + \sigma_t(X_t^{N,i}) dW_t^i, \quad \mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{N,i}} \quad \forall i = 1, \dots, N, \quad (3.12)$$

where $X_0^{N,i} = \xi^i$ for some i.i.d. $\{\xi^i\}_{i \in \mathbb{N}}$, $\{W^i\}_{i \in \mathbb{N}}$ are independent Brownian motions, σ satisfies properties $(\mathcal{H}_1^\sigma) - (\mathcal{H}_3^\sigma)$ from Assumption 1.1 and B is of the form considered in [13]. As $N \rightarrow \infty$, each particle $X^{i,N}$ of system (3.12) is expected to converge to an independent copy of the associated McKean-Vlasov SDE

$$dX_t^i = B_t(X_t^i, \mu_t) dt + \sigma_t(X_t^i) dW_t^i, \quad \mu_t = \mathcal{L}(X_t^i), \quad (3.13)$$

a property which is usually referred to as propagation of chaos; in [13] the authors show that such convergence holds in a strong sense, for suitable singular B , see Assumption (H)^b therein.

To explain how their techniques are related to ours, let us first recall some basic facts. Under (H)^b, equation (3.13) is wellposed and the associated drift $b_t^\mu(x) := B_t(x, \mu)$ satisfies an $L_t^{q_1} L_x^{p_1}$ integrability condition⁸. Moreover by exchangeability of (3.12), it suffices to show convergence for $i = 1$. Finally, by setting $X^{N,1} = Y$ and

$$R_t^1 := B_t(X_t^{N,1}, \mu_t^N) - b_t^\mu(X_t^{N,1}), \quad R_t^2 := 0,$$

we are in the setting of Proposition 2.12. The triple (Y, R^1, R^2) satisfies Assumption 2.10 thanks to an application of the *partial Girsanov transform*, cf. [13, Lemma 5.3 (ii)]; observe that $X^{1,N}$ is measurable w.r.t. to the filtration generated by $\{\xi^i, W^i\}_{i=1}^N$, which is covered by Assumption 2.10-iii) by taking $\tilde{W} = (\xi^i, W^i)_{i=2}^N$.⁹ Therefore by Proposition 2.12 and the usual properties of u we deduce that

$$\left\| \sup_{t \in [0, T]} |X_t^{N,1} - X_t^1| \right\|_{L_\omega^m} \lesssim \left\| \int_0^T |B_t(X_t^{N,1}, \mu_t^N) - b_t^\mu(X_t^{N,1})| dt \right\|_{L_\omega^{m\gamma}}.$$

The convergence of the quantity on the r.h.s. to 0 as $N \rightarrow \infty$ is established in [13, Lemma 6.1] for $m\gamma = 2$ (but the other cases follow similarly), which yields the desired propagation of chaos property.

Let us point out two interesting facts coming from our analysis:

- i) In order to apply our stability results, although some key information on the processes $X^{N,i}$ is needed, the main requirement is the regularity of b^μ . This can be better than the one of the original B , especially in the case of convolutional drifts $B_t(x, \mu) = (\phi_t * \mu)(x)$, thus there is hope to establish such results also in cases where ϕ does not satisfy the Krylov-Röckner condition, but b^μ does.

⁸Actually, the authors in [13] use the local spaces $L_t^{q_1} \tilde{L}_x^{p_1}$, but at the level of exposition this doesn't change much.

⁹This is indeed the only point in this paper where we need to use Assumption 2.10-iii) in its full generality, and the main reason for its abstract formulation.

- ii) We took σ measure independent as in [13], but this doesn't really play a role in our analysis; thus it might be possible to extend several result to the μ -dependent diffusion as well. The key problem would then be to establish some analogue of [13, Lemma 6.1] in this setting.

A More details on the proof of Theorem 3.14

In this appendix, we follow carefully the proof of [18, VI. Theorem 7.2] in order to indicate the precise dependence on the regularity of the coefficients (b, σ) and the resulting convergence as stated it Theorem 3.14.

It will be convenient in several passages to keep the same notations as in [18], based on the use of a parameter $\delta \ll 1$ and a function $N(\delta) : (0, 1] \rightarrow \mathbb{N}$ satisfying the hypothesis of [18, VI. Lemma 7.1], so let us explain shortly how it translates into our framework. The parameter δ corresponds to n^{-1} , while the function $N(\delta)$ is given by $N(\delta) := \lfloor \delta^{-\epsilon/4} \rfloor$ for a suitable parameter $\epsilon \in (0, 1)$ to be chosen later on. Observe that by construction, for any $\epsilon \in (0, 1)$ it holds

$$\lim_{\delta \rightarrow 0^+} N(\delta)^4 \delta = 0, \quad \lim_{\delta \rightarrow 0} N(\delta) = +\infty. \tag{A.1}$$

For such choice of $N(\delta)$, we set $\tilde{\delta} := N(\delta)\delta$ and

$$[s]^+(\tilde{\delta}) = (k + 1)\delta, \quad [s]^-(\tilde{\delta}) = k\tilde{\delta} \quad \text{if} \quad k\tilde{\delta} \leq s < (k + 1)\tilde{\delta}.$$

Several terms, previously indexed over n in Section 3.3, will now be indexed over δ , e.g. we will write $X^\delta, W^\delta, f_\delta$ and $s(t, \delta)$ in place of X^n, W^n, f_n and $s(t, n)$. Finally, throughout this appendix we will adopt the convention that, whenever the symbol \lesssim appears in the estimates, the hidden constant may depend on $T, \max_{ij} |c_{ij}|$ or d , but no other parameters.

Before going to the actual proof of Theorem 3.14, we need to collect a few basic observations coming from that of [18, VI. Lemma 7.1]. Therein it is shown that, for any $k \in \mathbb{N}$, it holds

$$k\delta s_{ij}(k\delta, \delta) = k\delta s_{ij}(\delta, \delta)$$

and moreover that

$$s_{ij}(\delta, \delta) - c_{ij}(k\delta, \delta) + \frac{1}{2}\delta_{ij} = \frac{1}{k\delta} \mathbb{E}[W_{0,i}^\delta(0)W_{k\delta,j} - W_{0,i}^\delta W_{0,j}^\delta]; \tag{A.2}$$

for the choice $k = N(\delta)$, this readily implies the estimate

$$|s_{ij}(\delta, \delta) - c_{ij}(\tilde{\delta}, \delta) + \frac{1}{2}\delta_{ij}| \lesssim \tilde{\delta}^{-1}[\tilde{\delta}^{\frac{1}{2}}\delta^{\frac{1}{2}} + \delta] \lesssim N(\delta)^{-\frac{1}{2}},$$

which combined with Assumption 3.13 implies

$$|c_{ij}(\tilde{\delta}, \delta) - c_{ij}|^2 \lesssim f_\delta^2 + N(\delta)^{-1}. \tag{A.3}$$

We can now proceed with the

Proof of Theorem 3.14. First by [18, VI. 7 (7.47)], for every $s \leq t$ we have

$$|X_t^\delta - X_s^\delta| \leq \|b\|_\infty(t - s) + \|\sigma\|_\infty \int_s^t |\dot{W}_s^\delta| ds. \tag{A.4}$$

As in [18, VI. 7 (7.48)], we decompose $X^n - X$ into the following four terms:

$$X_t^\delta - X_t := H_1(t) + H_2(t) + H_3(t) + H_4(t), \tag{A.5}$$

where

$$\begin{aligned}
 H_1(t) &= \int_{[t]^{-}(\tilde{\delta})}^t \sigma(X_s^\delta) dW_s^\delta - \int_{[t]^{-}(\tilde{\delta})}^t \sigma(X_s) dW_s - \int_{[t]^{-}(\tilde{\delta})}^t (c : \sigma \cdot \nabla \sigma)(X_s) ds, \\
 H_2(t) &= \int_{\tilde{\delta}}^{[t]^{-}(\tilde{\delta})} \sigma(X_s^\delta) dW_s^\delta - \int_{\tilde{\delta}}^{[t]^{-}(\tilde{\delta})} \sigma(X_s) dW_s - \int_{\tilde{\delta}}^{[t]^{-}(\tilde{\delta})} (c : \sigma \cdot \nabla \sigma)(X_s) ds, \\
 H_3(t) &= \int_0^{\tilde{\delta}} \sigma(X_s^\delta) dW_s^\delta - \int_0^{\tilde{\delta}} \sigma(X_s) dW_s - \int_0^{\tilde{\delta}} (c : \sigma \cdot \nabla \sigma)(X_s) ds
 \end{aligned}$$

and

$$H_4(t) = \int_0^t b(X_s^\delta) ds - \int_0^t b(X_s) ds.$$

Easily observe that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |H_4(t)|^2 \right] \leq \|\nabla b\|_\infty^2 \int_0^T \mathbb{E}[|X_s^\delta - X_s|^2] ds. \tag{A.6}$$

We rewrite

$$H_1(t) = H_{11}(t) - H_{12}(t) - H_{13}(t).$$

Following from [18, VI. 7 (7.54)] we get

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |H_{13}(t)|^2 \right] \leq \|\sigma \cdot \nabla \sigma\|_\infty^2 N(\delta)^2 \delta^2,$$

and [18, VI. 7 (7.55)] gives us

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |H_{11}(t)|^2 \right] \lesssim \|\sigma\|_\infty^2 N(\delta)^{\frac{3}{2}} \delta^{\frac{1}{2}}.$$

Note that

$$H_{12} = \sigma(X_{[t]^{-}(\tilde{\delta})})(W_t - W_{[t]^{-}(\tilde{\delta})}) + \int_{[t]^{-}(\tilde{\delta})}^t [\sigma(X_s) - \sigma(X_{[t]^{-}(\tilde{\delta})})] dW_s =: H_{121}(t) + H_{122}(t).$$

Then [18, VI. 7 (7.56)] yields

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |H_{121}(t)|^2 \right] \lesssim \|\sigma\|_\infty^2 N(\delta)^{\frac{1}{2}} \delta^{\frac{1}{2}}$$

and [18, VI. 7 (7.57)] implies

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |H_{122}(t)|^2 \right] \lesssim \|\nabla \sigma\|_\infty^2 (\|\sigma\|_\infty^2 + \|b\|_\infty^2) \tilde{\delta},$$

where inside these passages we used the inequality

$$\mathbb{E}[|X_s(x) - x|^2] \lesssim \|\sigma\|_\infty^2 s + \|b\|_\infty^2 s^2.$$

Putting all these estimates together we get

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |H_1(t)|^2 \right] \lesssim (1 + \|\sigma\|_{C_x^1}^4 + \|b\|_\infty^4) N(\delta)^{\frac{3}{2}} \delta^{\frac{1}{2}}. \tag{A.7}$$

Since $|H_3| \leq \sup_{0 \leq t \leq T} |H_1(t)|$,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |H_3(t)|^2 \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |H_1(t)|^2 \right] \lesssim (1 + \|\sigma\|_{C_x^1}^4 + \|b\|_\infty^4) N(\delta)^{\frac{3}{2}} \delta^{\frac{1}{2}}. \tag{A.8}$$

Stability estimates

We are only left with estimating H_2 . By integration by parts, we have

$$\begin{aligned} \int_{k\bar{\delta}}^{(k+1)\bar{\delta}} \sigma(X_s^\delta) dW_s^\delta &= \sigma(X_{k\bar{\delta}}^\delta) [W_{(k+1)\bar{\delta}}^\delta - W_{k\bar{\delta}}^\delta] \\ &\quad + \int_{k\bar{\delta}}^{(k+1)\bar{\delta}} \nabla \sigma(X_s^\delta) [\sigma(X_s^\delta) \dot{W}_s^\delta + b(X_s^\delta)] \cdot [W_{(k+1)\bar{\delta}}^\delta - W_{k\bar{\delta}}^\delta] ds \\ &=: J_1(k) + J_2(k). \end{aligned}$$

Also

$$\begin{aligned} J_1(k) &= \sigma(X_{k\bar{\delta}-\delta}^\delta) (W_{(k+1)\bar{\delta}}^\delta - W_{k\bar{\delta}}^\delta) + [\sigma(X_{k\bar{\delta}}^\delta) - \sigma(X_{k\bar{\delta}-\delta}^\delta)] (W_{(k+1)\bar{\delta}}^\delta - W_{k\bar{\delta}}^\delta) \\ &\quad + \sigma(X_{k\bar{\delta}-\delta}^\delta) (W_{(k+1)\bar{\delta}}^\delta - W_{(k+1)\bar{\delta}}^\delta) + \sigma(X_{k\bar{\delta}-\delta}^\delta) (W_{k\bar{\delta}}^\delta - W_{k\bar{\delta}}^\delta) \\ &=: J_{11}(k) + J_{12}(k) + J_{13}(k) + J_{14}(k). \end{aligned}$$

Write

$$H_2(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t)$$

where

$$I_1(t) = \sum_{k=1}^{m(t)-1} J_{11}(k) + \int_{\bar{\delta}}^{[t]^- (\bar{\delta})} \sigma(X_s) dW_s,$$

with $m(t)$ defined by $m(t) := \lfloor t/\bar{\delta} \rfloor$, and

$$I_i(t) = \sum_{k=1}^{m(t)-1} J_{1i}(k), \quad i = 2, 3, 4,$$

$$I_5(t) = \sum_{k=1}^{m(t)-1} J_2(k) - \int_{\bar{\delta}}^{[t]^- (\bar{\delta})} (c : \sigma \cdot \nabla \sigma)(X_s) ds.$$

From [18, VI. 7 (7.63)] we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_1(t)|^2 \right] \lesssim \|\nabla \sigma\|_\infty^2 \int_0^T \mathbb{E} [|X_s^\delta - X_s|^2] ds + (\|\sigma\|_\infty^2 + \|b\|_\infty^2) N(\delta)^2 \delta$$

and [18, VI. 7 (7.64)] shows

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_2(t)|^2 \right] \lesssim \|\nabla \sigma\|_\infty^2 N(\delta)^{-1}.$$

[18, VI. 7 (7.65), (7.66)] give us

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_3(t)|^2 \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |I_4(t)|^2 \right] \lesssim \|\sigma\|_\infty^2 N(\delta)^{-1}.$$

For I_5 we rewrite

$$I_5(t) = I_{51}(t) + I_{52}(t) + I_{53}(t) + I_{54}(t) + I_{55}(t)$$

with

$$I_{51}^{(h)}(t) = \sum_{k=1}^{m(t)-1} \int_{k\bar{\delta}}^{(k+1)\bar{\delta}} \sum_{i,l,j=1}^d [(\sigma_{il} \partial_l \sigma_{jh})(X_s^\delta) - (\sigma_{il} \partial_l \sigma_{jh})(X_{k\bar{\delta}}^\delta)] \dot{W}_{s,i}^\delta [W_{(k+1)\bar{\delta},j}^\delta - W_{s,j}^\delta] ds,$$

Stability estimates

$$\begin{aligned}
 I_{52}^{(h)}(t) &= \sum_{k=1}^{m(t)-1} \int_{k\bar{\delta}}^{(k+1)\bar{\delta}} \sum_{j,l=1}^d (b_l \partial_l \sigma_{jh})(X_s^\delta) [W_{(k+1)\bar{\delta},j}^\delta - W_{s,i}^\delta] ds, \\
 I_{53}^{(h)}(t) &= \sum_{k=1}^{m(t)-1} \int_{k\bar{\delta}}^{(k+1)\bar{\delta}} \sum_{i,l,j=1}^d [(\sigma_{il} \partial_l \sigma_{jh})(X_{k\bar{\delta}}^\delta) [\dot{W}_{s,i}^\delta (W_{(k+1)\bar{\delta},j}^\delta - W_{s,j}^\delta) - c_{ij}(\bar{\delta}, \delta)]] ds, \\
 I_{54}^{(h)}(t) &= \sum_{k=1}^{m(t)-1} \int_{k\bar{\delta}}^{(k+1)\bar{\delta}} \sum_{i,l,j=1}^d c_{ij} [(\sigma_{il} \partial_l \sigma_{jh})(X_{k\bar{\delta}}^\delta) - (\sigma_{il} \partial_l \sigma_{jh})(X_s)] ds, \\
 I_{55}^{(h)}(t) &= \sum_{k=1}^{m(t)-1} \sum_{i,l,j=1}^d [\bar{\delta} (\sigma_{il} \partial_l \sigma_{jh})(X_{k\bar{\delta}}^\delta) - c_{ij}].
 \end{aligned}$$

First we get from [18, VI. 7 (7.67)]

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_{55}(t)|^2 \right] \lesssim \sum_{i,j=1}^d \|\sigma \cdot \nabla \sigma\|_\infty^2 (c_{ij}(\bar{\delta}, \delta) - c_{ij})^2$$

and by [18, VI. 7 (7.68)]

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_{54}(t)|^2 \right] \lesssim \|\nabla(\sigma \cdot \nabla \sigma)\|_\infty^2 \left[\int_0^T \mathbb{E}[|X_s^\delta - X_s|^2] ds + (\|\sigma\|_\infty^2 + \|b\|_\infty^2) N(\delta)^2 \delta \right].$$

Furthermore by [18, VI. 7 (7.69)]

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_{52}(t)|^2 \right] \lesssim \|b \cdot \nabla \sigma\|_\infty^2 N(\delta)^2 \delta$$

and for I_{51} [18, VI. 7 (7.70)] yields

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_{51}(t)|^2 \right] \lesssim (\|\sigma\|_\infty^2 + \|b\|_\infty^2) N(\delta)^4 \delta.$$

[18, VI. 7 (7.71)] shows that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_{53}(t)|^2 \right] \lesssim \|\sigma\|_\infty^2 \left(N(\delta)^3 \delta + \left(\sum_{i,j=1}^d N(\delta)^{-1} c_{ij}^*(\bar{\delta}, \delta) \right)^2 + N(\delta)^{-1} \right),$$

for c_{ij}^* defined by

$$c_{ij}^*(\bar{\delta}, \delta) := N(\delta) c_{ij}(N(\delta)\delta, \delta) - (N(\delta) - 1) c_{ij}((N(\delta) - 1)\delta, \delta);$$

Applying identity (A.2) for the choices $k = N(\delta)$, $k = N(\delta) - 1$, one can see that

$$\begin{aligned}
 |N(\delta)^{-1} c_{ij}^*(\bar{\delta}, \delta)| &= \left| N(\delta)^{-1} \left[s_{ij}(\delta, \delta) + \frac{1}{2} \delta_{ij} \right] - (\bar{\delta})^{-1} \mathbb{E} \left[W_{0,i}^\delta (W_{N(\delta)\delta,j} - W_{(N(\delta)-1)\delta,j}) \right] \right| \\
 &\lesssim N(\delta)^{-1} + (\bar{\delta})^{-1} \delta^{\frac{1}{2}} \delta^{\frac{1}{2}} = N(\delta)^{-1}.
 \end{aligned}$$

Therefore

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_{53}(t)|^2 \right] \lesssim \|\sigma\|_\infty^2 (N(\delta)^3 \delta + N(\delta)^{-1}).$$

Collecting all the terms related to I_5 and applying (A.3), we get the following:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |I_5(t)|^2 \right] \lesssim \|\nabla(\sigma \cdot \nabla \sigma)\|_\infty^2 \int_0^T \mathbb{E}[|X_s^\delta - X_s|^2] ds$$

$$+ \left(1 + \|\sigma\|_{C_x^2}^8 + \|b\|_\infty^4\right) \left(f_\delta^2 + N(\delta)^4\delta + N(\delta)^{-1}\right).$$

Therefore

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |H_2(t)|^2 \right] &\lesssim \left(\|\nabla\sigma\|_\infty^2 + \|\nabla(\sigma \cdot \nabla\sigma)\|_\infty^2 \right) \int_0^T \mathbb{E}[|X_s^\delta - X_s|^2] \, ds \\ &+ \left(1 + \|\sigma\|_{C_x^2}^8 + \|b\|_\infty^4\right) \left(f_\delta^2 + N(\delta)^4\delta + N(\delta)^{-1}\right). \end{aligned} \quad (\text{A.9})$$

By combining (A.5), (A.6), (A.7), (A.8) and (A.9) we can conclude that

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t - X_t^\delta|^2 \right] &\lesssim (1 + \|b\|_{C_x^1}^2 + \|\sigma\|_{C_x^1}^4 + \|\sigma\|_\infty \|\sigma\|_{C_x^2}^2) \int_0^T \mathbb{E}[|X_s^\delta - X_s|^2] \, ds \\ &+ \left(1 + \|\sigma\|_{C_x^2}^8 + \|b\|_\infty^4\right) \left(f_\delta^2 + N(\delta)^{\frac{3}{2}}\delta^{\frac{1}{2}} + N(\delta)^4\delta + N(\delta)^{-1}\right). \end{aligned}$$

Grönwall's inequality then implies the estimate

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t - X_t^\delta|^2 \right] &\leq \exp \left(C' (1 + \|b\|_{C_x^1}^2 + \|\sigma\|_{C_x^1}^4 + \|\sigma\|_\infty \|\sigma\|_{C_x^2}^2) \right) \times \\ &\times \left(1 + \|\sigma\|_{C_x^2}^8 + \|b\|_\infty^4\right) \left(f_\delta^2 + N(\delta)^{\frac{3}{2}}\delta^{\frac{1}{2}} + N(\delta)^4\delta + N(\delta)^{-1}\right). \end{aligned} \quad (\text{A.10})$$

Observe that, up to relabelling the constant C' , the polynomial term $1 + \|\sigma\|_{C_x^2}^8 + \|b\|_\infty^4$ in (A.10) can be reabsorbed in the exponential. Moreover recall that $\delta = n^{-1}$ and $N(\delta) \sim n^{\epsilon/4}$, so that

$$N(\delta)^{\frac{3}{2}}\delta^{\frac{1}{2}} + N(\delta)^4\delta + N(\delta)^{-1} \sim n^{\frac{3\epsilon}{8} - \frac{1}{2}} + n^{\epsilon-1} + n^{-\frac{\epsilon}{4}};$$

optimizing it w.r.t. ϵ yields the choice $\epsilon = 4/5$ and a corresponding decay rate $n^{-1/5}$. Combining these two last observations with (A.10) yields the desired estimate (3.6). \square

References

- [1] Richard F. Bass and Zhen-Qing Chen. Brownian motion with singular drift. *The Annals of Probability*, 31(2):791–817, 2003. MR1964949
- [2] Ilya Bitter and Valentin Konakov. L^1 and L^∞ stability of transition densities of perturbed diffusions. *Random Operators and Stochastic Equations*, 29(4):287–308, 2021. MR4343186
- [3] Vladimir I. Bogachev, Michael Röckner, and Stanislav V. Shaposhnikov. Distances between transition probabilities of diffusions and applications to nonlinear Fokker–Planck–Kolmogorov equations. *Journal of Functional Analysis*, 271(5):1262–1300, 2016. MR3522009
- [4] Laura Caravenna and Gianluca Crippa. A directional Lipschitz extension lemma, with applications to uniqueness and Lagrangianity for the continuity equation. *Comm. Partial Differential Equations*, 46(8):1488–1520, 2021. MR4286465
- [5] Rémi Catellier and Massimiliano Gubinelli. Averaging along irregular curves and regularisation of ODEs. *Stochastic Processes and their Applications*, 126(8):2323–2366, 2016. MR3505229
- [6] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. A numerical scheme for stochastic differential equations with distributional drift. *Stochastic Process. Appl.*, 154:55–90, 2022. MR4490482
- [7] E. Fedrizzi and F. Flandoli. Hölder flow and differentiability for SDEs with nonregular drift. *Stoch. Anal. Appl.*, 31(4):708–736, 2013. MR3175794
- [8] Franco Flandoli, Massimiliano Gubinelli, and Enrico Priola. Well-posedness of the transport equation by stochastic perturbation. *Inventiones mathematicae*, 180(1):1–53, 2010. MR2593276

- [9] Franco Flandoli, Elena Issoglio, and Francesco Russo. Multidimensional stochastic differential equations with distributional drift. *Transactions of the American Mathematical Society*, 369(3):1665–1688, 2017. MR3581216
- [10] Lucio Galeati, Fabian A. Harang, and Avi Mayorcas. Distribution dependent SDEs driven by additive continuous noise. *Electronic Journal of Probability*, 27:1–38, 2022. MR4388460
- [11] Lucio Galeati, Fabian A. Harang, and Avi Mayorcas. Distribution dependent SDEs driven by additive fractional brownian motion. *Probability Theory and Related Fields*, pages 1–59, 2022. MR4528970
- [12] Sarah Geiss. Concave and other generalizations of stochastic Gronwall inequalities. *arXiv preprint arXiv:2204.06042*, 2022.
- [13] Zimo Hao, Michael Röckner, and Xicheng Zhang. Strong convergence of propagation of chaos for McKean–Vlasov SDEs with singular interactions. *arXiv preprint arXiv:2204.07952*, 2022.
- [14] Yaozhong Hu and Khoa Lê. Nonlinear Young integrals and differential systems in Hölder media. *Transactions of the American Mathematical Society*, 369(3):1935–2002, 2017. MR3581224
- [15] Xing Huang, Panpan Ren, and Feng-Yu Wang. Distribution dependent stochastic differential equations. *Frontiers of Mathematics in China*, 16(2):257–301, 2021. MR4254653
- [16] Xing Huang and Feng-Yu Wang. Distribution dependent SDEs with singular coefficients. *Stochastic Processes and their Applications*, 129(11):4747–4770, 2019. MR4013879
- [17] Xing Huang and Feng-Yu Wang. McKean–Vlasov SDEs with drifts discontinuous under Wasserstein distance. *Discrete Contin. Dyn. Syst.*, 41(4):1667–1679, 2021. MR4211198
- [18] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic Differential Equations and Diffusion Processes*. NorthHolland Publishing Company, Amsterdam Oxford New York, 1981. MR0637061
- [19] Elena Issoglio and Francesco Russo. McKean SDEs with singular coefficients. *arXiv:2107.14453*, to appear on *Annales de l’Institut Henri Poincaré*, 2021.
- [20] Arturo Kohatsu-Higa, Antoine Lejay, and Kazuhiro Yasuda. Weak rate of convergence of the Euler–Maruyama scheme for stochastic differential equations with non-regular drift. *Journal of Computational and Applied Mathematics*, 326:138–158, 2017. MR3668566
- [21] Valentin Konakov, Anna Kozhina, and Stéphane Menozzi. Stability of densities for perturbed diffusions and Markov chains. *ESAIM: Probability and Statistics*, 21:88–112, 2017. MR3630604
- [22] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005. MR2117951
- [23] Chaman Kumar, Neelima, Christoph Reisinger, and Wolfgang Stockinger. Well-posedness and tamed schemes for McKean–Vlasov equations with common noise. *Ann. Appl. Probab.*, 32(5):3283–3330, 2022. MR4497846
- [24] Daniel Lacker. On a strong form of propagation of chaos for McKean–Vlasov equations. *Electronic Communications in Probability*, 23:1–11, 2018. MR3841406
- [25] Khoa Lê and Chengcheng Ling. Taming singular stochastic differential equations: A numerical method. *arXiv preprint arXiv:2110.01343*, 2021.
- [26] Gunther Leobacher, Christoph Reisinger, and Wolfgang Stockinger. Well-posedness and numerical schemes for one-dimensional McKean–Vlasov equations and interacting particle systems with discontinuous drift. *BIT*, 62(4):1505–1549, 2022. MR4512614
- [27] Chengcheng Ling, Sebastian Riedel, and Michael Scheutzow. A Wong–Zakai theorem for SDEs with singular drift. *J. Differential Equations*, 326:344–363, 2022. MR4413292
- [28] Chengcheng Ling and Longjie Xie. Strong solutions of stochastic differential equations with coefficients in mixed-norm spaces. *Potential Analysis*, pages 1–15, 2021. MR4443124
- [29] Yuliya Mishura and Alexander Veretennikov. Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. *Theory of Probability and Mathematical Statistics*, 2020. MR4421344
- [30] Panpan Ren, Michael Röckner, and Feng-Yu Wang. Linearization of nonlinear Fokker–Planck equations and applications. *Journal of Differential Equations*, 322:1–37, 2022. MR4398417

- [31] Michael Röckner and Xicheng Zhang. Well-posedness of distribution dependent SDEs with singular drifts. *Bernoulli*, 27(2):1131–1158, 2021. MR4255229
- [32] Jacques Simon. Compact sets in the space $L^p(O, T; B)$. *Annali di Matematica pura ed applicata*, 146(1):65–96, 1986. MR0916688
- [33] Pietro Siorpaes. Applications of pathwise Burkholder–Davis–Gundy inequalities. *Bernoulli*, 24(4B):3222–3245, 2018. MR3788172
- [34] Elias M. Stein. *Singular integrals and differentiability properties of functions*, volume 2. Princeton university press, 1970. MR0290095
- [35] Veretennikov. On the strong solutions of stochastic differential equations. *Theory Probab. Appl.*, 24:354–366, 1979. MR0532447
- [36] Cédric Villani. *Optimal transport: old and new*, volume 338. Springer, 2009. MR2459454
- [37] Pengcheng Xia, Longjie Xie, Xicheng Zhang, and Guohuan Zhao. $L^q(L^p)$ -theory of stochastic differential equations. *Stochastic Process. Appl.*, 130(8):5188–5211, 2020. MR4108486
- [38] Longjie Xie and Xicheng Zhang. Ergodicity of stochastic differential equations with jumps and singular coefficients. *Ann. Inst. H. Poincaré Probab. Statist.*, 56:175–229, 2020. MR4058986
- [39] Shao-Qin Zhang and Chenggui Yuan. A Zvonkin’s transformation for stochastic differential equations with singular drift and applications. *Journal of Differential Equations*, 297:277–319, 2021. MR4280266
- [40] Xicheng Zhang. Stochastic differential equations with Sobolev diffusion and singular drift and applications. *Ann. Appl. Probab.*, 26(5):2697–2732, 2016. MR3563191
- [41] Xicheng Zhang and Guohuan Zhao. Heat kernel and ergodicity of SDEs with distributional drifts. *arXiv preprint arXiv:1710.10537*, 2017.
- [42] Guohuan Zhao. On distribution dependent SDEs with singular drifts. *arXiv preprint arXiv:2003.04829*, 2020.
- [43] Zvonkin. A transformation of the phase space of a diffusion process that removes the drift. *Math. Sbornik.*, 135:129–149, 1974. MR0336813

Electronic Journal of Probability

Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS¹)
- Easy interface (EJMS²)

Economical model of EJP-ECP

- Non profit, sponsored by IMS³, BS⁴, ProjectEuclid⁵
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund⁶ (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

¹LOCKSS: Lots of Copies Keep Stuff Safe <http://www.lockss.org/>

²EJMS: Electronic Journal Management System: <https://vtex.lt/services/ejms-peer-review/>

³IMS: Institute of Mathematical Statistics <http://www.imstat.org/>

⁴BS: Bernoulli Society <http://www.bernoulli-society.org/>

⁵Project Euclid: <https://projecteuclid.org/>

⁶IMS Open Access Fund: <https://imstat.org/shop/donation/>