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Tightness for thick points in two dimensions*

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Abstract

Let W_t be Brownian motion in the plane started at the origin and let θ be the first exit time of the unit disk D_1. Let

$$\mu_\theta(x,\epsilon)=\frac{1}{\pi\epsilon^2}\int_0^\theta 1_{\{B(x,\epsilon)\}}(W_t)\,dt,$$

and set $\mu^*_\theta(\epsilon)=\sup_{x\in D_1}\mu_\theta(x,\epsilon)$. We show that

$$\sqrt{\mu^*_\theta(\epsilon)}-\sqrt{2/\pi}\left(\log\epsilon^{-1}-\frac{1}{2}\log\log\epsilon^{-1}\right)$$

is tight.

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1 Introduction

Let W_t be Brownian motion in the plane started at the origin and let θ be the first exit time of the unit disk D_1. In [12] we showed that

$$\lim_{\epsilon\to 0}\sup_{x\in D_1}\frac{1}{\epsilon^2\log^2(\epsilon)}\int_0^\theta 1_{\{B(x,\epsilon)\}}(W_t)\,dt=2,\quad a.s.,\tag{1.1}$$

where $B(x,\epsilon)$ is the ball of radius ϵ centered at x. The integral above is the occupation measure of $B(x,\epsilon)$, and points x with large occupation measure are referred to as thick points. Taking square roots we can write this as

$$\lim_{\epsilon\to 0}\frac{1}{\log(\epsilon^{-1})}\sqrt{\sup_{x\in D_1}\frac{1}{\pi\epsilon^2}\int_0^\theta 1_{\{B(x,\epsilon)\}}(W_t)\,dt}=\sqrt{2/\pi},\quad a.s.\tag{1.2}$$

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Let

$$\mu_\theta(x,\epsilon)=\frac{1}{\pi\epsilon^2}\int_0^\theta 1_{\{B(x,\epsilon)\}}(W_t)\,dt, \tag{1.3}$$

and set $\mu^*_\theta(\epsilon)=\sup_{x\in D_1}\mu_\theta(x,\epsilon)$. Then (1.2) says that $\sqrt{\mu^*_\theta(\epsilon)}\sim\sqrt{2/\pi}\log\epsilon^{-1}$, as $\epsilon\to 0$. In this paper we obtain more detailed asymptotics. Let

$$m_\epsilon=\sqrt{2/\pi}\left(\log\epsilon^{-1}-\frac{1}{2}\log\log\epsilon^{-1}\right). \tag{1.4}$$

We will say that the thick points in D_1 are tight if $\sqrt{\mu^*_\theta(\epsilon)}-m_\epsilon$ is a tight family of random variables. That is,

$$\lim_{K\to\infty}\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\left|\sqrt{\mu^*_\theta(\epsilon)}-m_\epsilon\right|>K\right)=0. \tag{1.5}$$

Theorem 1.1. *The thick points in D_1 are tight.*

In fact we obtain the following improvement on the right tail of (1.5).

Theorem 1.2. *On D_1, for some $0<C,C',z_0<\infty$ and all $z\geq z_0$,*

$$\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sqrt{\mu^*_\theta(\epsilon)}-m_\epsilon\geq z\right)\leq Cze^{-2\sqrt{2\pi}\,z}, \tag{1.6}$$

and

$$\underline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sqrt{\mu^*_\theta(\epsilon)}-m_\epsilon\geq z\right)\geq C'ze^{-2\sqrt{2\pi}\,z}. \tag{1.7}$$

It follows from Brownian scaling that Theorems 1.1 and 1.2 hold if D_1 is replaced by any disc centered at the origin.

For reasons of symmetry it is easier to work on the sphere $\mathbf{S}^2$, and derive our results for thick points in D_1 from results for thick points on $\mathbf{S}^2$. We use $B_d(x,r)$ for the ball centered at x of radius r, in the spherical metric d. To distinguish this, we use $B_e(x,r)$ for the Euclidean ball in R^2 centered at x of radius r.

Let X_t be Brownian motion on $\mathbf{S}^2$, see for example [11], started at some point v (the 'South Pole'). For some (small) r^* let τ be the first hitting time of $\partial B_d(v,r^*)$ (the 'Antarctic Circle'). Let $\omega_\epsilon=2\pi(1-\cos\epsilon)$, the area of $B_d(x,\epsilon)$, and set

$$\bar{\mu}_\tau(x,\epsilon)=\frac{1}{\omega_\epsilon}\int_0^\tau 1_{\{B_d(x,\epsilon)\}}(X_t)\,dt, \tag{1.8}$$

With $\bar{\mu}^*_{\tau,\epsilon}=\sup_{x\in\mathbf{S}^2}\bar{\mu}_\tau(x,\epsilon)$ we will say that the thick points on $\mathbf{S}^2$ are tight if $\sqrt{\bar{\mu}^*_{\tau,\epsilon}}-m_\epsilon$ is a tight family of random variables.

Theorem 1.3. *The thick points on $\mathbf{S}^2$ are tight.*

As in Theorem 1.2 we obtain the following improvement for the right tail.

Theorem 1.4. *On $\mathbf{S}^2$, for some $0<C,C',z_0<\infty$ and all $z\geq z_0$,*

$$\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sqrt{\bar{\mu}^*_{\tau,\epsilon}}-m_\epsilon\geq z\right)\leq Cze^{-2\sqrt{2\pi}\,z}, \tag{1.9}$$

and

$$\underline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sqrt{\bar{\mu}^*_{\tau,\epsilon}}-m_\epsilon\geq z\right)\geq C'ze^{-2\sqrt{2\pi}\,z}. \tag{1.10}$$

Theorems 1.3 and 1.4 are stated and first proven for r^* sufficiently small. In Section 8 we show that they hold for any $0<r^*<\pi$.

In analogy with [12], rather than work directly with occupation measures, we work with excursion counts. To define this let $h_l=2\arctan(r_0e^{-l}/2)$ with r_0 small, see (2.5). For some $d_0\leq 1/1000$ let F_l be the centers of a d_0h_l covering of $\mathbf{S}^2$.

Let $\mathcal{T}^\tau_{x,l}$ be the number of excursions from $\partial B_d(x,h_{l-1})$ to $\partial B_d(x,h_l)$ prior to τ. We will obtain the following result for $\sup_{x\in F_L}\mathcal{T}^\tau_{x,L}$.

Theorem 1.5. *On* $\mathbf{S}^2$*, for some* $0 < z_0, C, C' < \infty$*, all* L *large and all* $z_0 \le z \le \log L$*,*

$$C' z e^{-2z} \le \mathbb{P}\left(\sup_{x\in F_L} \sqrt{2\mathcal{T}^{\tau}_{x,L}} - (2L - \log L) \ge z\right) \le C z e^{-2z}. \tag{1.11}$$

Equivalently

$$C' z e^{-2z} \le \mathbb{P}\left(\sup_{x\in F_L} \mathcal{T}^{\tau}_{x,L} \ge 2L\,(L - \log L + z)\right) \le C z e^{-2z}. \tag{1.12}$$

Since $L \sim \log h_L^{-1}$, Theorem 1.5 is then suggestive of Theorem 1.4 if we knew that on average the occupation measure of $B_d\,(x, h_L)$ during an excursion from $\partial B_d\,(x, h_L)$ to $\partial B_d\,(x, h_{L-1})$ was 'about' h_L^2. While this is basically known for our choices of h_L, h_{L-1}, see [12, Lemma 6.2], it is more delicate to get the precision necessary to show the equivalence of Theorem 1.5 with (1.9).

We now write (1.11) in a more convenient form. Set

$$\rho_L = 2 - \frac{\log L}{L}. \tag{1.13}$$

We will prove the following version of Theorem 1.5.

Theorem 1.6. *On* $\mathbf{S}^2$*, for some* $0 < z_0, C, C' < \infty$*, all* L *large and all* $z_0 \le z \le \log L$*,*

$$C' z e^{-2z} \le \mathbb{P}\left[\sup_{x\in F_L} \sqrt{2\mathcal{T}^{\tau}_{x,L}} \ge \rho_L L + z\right] \le C z e^{-2z}. \tag{1.14}$$

1.1 Background

This paper is based in many ways on my work [7] with Belius and Zeituni on tightness for the cover time of $\mathbf{S}^2$. The general approach is similar, and whenever results of that paper could be used directly I did so. However, the mathematics often necessitated different arguments.

The family

$$\{\bar{\mu}_\tau(x,\epsilon); x \in B_d(v, r^*), \epsilon > 0\}$$

is associated with a second order Gaussian chaos $H(x,\epsilon)$, $x \in B_d(v,r^*), \epsilon > 0$ by an isomorphism theorem of Dynkin [18]. Intuitively,

$$H(x,\epsilon) = \int_{B_d(x,\epsilon)} G_y^2\, dm(y) - E\left(\int_{B_d(x,\epsilon)} G_y^2\, dm(y)\right) \tag{1.15}$$

where G_x is the mean zero Gaussian process with covariance $u(x,y)$, the Green's function for $B_d(v,r^*)$ and m denotes the standard surface measure on S^2. Since $u(x,x) = \infty$ for all x, (1.15) is not a priori well defined. Nevertheless, this would suggest that there is a close relationship between $\bar{\mu}^*_{\tau,\epsilon} = \sup_{x\in R^2} \bar{\mu}_\tau(x,\epsilon)$ and the supremum of Gaussian fields. For details on H and the isomorphism theorem see [25, Section 2].

1.2 Open problems

1. Based on the analogy with the extrema of Branching random walks and log-correlated Gaussian fields, one expects that Theorem 1.1 should be replaced by the statement that the sequence of random variables $\sqrt{\mu^*_\theta(\epsilon)} - m_\epsilon$ converges in distribution to a randomly shifted Gumbel random variable. The recent paper [24] contains a much more precise conjecture about this limit. Let $A^{x,\epsilon}_t$ be the continuous additive functional for planar Brownian motion started at the origin with Revuz measure γ_ϵ which is uniform

measure on $\partial B_\epsilon(x,\epsilon)$. Planar Brownian motion does not have local times, but $A_t^{x,\epsilon}$ can be thought of as an approximate local time at x. It is shown in [24] that

$$\mu_\epsilon(F) = \log(1/\epsilon)\,\epsilon^2 \int_F e^{2\sqrt{2\pi\,A_\theta^{x,\epsilon}}}\,dx \tag{1.16}$$

converges in probability to a random Borel measure $\mu(F)$ called the critical Brownian multiplicative chaos. The conjecture is that for some $c_1, c_2 > 0$

$$\lim_{\epsilon\to 0} \mathbb{P}\left(\sqrt{\mu_\theta^*(\epsilon)} \le m_\epsilon + z\right) = \mathbb{E}\left(\exp\left[-c_1\mu(D_1)\,e(-c_2 z)\right]\right). \tag{1.17}$$

A key step in proving such convergence would be the improvement of the tail estimates in Theorems 1.2 and 1.4 for z large, which in turn would require a corresponding improvement of Theorem 1.6.

2. In [12] we also proved a conjecture of Erdös and Taylor concerning the number L_n^* of visits to the most visited site for simple random walk in $\mathbf{Z}^2$ up to step n. It was shown there that

$$\lim_{n\to\infty} \frac{L_n^*}{(\log n)^2} = 1/\pi \quad \text{a.s.} \tag{1.18}$$

The approach in that paper was to first prove (1.1) for planar Brownian motion and then to use strong approximation. Subsequently, in [29], we presented a purely random walk method to prove (1.18) for simple random walk. See also [5] and more recently [23]. A natural problem is to prove tightness for $\sqrt{L_n^*}$. In fact, the conjecture in [24] mentioned above was actually stated for the random walk, and also conjectures a complete description for the landscape at high values of the field.

See [1, 9] for random walks on trees, and [2, 3] for planar random walks.

3. Following [12] we analyzed thick points for several other processes. See [13] for transient symmetric stable process, [14] for spatial Brownian motion and [15] intersections of planar Brownian motion. One can ask about tightness or some analog for these processes.

1.3 Structure of the paper

In Section 2 we obtain the upper bounds for excursion counts in Theorem 1.6, and in Section 3 we derive the lower bounds. These sections employ many of the tools developed in [7]. In Section 4 we show how to go from results on excursion counts to Theorems 1.3 and 1.4 which involve $\bar\mu_\tau(x,\epsilon)$ in $\mathbf{S}^2$. Here we have to deal with a new problem for the upper bounds: $\bar\mu_\tau(x,\epsilon)$ in $\mathbf{S}^2$ is not in general monotone in ϵ. This requires interpolation and a continuity estimate which are developed in Sections 5 and 7. In the short Section 8 we derive our results on thick points for the unit disc in the plane from our results on thick points for $\mathbf{S}^2$, and use this to show that Theorems 1.3 and 1.4 hold for any $0 < r^* < \pi$. The last section is an Appendix containing the barrier estimates we need for Sections 2 and 3.

1.4 Index of notation

The following are frequently used notation, and a pointer to the location where the definition appears.

$\mu_\theta(x,\epsilon)$	(1.3)
m_ϵ	(1.4)
c^*	(1.6)
$\bar{\mu}_\tau(x,\epsilon)$	(1.8)
$B_d(x,r), B_e(x,r)$	page 2
ρ_L	(1.13)
$\mathcal{T}^{\tau}_{x,l}$	(1.11)
r_l, h_l	(2.2)
F_l	(2.4)
$\mathcal{T}^{k\to 0}_{y,l}$	(2.6)
l_L	(2.14)
$\alpha_{z,+}(l)$	(2.15)
k_y	(2.16)
F^*_L	(2.17)
$F^m_L, \mathcal{H}_{m,l}$	(2.47)
$\mathcal{B}_{m,l}$	(2.49)
$\mathcal{C}_{m,l}$	(2.52)
$\mathcal{D}_{m,l}(j)$	(2.55)
$\mathcal{B}^{\gamma,k}_{m,l}$	(2.63)
$\mathcal{T}^{u_m,r_{l-2},n}_{y,\widetilde{r}_l}$	(2.66)
$\beta_z\,(l)$	(3.2)
$\alpha_{z,-}\,(l)$	(3.3)
$\mathcal{T}^1_{y,l}, \mathcal{T}^{1,x^2}_{y,l}$	(3.3)
F^0_L	(3.4)
$\mathcal{W}_{y,k}(n)$	(3.10)
$N_{k,a}$	(3.11)
N_k, I_u	(3.12)
$\mathcal{H}_{k,a}$	(3.18)
k^+, k^{++}	(3.52)
$\widehat{\mathcal{I}}_{y,z}$	(3.11)
$N_{k,a}$	(3.11)
N_k	(3.12)
$\mathcal{I}_{y,z}$	(3.13)
$\mathcal{H}_{k,a}$	(3.18)
$J^{\uparrow}_{y,k}$	(3.30)
$\mathcal{B}_{y,k,a}$	(3.31)
$\overline{\mathcal{M}}_{x,\epsilon,a,b}(n)$	(4.1)
$\underline{t_L}\,(z)$	(4.5)
$\overline{\mathcal{M}}_{y,\bar{\epsilon}_y,y_0,a,b}(n)$	(4.27)
D_*	(8.2)

2 Upper bounds for excursions

Let

$$h(r) = 2\arctan(r/2) \tag{2.1}$$

and let

$$r_l = r_0 e^{-l}, l = 0, 1, \ldots, \quad \text{and} \quad h_l = h(r_l) \tag{2.2}$$

for some $r_0 < 1$. We can take $r_0 < 1$ sufficiently small that for all $0 \le x \le r_0$

$$x - x^3 \le h(x) \le x \quad \text{and} \quad |h'(x) - 1| \le x^2. \tag{2.3}$$

For some $d_0 \leq 1/1000$ let F_l be the centers of an $d_0 h_l$ covering of S^2. It follows from the above that

$$|F_l| \asymp c r_l^{-2} = c r_0^{-2} e^{2l}, l \geq 0. \tag{2.4}$$

Recall that $\mathcal{T}_{x,l}^{\tau}$ is the number of excursions from $\partial B_d(x, h_{l-1})$ to $\partial B_d(x, h_l)$ prior to τ. In this section we will assume that $2r^* \leq h_0$ so that for all $y \in B_d(v, r^*)$ we have $B_d(v, r^*) \subseteq B_d(y, h_0)$.

The reason for using $h(r)$ is due to the following result for $\mathbf{S}^2$, see [7, (2.6)]. If H_A is the first hitting time of A, then for any $u_1 < u_2 < u_3$

$$\mathbb{P}^{x \in \partial B_d(0, h(u_2))} \left[H_{\partial B_d(0, h(u_1))} < H_{\partial B_d(0, h(u_3))} \right] = \frac{\log\left(\frac{u_2}{u_3}\right)}{\log\left(\frac{u_1}{u_3}\right)}. \tag{2.5}$$

The next Lemma provides simple bounds which will be adequate to handle points which are close to the 'South Pole' v.

Lemma 2.1. *For L large, any $y \in B_d^c(v, h_k)$ and all $|z| \leq \log L$,*

$$\mathbb{P}\left[\sqrt{2\mathcal{T}_{y,L}^{\tau}} \geq \rho_L L + z \right] \leq c k e^{-2L} L e^{-2z}, \tag{2.6}$$

for some $c < \infty$ independent of $1 \leq k \leq L-1$.

If $y \in B_d(v, h_{L-1})$

$$\mathbb{P}\left[\sqrt{2\mathcal{T}_{y,L}^{\tau}} \geq \rho_L L + z \right] \leq c e^{-2L} L^2 e^{-2z}. \tag{2.7}$$

Proof. For $k \leq l-1$, let $\mathcal{T}_{y,l}^{k\to 0}$ be the number of excursions from $\partial B_d(y, h_{l-1})$ to $\partial B_d(y, h_l)$ between $H_{\partial B_d(y,h_k)}$ and $H_{\partial B_d(y,h_0)}$. We first estimate probabilities involving $\mathcal{T}_{y,l}^{k\to 0}$. Using (2.5), an excursion from $\partial B_d(y, h_k)$ hits $B_d(y, h_{l-1})$ before exiting $B_d(y, h_0)$ with probability $k/(l-1)$, and then the probability to hit $\partial B_d(y, h_l)$ before exiting $B_d(y, h_0)$ is $1 - \frac{1}{l}$. Thus, using the strong Markov property,

$$\begin{aligned} \mathbb{P}\left[\mathcal{T}_{y,l}^{k\to 0} \geq n \right] &= \frac{k}{l-1} \left(1 - \frac{1}{l} \right)^n \\ &\leq \frac{k}{l-1} e^{-\frac{n}{l}}, \end{aligned} \tag{2.8}$$

for n large. Since (recall (1.13))

$$\begin{aligned} (\rho_L L + z)^2 &= (\rho_L L)^2 + 2z\rho_L L + z^2 \\ &= 4L^2 - 4L\log L + 4zL + z^2 - 2z\log L + \log^2 L, \end{aligned} \tag{2.9}$$

it follows that for L large

$$\mathbb{P}\left[\sqrt{2\mathcal{T}_{y,L}^{k\to 0}} \geq \rho_L L + z \right] \leq c k e^{-2L} L e^{-2z}. \tag{2.10}$$

(2.6) follows since for $y \in B_d^c(v, h_k)$ we have $\mathcal{T}_{y,L}^{\tau} \leq \mathcal{T}_{y,L}^{k\to 0}$.

For (2.7) we note that for $y \in B_d(v, h_{L-1})$ we have $\mathcal{T}_{y,L}^{\tau} \leq \mathcal{T}_{y,L}^{L-1\to 0}$. □

Lemma 2.2. *For L large and all $0 \leq z \leq \log L$,*

$$\mathbb{P}\left[\sup_{y \in F_L \cap B_d(v, h_{\log L})} \sqrt{2\mathcal{T}_{y,L}^{\tau}} \geq \rho_L L + z \right] \leq c e^{-2z}, \tag{2.11}$$

for some $c < \infty$.

Proof. By Lemma 2.1 the probability in (2.11) is bounded by

$$\begin{aligned} &\sum_{k=\log L}^{L-2} \mathbb{P}\left[\sup_{y\in F_L\cap B_d(v,h_k)\cap B_d^c(v,h_{k+1})} \sqrt{2\mathcal{T}_{y,L}^{\tau}} \geq \rho_L L + z\right] \\ &\quad + \mathbb{P}\left[\sup_{y\in F_L\cap B_d(v,h_{L-1})} \sqrt{2\mathcal{T}_{y,L}^{\tau}} \geq \rho_L L + z\right] \\ &\leq \sum_{k=\log L}^{L-2} |F_L \cap B_d(v,h_k) \cap B_d^c(v,h_{k+1})| cke^{-2L} Le^{-2z} \\ &\quad + |F_L \cap B_d(v,h_{L-1})| ce^{-2L} L^2 e^{-2z} \\ &\leq cLe^{-2z} \sum_{k=\log L}^{\infty} ke^{-2k} \leq ce^{-2z}. \end{aligned} \tag{2.12}$$

□

Thus we only need deal with $y \in B_d^c(v, h_{\log L})$. However, Lemma 2.1 would give, for example, that

$$\mathbb{P}\left[\sup_{y\in F_L\cap B_d^c(v,h_1)} \sqrt{2\mathcal{T}_{y,L}^{\tau}} \geq \rho_L L + z\right] \leq CLe^{-2z}, \tag{2.13}$$

which would be disastrous if we let $L \to \infty$. To deal with this we introduce a barrier.

Let

$$l_L = l \wedge (L - l). \tag{2.14}$$

Fix z and set

$$\alpha_{z,+}(l) = \alpha(l, L, z) = \rho_L l + z + l_L^{1/4}. \tag{2.15}$$

Let

$$k_y = \inf\{k \mid y \in B_d^c(v, h_k)\}, \tag{2.16}$$

and

$$F_L^* = F_L \cap B_d^c(v, h_{\log L}). \tag{2.17}$$

Since $\alpha_{z,+}(L) = \rho_L L + z$ and $k_y \leq \log L$ for $y \in F_L^*$, our desired upper bound will follow from the next Lemma.

Lemma 2.3. *There exists $z_0 > 0$ such that for all $z_0 \leq z \leq \log L$ and all L large*

$$\mathbb{P}\left[\exists y \in F_L^*, l \in \{k_y + 1, \ldots, L\} \text{ s.t. } \mathcal{T}_{y,l}^{\tau} \geq \alpha_{z,+}^2(l)/2\right] \leq cze^{-2z}. \tag{2.18}$$

Although this formulation looks more complicated and demanding than our desired upper bound, it will allow us to proceed level by level and to eventually use a barrier estimate. The next Lemma will be used in our proof.

Lemma 2.4. *For L large, any $y \in B_d^c(v, h_k)$ and all $0 \leq z \leq \log L$,*

$$\mathbb{P}\left[\mathcal{T}_{y,l}^{\tau} \geq \alpha_{z,+}^2(l)/2\right] \leq ckle^{-2l} e^{-2(z+l_L^{1/4}) - (z+l_L^{1/4})^2/2l}, \tag{2.19}$$

for some $c < \infty$ independent of $k \geq 1$ and $l \in \{k+1, \ldots, L\}$.

Proof. As in (2.8)

$$\mathbb{P}\left[\mathcal{T}_{y,l}^{\tau} \geq \alpha_{z,+}^2(l)/2\right] \leq c\frac{k}{l-1} e^{-\frac{\alpha_{z,+}^2(l)}{2l}} \tag{2.20}$$

and

$$\alpha_{z,+}^2(l) = l^2\rho_L^2 + 2(z+l_L^{1/4})l\rho_L + (z+l_L^{1/4})^2$$
$$= l^2\left(4 - 4\frac{\log L}{L} + \frac{\log^2 L}{L^2}\right) + 2(z+l_L^{1/4})l\left(2 - \frac{\log L}{L}\right)$$
$$+(z+l_L^{1/4})^2 \qquad (2.21)$$

Hence

$$\alpha_{z,+}^2(l)/2l \;=\; \left(2l - 2\frac{l}{L}\log L\right) + 2(z+l_L^{1/4}) + (z+l_L^{1/4})^2/2l + o_L(1)$$
$$\geq (2l - 2\log l) + 2(z+l_L^{1/4}) + (z+l_L^{1/4})^2/2l + o_L(1), \qquad (2.22)$$

using the concavity of the logarithm. Our result follows. □

The proof of (2.18) will be provided in Sections 2.1-2.3, and is split into two cases. For l which are not too large, i.e. $l \leq L - (4\log L)^4$, we can deal with (2.18) one level at a time. This is the content of Section 2.1. For larger l's, which are handled in Section 2.2, and in particular for $l = L$, we need to proceed inductively and make use of the facts established for lower levels. This method can be traced back to Bramson's work [10]. Some crucial auxiliary estimates are postponed to Section 2.3.

2.1 Proof of (2.18) for l not too large

Proposition 2.5. *There exists $z_0 > 0$ such that for all $z_0 \leq z \leq \log L$ and all L large*

$$\mathbb{P}\left[\exists y \in F_L^*, l \in \left\{k_y+1, \ldots, L-(4\log L)^4\right\}\right.$$
$$\left.\text{s.t. } \mathcal{T}_{y,l}^{\tau} \geq \alpha_{z,+}^2(l)/2\right] \leq ce^{-2z}. \qquad (2.23)$$

Proof. We use a packing argument. Let $\phi(l) = e^{.25\, l_L^{1/4}}$. Considering separately the case of $l \leq L/2$ and $L/2 < l \leq L - (4\log L)^4$, we see that for some m_0

$$l_L^{1/4} \geq 4\log l, \qquad m_0 \leq l \leq L - (4\log L)^4, \qquad (2.24)$$

so that

$$\frac{l}{\phi(l)} = \frac{l}{e^{.25\, l_L^{1/4}}} \leq 1, \qquad m_0 \leq l \leq L - (4\log L)^4. \qquad (2.25)$$

We define modified radii by

$$r_{l-1}^- = \left(1 - \frac{1}{\phi(l-1)}\right) r_{l-1} \text{ and } r_l^+ = \left(1 + \frac{1}{\phi(l)}\right) r_l \text{ for } l \geq 1. \qquad (2.26)$$

Note that

$$h(r_{l+\log\phi(l)}) \overset{(2.3)}{\leq} r_{l+\log\phi(l)} \overset{(2.2)}{=} \frac{r_l}{\phi(l)} = \frac{r_{l-1}}{e\,\phi(l)} \leq \frac{r_{l-1}}{\phi(l-1)}. \qquad (2.27)$$

Using this and (2.3) we have for $\phi(l)$ large enough

$$h(r_{l-1}^-) + \frac{1}{10^3}h(r_{l+\log\phi(l)}) \leq h(r_{l-1}) \text{ and } h(r_l) + \frac{1}{10^3}h(r_{l+\log\phi(l)}) \leq h(r_l^+). \qquad (2.28)$$

For each $y \in \mathbf{S}^2$ let y_l denote the point in F_l closest to y (breaking ties in some arbitrary way). By the definition of $F_{l+\log\phi(l)}$, recalling that $d_0 \leq 10^{-3}$, we have

$$d\left(y, y_{l+\log\phi(l)}\right) \leq \frac{1}{10^3}h(r_{l+\log\phi(l)}), \qquad (2.29)$$

so that using (2.28) we see that for all $y \in \mathbf{S}^2$

$$B_d\left(y, h_l\right) \subset B_d\left(y_{l+\log\phi(l)}, h(r_l^+)\right) \subset B_d\left(y_{l+\log\phi(l)}, h(r_{l-1}^-)\right) \subset B_d\left(y, h_{l-1}\right). \tag{2.30}$$

Now for $k \leq l-1$ set

$$r_{k,l}^- = \left(1 - \frac{1}{\phi(l)}\right) r_k \text{ and } r_{0,l}^+ = \left(1 + \frac{1}{\phi(l)}\right) r_0 \text{ for } l \geq 0. \tag{2.31}$$

As in the proof of (2.28) we have

$$h(r_{k,l}^-) + \frac{1}{10^3} h(r_{l+\log\phi(l)}) \leq h(r_k) \text{ and } h(r_0) + \frac{1}{10^3} h(r_{l+\log\phi(l)}) \leq h(r_{0,l}^+),$$

so that (2.29) also implies that

$$B_d\left(y_{l+\log\phi(l)}, h(r_{k,l}^-)\right) \subset B_d\left(y, h(r_k)\right) \subset B_d\left(y, h(r_0)\right) \subset B_d\left(y_{l+\log\phi(l)}, h(r_{0,l}^+)\right). \tag{2.32}$$

For each $y \in F_{l+\log\phi(l)}$ let $\widetilde{\mathcal{T}}_{y,l}^{k\to 0}$ be the number of excursions from $\partial B_d\left(y, h(r_{l-1}^-)\right)$ to $\partial B_d\left(y, h(r_l^+)\right)$ prior to the first excursion from $\partial B_d\left(y, h(r_{k,l}^-)\right)$ to $\partial B_d\left(y, h(r_{0,l}^+)\right)$. Then define

$$\widetilde{\mathcal{T}}_{y,l}^{k\to 0} = \widetilde{\mathcal{T}}_{y_{l+\log\phi(l)},l}^{k\to 0}, \text{ for } y \in \mathbf{S}^2 \backslash F_{l+\log\phi(l)} \text{ for all } l \geq k+1.$$

It follows from (2.30) and (2.32) that $\widetilde{\mathcal{T}}_{y,l}^{k_y\to 0} \geq \mathcal{T}_{y,l}^{k_y\to 0} \geq \mathcal{T}_{y,l}^{\tau}$ for all $l \geq k_y+1$. Thus

Lemma 2.6. *For all $y \in \mathbf{S}^2, l \geq k_y+1$ we have that*

$$\widetilde{\mathcal{T}}_{y,l}^{k_y\to 0} \geq \mathcal{T}_{y,l}^{\tau}. \tag{2.33}$$

Because of this the probability in (2.23) is bounded above by

$$\begin{aligned} &\sum_{k=1}^{\log L} \sum_{l=k+1}^{L-(4\log L)^4} \sum_{y \in B_d(v, h(r_{k-1})) \cap F_{l+\log\phi(l)}} \mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \geq \alpha_{z,+}^2(l)/2\right] \qquad (2.34)\\ = &\sum_{k=1}^{\log L} \sum_{l=k+1}^{L-(4\log L)^4} \left|B_d\left(v, h(r_{k-1})\right) \cap F_{l+\log\phi(l)}\right| \mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \geq \alpha_{z,+}^2(l)/2\right] \\ \leq\; &c \sum_{k=1}^{\log L} \sum_{l=k+1}^{L-(4\log L)^4} e^{.5\, l_L^{1/4}} e^{2(l-k)} \mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \geq \alpha_{z,+}^2(l)/2\right], \end{aligned}$$

for some arbitrary $y \in F_{l+\log\phi(l)}$. We show below that for all $k \leq l$

$$\mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \geq \alpha_{z,+}^2(l)/2\right] \leq c e^{-2l - l_L^{1/4}} e^{-2z}, \tag{2.35}$$

and since

$$\sum_{k=1}^{\infty} \sum_{l=1}^{L} e^{.5\, l_L^{1/4}} e^{2(l-k)} e^{-2l - l_L^{1/4}} e^{-2z} \leq c e^{-2z},$$

this will complete the proof of (2.23).

We now turn to the proof of (2.35). Let

$$\begin{aligned} p_l &= \frac{\log\left(r_{l-1}^-/r_{0,l}^+\right)}{\log\left(r_l^+/r_{0,l}^+\right)} = \frac{\log\left(\left(1 - \frac{1}{\phi(l-1)}\right)\left(1 + \frac{1}{\phi(l)}\right)^{-1} e^{-(l-1)}\right)}{\log\left(e^{-l}\right)} \\ &= \frac{l - 1 + 2/\phi(l) + O(\phi(l)^{-2})}{l} = 1 - \frac{1 - 2/\phi(l) + O(\phi(l)^{-2})}{l}, \qquad (2.36) \end{aligned}$$

and

$$\begin{aligned} q_l &= \frac{\log\left(r_{k,l}^-/r_{0,l}^+\right)}{\log\left(r_{l-1}^-/r_{0,l}^+\right)} = \frac{\log\left(\left(1-\frac{1}{\phi(l)}\right)\left(1+\frac{1}{\phi(l)}\right)^{-1}e^{-k}\right)}{\log\left(\left(1-\frac{1}{\phi(l-1)}\right)\left(1+\frac{1}{\phi(l)}\right)^{-1}e^{-(l-1)}\right)} \\ &= \frac{k+2/\phi(l)+O(\phi(l)^{-2})}{l-1+2/\phi(l)+O(\phi(l)^{-2})} = \frac{k+O(\phi(l)^{-1})}{l-1}. \end{aligned}$$

Using the fact that $p_l < 1$ together with (2.36) we can write

$$p_l = 1 - \frac{1-b_l/\phi(l)}{l} \tag{2.37}$$

with $1 - b_l/\phi(l) > 0$. In addition, using (2.25) and possibly increasing m_0,

$$\frac{lb_l}{\phi(l)} \le 3, \qquad l \ge m_0. \tag{2.38}$$

Since q_l is the probability for an excursion from $\partial B_d\left(y, h(r_{k,l}^-)\right)$ to hit $B_d\left(y, h(r_{l-1}^-)\right)$ before $\partial B_d\left(y, h(r_{0,l}^+)\right)$, and p_l is the probability for an excursion from $B_d\left(y, h(r_{l-1}^-)\right)$ to hit $\partial B_d\left(y, h(r_l^+)\right)$ before $\partial B_d\left(y, h(r_{0,l}^+)\right)$, we see that as in (2.8)

$$\mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \ge \alpha_{z,+}^2(l)/2\right] \le \frac{ck}{l} e^{-\frac{\alpha_{z,+}^2(l)}{2l}\left(1-\frac{b_l}{\phi(l)}\right)}. \tag{2.39}$$

By (2.22) we have that

$$\frac{\alpha_{z,+}^2(l)}{2l} \quad \ge \quad (2l - 2\log l) + 2(z + l_L^{1/4}) + z^2/2l + o_L(1), \tag{2.40}$$

so that, for $k \le l$

$$\mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \ge \alpha_{z,+}^2(l)/2\right] \le cl^2 e^{-\left(2l+2(z+l_L^{1/4})+z^2/2l\right)\left(1-\frac{b_l}{\phi(l)}\right)}. \tag{2.41}$$

We claim that

$$\frac{z^2}{2l}\left(1-\frac{b_l}{\phi(l)}\right) - 2zb_l/\phi(l) > 0,$$

that is

$$z\left(1-\frac{b_l}{\phi(l)}\right) > 4lb_l/\phi(l)$$

for $z \ge z_0$ sufficiently large. For $l > m_0$, this follows from (2.38), and for $l \le m_0$ we can just increase z further. Thus for such z

$$\mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \ge \alpha_{z,+}^2(l)/2\right] \le cl^2 e^{-\left(2l+2l_L^{1/4}\right)\left(1-\frac{b_l}{\phi(l)}\right)} e^{-2z}. \tag{2.42}$$

For $k \le l \le m_0$ this already proves (2.35) with c sufficiently large. For $l > m_0$, using (2.38) again we now have

$$\mathbb{P}\left[\widetilde{\mathcal{T}}_{y,l}^{k\to 0} \ge \alpha_{z,+}^2(l)/2\right] \le cl^2 e^{-\left(2l+2l_L^{1/4}\right)} e^{-2z}. \tag{2.43}$$

and (2.24) completes the proof of (2.35). □

2.2 Proof of (2.18) for l very large

We will show that for some small but fixed constant $\tilde{c}$ to be chosen later we have that for all L sufficiently large and all $z_0 \le z \le \log L$

$$\mathbb{P}\left[\begin{array}{c}\exists y \in F_L^* \cap B_d(0,\tilde{c}h_0) \text{ and } k_y+1 \le l \le L \\ \text{such that } \sqrt{2\mathcal{T}_{y,l}^{\tau}} \ge \alpha_{z,+}(l)\end{array}\right] \le cze^{-2z}. \tag{2.44}$$

Here 0, the center of $B_d(0,\tilde{c}h_0)$, is used to denote an arbitrary point in $\mathbf{S}^2$. A simple union bound (over $\sim (1/\tilde{c}h_0)^2$ balls) then completes the proof of (2.18).

Now consider

$$\mathcal{G}_l = \left\{\sqrt{2\mathcal{T}_{y,l'}^{\tau}} \le \alpha_{z,+}(l') \text{ for all } l' = k_y+1,\dots,l \text{ and } \forall y \in F_L^* \cap B_d(0,\tilde{c}h_0)\right\}.$$

Let $L' = L - (4\log L)^4$. With

$$\mathcal{H}_l = \left\{\exists y \in F_L^* \cap B_d(0,\tilde{c}h_0) \text{ s.t. } \sqrt{2\mathcal{T}_{y,l}^{\tau}} \ge \alpha_{z,+}(l),\ k_y < l\right\}, \tag{2.45}$$

we will prove that for all $l > L'$

$$\mathbb{P}\left[\mathcal{H}_l \cap \mathcal{G}_{l-2}\right] \le cze^{-l_L^{1/4}-2z}, \tag{2.46}$$

so that we have

$$\begin{aligned}\mathbb{P}\left[\mathcal{G}_L^c\right] &\le \sum_{l=L'+1}^{L} \mathbb{P}\left[\mathcal{G}_l^c \cap \mathcal{G}_{l-1}\right] + \mathbb{P}\left[\mathcal{G}_{L'}^c\right] \\ &\le \sum_{l=L'+1}^{L} \mathbb{P}\left[\mathcal{H}_l \cap \mathcal{G}_{l-2}\right] + \mathbb{P}\left[\mathcal{G}_{L'}^c\right] \\ &\le \sum_{l=L'+1}^{L} cze^{-l_L^{1/4}-2z} + \mathbb{P}\left[\mathcal{G}_{L'}^c\right] \le cze^{-2z},\end{aligned}$$

by Proposition 2.5, which will prove (2.44).

Setting $F_L^m = F_L \cap B_d^c(v,h_m) \cap B_d(v,h_{m-1})$, so that $k_y = m$ for $y \in F_L^m$, and for any $l > m$

$$\mathcal{H}_{m,l} = \left\{\exists y \in F_L^m \cap B_d(0,\tilde{c}h_0) \text{ s.t. } \sqrt{2\mathcal{T}_{y,l}^{\tau}} \ge \alpha_{z,+}(l)\right\}, \tag{2.47}$$

we will prove that for all $l > L'$

$$\sum_{m=1}^{\log L} \mathbb{P}\left[\mathcal{H}_{m,l} \cap \mathcal{G}_{l-2}\right] \le cze^{-l_L^{1/4}-2z}, \tag{2.48}$$

which gives (2.46) since, recall (2.17), $F_L^* = F_L \cap B_d^c(v,h_{\log L})$.

To prove (2.48) we need to work with the following localized version of $\mathcal{H}_{m,l}$. For any $l > m$ let

$$\mathcal{B}_{m,l} = \left\{\exists x \in F_L^m \cap B_d(u_m,\tilde{c}h_l) \text{ s.t. } \sqrt{2\mathcal{T}_{x,l}^{\tau}} \ge \alpha_{z,+}(l)\right\}, \tag{2.49}$$

where u_m is used to denote an arbitrary point in F_L^m. By a union bound, $\mathbb{P}\left[\mathcal{H}_{m,l} \cap \mathcal{G}_{l-2}\right]$ is bounded above by

$$ce^{2(l-m)} \times \mathbb{P}\left[\mathcal{B}_{m,l} \cap \mathcal{G}_{l-2}\right]. \tag{2.50}$$

Hence it suffices to show that

$$\sum_{m=1}^{\log L} e^{-2m}\mathbb{P}\left[\mathcal{B}_{m,l} \cap \mathcal{G}_{l-2}\right] \le cze^{-2l-l_L^{1/4}-2z}. \tag{2.51}$$

Since $\mathcal{G}_{l-2} \subset \mathcal{C}_{m,l}$, where

$$\mathcal{C}_{m,l} = \left\{ \sqrt{2\mathcal{T}^{\tau}_{u_m,l'}} \leq \alpha_{z,+}(l') \text{ for all } l' = m+1, \ldots, l-2 \right\}, \tag{2.52}$$

it suffices to show that

$$\sum_{m=1}^{\log L} e^{-2m} \mathbb{P}\left[\mathcal{B}_{m,l} \cap \mathcal{C}_{m,l}\right] \leq cze^{-2l - l_L^{1/4} - 2z}. \tag{2.53}$$

We show in the next Section that for all $l \geq L - (4\log L)^4$

$$\mathbb{P}\left[\mathcal{B}_{m,l} \cap \left\{ \sqrt{2\mathcal{T}^{\tau}_{u_m,l-2}} \leq \frac{1}{2}\alpha_{z,+}(l-2) \right\}\right] \leq ce^{-c'L^2}. \tag{2.54}$$

It follows from this that with

$$\mathcal{D}_{m,l}(j) = \left\{ \sqrt{2\mathcal{T}^{\tau}_{u_m,l-2}} \in I_{\alpha_{z,+}(l-2)+j} \right\}, \tag{2.55}$$

where $I_s = [s, s+1]$, it suffices to show that

$$\sum_{m=1}^{\log L} e^{-2m} \sum_{j=0}^{\frac{1}{2}\alpha_{z,+}(l-2)} \mathbb{P}\left[\mathcal{B}_{m,l} \cap \mathcal{C}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right] \leq cze^{-2l - l_L^{1/4} - 2z}. \tag{2.56}$$

We also show in the next Section that we can find a fixed j_0 such that for all $j_0 \leq j \leq \frac{1}{2}\alpha_{+,z}(l-2)$, uniformly in $1 \leq m \leq \log L$ and $z_0 \leq z \leq \log L$, for any $\widetilde{\mathcal{C}}_{m,l} \in \mathcal{F}\left(\mathcal{T}^{\tau}_{u_m,k}, k = 1, \ldots, l-2\right)$

$$\mathbb{P}\left[\mathcal{B}_{m,l} \,\middle|\, \widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right] \leq Ce^{-4j}, \tag{2.57}$$

by taking $\tilde{c} > 0$ sufficiently small.

It follows from the barrier estimate (9.5) that for $0 \leq j \leq \frac{1}{2}\alpha_{z,+}(l-2)$,

$$\mathbb{P}\left[\mathcal{C}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right] \leq ce^{-2l-2z-2l_L^{1/4}+2j} \times m^2 \left(1 + z + m + l_L^{1/4}\right)(1+j). \tag{2.58}$$

Combining the last 2 displays we can bound the left hand side of (2.56) by

$$Cze^{-2l-2z-l_L^{1/4}} \sum_{j=0}^{\infty} e^{-4j1_{\{j \geq j_0\}} + 2j}(1+j), \tag{2.59}$$

which proves (2.56).

2.3 Proof of the continuity estimate (2.57) and the bound (2.54)

We first prove that for some j_0 fixed and all $j_0 \leq j \leq \frac{1}{2}\alpha_{+,z}(l)$, uniformly in $1 \leq m \leq \log L$ and $z_0 \leq z \leq \log L$, for any $\widetilde{\mathcal{C}}_{m,l} \in \mathcal{F}\left(\mathcal{T}^{\tau}_{u_m,k}, k = 1, \ldots, l-2\right)$

$$\mathbb{P}\left[\mathcal{B}_{m,l} \,\middle|\, \widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right] \leq Ce^{-4j}. \tag{2.60}$$

Proof. For each $\gamma \in (0,1]$ and y, let $\mathcal{T}^{\tau}_{y,\widetilde{r}_l}$ be the number of excursions from $\partial B\left(y, h(\widetilde{r}_{l-1})\right)$ to $\partial B\left(y, h(\widetilde{r}_l)\right)$ prior to τ, where

$$\widetilde{r}_{l-1} = r_{l-1}\left(1-\gamma\right), \qquad \widetilde{r}_l = r_l\left(1+\gamma\right). \tag{2.61}$$

Note that

$$\mathcal{T}^{\tau}_{y',\widetilde{r}_l} \geq \mathcal{T}^{\tau}_{y,l} \text{ for all } y' \text{ such that } d\left(y,y'\right) \leq \frac{\gamma r_l}{2}, \tag{2.62}$$

since then

$$B_d\left(y,h_{l-1}\right) \supset B_d\left(y',h(r_{l-1}\left(1-\gamma\right))\right) \supset B_d\left(y',h(r_l\left(1+\gamma\right))\right) \supset B_d\left(y,h_l\right).$$

Let

$$\mathcal{B}^{\gamma,k}_{m,l} = \left\{\exists y \in F^m_k \cap B_d\left(u_m,\tilde{c}\,h_l\right) \text{ such that } \sqrt{2\mathcal{T}^{\tau}_{y,\widetilde{r}_l}} \geq \alpha_{+,z}(l)\right\}. \tag{2.63}$$

Note F^m_k not F^m_L. From now on we fix

$$\gamma = \frac{1}{\alpha_{+,z}(l)-j}, \qquad \text{and} \qquad k = \log\left(2(\alpha_{+,z}(l)-j)\right)+l. \tag{2.64}$$

We will show that with these values

$$\mathbb{P}\left[\mathcal{B}^{\gamma,k}_{m,l}\,\middle|\,\widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right] \leq Ce^{-4j}. \tag{2.65}$$

Using (2.62) this will imply (2.60), since for each $y \in F^m_L \cap B_d\left(u_m,\tilde{c}h_l\right)$ there exists a representative $y' \in F^m_k \cap B_d\left(u_m,\tilde{c}h_l\right)$ such that

$$d\left(y,y'\right) \leq r_k = \frac{1}{2(\alpha_{+,z}(l)-j)}\; r_l = \frac{\gamma r_l}{2}.$$

To show (2.65), we first show that for some $c_3 > 0$

$$\mathbb{P}\left[\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} \geq \alpha_{+,z}(l) - \frac{j}{2}\,\middle|\,\widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right] \leq c'e^{-c_3 j^2}. \tag{2.66}$$

Let $\mathcal{T}^{u_m,r_{l-2},n}_{y,\widetilde{r}_l}$ be the number of excursions from $\partial B\left(y,h(\widetilde{r}_{l-1})\right)$ to $\partial B\left(y,h(\widetilde{r}_l)\right)$ during the first n excursions from $\partial B\left(u_m,h_{l-2}\right)$ to $\partial B\left(u_m,h_{l-3}\right)$. Using the Markov property we have that

$$\begin{aligned}
&\mathbb{P}\left[\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} \geq \alpha_{+,z}(l) - \frac{j}{2}\,\middle|\,\widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right] \qquad (2.67)\\
=\;& \mathbb{P}\left[\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} \geq \alpha_{+,z}(l) - \frac{j}{2}\,\middle|\,\mathcal{D}_{m,l}(-j)\right]\\
=\;& \mathbb{P}\left[\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} \geq \alpha_{+,z}(l) - \frac{j}{2}\,\middle|\,\sqrt{2\mathcal{T}^{\tau}_{u_m,l-2}} \in I_{\alpha_{z,+}(l)-j}\right]\\
=\;& \mathbb{P}\left[\sqrt{2\mathcal{T}^{u_m,r_{l-2},\mathcal{T}^{\tau}_{u_m,l-2}}_{u_m,\widetilde{r}_l}} \geq \alpha_{+,z}(l) - \frac{j}{2}\,\middle|\,\sqrt{2\mathcal{T}^{\tau}_{u_m,l-2}} \in I_{\alpha_{z,+}(l)-j}\right].
\end{aligned}$$

To prove (2.66) it suffices to show that show that uniformly for $s \in I_{\alpha_{z,+}(l)-j}$

$$\mathbb{P}\left[\sqrt{2\mathcal{T}^{u_m,r_{l-2},s^2/2}_{u_m,\widetilde{r}_l}} \geq \alpha_{+,z}(l) - \frac{j}{2}\right] \leq c'e^{-c_3 j^2}. \tag{2.68}$$

To see this, let $s = \alpha_{z,+}(l) - j + \zeta$, where $0 \leq \zeta \leq 1$. Set $n = s^2/2$ and $\theta = \left(\alpha_{+,z}(l) - \frac{j}{2}\right)^2/2$,

$$\begin{aligned}
\bar{q} &= \mathbb{P}^{u\in\partial B_d(u_m,h_{l-2})}\left[H_{\partial B_d(u_m,h(r_l(1+\gamma)))} < H_{\partial B_d(u_m,h_{l-3})}\right]\\
&= \frac{\log r_{l-3} - \log r_{l-2}}{\log r_{l-3} - \log(r_l\left(1+\gamma\right))} = \frac{1}{3+O\left(\gamma\right)},
\end{aligned} \tag{2.69}$$

and

$$\begin{aligned} \bar{p} &= \mathbb{P}^{u\in\partial B_d(u_m,h(r_{l-1}(1-\gamma)))}\left[H_{\partial B_d(u_m,h_{l-3})} < H_{\partial B_d(u_m,h(r_l(1+\gamma)))}\right] \\ &= \frac{\log(r_{l-1}\,(1-\gamma)) - \log(r_l\,(1+\gamma))}{\log r_{l-3} - \log(r_l\,(1+\gamma))} = \frac{1+O\,(\gamma)}{3+O\,(\gamma)}. \end{aligned} \tag{2.70}$$

[6, Lemma 4.6] states that if $\theta \le n\bar{p}/\bar{q}$ then

$$\mathbb{P}\left[\mathcal{T}^{u_m,r_{l-2},n}_{u_m,\widetilde{r}_l} \le \theta\right] \le e^{-\left(\sqrt{\bar{q}n}-\sqrt{\bar{p}\theta}\right)^2}. \tag{2.71}$$

The same proof shows that if $\theta \ge n\bar{p}/\bar{q}$ then

$$\mathbb{P}\left[\mathcal{T}^{u_m,r_{l-2},n}_{u_m,\widetilde{r}_l} \ge \theta\right] \le e^{-\left(\sqrt{\bar{q}n}-\sqrt{\bar{p}\theta}\right)^2}. \tag{2.72}$$

Translating back this shows that

$$\mathbb{P}\left[\sqrt{2\mathcal{T}^{u_m,r_{l-2},s^2/2}_{u_m,\widetilde{r}_l}} \ge \alpha_{+,z}(l) - \frac{j}{2}\right] \le e^{-\left(\sqrt{\bar{q}}(\alpha_{+,z}(l)-j+\zeta)-\sqrt{\bar{p}}(\alpha_{+,z}(l)-\frac{j}{2})\right)^2/2} \tag{2.73}$$

once we have verified that

$$\alpha_{z,+}(l) - \frac{j}{2} \ge (\alpha_{+,z}(l) - j + 1)\sqrt{q/p}.$$

But the right hand side

$$= (\alpha_{+,z}(l) - j + 1)(1 + O\,(\gamma)) = (\alpha_{+,z}(l) - j) + O\,(1)\,,$$

since $\gamma\,(\alpha_{+,z}(l) - j) = 1$. Thus we can use (2.73) for all $j \ge c_3$ for some $c_3 < \infty$. For such j we therefore have

$$\mathbb{P}\left[\sqrt{2\mathcal{T}^{u_m,r_{l-2},s^2/2}_{u_m,\widetilde{r}_l}} \ge \alpha_{+,z}(l) - \frac{j}{2}\right] \le ce^{-\frac{1}{6}\left(-\frac{j}{2}+\zeta+O\left(\gamma\left(\alpha_{+,z}(l)-\frac{j}{2}\right)\right)\right)^2},$$

and since $\gamma\,(\alpha_{+,z}(l) - j) = 1$ and $j \le \frac{1}{2}\alpha_{+,z}(l)$ so that $j \le (\alpha_{+,z}(l) - j)$, it follows that

$$\gamma\left(\alpha_{+,z}(l) - \frac{j}{2}\right) = \gamma\,(\alpha_{+,z}(l) - j) + \gamma\frac{j}{2} \le 2,$$

so that we obtain (2.68) for all $j \ge c_3$. By enlarging c' we then have (2.68) for all j.

We now bound

$$\begin{aligned} \mathbb{P}\left[\mathcal{B}^{\gamma,k}_{m,l}\,\middle|\,\widetilde{\mathcal{C}}_{m,l}\cap\mathcal{D}_{m,l}(-j)\right] &\le \mathbb{P}\left[\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} \ge \alpha_{+,z}(l) - \frac{j}{2}\,\middle|\,\widetilde{\mathcal{C}}_{m,l}\cap\mathcal{D}_{m,l}(-j)\right] \\ &\quad + \mathbb{P}\left[\mathcal{B}^{\gamma,k}_{m,l}\cap\left\{\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} < \alpha_{+,z}(l) - \frac{j}{2}\right\}\,\middle|\,\widetilde{\mathcal{C}}_{m,l}\cap\mathcal{D}_{m,l}(-j)\right]. \end{aligned} \tag{2.74}$$

Because of the bound (2.66), to prove (2.65) it suffices to show that

$$\mathbb{P}\left[\mathcal{B}^{\gamma,k}_{m,l}\cap\left\{\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} < \alpha_{+,z}(l) - \frac{j}{2}\right\}\,\middle|\,\widetilde{\mathcal{C}}_{m,l}\cap\mathcal{D}_{m,l}(-j)\right] \le Ce^{-4j}. \tag{2.75}$$

We use a chaining argument. Assign to each $y \in F^m_{l+i} \cap B_d\,(u_m, \tilde{c}h_l)$ a unique "parent" $\tilde{y} \in F^m_{l+i-1} \cap B_d\,(u_m, \tilde{c}h_l)$ such that $d\,(\tilde{y}, y) \le r_{l+i}$. In particular, for $i = 1$ we set $\tilde{y} = u_m$. Let $q = q(\tilde{y}, y) = d\,(\tilde{y}, y)\,/r_l$ and set

$$A_i = \left\{\sup_{y\in F^m_{l+i}\cap B_d(u_m,\tilde{c}h_l)} \left|\mathcal{T}^{\tau}_{y,\widetilde{r}_l} - \mathcal{T}^{\tau}_{\tilde{y},\widetilde{r}_l}\right| \le d_0 ji\,(\alpha_{+,z}(l) - j)\,\sqrt{q}\right\}, \tag{2.76}$$

where d_0 will be chosen later, but small enough that $d_0 \sum_{i\geq 1} ie^{-i/2} \leq \frac{1}{8}$.

We note that as i increases y and $\tilde{y}$ will be closer together so we expect $\left|\mathcal{T}^{\tau}_{y,\widetilde{r}_l} - \mathcal{T}^{\tau}_{\tilde{y},\widetilde{r}_l}\right|$ to decrease, and on the right $i\sqrt{q}$ is also decreasing in i, but we are now taking the sup over a larger set. As we will see, this combination will allow us to complete the chaining argument to prove (2.75).

We now show that

$$\begin{aligned}&\bigcap_{i=1}^{k-l} A_i \cap \left\{\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} < \alpha_{+,z}(l) - \frac{j}{2}\right\} \\ &\subseteq \left\{\sqrt{2\mathcal{T}^{\tau}_{y,\widetilde{r}_l}} < \alpha_{+,z}(l),\ \forall y \in F^m_k \cap B_d\left(u_m, \tilde{c}h_l\right)\right\}.\end{aligned} \tag{2.77}$$

For this, we use $q = d\left(\tilde{y}, y\right)/r_l \leq r_{l+i}/r_l = e^{-i}$ for $y \in F^m_{l+i}$ to see that for any trajectory in the left hand side of (2.77) and all $y \in F^m_k \cap B_d\left(u_m, \tilde{c}h(r_l)\right)$

$$\mathcal{T}^{\tau}_{y,\widetilde{r}_l} \leq \left(\alpha_{+,z}(l) - \frac{j}{2}\right)^2/2 + j\left(\alpha_{+,z}(l) - j\right) d_0 \sum_{i\geq 1} ie^{-i/2},$$

which, since $d_0 \sum_{i\geq 1} ie^{-i/2} \leq \frac{1}{8}$, implies that

$$\begin{aligned}\mathcal{T}^{\tau}_{y,\widetilde{r}_l} &\leq \left(\alpha_{+,z}(l) - \tfrac{j}{2}\right)^2/2 + \tfrac{1}{8} j\left(\alpha_{+,z}(l) - j\right) \\ &= \alpha^2_{+,z}(l)/2 - \alpha_{+,z}(l)j/2 + (\tfrac{j}{2})^2/2 + \tfrac{1}{8}\alpha_{+,z}(l)j - \tfrac{j^2}{8} \\ &< \alpha^2_{+,z}(l)/2.\end{aligned}$$

This establishes (2.77) and taking complements we see that

$$\begin{aligned}\mathcal{B}^{\gamma,k}_{m,l} &= \left\{\exists y \in F^m_k \cap B_d\left(u_m, \tilde{c}h_l\right) \text{ such that } \sqrt{2\mathcal{T}^{\tau}_{y,\widetilde{r}_l}} \geq \alpha_{+,z}(l)\right\} \\ &\subseteq \bigcup_{i=1}^{k-l} A^c_i \cup \left\{\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} \geq \alpha_{+,z}(l) - \frac{j}{2}\right\}.\end{aligned} \tag{2.78}$$

It follows that

$$\mathcal{B}^{\gamma,k}_{m,l} \cap \left\{\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} < \alpha_{+,z}(l) - \frac{j}{2}\right\} \subseteq \bigcup_{i=1}^{k-l} A^c_i. \tag{2.79}$$

We can thus bound $\mathbb{P}\left[\mathcal{B}^{\gamma,k}_{m,l} \cap \left\{\sqrt{2\mathcal{T}^{\tau}_{u_m,\widetilde{r}_l}} < \alpha_{+,z}(l) - \frac{j}{2}\right\} \middle| \widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right]$ by

$$\sum_{i=1}^{k-l} \mathbb{P}\left[\sup_{y \in F^m_{l+i} \cap B_d(u_m, \tilde{c}h_l)} \left|\mathcal{T}^{\tau}_{y,\widetilde{r}_l} - \mathcal{T}^{\tau}_{\tilde{y},\widetilde{r}_l}\right| \geq d_0 ji\left(\alpha_{+,z}(l) - j\right)\sqrt{q} \middle| \widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right]. \tag{2.80}$$

Since $\left|F^m_{l+i} \cap B_d\left(u_m, \tilde{c}h_l\right)\right| \leq ce^{2i}$, a union bound gives that (2.80) is at most

$$\begin{aligned}&c \sum_{i=1}^{k-l} e^{2i} \sup_{y \in F^m_{l+i} \cap B_d(u_m, \tilde{c}h_l)} \\ &\qquad \mathbb{P}\left[\left|\mathcal{T}^{\tau}_{y,\widetilde{r}_l} - \mathcal{T}^{\tau}_{\tilde{y},\widetilde{r}_l}\right| \geq d_0 ji\left(\alpha_{+,z}(l) - j\right)\sqrt{q} \middle| \widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right].\end{aligned} \tag{2.81}$$

We can write the last probability as

$$\mathbb{P}\left[\left|\mathcal{T}^{u_m, r_{l-2}, \mathcal{T}^{\tau}_{u_m,l-2}}_{y,\widetilde{r}_l} - \mathcal{T}^{u_m, r_{l-2}, \mathcal{T}^{\tau}_{u_m,l-2}}_{\tilde{y},\widetilde{r}_l}\right| \geq d_0 ji\left(\alpha_{+,z}(l) - j\right)\sqrt{q} \middle| \widetilde{\mathcal{C}}_{m,l} \cap \mathcal{D}_{m,l}(-j)\right]. \tag{2.82}$$

Using [7, Lemma 5.6] with $\theta = d_0 ji$ and $n = (\alpha_{+,z}(l) - j)^2/2$, we find that for an appropriate choice of $d_0, \tilde{c}$, the last probability is bounded by $ce^{-8ji} \le ce^{-4(j+i)}$ since $i, j \ge 1$. To apply [7, Lemma 5.6] we must verify several points.

First, we need to verify that for some small $\bar{c}_0$ we have $\theta \le \bar{c}_0(n-1)$, that is $d_0 ji \le \bar{c}'_0(\alpha_{+,z}(l) - j)^2$. For this it suffices to note that for j, l in our range $i/(\alpha_{+,z}(l) - j) \le (k-l)/(\alpha_{+,z}(l) - j) = (\log 2(\alpha_{+,z}(l) - j))/(\alpha_{+,z}(l) - j)$ goes to 0 as $L \to \infty$.

Secondly, we need to show that $\theta \le ((n-1)q)^2$. Since we have already seen that $\theta \le \bar{c}_0(n-1)$, it suffices to show that $(n-1)q^2 \ge c_2^2$ for some $c_2 > 0$, or equivalently that $\sqrt{2n}\, q \ge c'_2 > 0$. That is, $(\alpha_{+,z}(l) - j) d\left(\tilde{y}, y\right)/r_l \ge c'_2$. Assume that $d\left(\tilde{y}, y\right) \ge c_3 r_k$ for a small $c_3 > 0$, so that, see (2.64),

$$(\alpha_{+,z}(l) - j) d\left(\tilde{y}, y\right)/r_l \ge c_3(\alpha_{+,z}(l) - j) e^{-(k-l)} = c_3/2.$$

With the F_l constructed appropriately we can indeed assume that either $d\left(\tilde{y}, y\right) \ge c_3 r_k$ for a small $c_3 > 0$, or that $y = \tilde{y}$, in which case the corresponding term in the sum in (2.81) is zero. Also, by taking $\tilde{c} = q_0/2$ we will have $d\left(\tilde{y}, y\right)/r_l \le q_0$.

Thus we see that (2.81) is at most

$$c \sum_{i=1}^{k-l} e^{2i} e^{-4(j+i)} \le C e^{-4j}. \tag{2.83}$$

This completes the proof of (2.75). □

Proof of (2.54). As in (2.62)

$$\mathcal{T}^{\tau}_{u_m, \tilde{r}_l} \ge \mathcal{T}^{\tau}_{y,l} \text{ for all } y \text{ such that } d\left(y, u_m\right) \le \frac{\gamma r_l}{2}, \tag{2.84}$$

where we take γ to be some fixed small number. Hence under $\mathcal{B}_{m,l}$ we have $\sqrt{2\mathcal{T}^{\tau}_{u_m, \tilde{r}_l}} \ge \alpha_{z,+}(l)$. The fact that for all $l \ge L - (4 \log L)^4$

$$\mathbb{P}\left[\sqrt{2\mathcal{T}^{\tau}_{u_m, l-2}} \le \frac{1}{2}\alpha_{z,+}(l-2),\ \sqrt{2\mathcal{T}^{\tau}_{u_m, \tilde{r}_l}} \ge \alpha_{z,+}(l)\right] \le ce^{-c' L^2} \tag{2.85}$$

then follows easily as in the proof of (2.68). (In fact, the proof uses the same ideas but is much easier). □

3 Lower bounds for excursions

In this section we will prove the following.

Lemma 3.1. *There exist $0 < c_1, c_2 < \infty$ such that for all L large and all $0 \le z \le \log L$,*

$$\mathbb{P}\left[\sup_{y \in F_L} \sqrt{2\mathcal{T}^{\tau}_{y,L}} \ge \rho_L L + z\right] \ge c_1 \frac{(1+z)e^{-2z}}{(1+z)e^{-2z} + c_2}. \tag{3.1}$$

This will immediately give the lower bounds in Theorems 1.6 and 1.5 and hence complete the proofs of those Theorems.

Note that for any z_0 it suffices to show that (3.1) holds for all $z_0 \le z \le \log L$, since by adjusting c_1 we then get (3.1) for all $0 \le z \le \log L$.

Let

$$\beta_z\left(l\right) = \rho_L l + z, \tag{3.2}$$

and

$$\alpha_{z,-}\left(l\right) = \alpha_{z,-}\left(l, L, z\right) = \rho_L l + z - l_L^{1/4}. \tag{3.3}$$

For each $k \geq 1$ we define $\mathcal{T}_{y,l}^{k;m}$ be the number of excursions from $\partial B_d\left(y, h_{l-1}\right)$ to $\partial B_d\left(y, h_l\right)$ during the first m excursions from $\partial B_d\left(y, h_k\right)$ to $\partial B_d\left(y, h_{k-1}\right)$. We abbreviate $\mathcal{T}_{y,l}^1 = \mathcal{T}_{y,l}^{1,x^2}$ with x fixed.

Choose r_0 in (2.2) sufficiently small that $4h(r_{-1}) \leq r^*$. (Recall that τ is the first hitting time of $\partial B_d(v, r^*)$.) Let $\widehat{r} = h_1/20$, and with $F^0 := B_d(v, \widehat{r})$ we set

$$F_L^0 = F^0 \cap F_L, \qquad \text{so that} \qquad c_1 e^{2L} \leq |F_L^0| \leq c_2 e^{2L}, \tag{3.4}$$

where we can take c_1, c_2 independent of r_0. Compare (2.4).

In this section we show that

Lemma 3.2. *There exists a $0 < c < \infty$ such that for all $0 < r_0$ sufficiently small, L large and all $0 \leq z \leq \log L$,*

$$\mathbb{P}\left[\sup_{y \in F_L^0} \sqrt{2\mathcal{T}_{y,L}^1} \geq \rho_L L + z\right] \geq \frac{(1+z)e^{-2z}}{(1+z)e^{-2z} + c}. \tag{3.5}$$

Since the probability of x^2 excursions from $\partial B_d(v, h_1 - \widehat{r})$ to $\partial B_d(v, h_0 + \widehat{r})$ before τ is greater than 0 and does not depend on L, (3.5) will imply Lemma 3.1. We note that the r_0 used in this Lemma, and hence all h_l, are smaller than the corresponding quantities used until now. This is for notational convenience and, as can easily be seen, does not affect Lemma 3.1 which concerns large L. We could have kept the original r_0 and in place of h_l used h_{l+k} for some fixed k, but this would have made the notation cumbersome.

The proof of Lemma 3.2 uses a modified second moment method and occupies the rest of this section.

We introduce the events $\mathcal{I}_{y,z}$, beginning with a barrier event. Let

$$\widehat{\mathcal{I}}_{y,z} = \left\{\sqrt{2\mathcal{T}_{y,l}^1} \leq \alpha_{z,-}(l) \text{ for } l = 1, \ldots, L-1 \text{ and } \sqrt{2\mathcal{T}_{y,L}^1} \geq \rho_L + z\right\}, \tag{3.6}$$

for $y \in F_L$. As discussed in [7], we need to augment $\widehat{\mathcal{I}}_{y,z}$ by information on the angular increments of the excursions. Instead of keeping track of individual excursions, we track the empirical measure of the increments, by comparing it in Wasserstein distance to a reference measure. This will suffice for the decoupling arguments used in [7, Section 4.5] which we will use. Recall that the Wasserstein L^1-distance between probability measures on $\mathbf{R}$ is given by

$$d_{\text{Wa}}^1(\mu, \nu) = \inf_{\xi \in \mathcal{P}^2(\mu, \nu)} \left\{\int |x - y| \, d\xi(x, y)\right\}, \tag{3.7}$$

where $\mathcal{P}^2(\mu, \nu)$ denotes the set of probability measures on $\mathbf{R} \times \mathbf{R}$ with marginals μ, ν. If μ is a probability measure on $\mathbf{R}$ with finite support and if θ_i, $1 \leq i \leq n$ denotes a sequence of i.i.d μ-distributed random variables then it follows from [19, Theorem 2] that for some $c_0 = c_0(\mu)$

$$\text{Prob}\left\{d_{\text{Wa}}^1\left(\frac{1}{n}\sum_{i=1}^n \delta_{\theta_i}, \mu\right) > \frac{c_0 x}{\sqrt{n}}\right\} \leq 2e^{-x^2}. \tag{3.8}$$

Let W_t be Brownian motion in the plane. For each k let ν_k be the probability measure on $[0, 2\pi]$ defined by

$$\nu_k(dx) = P^{(r_k, 0)}\left(\text{arg } W_{H_{\partial B(0, r_{k-1})}} \in dx\right), \tag{3.9}$$

where arg x for $x \in \mathbf{R}^2$ is the argument of x measured from the positive x-axis and P^w is the law of $W_\cdot$ started from w.

Returning to X_t, our Brownian motion on the sphere, and using isothermal coordinates, see [7, Section 2], let $0 \le \theta_{k,i} \le 2\pi$, $i = 1, 2, \ldots$ be the angular increments centered at y, mod 2π, from $X_{H^i_{\partial B(y,h_k)}}$ to $X_{H^i_{\partial B(y,h_{k-1})}}$, the endpoints of the i'th excursion between $\partial B(y, h_k)$ and $\partial B(y, h_{k-1})$. By the Markov property the $\theta_{k,i}$, $i = 1, 2, \ldots$ are independent, and using [7, Section 2] we see that each $\theta_{k,i}$ has distribution ν_k. We set, for n a positive integer,

$$\mathcal{W}_{y,k}(n) = \left\{ d^1_{\text{Wa}} \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_{k,i}}, \nu_k \right) \le \frac{c_0 \log(L-k)}{2\sqrt{n}} \right\}. \tag{3.10}$$

We are ready to define the good events $\mathcal{I}_{y,z}$. For $a \in \mathbf{Z}_+$ let

$$N_{k,a} = [(\rho_L k + z - a + 1)^2/2]. \tag{3.11}$$

We set

$$N_k = N_{k,a} \quad \text{if} \quad \sqrt{2\mathcal{T}^1_{y,k}} \in I_{\rho_L k + z - a}, \tag{3.12}$$

where $I_s = [s, s+1]$. With $L_+ = L - (500 \log L)^4$ and d^* a constant to be determined below, let

$$\mathcal{I}_{y,z} = \widehat{\mathcal{I}}_{y,z} \cap_{k=L_+}^{L-d^*} \mathcal{W}_{y,k}\left(N_k\right), \tag{3.13}$$

and define the count

$$J_z = \sum_{y \in F^0_L} \mathbf{1}_{\mathcal{I}_{y,z}}. \tag{3.14}$$

To obtain (3.5), we need a control on the first and second moments of J_z, which is provided by the next two lemmas. In fact, (3.5) will follow directly from these two Lemmas as in the proof of [7, Proposition 4.2], taking into account that $|F^0_L|$ does not depend on r_0. Most of this section is devoted to their proof. We emphasize that in the statements of the lemma, the implied constants are uniform in r_0 smaller than a fixed small threshold.

Lemma 3.3 (First moment estimate)**.** *There is a large enough d^*, such that for all L sufficiently large, all $0 \le z \le \log L$, and all $y \in F^0_L$,*

$$\mathbb{P}\left(\mathcal{I}_{y,z}\right) \asymp (1+z) e^{-2L} e^{-2z}. \tag{3.15}$$

Let

$$\begin{aligned} G_0 &= \left\{(y,y') : y, y' \in F_L \text{ s.t. } d\left(y,y'\right) > 2h_0\right\}, \\ G_k &= \left\{(y,y') : y, y' \in F_L \text{ s.t. } 2h_k < d\left(y,y'\right) \le 2h_{k-1}\right\} \text{ for } 1 \le k < L, \\ G_L &= \left\{(y,y') : y, y' \in F_L \text{ s.t. } 0 < d\left(y,y'\right) \le 2h_{L-1}\right\}. \end{aligned} \tag{3.16}$$

Recall from (2.14) that $k_L = k \wedge (L-k)$.

Lemma 3.4 (Second moment estimate)**.** *There are large enough d^*, c', such that for all L sufficiently large, all $0 \le z \le \log L$ and all $(y, y') \in G_k$, $1 \le k \le L$,*

$$\mathbb{P}\left(\mathcal{I}_{y,z} \cap \mathcal{I}_{y',z}\right) \le c'(1+z) e^{-4L+2k} e^{-2z} e^{-c k_L^{1/4}}. \tag{3.17}$$

Before turning to the proofs, we introduce some notation and record some simple estimates that will be useful in calculations. Recall (3.2), (3.11)-(3.12) and for $a \in \mathbf{Z}_+$ let

$$\mathcal{H}_{k,a} = \left\{ \sqrt{2\mathcal{T}^1_{y,k}} \in I_{\rho_L k + z - a} \right\} = \left\{ \sqrt{2\mathcal{T}^1_{y,k}} \in I_{\beta_z(k) - a} \right\}. \tag{3.18}$$

Note that on $\mathcal{H}_{k,a}$ we have $N_k = N_{k,a}$.

Before proceeding we need to state a deviation inequality of Gaussian type for the Galton-Watson process $T_l, l \geq 0$ under P_n^{GW}, the law of a critical Galton-Watson process with geometric offspring distribution with initial offspring n. The proof is very similar to [6, Lemma 4.6], and is therefore omitted.

Lemma 3.5. *For all $n = 1, 2, 3, \ldots$, and all l,*

$$P_n^{\mathrm{GW}}\left(\left|\sqrt{2T_l} - \sqrt{2T_0}\right| \geq \theta\right) \leq ce^{-\frac{\theta^2}{2l}}, \quad \theta \geq 0. \tag{3.19}$$

Using (2.5) and the strong Markov property, it is easy to see that

$$\text{the } \mathbb{P}\text{-law of } T_l^{x,n}, l \geq 0, \text{ is } P_n^{\mathrm{GW}}. \tag{3.20}$$

Therefore, we obtain the following estimates from Lemma 3.5, for $\theta \in \mathbf{R}$:

$$\mathbb{P}\left(\sqrt{2T_l^{x,n^2/2}} \leq \theta\right) \leq ce^{-(n-\theta)^2/2l}, \text{ if } \theta \leq n, \tag{3.21}$$

and

$$\mathbb{P}\left(\sqrt{2T_l^{x,n^2/2}} \geq \theta\right) \leq ce^{-(n-\theta)^2/2l}, \text{ if } \theta \geq n. \tag{3.22}$$

In the proof of our moment estimates we will need the following.

Lemma 3.6. *For any $a, b \leq L/\log L$ and $k < L$*

$$\mathbb{P}\left[\sqrt{2\mathcal{T}_{y,L}^{k,(\beta_z(k)-a)^2/2}} \geq \rho_L L + z - b\right] \tag{3.23}$$
$$\leq ce^{-2(L-k)-2(a-b)-\frac{(a-b)^2}{2(L-k)}} L^{2\frac{(L-k)}{L}}.$$

Proof. By (3.22) we have that for all $\theta > n \geq 1$

$$\mathbb{P}\left[\mathcal{T}_{y,L}^{k,n^2/2} \geq \theta^2/2\right] \leq c\exp\left(-\frac{(\theta-n)^2}{2(L-k)}\right). \tag{3.24}$$

We apply this with $\theta = \rho_L L + z - b$ and

$$n = \beta_z(k) - a = \rho_L k + z - a$$

so that

$$\theta - n = \rho_L (L-k) + a - b,$$

and hence

$$\frac{(\theta-n)^2}{2(L-k)} \geq 2(L-k) - 2\frac{(L-k)}{L}(\log L) + 2(a-b) + \frac{(a-b)^2}{2(L-k)} + O_L(1).$$

This gives (3.23). □

3.1 First moment estimate

In this subsection we prove Lemma 3.3.

For the lower bound we have that

$$\mathbb{P}\left[\mathcal{I}_{y,z}\right] \geq \mathbb{P}\left[\widehat{\mathcal{I}}_{y,z}\right] - \sum_{k=L_+}^{L-d^*} \mathbb{P}\left[\widehat{\mathcal{I}}_{y,z} \bigcap W_{y,k}^c(N_k)\right] \tag{3.25}$$
$$\geq c(1+z)e^{-2L}e^{-2z} - \sum_{k=L_+}^{L-d^*} \mathbb{P}\left[\widehat{\mathcal{I}}_{y,z} \bigcap W_{y,k}^c(N_k)\right],$$

where for $P\left[\widehat{\mathcal{I}}_{y,z}\right]$ we have used the barrier estimate (9.12) of Appendix I. We note that

$$\mathbb{P}\left[\widehat{\mathcal{I}}_{y,z}\bigcap W^c_{y,k}\left(N_k\right)\right]\leq\sum_{a\geq k_L^{1/4}}\mathbb{P}\left(\widehat{\mathcal{I}}^{k,a}_{y,z}\right), \tag{3.26}$$

where

$$\widehat{\mathcal{I}}^{k,a}_{y,z}=\widehat{\mathcal{I}}_{y,z}\bigcap\mathcal{H}_{k,a}\bigcap W^c_{y,k}\left(N_{k,a}\right). \tag{3.27}$$

We show below that for all $L_+\leq k\leq L-d^*$ and $0\leq z\leq\log L$,

$$\sum_{a\geq k_L^{1/4}}\mathbb{P}\left(\widehat{\mathcal{I}}^{k,a}_{y,z}\right)\leq c'(1+z)\left(e^{-2L}e^{-2z}\right)e^{-c\log^2(L-k)}, \tag{3.28}$$

which will finish the proof of the lower bound for (3.15) for d^* sufficiently large.

Furthermore, it is easily seen using (3.23) and the fact that $L-k\leq(500\log L)^4$ that the sum in (3.28) over $a\geq L^{3/4}$ is much smaller than the right hand side of (3.28), hence it suffices to show that

$$\sum_{a\geq k_L^{1/4}}^{L^{3/4}}\mathbb{P}\left(\widehat{\mathcal{I}}^{k,a}_{y,z}\right)\leq c'(1+z)\left(e^{-2L}e^{-2z}\right)e^{-c\log^2(L-k)}, \tag{3.29}$$

We now turn to the proof of (3.29). Let

$$J^{\uparrow}_{y,k}=\left\{\sqrt{2\mathcal{T}^1_{y,l}}\leq\rho_L l+z\text{ for }l=1,\ldots,k\right\}, \tag{3.30}$$

and

$$\mathcal{B}_{y,k,a}=\left\{\sqrt{2\mathcal{T}^{k,(\beta_z(k)-a)^2/2}_{y,L}}\geq\rho_L L+z\right\}. \tag{3.31}$$

Then with

$$\mathcal{K}_{k,p,a}=J^{\uparrow}_{y,k-3}\bigcap\mathcal{H}_{k-3,p}\bigcap\mathcal{H}_{k,a}\bigcap W^c_{y,k}\left(N_{k,a}\right)\bigcap\mathcal{B}_{y,k,a}$$

we have

$$\mathbb{P}\left(\widehat{\mathcal{I}}^{k,a}_{y,z}\right)\leq\sum_{p\geq(k-3)_L^{1/4}}^{L^{3/4}}\mathbb{P}\left(\mathcal{K}_{k,p,a}\right), \tag{3.32}$$

plus a term which is much smaller than the right hand side of (3.28).

Let

$$\mathcal{W}^{\in x}_{y,k}(n)=\left\{d^1_{\mathrm{Wa}}\left(\frac{1}{n}\sum_{i=1}^n\delta_{\theta_{k,i}},\nu_k\right)\in\frac{c_0}{\sqrt{n}}I_x\right\}, \tag{3.33}$$

so that

$$\mathcal{W}^c_{y,k}(N_{k,a})\subseteq\cup_{m=\log(L-k)}^{\infty}\mathcal{W}^{\in m}_{y,k}(N_{k,a}), \tag{3.34}$$

and consequently, setting

$$\mathcal{L}_{k,m,p,a}=\mathcal{K}_{k,p,a}\cap\mathcal{W}^{\in m}_{y,k}(N_{k,a}), \tag{3.35}$$

we have

$$\mathbb{P}\left(\mathcal{K}_{k,p,a}\right)\leq\sum_{m=\log(L-k)}^{\infty}\mathbb{P}\left(\mathcal{L}_{k,m,p,a}\right). \tag{3.36}$$

Let

$$\mathcal{L}'_{k,m,p,a}=:J^{\uparrow}_{y,k-3}\bigcap\mathcal{H}_{k-3,p}\bigcap\mathcal{H}_{k,a}\bigcap W^{\in m}_{y,k}\left(N_{k,a}\right).$$

To prove (3.28) it suffices to prove that for all $m\geq\log(L-k)$,

$$\sum_{a\geq k_L^{1/4}}^{L^{3/4}}\sum_{p\geq(k-3)_L^{1/4}}^{L^{3/4}}\mathbb{P}\left(\mathcal{B}_{y,k,a}\cap\mathcal{L}'_{k,m,p,a}\right)\leq c'(1+z)\left(e^{-2L}e^{-2z}\right)e^{-cm^2}. \tag{3.37}$$

Lemma 3.7.

$$\mathbb{P}\left(\mathcal{L}'_{k,m,p,a}\right) = \mathbb{P}\left(J^{\uparrow}_{y,k-3}\bigcap\mathcal{H}_{k-3,p}\bigcap\mathcal{H}_{k,a}\bigcap W^{\in m}_{y,k}\left(N_{k,a}\right)\right) \tag{3.38}$$
$$\leq C(1+z)\left(1+p\right)e^{-2k-2(z-p)}e^{-c(p-a)^2}e^{-m^2}.$$

Proof. By (3.8)

$$\mathbb{P}\left(W^{\in m}_{y,k}\left(N_{k,a}\right)\,|\,\mathcal{H}_{k,a}\right) \leq e^{-m^2}. \tag{3.39}$$

By (3.23)

$$\mathbb{P}\left(\mathcal{H}_{k,a}\,|\,\mathcal{H}_{k-3,p}\right) \leq ce^{-c(p-a)^2}, \tag{3.40}$$

and by (9.14) we see that

$$\mathbb{P}\left(J^{\uparrow}_{y,k-3}\bigcap\mathcal{H}_{k-3,p}\right) \leq C(1+z)\left(1+p\right)e^{-2k-2(z-p)}. \tag{3.41}$$

□

The presence of $W^{\in m}_{y,k}\left(N_{k,a}\right)$ in $\mathcal{L}'_{k,m,p,a}$ will allow us to effectively decouple $\mathcal{B}_{y,k,a}$ from $\mathcal{L}'_{k,m,p,a}$. More precisely, it follows as in the proof of [7, Lemma 4.7] that for some $M_0 < \infty$

$$\mathbb{P}\left(\mathcal{B}_{y,k,a}\cap\mathcal{L}'_{k,m,p,a}\right) \leq \mathbb{P}\left\{\sqrt{2\mathcal{T}^{k,(\beta_z(k)-a)^2/2}_{y,L}} \geq \rho_L L + z - M_0 m\right\} \tag{3.42}$$
$$\times\mathbb{P}\left(\mathcal{L}'_{k,m,p,a}\right) + e^{-4L}.$$

We note that by (3.23)

$$\mathbb{P}\left\{\sqrt{2\mathcal{T}^{k,(\beta_z(k)-a)^2/2}_{y,L}} \geq \rho_L L + z - M_0 m\right\} \leq ce^{-2(L-k)-2(a-M_0 m)-\frac{(a-M_0 m)^2}{2(L-k)}}, \tag{3.43}$$

Putting this all together with (3.38), and using $|a-p| \leq 1+(p-a)^2$ we find that

$$\mathbb{P}\left(\mathcal{B}_{y,k,a}\cap\mathcal{L}'_{k,m,p,a}\right) \leq C(1+z)e^{-2L-2z}e^{-m^2/2}\left(1+p\right)e^{-c(p-a)^2}e^{-\frac{a^2}{2(L-k)}}. \tag{3.44}$$

Summing first over p and then over a it is easy to see, using a fraction of the exponent $m^2/2$, that (3.37) holds for all $m \geq \log(L-k)$. This completes the proof of the lower bound in (3.15).

Since $\mathcal{I}_{y,z} \subseteq \widehat{\mathcal{I}}_{y,z}$ the upper bound in (3.15) follows from the barrier estimate (9.11) of Appendix I. □

3.2 Second moment estimate: branching in the bulk

We prove the second moment estimate for $y, y' \in F^0_L$ with

$$2h_{k-1} < d(y,y') \leq 2h_{k-2}.$$

In this subsection we prove Lemma 3.4 for

$$(500\log L)^4 < k \leq L - (500\log L)^4. \tag{3.45}$$

Here we will not have to keep track of the angles.

We need to “give ourselves a bit of space”, and we therefore define

$$k^+ = k + \lceil 100\log L\rceil. \tag{3.46}$$

Let

$$\widehat{\mathcal{I}}_{y,z;k\pm3} = \left\{\sqrt{2\mathcal{T}^1_{y,l}} \leq \rho_L l + z;\; l = 1,\ldots,k-4,k+4,\ldots,L-1\right\}$$

$$\cap\left\{\sqrt{\mathcal{T}^{1}_{y,L}}\geq \rho_L L+z\right\}, \tag{3.47}$$

where we have skipped the barrier condition for $k-3,\dots,k+3$. To obtain the two point bound for the range (3.45) we will bound the probability of

$$\widehat{\mathcal{I}}_{y,z;k\pm 3}\cap\left\{\sqrt{2\mathcal{T}^{k^+,\alpha^2_{z,-}(k^+)/2}_{y',L}}\geq \rho_L L+z\right\}, \tag{3.48}$$

which contains the event $\mathcal{I}_{y,z}\cap\mathcal{I}_{y',z}$.

Let $\mathcal{G}^{y'}$ denote the σ-algebra generated by the excursions from $\partial B_d(y',h_{k-1})$ to $\partial B_d(y',h(r_{k^+}))$. Note that $\widehat{\mathcal{I}}_{y,z;k\pm 3}\in\mathcal{G}^{y'}$. Since

$$\left\{\sqrt{2\mathcal{T}^{k^+,\alpha^2_{z,-}(k^+)/2}_{y',L}}\geq \rho_L L+z\right\}$$

is measurable with respect to the first $\alpha^2_{z,-}(k^+)$ excursions from $\partial B_d(y',h(r_{k^+}))$ to $\partial B_d(y',h(r_{k^+-1}))$, we can effectively decouple $\widehat{\mathcal{I}}_{y,z;k\pm 3}$ from $\left\{\sqrt{2\mathcal{T}^{k^+,\alpha^2_{z,-}(k^+)/2}_{y',L}}\geq \rho_L L+z\right\}$. More precisely, it follows from the basic ideas in [6, sub-section 6.2] that

$$\begin{aligned}&\mathbb{P}\left[\widehat{\mathcal{I}}_{y,z;k\pm 3},\ \sqrt{2\mathcal{T}^{k^+,\alpha^2_{z,-}(k^+)/2}_{y',L}}\geq \rho_L L+z\right] \\ &\leq c\mathbb{P}\left[\widehat{\mathcal{I}}_{y,z;k\pm 3}\right]P\left[\sqrt{2\mathcal{T}^{k^+,\alpha^2_{z,-}(k^+)/2}_{y',L}}\geq \rho_L L+z\right].\end{aligned} \tag{3.49}$$

By Lemma 9.3

$$\mathbb{P}\left[\widehat{\mathcal{I}}_{y,z;k\pm 3}\right]\leq c(1+z)e^{-2L}e^{-2z}. \tag{3.50}$$

Using (3.23) for the last term in (3.49) together with the fact that in the range (3.45) we have $(k^+)^{1/4}_L\geq 500\log L$, we find that (3.49) is bounded by

$$\begin{aligned}&c(1+z)e^{-2L}e^{-2z}e^{-2(L-k^+)-2(k^+)^{1/4}_L}L^2, \\ &\leq c(1+z)e^{-2L}L^{202}e^{-2(L-k)-2(k^+)^{1/4}_L}e^{-2z}. \\ &\leq c(1+z)e^{-2L}e^{-2(L-k)-k^{1/4}_L}e^{-2z}.\end{aligned} \tag{3.51}$$

3.3 Second moment estimate: early branching

In this subsection we prove Lemma 3.4 for

$$1\leq k<(500\log L)^4.$$

Since we no longer have $k^{1/4}_L\geq \log L$ we will have to use barrier estimates to control the factors of L such as arise in the first line of (3.51). On the other hand, since the number of excursions at lower levels is not so great we don't need such a large separation. Let

$$\widetilde{k}=k+\lceil 100\log k\rceil,\qquad k_z=k+\lceil 100\log z\rceil. \tag{3.52}$$

For $v\in\{y,y'\}$

$$\begin{aligned}J^{\downarrow}_{v,s,\widetilde{k}}\ =\ &\left\{\sqrt{2\mathcal{T}^{\widetilde{k},s^2/2}_{v,l}}\leq \rho_L l+z \text{ for } l=\widetilde{k}+1,\dots,L-1;\right. \\ &\left.\sqrt{2\mathcal{T}^{\widetilde{k},s^2/2}_{v,L}}\geq \rho_L L+z\right\},\end{aligned}$$

with the barrier condition applied only for $l \geq \widetilde{k}$.

We first consider the case where $z \leq 100k$. Then

$$\mathbb{P}\left(I_{y,z} \cap I_{y',z}\right) \leq \sum_{n=1}^{\alpha_{z,-}(\widetilde{k})} \mathbb{P}\left(J^{\downarrow}_{y,n,\widetilde{k}} \cap \widehat{\mathcal{I}}_{y',z;k\pm 3}\right). \tag{3.53}$$

Let $\mathcal{G}^y$ denote the σ-algebra generated by the excursions from $\partial B_d(y, h_{k-1})$ to $\partial B_d(y, h(r_{\widetilde{k}}))$. Note that $\widehat{\mathcal{I}}_{y',z;k\pm 3} \in \mathcal{G}^y$. Since, under our assumption that $z \leq 100k$, the number of excursions from $\partial B_d(y, h_{k-1})$ to $\partial B_d(y, h(r_{\widetilde{k}}))$ is dominated by $n = O(k^2)$, it follows as in (3.49) that

$$\mathbb{P}\left(J^{\downarrow}_{y,n,\widetilde{k}} \cap \widehat{\mathcal{I}}_{y',z;k\pm 3}\right) \leq cP\left(J^{\downarrow}_{y,n,\widetilde{k}}\right)\mathbb{P}\left(\widehat{\mathcal{I}}_{y',z;k\pm 3}\right). \tag{3.54}$$

By the barrier estimate (9.16) of Appendix I, with $n = \beta_z(\widetilde{k}) - t$

$$\mathbb{P}\left(J^{\downarrow}_{y,n,\widetilde{k}}\right) \leq ct\, n^{1/2} e^{-2(L-\widetilde{k})-2t} \leq ck^{202} e^{-2(L-k)-2k_L^{1/4}}. \tag{3.55}$$

where the last step followed from the fact that $k_L^{1/4} \leq \widetilde{k}_L^{1/4} \leq t \leq \beta_z(\widetilde{k}) \leq ck$. Since, under our assumption that $z \leq 100k$, the number of terms in (3.53) is $\leq ck^2$, and using (3.50), we find that (3.53) is bounded by

$$\begin{aligned} & ck^{204} e^{-2(L-k)-2k_L^{1/4}} (1+z) e^{-2L} e^{-2z} \\ \leq\ & c(1+z) e^{-4L+2k-k_L^{1/4}} e^{-2z}. \end{aligned} \tag{3.56}$$

Thus we can assume that

$$z \geq 100k. \tag{3.57}$$

We have

$$\begin{aligned} & \mathbb{P}\left(I_{y,z} \cap I_{y',z}\right) \\ &= \sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z)-t;\, t\geq z/2\}} \mathbb{P}\left(\left\{\sqrt{2\mathcal{T}^1_{y,k_z}} = n\right\} \cap I_{y,z} \cap I_{y',z}\right) \\ &+ \sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z)-t;\, t< z/2\}} \mathbb{P}\left(\left\{\sqrt{2\mathcal{T}^1_{y,k_z}} = n\right\} \cap I_{y,z} \cap I_{y',z}\right) \end{aligned} \tag{3.58}$$

Since in the above sums $n \leq cz$ in view of (3.57), we can bound the first sum in (3.58) by

$$\begin{aligned} & \sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z)-t;\, t\geq z/2\}} \mathbb{P}\left(J^{\downarrow}_{y,n,k_z} \cap \widehat{\mathcal{I}}_{y',z;k\pm 3}\right) \\ &\leq c \sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z)-t;\, t\geq z/2\}} \mathbb{P}\left(J^{\downarrow}_{y,n,k_z}\right)\mathbb{P}\left(\widehat{\mathcal{I}}_{y',z;k\pm 3}\right) \\ &\leq c \sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z)-t;\, t\geq z/2\}} \mathbb{P}\left(J^{\downarrow}_{y,n,k_z}\right)(1+z)e^{-2L}e^{-2z}, \end{aligned} \tag{3.59}$$

as before. Instead of (3.55) we now have

$$\mathbb{P}\left(J^{\downarrow}_{y,n,k_z}\right) \leq ct\, z^{1/2} e^{-2(L-k_z)-2t} \leq cz^{202} e^{-2(L-k)-z/2}, \tag{3.60}$$

where the last inequality used $t \geq z/2$. In view of (3.57) and the fact that the number of terms in the sum is $\leq cz$, this gives the desired bound for the first sum in (3.58).

Note next that if $t < z/2$ then we must have $n = \beta_z(k_z) - t \geq z/2$, (but we still have $n \leq cz$ by (3.57)). Thus we can bound the second sum in (3.58) by

$$\sum_{n,n'=1}^{\alpha_{z,-}(k_z)} 1_{\{n \geq z/2\}} \mathbb{P}\left(\left\{\sqrt{2\mathcal{T}^1_{y,k_z}} = n\right\} \cap J^{\downarrow}_{y,n,k_z} \cap J^{\downarrow}_{y',n',k_z}\right) \tag{3.61}$$
$$\leq c \sum_{n,n'=1}^{\alpha_{z,-}(k_z)} 1_{\{n \geq z/2\}} \mathbb{P}\left(\left\{\sqrt{2\mathcal{T}^1_{y,k_z}} = n\right\} \cap J^{\downarrow}_{y,n,k_z}\right) \mathbb{P}\left(J^{\downarrow}_{y',n',k_z}\right).$$

as before. Then by the Markov property, this is bounded by

$$c \sum_{n,n'=1}^{\alpha_{z,-}(k_z)} 1_{\{n \geq z/2\}} \mathbb{P}\left(\left\{\sqrt{2\mathcal{T}^1_{y,k_z}} = n\right\}\right) \mathbb{P}\left(J^{\downarrow}_{y,n,k_z}\right) \mathbb{P}\left(J^{\downarrow}_{y',n',k_z}\right). \tag{3.62}$$

By (3.21)-(3.22) with $n \geq z/2$ and then (3.57)

$$\mathbb{P}\left(\left\{\sqrt{2\mathcal{T}^1_{y,k_z}} = n\right\}\right) \leq e^{-z^2/4k_z} \leq e^{-10z}, \tag{3.63}$$

while now, instead of (3.60), we use

$$\mathbb{P}\left(J^{\downarrow}_{y,n,k_z}\right) \leq ct\, z^{1/2} e^{-2(L-k_z)-2t} \leq cz^{202} e^{-2(L-k)}, \tag{3.64}$$

and a similar bound for $\mathbb{P}\left(J^{\downarrow}_{y',n',k_z}\right)$. Thus (3.62) is bounded by

$$c \sum_{n,n'=1}^{\alpha_{z,-}(k_z)} e^{-10z} z^{404} e^{-4(L-k)} \leq ce^{-10z} z^{408} e^{2k} e^{-4L+2k}.$$

In view of (3.57), this gives the desired bound for the second sum in (3.58).

3.4 Second moment estimate: late branching

In this subsection we prove Lemma 3.4 for $L - (500 \log L)^4 \leq k < L - 1$.

Consider first the case

$$L - (500 \log L)^4 \leq k < L - d^*.$$

We will bound the probability of

$$\mathcal{A} = \left\{\sqrt{2\mathcal{T}^{k,\alpha^2_{z,-}(k)/2}_{y,L}} \geq \rho_L L + z\right\} \cap \mathcal{W}_{y,k}(N_k) \cap \widehat{\mathcal{I}}_{y',z;k\pm 3}, \tag{3.65}$$

(which contains the event $\mathcal{I}_{y,z} \cap \mathcal{I}_{y',z}$).

The presence of $\mathcal{W}_{y,k}(N_k)$ in $\mathcal{A}$ will allow us to effectively decouple the event $\left\{\sqrt{2\mathcal{T}^{k,\alpha^2_{z,-}(k)/2}_{y,L}} \geq \rho_L L + z\right\}$ from $\widehat{\mathcal{I}}_{y',z;k\pm 3}$. More precisely, it follows as in the proof of [7, Lemma 4.7] that for some $M_0 < \infty$

$$\mathbb{P}(\mathcal{A}) \leq \mathbb{P}\left\{\sqrt{2\mathcal{T}^{k,\alpha^2_{z,-}(k)/2}_{y,L}} \geq \rho_L L + z - M_0 \log(L-k)\right\} \tag{3.66}$$
$$\times \mathbb{P}\left(\widehat{\mathcal{I}}_{y',z;k\pm 3}\right) + e^{-4L}.$$

Using (3.23) and (3.50) this shows that

$$\mathbb{P}(\mathcal{A}) \le ce^{-2(L-k)+2M_0\log(L-k)-2k_L^{1/4}}(1+z)e^{-2L}e^{-2z} + e^{-4L} \tag{3.67}$$

By taking d^* sufficiently large we will have $M_0 \log(L-k) \le k_L^{1/4}/2$, which then gives (3.15). □

For $L - d^* \le k < L-1$ we simply bound the term $\mathbb{P}(\mathcal{I}_{y,z} \cap \mathcal{I}_{y',z})$ by $\mathbb{P}(\mathcal{I}_{y,z})$ and obtain from (3.15) the following upper bound

$$\mathbb{P}(\mathcal{I}_{y,z} \cap \mathcal{I}_{y',z}) \le c(1+z)e^{-2L}e^{-2z} \le c_{d^*}(1+z)e^{-(4L-2k)-ck_L^{1/4}}e^{-2z}. \tag{3.68}$$

□

4 Excursion counts and occupation measure on S^2

In this section we prove Theorems 1.3 and 1.4.

For $0 < \epsilon < a < b < \pi$, let $\mathcal{M}_{x,\epsilon,a,b}(n)$ be the total occupation measure of $B_d(x,\epsilon)$ until the end of the first n excursions from $\partial B_d(x,a)$ to $\partial B_d(x,b)$. With $\omega_\epsilon = 2\pi(1-\cos(\epsilon))$, the area of $B_d(x,\epsilon)$, let

$$\overline{\mathcal{M}}_{x,\epsilon,a,b}(n) = \frac{1}{\omega_\epsilon}\mathcal{M}_{x,\epsilon,a,b}(n). \tag{4.1}$$

In particular, when starting from $\partial B_d(x,a)$,

$$\overline{\mathcal{M}}_{x,\epsilon,a,b}(1) = \frac{1}{\omega_\epsilon}\int_0^{H_{\partial B_d(x,b)}} 1_{\{B_d(x,\epsilon)\}}(X_t)\, dt. \tag{4.2}$$

The following Lemma is proven in Section 6.

Lemma 4.1. *For some $c > 0$, uniformly in $x \in S^2$, and $h_k/100 \le \epsilon \le h_k$,*

$$\mathbb{P}\left(\overline{\mathcal{M}}_{x,\epsilon,h_k,h_{k-1}}(n) \le \frac{1}{\pi}(1-\delta)\, n\right) \le e^{-c\delta^2 n} \tag{4.3}$$

and

$$\mathbb{P}\left(\overline{\mathcal{M}}_{x,\epsilon,h_k,h_{k-1}}(n) \ge \frac{1}{\pi}(1+\delta)\, n\right) \le e^{-c\delta^2 n} \tag{4.4}$$

Recall $\bar{\mu}_\tau(y,\epsilon)$ from (1.8) and set

$$t_L(z) = 2L(L - \log L + z). \tag{4.5}$$

Lemma 4.2. *We can find $0 < c, c', z_0 < \infty$ such that for L large, all $z_0 \le z \le \log L$, and all $\epsilon_y, y \in F_L$ such that $h_L/100 \le \epsilon_y \le h_L$,*

$$c' z e^{-2z} \le \mathbb{P}\left(\exists y \in F_L \text{ s.t. } \bar{\mu}_\tau(y,\epsilon_y) \ge \frac{1}{\pi}t_L(z)\right) \le cze^{-2z}. \tag{4.6}$$

For the sphere, it suffices to take $\epsilon_y = \epsilon$ independent of y. The present formulation is needed for the plane, as we will see in Section 8. To clarify the connection with (1.9)-(1.10) we note that for some $0 < c_* = c_*(r_0) < \infty$,

$$(m_{h_L} + z)^2 = \frac{1}{\pi}t_L\left(\sqrt{2\pi}z + c_* + o_L(1)\right). \tag{4.7}$$

In fact, using the last two displays for $h_{L+1} \le \epsilon \le h_L$ would prove the lower bound (1.10), but for the upper bound (1.9) we need the sup over all y not just $y \in F_L$. We will deal with this in Lemma 4.4.

4.1 The upper bound for (4.6)

We first show that, with $F_L^+ = F_L \cap B_d(v, h_{\log L})$,

$$\mathcal{P}_1 =: \mathbb{P}\left(\exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \leq cze^{-2z}. \tag{4.8}$$

If

$$\widehat{\mathcal{A}}_{L,z} = \left\{ \sup_{y \in F_L^+} \sqrt{2\mathcal{T}_{y,L}^\tau} \geq \rho_L L + z \right\},$$

then by (2.11)

$$\begin{aligned} \mathcal{P}_1 &\leq \mathbb{P}\left(\widehat{\mathcal{A}}_{L,z}\right) + \mathbb{P}\left(\widehat{\mathcal{A}}_{L,z}^c, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &\leq ce^{-2z} + \mathbb{P}\left(\widehat{\mathcal{A}}_{L,z}^c, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right). \end{aligned}$$

Recalling the notation $F_L^m = F_L \cap B_d^c(v, h_m) \cap B_d(v, h_{m-1})$, we then bound

$$\begin{aligned} &\mathbb{P}\left(\widehat{\mathcal{A}}_{L,z}^c, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &\leq \sum_{m=\log L}^{L-2} ce^{2(L-m)} \\ &\qquad \sup_{y \in F_L^m} \mathbb{P}\left(\sqrt{2\mathcal{T}_{y,L}^\tau} \leq \rho_L L + z, \, \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &\quad + c \sup_{y \in F_L \cap B_d(v, h_{L-1})} \mathbb{P}\left(\sqrt{2\mathcal{T}_{y,L}^\tau} \leq \rho_L L + z, \, \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right). \end{aligned} \tag{4.9}$$

We treat the case in the sum. The case of $y \in F_L \cap B_d(v, h_{L-1})$ can be treated similarly.

We can write

$$\begin{aligned} &\mathbb{P}\left(\sqrt{2\mathcal{T}_{y,L}^\tau} \leq \rho_L L + z, \, \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &= \sum_{j=1}^{z + ML^{1/2}} P\left(\sqrt{2\mathcal{T}_{y,L}^\tau} \in I_{\rho_L L + z - j} \text{ and } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &+ \mathbb{P}\left(\sqrt{2\mathcal{T}_{y,L}^\tau} \leq \rho_L L - ML^{1/2} \text{ and } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right). \end{aligned} \tag{4.10}$$

Lemma 4.3. *For all $y \in F_L^m$, $\log L \leq m \leq L$ and $j \leq z + ML^{1/2}$*

$$\begin{aligned} &P\left(\sqrt{2\mathcal{T}_{y,L}^\tau} \in I_{\rho_L L + z - j} \text{ and } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &\leq cme^{-2L} Le^{-2(z-j)} e^{-c' j^2}, \end{aligned} \tag{4.11}$$

and

$$\mathbb{P}\left(\sqrt{2\mathcal{T}_{y,L}^\tau} \leq \rho_L L - ML^{1/2} \text{ and } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \leq ce^{-4L}. \tag{4.12}$$

Proof of Lemma 4.3. By (2.9)

$$(\rho_L L + z - j)^2 / 2 \leq t_L\left(z - j + 2M^2\right) \tag{4.13}$$

for all $j \leq z + ML^{1/2}$. Hence for such j

$$\begin{aligned}&\mathbb{P}\left(\sqrt{2\mathcal{T}^{\tau}_{y,L}} \in I_{\rho_L L+z-j} \text{ and } \bar{\mu}_{\tau}\left(y,\epsilon_y\right) \geq \frac{1}{\pi} t_L\left(z\right)\right)\\ &\leq \mathbb{P}\left(\sqrt{2\mathcal{T}^{\tau}_{y,L}} \in I_{\rho_L L+z-j},\ \overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right).\end{aligned}$$

Using the Markov property and then (2.6), we have for $y \in F_L^m$ this is

$$\begin{aligned}&= \mathbb{P}\left(\sqrt{2\mathcal{T}^{\tau}_{y,L}} \in I_{\rho_L L+z-j}\right)\mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right)\\ &\leq cme^{-2L}Le^{-2(z-j)}\mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right).\end{aligned}$$

Consider first the case of $4M^2 \leq j$. We now apply (4.4) with

$$n = t_L\left(z-j+2M^2\right) = t_L\left(z\right) - 2(j-2M^2)L \sim L^2$$

and

$$\delta = 2(j-2M^2)L/t_L\left(z-j+2M^2\right) \ll 1$$

for $4M^2 \leq j \leq z + ML^{1/2}$ to see that

$$\begin{aligned}&\mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \qquad (4.14)\\ &= \mathbb{P}\Big(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right)\\ &\qquad\qquad \geq \frac{1}{\pi}\left(t_L\left(z-j+2M^2\right) + 2(j-2M^2)L\right)\Big)\\ &= \mathbb{P}\Big(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right)\\ &\qquad\qquad \geq \frac{1}{\pi}\left(1 + \frac{2(j-2M^2)L}{t_L\left(z-j+2M^2\right)}\right) t_L\left(z-j+2M^2\right)\Big)\\ &\leq e^{-c\frac{\left((j-2M^2)L\right)^2}{t_L\left(z-j+2M^2\right)}} \leq e^{-c'j^2}.\end{aligned}$$

For $j < 4M^2$ we simply bound the probability in the first line of (4.14) by 1 which we can bound by $Ce^{-c'j^2}$ for C sufficiently large.

Similarly, for (4.12) we use

$$\begin{aligned}&\mathbb{P}\left(\sqrt{2\mathcal{T}^{\tau}_{y,L}} \leq \rho_L L - ML^{1/2} \text{ and } \bar{\mu}_{\tau}\left(y,\epsilon_y\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \qquad (4.15)\\ &\leq \mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(-ML^{1/2}+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \leq e^{-4L}\end{aligned}$$

by (4.14) with $z - j = -ML^{1/2}$, for M sufficiently large. □

Then using (4.9) and Lemma 4.3 we see that

$$\begin{aligned}&\mathbb{P}\left(\widehat{\mathcal{A}}^{c}_{L,z}, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_{\tau}\left(y,\epsilon_y\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \qquad (4.16)\\ &\leq C\sum_{m=\log L}^{L} cmLe^{2(L-m)}\sum_{j=1}^{z+ML^{1/2}} e^{-2L}e^{-2(z-j)}e^{-c'j^2} + \sum_{m=\log L}^{L} ce^{-4L}.\end{aligned}$$

This is easily seen to be bounded by the right hand side of (4.8).

Recalling the notation $F_L^* = F_L \cap B_d^c(v, h_{\log L})$ from (2.17), to complete the proof of the upper bound for (4.6) it remains to show that

$$\mathcal{P}_2 =: \mathbb{P}\left(\exists y \in F_L^* \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \leq cze^{-2z}. \tag{4.17}$$

Note that with k_y as in (2.16), if

$$\mathcal{A}_{L,z} = \left\{\exists y \in F_L^*, l \in \{k_y + 1, \ldots, L\} \text{ s.t. } \mathcal{T}_{y,l}^\tau \geq \alpha_{z,+}^2(l)/2\right\}, \tag{4.18}$$

then

$$\begin{aligned} \mathcal{P}_2 &\leq \mathbb{P}(\mathcal{A}_{L,z}) + \mathbb{P}\left(\mathcal{A}_{L,z}^c, \exists y \in F_L^* \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &\leq cze^{-2z} + \mathbb{P}\left(\mathcal{A}_{L,z}^c, \exists y \in F_L^* \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right), \end{aligned}$$

by (2.18). Recalling again the notation $F_L^m = F_L \cap B_d^c(v, h_m) \cap B_d(v, h_{m-1})$, we have that

$$\begin{aligned} &\mathbb{P}\left(\mathcal{A}_{L,z}^c, \exists y \in F_L^* \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &= \sum_{m=1}^{\log L} \mathbb{P}\left(\mathcal{A}_{L,z}^c, \exists y \in F_L^m \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \end{aligned} \tag{4.19}$$

Since

$$\mathcal{A}_{L,z}^c = \left\{\sup_{y \in F_L^*} \mathcal{T}_{y,l}^\tau \leq \alpha_{z,+}^2(l)/2, k_y + 1 \leq l \leq L\right\} \tag{4.20}$$

and $k_y = m$ for $y \in F_L^m$, we see that

$$\begin{aligned} &\mathbb{P}\left(\mathcal{A}_{L,z}^c, \exists y \in F_L^m \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &\leq ce^{2(L-m)} \\ &\sup_{y \in F_L^m} \mathbb{P}\left(\mathcal{T}_{y,l}^\tau \leq \alpha_{z,+}^2(l)/2, m + 1 \leq l \leq L \text{ and } \bar{\mu}_\tau(y, h_L) \geq \frac{1}{\pi} t_L(z)\right). \end{aligned} \tag{4.21}$$

With

$$\mathcal{B}_{L,m,z}^y = \left\{\mathcal{T}_{y,l}^\tau \leq \alpha_{z,+}^2(l)/2, m + 1 \leq l \leq L - 1\right\} \tag{4.22}$$

we have for $y \in F_L^m$,

$$\begin{aligned} &\mathbb{P}\left(\mathcal{T}_{y,l}^\tau \leq \alpha_{z,+}^2(l)/2, m + 1 \leq l \leq L \text{ and } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &= \sum_{j=1}^{z+ML^{1/2}} P\left(\mathcal{B}_{L,m,z}^y, \sqrt{2\mathcal{T}_{y,L}^\tau} \in I_{\alpha_{z,+}(L)-j} \text{ and } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \\ &+ \mathbb{P}\left(\mathcal{B}_{L,m,z}^y, \sqrt{2\mathcal{T}_{y,L}^\tau} \leq \alpha_{z,+}(L) - z - ML^{1/2} \text{ and } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right). \end{aligned} \tag{4.23}$$

Here, $M \geq 1$ is a fixed constant to be chosen shortly.

Recalling, see (2.15), that $\alpha_{z,+}(L) = \rho_L L + z$, and using (4.13) we see that

$$(\alpha_{z,+}(L) - j)^2/2 \leq t_L\left(z - j + 2M^2\right)$$

for all $j \leq z + ML^{1/2}$. It follows that for such j

$$
\begin{aligned}
&P\left(\mathcal{B}^{y}_{L,m,z},\ \sqrt{2\mathcal{T}^{\tau}_{y,L}} \in I_{\alpha_{z,+}(L)-j} \text{ and } \bar{\mu}_{\tau}\left(y,\epsilon_y\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \\
&\leq \mathbb{P}\left(\mathcal{B}^{y}_{L,m,z},\ \sqrt{2\mathcal{T}^{\tau}_{y,L}} \in I_{\alpha_{z,+}(L)-j}, \overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \\
&= \mathbb{P}\left(\sqrt{2\mathcal{T}^{\tau}_{y,l}} \leq \alpha_{z,+}\left(l\right), m+1 \leq l \leq L-1,\ \sqrt{2\mathcal{T}^{\tau}_{y,L}} \in I_{\alpha_{z,+}(L)-j}\right) \\
&\qquad\qquad \times \mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right),
\end{aligned}
$$

by the Markov property. Using the barrier estimate (9.5) of Appendix I, and recalling that $m = k_y < \log L$, this is bounded by

$$
ce^{-2L} e^{-2(z-j)} \times m^2 j\left(z+m\right) \mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right). \tag{4.24}
$$

The rest of the proof of (4.17) follows as in the proof of (4.8). This completes the proof of the upper bound in Lemma 4.2.

We now remove the restriction that $y \in F_L$ in the upper bound, subject to a continuity restriction on ϵ_y. As mentioned, this will complete the proof of the upper bound (1.9).

Lemma 4.4. *We can find $0 < c, C, z_0 < \infty$ such that for L large, all $z_0 \leq z \leq \log L$, and all $h_L/20 \leq \epsilon_y \leq h_{L+1}$ such that $|\epsilon_y - \epsilon_{y'}| \leq C\, d(y,y')/L$ for all $y, y' \in \mathbf{S}^2$,*

$$
\mathbb{P}\left(\exists y \text{ s.t. } \bar{\mu}_{\tau}\left(y,\epsilon_y\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \leq cze^{-2z}. \tag{4.25}
$$

Proof of Lemma 4.4. Let F'_L be the centers of a $\frac{d_0}{L} h_L$ covering of $\mathbf{S}^2$ which contains F_L. For any $y \in \mathbf{S}^2$ we can find $y' \in F'_L$ such that $d(y,y') \leq \frac{d_0}{L} h_L$, so that by our assumptions $|\epsilon_y - \epsilon_{y'}| \leq C \frac{d_0}{L^2} h_L$. If we set $\bar{\epsilon}_y = \left(1 + \frac{1}{L}\right)\epsilon_y$ for all $y \in \mathbf{S}^2$ we see that for L large $h_L/30 \leq \bar{\epsilon}_y \leq 2h_{L+1}$ and $|\bar{\epsilon}_y - \bar{\epsilon}_{y'}| \leq \frac{d_0}{L} h_L$. It follows from Lemma 5.1 below that it suffices to prove that

$$
\mathbb{P}\left(\exists y \in F'_L \text{ s.t. } \bar{\mu}_{\tau}\left(y,\bar{\epsilon}_y\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \leq cze^{-2z}. \tag{4.26}
$$

We note that there are too many points in F'_L to prove (4.26) using the methods used to prove (4.6). We will need to use the continuity estimates of Section 7.

For $0 < \epsilon < a < b < \pi$, let $\mathcal{M}_{y,\bar{\epsilon}_y,y_0,a,b}(n)$ be the total occupation measure of $B_d(y,\bar{\epsilon}_y)$ during the first n excursions from $\partial B_d\left(y_0,a\right)$ to $\partial B_d\left(y_0,b\right)$. With $\omega_\epsilon = 2\pi(1-\cos(\epsilon))$, the area of $B_d\left(y,\epsilon\right)$, let

$$
\overline{\mathcal{M}}_{y,\bar{\epsilon}_y,y_0,a,b}(n) = \frac{1}{\omega_{\bar{\epsilon}_y}} \mathcal{M}_{y,\bar{\epsilon}_y,y_0,a,b}(n). \tag{4.27}
$$

For $y_0 \in F_L$ let

$$
D_{y_0} = \{y \in F'_L \,|\, d\left(y,y_0\right) \leq d_0 h_L/2\}. \tag{4.28}
$$

Following the proof of the upper bound for Lemma 4.2, to prove (4.26) it suffices to show that

$$
\mathbb{P}\left(\sup_{y \in D_{y_0}} \overline{\mathcal{M}}_{y,\bar{\epsilon}_y,y_0,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z\right)\right) \leq ce^{-c'j^2} \tag{4.29}
$$

for $j \leq z + ML^{1/2}$ sufficiently large. Setting $\epsilon = \sup_{y \in D_{y_0}} \bar{\epsilon}_y$ and using our condition on $|\bar{\epsilon}_y - \bar{\epsilon}_{y'}|$ to control the denominator in (4.27), we see that it suffices to show that

$$
\mathbb{P}\left(\sup_{y \in D_{y_0}} \overline{\mathcal{M}}_{y,\epsilon,y_0,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z - M^2/2\right)\right) \leq ce^{-c'j^2}. \tag{4.30}
$$

Abbreviating $Y_y^{(n)} = \overline{\mathcal{M}}_{y,\epsilon,y_0,h_L,h_{L-1}}(n)$ where $n = t_L\left(z - j + 2M^2\right)$ we have that

$$\begin{aligned} &\mathbb{P}\left(\sup_{y\in D_{y_0}} \overline{\mathcal{M}}_{y,\epsilon,y_0,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z - M^2/2\right)\right) \\ &\leq \mathbb{P}\left(\overline{\mathcal{M}}_{y_0,\epsilon,h_L,h_{L-1}}\left(t_L\left(z-j+2M^2\right)\right) \geq \frac{1}{\pi} t_L\left(z - j/2 - M^2/2\right)\right) \\ &+\mathbb{P}\left(\sup_{y\in D_{y_0}} |Y_y^{(n)} - Y_{y_0}^{(n)}| \geq jL/2\right). \end{aligned} \tag{4.31}$$

As in the proof of Lemma 4.3, the first term on the right hand side is bounded by $ce^{-c'j^2}$ for $j \leq z + ML^{1/2}$ sufficiently large. We then bound

$$\begin{aligned} &\mathbb{P}\left(\sup_{y\in D_{y_0}} |Y_y^{(n)} - Y_{y_0}^{(n)}| \geq jL/2\right) \\ &\leq \sum_{l=1}^{\log_2 L} \mathbb{P}\left(\sup_{y,y'\in D_{y_0},\, d(y,y')\approx 2^{-l} d_0 h_L} |Y_y^{(n)} - Y_{y'}^{(n)}| \geq jL/2l^2\right) \\ &\leq \sum_{l=1}^{\log_2 L} 2^{2l} \sup_{y,y'\in D_{y_0},\, d(y,y')\approx 2^{-l} d_0 h_L} \mathbb{P}\left(|Y_y^{(n)} - Y_{y'}^{(n)}| \geq jL/2l^2\right). \end{aligned} \tag{4.32}$$

It follows from Lemma 7.2 with $n = t_L\left(z - j + 2M^2\right) \sim 2L^2$ as above and $\theta = j/2^{3/2}l^2, \bar{d}(y,y') = 2^{-l}d_0$ that for some $C_0 > 0$

$$\begin{aligned} &2^{2l} \sup_{y,y'\in D_{y_0},\, d(y,y')\approx 2^{-l} d_0 h_L} \mathbb{P}\left(|Y_y^{(n)} - Y_{y'}^{(n)}| \geq jL/2l^2\right) \\ &\leq 2^{2l} \exp\left(-C_0 j^2 2^{l/2}/8 d_0^{1/2} l^4\right) \end{aligned} \tag{4.33}$$

whose sum over l is bounded by $ce^{-c'j^2}$. In order to apply Lemma 7.2 we have to verify that $\theta \leq \sqrt{\bar{d}(y,y')n}/2$. In our situation this means that $j/2^{3/2}l^2 \leq 2^{-l/2}d_0^{1/2}L/2$, for all $j \leq 2ML^{1/2}$. Thus we have to verify that $2^{1/2}M2^{l/2}/l^2 \leq d_0^{1/2}L^{1/2}$, which follows from the fact that $l \leq \log_2 L$, L is large and d_0, M are fixed.

4.2 The lower bound for (4.6)

Recall the notation $\mathcal{T}_{y,l}^1 = \mathcal{T}_{y,l}^{x^2,1}$ from the beginning of Section 3. Let τ_y be the time needed to complete x^2 excursions from $\partial B_d(y,h_1)$ to $\partial B_d(y,h_0)$, and set

$$\bar{\mu}_{\tau_y}(y,\epsilon) = \frac{1}{\omega_\epsilon}\int_0^{\tau_y} 1_{\{B_d(y,\epsilon)\}}(X_t)\, dt. \tag{4.34}$$

Recall F_L^0 from (3.4). We will prove the following analogue of Lemma 3.2.

Lemma 4.5. *There exists a* $0 < c < \infty$ *such that for all* $0 < r_0$ *sufficiently small,* L *large, all* $0 \leq z \leq \log L$*, and all* $h_L/100 \leq \epsilon_y \leq h_L$

$$\mathbb{P}\left[\sup_{y\in F_L^0} \bar{\mu}_{\tau_y}(y,\epsilon_y) \geq \frac{1}{\pi} t_L(z)\right] \geq \frac{(1+z)e^{-2z}}{(1+z)e^{-2z} + c}. \tag{4.35}$$

As before, the lower bound in (4.6) will follow from this, and hence combined with (4.25) we see that for some $0 < z_0$, and all $z_0 \leq z \leq \log L$

$$c' z e^{-2z} \leq \mathbb{P}\left(\exists y \text{ s.t. } \bar{\mu}_\tau(y,\epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \leq cze^{-2z}. \tag{4.36}$$

Combined with (4.7) it is easy to check that this implies Theorem 1.4.

To prove (4.35) set

$$\widetilde{\mathcal{I}}_{y,z+d}=\mathcal{I}_{y,z+d}\cap\{\bar{\mu}_{\tau_y}(y,\epsilon_y)\geq\frac{1}{\pi}t_L(z)\} \tag{4.37}$$

for some $d<\infty$ to be chosen shortly. We use the second moment method used in the proof of Lemma 3.2. Indeed, since $\widetilde{\mathcal{I}}_{y,z+d}\subseteq\mathcal{I}_{y,z+d}$ all upper bounds needed follow from those used in the proof of Lemma 3.2, and it only remains to prove the appropriate lower bound for $\widetilde{\mathcal{I}}_{y,z+d}$.

As in (3.25)-(3.26) we have

$$\begin{aligned}\mathbb{P}\left(\widetilde{\mathcal{I}}_{y,z+d}\right)\geq\mathbb{P}&\left(\widehat{\mathcal{I}}_{y,z+d},\,\bar{\mu}_{\tau_y}(y,h_L)\geq\frac{1}{\pi}t_L(z)\right)\\&-\sum_{k=L_+}^{L-d^*}\sum_{a\geq k_L^{1/4}}\mathbb{P}\left[\widehat{\mathcal{I}}_{y,z+d}\bigcap\mathcal{H}_{k,a}\bigcap W^c_{y,k}(N_{k,a})\right].\end{aligned} \tag{4.38}$$

Using the Markov property and then the barrier estimate (9.12) of Appendix I,

$$\begin{aligned}&\mathbb{P}\left(\widehat{\mathcal{I}}_{y,z+d},\,\bar{\mu}_{\tau_y}(y,\epsilon_y)\geq\frac{1}{\pi}t_L(z)\right)\\&\geq\mathbb{P}\left(\widehat{\mathcal{I}}_{y,z+d}\text{ and }\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}(t_L(z+d))\geq\frac{1}{\pi}t_L(z)\right)\\&=\mathbb{P}\left(\widehat{\mathcal{I}}_{y,z+d}\right)\mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}(t_L(z+d))\geq\frac{1}{\pi}t_L(z)\right)\\&\geq\bar{c}(1+z)e^{-2L}e^{-2(z+d)}\\&\qquad\mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}(t_L(z+d))\geq\frac{1}{\pi}(t_L(z+d)-dL)\right)\\&=\bar{c}(1+z)e^{-2L}e^{-2(z+d)}\\&\qquad\mathbb{P}\left(\overline{\mathcal{M}}_{y,\epsilon_y,h_L,h_{L-1}}(t_L(z+d))\geq\frac{1}{\pi}\left(1-\frac{dL}{t_L(z+d)}\right)t_L(z+d)\right)\\&\geq\bar{c}(1+z)e^{-2L}e^{-2(z+d)}(1-e^{-c''\frac{(dL)^2}{t_L(z+d)}}),\end{aligned} \tag{4.39}$$

where the last line used (4.3). It should be clear from the structure of $t_L(z+d)$ that we can choose some $d<\infty$ so that $e^{-c''\frac{(dL)^2}{t_L(z+d)}}\leq 1/2$ uniformly in $0\leq z\leq\log L$. Finally, after fixing such a d, we can show as in the proof of the first moment estimate in Section 3.1, that for d^* large enough, the last line in (4.38) is much smaller than the last line of (4.39). ☐

Thus we have completed the proof of Theorem 1.4.

4.3 The left tail

Lemma 4.6. *There exists a $0<c<\infty$ such that for all $0<r_0$ sufficiently small, L large, all $0\leq z\leq\log L$, and all $h_L/100\leq\epsilon_y\leq h_L$*

$$\mathbb{P}\left[\sup_{y\in F_L^0}\bar{\mu}_{\tau_y}(y,\epsilon_y)\geq\frac{1}{\pi}t_L(-z)\right]\geq\frac{e^{2z}}{e^{2z}+c}. \tag{4.40}$$

This will complete the proof of Theorem 1.3 since, as discussed right after the statement of Lemma 3.2, the probability of completing x^2 excursions from $\partial B_d(y,h_1)$ to

$\partial B_d\,(y, h_0)$ before time τ for all $y \in F_L^0$ is a strictly positive function of r_0 which goes to 1 as $r_0 \to 0$.

The proof of Lemma 4.6 is very similar to our proof of the lower bound on the right tail, except we now have to change the upper barrier to allow for negative z. Fix $|z| \le \log L$. We fix $\widehat{x} > 0$ once and for all. We abbreviate,

$$\widehat{\beta}_z\,(l) = f_{\widehat{x},\rho_L L+z}\,(l; L) = \widehat{x}\left(1 - \frac{l}{L}\right) + \left(\rho_L l + z\frac{l}{L}\right), \tag{4.41}$$

and

$$\widehat{\gamma}_{z,-}\,(l) = \widehat{\gamma}_{z,-}\,(l, L, z) = \widehat{\beta}_z\,(l) - l_L^{1/4}. \tag{4.42}$$

The barrier estimates needed are given in Lemma 9.6. We point out that the factors $(1 + z)$ which appear on the right hand side of (4.35) but not (4.40) come from the difference in the initial points of the barriers. □

5 Interpolation used to reduce (4.25) to (4.26)

Recall, (1.8), that

$$\bar{\mu}_\tau(y, \epsilon_y) = \frac{1}{\omega_{\epsilon_y}}\int_0^\tau 1_{\{B_d(y,\epsilon_y)\}}(X_t)\,dt, \tag{5.1}$$

where $\omega_{\epsilon_y} = 2\pi(1 - \cos\epsilon_y)$, the area of $B_d(y, \epsilon_y)$, and, (4.5),

$$t_L\,(z) = 2L\,(L - \log L + z)\,. \tag{5.2}$$

Lemma 5.1. *Assume that $d(y, y') \le a\frac{h_L}{L}$, $|\epsilon_y - \epsilon_{y'}| \le b\,\frac{h_L}{L}$, and $h_L/30 \le \epsilon_y, \epsilon_{y'} \le 2h_{L+1}$. We can find a $d_1 < \infty$ such that for all L large and $z \le \log L$, if*

$$\bar{\mu}_\tau\,(y', \epsilon_{y'}) \ge \frac{1}{\pi}t_L\,(z)\,, \tag{5.3}$$

then for any $c_1 \ge 30(a + b)$,

$$\bar{\mu}_\tau\,(y, (1 + c_1/L)\,\epsilon_y) \ge \frac{1}{\pi}t_L\,(z - d_1)\,. \tag{5.4}$$

Proof. Under our assumptions, for any $z \in B_d(y', \epsilon_{y'})$ we have $d(z, y) \le d(z, y') + d(y, y') \le \epsilon_{y'} + a\frac{h_L}{L} \le \left(1 + \frac{c_1}{L}\right)\epsilon_y$ so that

$$B_d(y', \epsilon_{y'}) \subseteq B_d\,(y, (1 + c_1/L)\,\epsilon_y)\,. \tag{5.5}$$

It follows that

$$\begin{aligned} \bar{\mu}_\tau(y', \epsilon_{y'}) &= \frac{1}{\omega_{\epsilon_{y'}}}\int_0^\tau 1_{\{B_d(y',\epsilon_{y'})\}}(X_t)\,dt \\ &\le \frac{1}{\omega_{\epsilon_{y'}}}\int_0^\tau 1_{\{B_d(y,(1+c_1/L)\epsilon_y)\}}(X_t)\,dt \\ &= \frac{\omega_{(1+c_1/L)\epsilon_y}}{\omega_{\epsilon_{y'}}}\bar{\mu}_\tau(y, (1 + c_1/L)\,\epsilon_y). \end{aligned} \tag{5.6}$$

Hence

$$\bar{\mu}_\tau\,(y', \epsilon_{y'}) \ge \frac{1}{\pi}t_L\,(z) \tag{5.7}$$

implies that

$$\bar{\mu}_\tau(y, (1 + c_1/L)\,\epsilon_y) \ge \frac{\omega_{\epsilon_{y'}}}{\omega_{(1+c_1/L)\epsilon_y}}\frac{1}{\pi}t_L\,(z)\,. \tag{5.8}$$

But under our assumptions

$$\frac{\omega_{\epsilon_{y'}}}{\omega_{(1+c_1/L)\epsilon_y}} = 1 + O\,(1/L)\,. \tag{5.9}$$

This gives (5.4). □

6 Green's functions and proof of Lemma 4.1

Let $G_a(x,y)$ denote the potential density for Brownian motion killed the first time it leaves $B_e(0,a)$, that is, the Green's function for $B_e(0,a)$. Recall that $B_e(x,r)$ is the Euclidean ball in R^2 centered at x of radius r. We have, see [15, Section 2] or [17, Chapter 2, (1.1)],

$$G_a(x,y) = -\frac{1}{\pi}\log|x-y| + \frac{1}{\pi}\log\left(\frac{|y|}{a}|x-y_a^*|\right), \qquad y \neq 0, \tag{6.1}$$

where

$$y_a^* = \frac{a^2 y}{|y|^2}, \tag{6.2}$$

and

$$G_a(x,0) = -\frac{1}{\pi}\log|x| + \frac{1}{\pi}\log a. \tag{6.3}$$

Let v denote the south pole of $\mathbf{S}^2$. If σ denotes stereographic projection, then $\sigma\left(B_d(v,h(a))\right) = B_e(0,a)$, see [7, (2.4)]. We claim that in the isothermal coordinates induced by stereographic projection σ, the Green's function for $\sigma\left(B_d(v,h(a))\right) = B_e(0,a)$ is just $G_a(x,y)$. To see this we must show that if $\Delta_{\mathbf{S}^2}$ is the Laplacian for $\mathbf{S}^2$ in isothermal coordinates and $dV(y)$ is the volume measure, then

$$\frac{1}{2}\Delta_{\mathbf{S}^2}\int G_a(x,y)f(y)\,dV(y) = -f(x) \tag{6.4}$$

for all continuous f compactly supported in $B_e(0,a)$.

For $x=(x_1,x_2)$, let

$$g(x) = \frac{1}{(1+\frac{1}{4}(x_1^2+x_2^2))^2}. \tag{6.5}$$

As shown in [30, Chapter 7, p. 6-9], the stereographic projection σ is an isometry if we give R^2 the metric

$$g(x)\left(dx_1\otimes dx_1 + dx_2\otimes dx_2\right). \tag{6.6}$$

Because of (6.6) the Laplace-Beltrami operator takes the form

$$\frac{1}{g(x)}\left(\partial_{x_1}^2 + \partial_{x_2}^2\right). \tag{6.7}$$

Thus, $\Delta_{\mathbf{S}^2} = \frac{1}{g}\Delta$ and $dV(y) = g(y)\,dy$, so that (6.4) holds.

Proof of Lemma 4.1. Let $\epsilon = h(\alpha)$ so that $h(\alpha) \leq h_k$. If $\tau_{h_{k-1}}$ is the first exit time of $B_d(v,h_{k-1})$ and ρ_{h_k} is uniform measure on $\partial B_d(v,h_k)$, then by symmetry, for any $z \in \partial B_d(v,h_k)$

$$\begin{aligned} J_1 \quad &=: \quad \mathbb{E}^z\left(\int_0^{\tau_{h_{k-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\,dt\right) \\ &= \quad \mathbb{E}^{\rho_{h_k}}\left(\int_0^{\tau_{h_{k-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\,dt\right). \end{aligned} \tag{6.8}$$

Since uniform measure ρ_{h_k} on $\partial B_d(v,h_k)$ goes over to uniform measure γ_{r_k} on $\partial B_e(0,r_k)$, using the discussion at the beginning of this section we have

$$J_1 = \int_{B_e(0,\alpha)}\int_{\partial B_e(0,r_k)} G_{r_{k-1}}(x,y)\,d\gamma_{r_k}(x)g(y)\,dy. \tag{6.9}$$

We recall, [27, Chapter 2, Prop. 4.9] or [17, Chapter 1, (5.4), (5.5)], that

$$\int_{\partial B_e(0,b)} \log\left(|x-y|\right)\, d\gamma_b(x) = \log\left(b \vee |y|\right). \tag{6.10}$$

This shows that for $y \in B_e(0, r_k)$

$$\begin{aligned} &\int_{\partial B_e(0,r_k)} G_{r_{k-1}}(x,y)\, d\gamma_{r_k}(x) \qquad (6.11)\\ &= \frac{1}{\pi}\int_{\partial B_e(0,r_k)} \left(-\log|x-y| + \log\left(\frac{|y|}{r_{k-1}}|x-y^*_{r_{k-1}}|\right)\right)\, d\gamma_{r_k}(x)\\ &= \frac{1}{\pi}\left(-\log r_k + \log\left(\frac{|y|}{r_{k-1}}|y^*_{r_{k-1}}|\right)\right)\\ &= \frac{1}{\pi}\left(-\log r_k + \log r_{k-1}\right) = \frac{1}{\pi}\log(r_{k-1}/r_k) = \frac{1}{\pi}. \end{aligned}$$

Thus

$$J_1 = \frac{1}{\pi}\int_{B_e(0,\alpha)} g(y)\, dy = \frac{1}{\pi}\text{Area}\ \left(B_d(v, h(\alpha))\right) = \frac{1}{\pi}\omega_{h(\alpha)} = \frac{1}{\pi}\omega_\epsilon. \tag{6.12}$$

It follows that for any $z \in \partial B_d(v, h_k)$

$$\mathbb{E}^z\left(\overline{\mathcal{M}}_{v,\epsilon,h_k,h_{k-1}}(1)\right) = \frac{1}{\pi}. \tag{6.13}$$

By the Kac moment formula, for any $z \in \partial B_d(v, h_k)$, with $x = \sigma(z)$

$$\begin{aligned} &\mathbb{E}^z\left(\left(\int_0^{\tau_{h_{k-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt\right)^n\right) \qquad (6.14)\\ &= n!\int_{B_e^n(0,\alpha)} G_{r_{k-1}}(x,y_1)\prod_{j=2}^n G_{r_{k-1}}(y_{j-1},y_j)\prod_{i=1}^n g(y_i)\, dy_i\\ &\le c^n n!\int_{B_e^n(0,\alpha)} G_{r_{k-1}}(x,y_1)\prod_{j=2}^n G_{r_{k-1}}(y_{j-1},y_j)\prod_{i=1}^n dy_i\\ &\le c^n n!\alpha^{2n}\left(\log\left(r_{k-1}/\alpha\right) + c_0\right)^n \end{aligned}$$

where the last inequality follows as in the proof of [15, Lemma 2.1]. It follows that for any $z \in \partial B_d(v, h_k)$

$$\mathbb{E}^z\left(\left(\overline{\mathcal{M}}_{v,\epsilon,h_k,h_{k-1}}(1)\right)^n\right) \le c^n n!\left(\log\left(r_{k-1}/\alpha\right) + c_0\right)^n. \tag{6.15}$$

By (2.3), our assumption that $h_k/100 \le \epsilon \le h_k$ implies that $e \le r_{k-1}/\alpha \le 200e$. Using (6.13) and (6.15), our Lemma then follows as in the proof of [16, Lemma 2.2] which uses moment inequalities to show that excursion times are concentrated around their mean. □

7 Continuity estimates

The goal of this Section is to prove the continuity estimate Lemma 7.2 which was used in the proof of (4.26).

For fixed $u \in \mathbf{S}^2$, let τ_a be the first exit time of $B_d(u, a)$ and let ρ_m be uniform measure on $\partial B_d(u, m)$. Recall that for some $d_0 \le 1/1000$, we take F_l to be the centers of an $d_0 h_l$ covering of $\mathbf{S}^2$.

Lemma 7.1. *If $d(u,v), d(u,\widetilde{v}) \le d_0 h_L/2$, $d_0/L \le \bar{d} =: d(v,\widetilde{v})/h_L \le d_0$, and $h_L/20 \le \epsilon \le h_{L+1}$, then*

$$\mathbb{E}^{\rho_{h_L}} \left(\int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt - \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\widetilde{v},\epsilon)\}}(X_t)\, dt \right) = 0, \tag{7.1}$$

$$\mathbb{E}^{\rho_{h_L}} \left(\left(\int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt - \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\widetilde{v},\epsilon)\}}(X_t)\, dt \right)^2 \right) \le c\epsilon^4 \bar{d}^2, \tag{7.2}$$

and

$$\sup_{x \in \partial B_d(u,h_L)} \mathbb{E}^{x} \left(\left(\int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt - \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\widetilde{v},\epsilon)\}}(X_t)\, dt \right)^2 \right) \le c\epsilon^4 \bar{d}^2. \tag{7.3}$$

Proof of Lemma 7.1. As in (6.8)-(6.9) we have

$$\begin{aligned} J_2 &= \mathbb{E}^{\rho_{h_L}} \left(\int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt - \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\widetilde{v},\epsilon)\}}(X_t)\, dt \right) \\ &= \int \int G_{r_{L-1}}(x,y)\, d\gamma_{r_L}(x)\, d\mu_{v,\widetilde{v}}(y), \end{aligned} \tag{7.4}$$

where

$$d\mu_{v,\widetilde{v}}(y) = \left(1_{\{\sigma(B_d(v,\epsilon))\}} - 1_{\{\sigma(B_d(\widetilde{v},\epsilon))\}} \right)(y) g(y)\, dy. \tag{7.5}$$

Then by (6.11)-(6.12) we have that

$$\begin{aligned} J_2 &= \frac{1}{\pi} \int d\mu_{v,\widetilde{v}}(y) \\ &= \frac{1}{\pi} \left(\text{Area } (B_d(v,\epsilon)) - \text{Area } (B_d(\widetilde{v},\epsilon)) \right) = 0, \end{aligned} \tag{7.6}$$

since all balls of radius ϵ on the sphere have area $\omega_\epsilon = 2\pi(1-\cos\epsilon)$. This completes the proof of (7.1).

We next observe that

$$\begin{aligned} &\mathbb{E}^{\rho_{h_L}} \left(\left(\int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt - \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\widetilde{v},\epsilon)\}}(X_t)\, dt \right)^2 \right) \\ &= 2 \int \int \int G_{r_{L-1}}(x,y) G_{r_{L-1}}(y,z)\, d\gamma_{r_L}(x)\, d\mu_{v,\widetilde{v}}(y)\, d\mu_{v,\widetilde{v}}(z) \\ &= \frac{2}{\pi} \int \int G_{r_{L-1}}(y,z)\, d\mu_{v,\widetilde{v}}(y)\, d\mu_{v,\widetilde{v}}(z) \end{aligned} \tag{7.7}$$

as above.

We note that for $b < a$

$$G_a(bx,by) = -\frac{1}{\pi} \log(b\,|x-y|) + \frac{1}{\pi} \log\left(b \frac{|y|}{a/b} |x - y^*_{a/b}| \right) = G_{a/b}(x,y), \tag{7.8}$$

since

$$(by)^*_a = \frac{a^2 by}{b^2 |y|^2} = b y^*_{a/b}. \tag{7.9}$$

Using this to scale by r_L we see that

$$\int \int G_{r_{L-1}}(y,z)\, d\mu_{v,\widetilde{v}}(y)\, d\mu_{v,\widetilde{v}}(z) = r_L^4 \int \int G_e(y,z)\, d\mu_{L,v,\widetilde{v}}(y)\, d\mu_{L,v,\widetilde{v}}(z), \tag{7.10}$$

where

$$\begin{aligned} d\mu_{L,v,\widetilde{v}}(y) &= \left(1_{\{\sigma(B_d(v,\epsilon))\}} - 1_{\{\sigma(B_d(\widetilde{v},\epsilon))\}}\right)(r_L y) g(r_L y)\, dy \\ &= \left(1_{\{\frac{1}{r_L}\sigma(B_d(v,\epsilon))\}} - 1_{\{\frac{1}{r_L}\sigma(B_d(\widetilde{v},\epsilon))\}}\right)(y) g(r_L y)\, dy. \end{aligned} \tag{7.11}$$

For y in our range we have $g(r_L y) = 1 + O(\epsilon)$, and it is easy to check that up to errors of order ϵ, $\frac{1}{r_L}\sigma(B_d(v,\epsilon))$ and $\frac{1}{r_L}\sigma(B_d(\widetilde{v},\epsilon))$ can be replaced by $B_e(v',\epsilon/r_L)$ and $B_e(v' - (0,\delta\epsilon/r_L),\epsilon/r_L)$ for some v' with $|v'| \le d_0$ and $0 < \delta \le c_2\bar{d}$.

Hence with

$$d\nu_{v'}(y) = \left(1_{\{B_e(v',\epsilon/r_L)\}} - 1_{\{B_e(v'-(0,\delta\epsilon/r_L),\epsilon/r_L)\}}\right)(y)\, dy, \tag{7.12}$$

it remains to show that

$$\int\int G_e(y,z)\, d\mu_{v'}(y)\, d\mu_{v'}(z) \le C\delta^2. \tag{7.13}$$

The symmetric difference of $B_e(v',s)$ and $B_e(v'-(0,\delta s),s)$ consist of two disjoint pieces we denote by A, B. They have the same area

$$\text{Area}\,(A) = 2s^2 \arcsin\left(\frac{\delta}{2}\right) + \frac{\delta s}{2}\sqrt{(4-\delta^2)s^2} \asymp \delta s^2. \tag{7.14}$$

We observe that

$$\lim_{y\to 0}\frac{|y|}{a}|z - y_a^*| = \lim_{y\to 0}\frac{|y|}{a}\left(|y_a^*| + O(1)\right) = \lim_{y\to 0}\frac{|y|}{a}\left(\frac{a^2|y|}{|y|^2} + O(1)\right) = a. \tag{7.15}$$

It follows that for y, z in our range, $\log\left(\frac{|y|}{e}|z - y_e^*|\right)$ is bounded, hence to prove (7.13) it suffices to show that

$$\int\int \big|\log|y-z|\big|\, d\mu_{v'}(y)\, d\mu_{v'}(z) \le C\delta^2. \tag{7.16}$$

It is then easy to see that we need only show that

$$\int_A\int_A \big|\log|y-z|\big|\, dy\, dz \le C\delta^2. \tag{7.17}$$

It is also clear that we only need to consider $|y-z| \le 1/2$. Writing $y = (y_1,y_2), z = (z_1,z_2)$ we see that

$$\begin{aligned} &\int_A\int_A \big|\log|y-z|\big| 1_{\{|y-z|\le 1/2\}}\, dy\, dz \\ &\le \int_{[0,1]\times[0,\delta]}\int_{[0,1]\times[0,\delta]} \big|\log|y-z|\big| 1_{\{|y-z|\le 1/2\}}\, dy_1\, dy_2\, dz_1\, dz_2 \\ &\le \int_{[0,1]\times[0,\delta]}\int_{[0,1]\times[0,\delta]} \big|\log|y_1-z_1|\big|\, dy_1\, dy_2\, dz_1\, dz_2 \le C\delta^2, \end{aligned} \tag{7.18}$$

which completes the proof of (7.17).

To obtain (7.3), arguing as before we need to show that

$$K_1 = \sup_{x\in\partial B_e(0,r_L)}\int\int G_{r_{L-1}}(x,y) G_{r_{L-1}}(y,z)\, d\mu_{v,\widetilde{v}}(y)\, d\mu_{v,\widetilde{v}}(z) \le c\epsilon^4\delta^2. \tag{7.19}$$

Scaling in r_L as before shows that

$$K_1 = r_L^4 \sup_{x\in\partial B_e(0,1)}\int\int G_e(x,y) G_e(y,z)\, d\mu_{L,v,\widetilde{v}}(y)\, d\mu_{L,v,\widetilde{v}}(z) \tag{7.20}$$

But for y in our range, $G_e(x,y)$ is bounded uniformly in $x \in \partial B_e(0,1)$, so that (7.3) follows as before. □

The same proof shows that

$$\sup_{x\in\partial B_d(u,h_L)} \mathbb{E}^x \left(\left(\int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt - \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\widetilde{v},\epsilon)\}}(X_t)\, dt \right)^{2n} \right) \le (2n)! c_1^{2n} \epsilon^{4n} \bar{d}^{2n}. \tag{7.21}$$

and hence by the Cauchy-Schwarz inequality

$$\sup_{x\in\partial B_d(u,h_L)} \mathbb{E}^x \left(\Big| \frac{1}{\omega_\epsilon} \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_t)\, dt - \frac{1}{\omega_\epsilon} \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\widetilde{v},\epsilon)\}}(X_t)\, dt \Big|^{n} \right) \le n! c_2^{n} \bar{d}^{n}. \tag{7.22}$$

Recall (4.27) and set

$$Y_y^{(n)} = \overline{\mathcal{M}}_{y,\epsilon,u,h_L,h_{L-1}}(n)$$

Lemma 7.2. *For some* $d_0 > 0$ *we can find* $C_0 > 0$ *such that, if* $d(u,v), d(u,\widetilde{v}) \le d_0 h_L/2$, $d_0/L \le \bar{d}(v,\widetilde{v}) =: d(v,\widetilde{v})/h_L \le d_0$, $h_L/20 \le \epsilon \le h_{L+1}$, *and* $\theta \le \sqrt{\bar{d}(v,\widetilde{v})n}/2$, *then*

$$\mathbb{P}\left(|Y_v^{(n)} - Y_{\widetilde{v}}^{(n)}| \ge \theta\sqrt{n} \right) \le e^{-C_0\theta^2/\bar{d}^{1/2}(v,\widetilde{v})}. \tag{7.23}$$

Proof of Lemma 7.2. We follow the proof of [7, Lemma 5.1].

Let T_i denote the successive excursion times $T_{\partial B_d(u,h_L)}\circ\theta_{T_{\partial B_d(u,h_{L-1})}}$ from $\partial B_d(u,h_{L-1})$ to $\partial B_d(u,h_L)$ and set

$$Y_{v,i} = \frac{1}{\omega_\epsilon} \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v,\epsilon)\}}(X_{t+T_i})\, dt, \tag{7.24}$$

so that

$$Y_v^{(n)} = \sum_{i=1}^{n} Y_{v,i}. \tag{7.25}$$

Let J be a geometric random variable with success parameter $p_3 > 0$, independent of $\{Y_{v,i}, Y_{\widetilde{v},i}\}$. It follows from (7.22) and the proof of [7, Corollary 5.3] that, abbreviating $\bar{d} = \bar{d}(v,\widetilde{v})$, if $c_2\bar{d}\lambda \le p_3/2$ then for some c_4

$$\sup_{x\in\partial B_d(u,h_L)} \mathbb{E}^x \left(\exp\left(\lambda \sum_{i=1}^{J-1} |(Y_{v,i} - Y_{\widetilde{v},i})| \right) \right) \le e^{c_4\bar{d}\lambda/p_3}, \tag{7.26}$$

and from (7.22) together with (7.1) and the proof and notation of [7, Lemma 5.5] it follows that, after perhaps enlarging c_4

$$\mathbb{E}\left(\exp\left(\lambda \sum_{i=J_1}^{J_2-1} (Y_{v,i} - Y_{\widetilde{v},i}) \left(X_\cdot^i\right) \right) \right) \le e^{c_4\left(\bar{d}\lambda/p_3\right)^2}. \tag{7.27}$$

The essence of [7, Lemma 5.5] is to use a renewal argument to allow one to take advantage of (7.1) to eliminate the linear term in the expansion of the exponential so that, as opposed to (7.26), we now have a quadratic term in the exponential.

Then, instead of [7, (5.33)] we set

$$\delta = \theta/\sqrt{\bar{d}n} \le 1/2.$$

With this it follows from the proof of [7, Lemma 5.1] that for $c_2\bar{d}\lambda \le p_3/2$

$$\mathbb{P}\left(|Y_v^{(n)} - Y_{\widetilde{v}}^{(n)}| \ge \theta\sqrt{n} \right) \tag{7.28}$$
$$\le e^{-\bar{c}\theta^2/\bar{d}} + \exp\left(c_4\lambda^2\bar{d}^2 n/p_3 + 2c_4\lambda\theta\sqrt{\bar{d}n} - \lambda\theta\sqrt{n} \right)$$

By taking d_0 sufficiently small we can be sure that $c_2\bar{d} \le 1$, so the above holds for any $\lambda \le p_3/2$. If we set

$$\lambda = p_3\theta/\sqrt{\bar{d}n} \le p_3/2$$

we see that

$$\exp\left(c_4\lambda^2\bar{d}^2 n/p_3 + 2c_4\lambda\theta\sqrt{\bar{d}n} - \lambda\theta\sqrt{n}\right)$$
$$= \exp\left(c_4 p_3\theta^2\bar{d} + 2c_4 p_3\theta^2 - p_3\theta^2/\sqrt{\bar{d}}\right),$$

which completes the proof of (7.23) for d_0 sufficiently small. □

8 From the sphere to the plane, and back

Using (6.7) it follows from [28, Chapter 5, Theorem 1.9], that we can find a planar Brownian motion W_t such that in the isothermal coordinates induced by stereographic projection,

$$X_t = W_{U_t}, \quad \text{where} \quad U_t = \int_0^t \frac{1}{g(X_s)}\,ds. \tag{8.1}$$

where g is defined in (6.5).

We take the v of this paper to be $v = (0,0,0)$. Let

$$D_* = \sigma\left(B_d\left(v, r^*\right)\right) = B_e\left((0,0), 2\tan(r^*/2)\right). \tag{8.2}$$

For the last equality see [7, (2.4)]. If θ is the first hitting time of ∂D_* by W_t, then under the coupling (8.1) we see that $\theta = U_\tau$. Set

$$\mu_\theta(x,\epsilon) = \frac{1}{\pi\epsilon^2}\int_0^\theta 1_{\{B_e(x,\epsilon)\}}(W_t)\,dt. \tag{8.3}$$

Lemma 8.1. *For some $-\infty < d_1, d_2, d_3, d_4 < \infty$, all $x \in D_*$ and all ϵ sufficiently small*

$$\mu_\theta(x,\epsilon) \le (1 + d_1\ \epsilon)\ \bar{\mu}_\tau(x, g^{1/2}(x)\epsilon(1 + d_2\epsilon)), \tag{8.4}$$

and

$$\mu_\theta(x,\epsilon) \ge (1 + d_3\ \epsilon)\ \bar{\mu}_\tau(x, g^{1/2}(x)\epsilon(1 + d_4\epsilon)). \tag{8.5}$$

Proof of Lemma 8.1. We first note that for ϵ sufficiently small, we can find $c_1 < c_2$ such that uniformly in $x' \in B_e(x, 2\epsilon)$ and $x \in D_*$

$$g(x)(1 + c_1\epsilon) \le g(x') \le g(x)(1 + c_2\epsilon). \tag{8.6}$$

For $x' \in B_e(x,\epsilon)$, with $x_t = x + t(x' - x)$

$$d(x,x') \le \int_0^1 g^{1/2}(x_t)|x' - x|\,dt \le g^{1/2}(x)|x' - x|(1 + c_3\epsilon). \tag{8.7}$$

Hence

$$B_e(x,\epsilon) \subseteq B_d(x, g^{1/2}(x)\epsilon(1 + c_3\epsilon)). \tag{8.8}$$

Similarly, for some $c_4 < c_3$

$$B_e(x,\epsilon) \supseteq B_d(x, g^{1/2}(x)\epsilon(1 + c_4\epsilon)). \tag{8.9}$$

Consider

$$\int_0^\tau 1_{\{B_d(x, g^{1/2}(x)\epsilon(1+c_3\epsilon))\}}(W_{U_t})\,dt. \tag{8.10}$$

By the nature of U_t in (8.1) it follows that whenever the path W_{U_t} enters $B_d(x, g^{1/2}(x)\epsilon(1+c_3\epsilon))$ it is slowed by a variable factor between $\frac{1}{g(x)(1+c_5\epsilon)}$ and $\frac{1}{g(x)(1+c_6\epsilon)}$. Hence the amount of time spent in $B_d(x, g^{1/2}(x)\epsilon(1+c_3\epsilon))$ during each incursion is multiplied by a variable factor between $g(x)(1+c_7\epsilon)$ and $g(x)(1+c_8\epsilon)$. Thus

$$\int_0^\tau 1_{\{B_d(x,g^{1/2}(x)\epsilon(1+c_3\epsilon))\}}(W_{U_t})\,dt \geq g(x)(1+c_7\epsilon)\int_0^\theta 1_{\{B_d(x,g^{1/2}(x)\epsilon(1+c_3\epsilon))\}}(W_t)\,dt, \tag{8.11}$$

and

$$\int_0^\tau 1_{\{B_d(x,g^{1/2}(x)\epsilon(1+c_3\epsilon))\}}(W_{U_t})\,dt \leq g(x)(1+c_8\epsilon)\int_0^\theta 1_{\{B_d(x,g^{1/2}(x)\epsilon(1+c_3\epsilon))\}}(W_t)\,dt, \tag{8.12}$$

It follows from (8.8) and (8.11) that

$$\begin{aligned}
&\int_0^\theta 1_{\{B_e(x,\epsilon)\}}(W_t)\,dt \qquad\qquad (8.13)\\
&\leq \int_0^\theta 1_{\{B_d(x,g^{1/2}(x)\epsilon(1+c_3\epsilon))\}}(W_t)\,dt\\
&\leq \frac{1}{g(x)(1+c_7\epsilon)}\int_0^\tau 1_{\{B_d(x,g^{1/2}(z)\epsilon(1+c_3\epsilon))\}}(W_{U_t})\,dt.
\end{aligned}$$

Since $\omega_\delta = 2\pi\,(1-\cos(\delta))$, we see that if we set $\delta_x = g^{1/2}(x)\epsilon(1+c_3\epsilon)$, then uniformly in $x \in D_*$ and sufficiently small ϵ

$$(1+f_0'\epsilon) \leq \frac{\omega_{\delta_x}}{\pi g\,(x)\,\epsilon^2} \leq (1+f_0\epsilon),$$

so that by (8.13)

$$\begin{aligned}
\mu_\theta(x,\epsilon) &= \frac{1}{\pi\epsilon^2}\int_0^\theta 1_{\{B_e(x,\epsilon)\}}(W_t)\,dt\\
&\leq \frac{1}{\pi\epsilon^2 g(x)(1+c_7\epsilon)}\int_0^\tau 1_{\{B_d(x,g^{1/2}(z)\epsilon(1+c_3\epsilon))\}}(W_{U_t})\,dt\\
&= \frac{1}{(1+c_7\epsilon)}\;\frac{\omega_{\delta_x}}{\pi g\,(x)\,\epsilon^2}\;\bar{\mu}_\tau(x, g^{1/2}(x)\epsilon(1+c_3\epsilon))\\
&\leq (1+\widehat{d}\;\epsilon)\;\bar{\mu}_\tau(x, g^{1/2}(x)\epsilon(1+c_3\epsilon)), \qquad (8.14)
\end{aligned}$$

where

$$(1+\widehat{d}\;\epsilon) = \frac{1+f_0\epsilon}{1+c_7\epsilon}.$$

This completes the proof of (8.4).

The lower bound (8.5) is proven similarly using (8.9) and (8.12). □

Lemma 8.2. *We can find $0 < c, c', z_0 < \infty$ such that for L large and all $\frac{1}{12}h_L \leq \epsilon \leq \frac{1}{3}h_L$ and $z_0 \leq z \leq \log L$,*

$$c' z e^{-2\sqrt{2\pi}z} \leq \mathbb{P}\left(\sqrt{\sup_y \mu_\theta(y,\epsilon)} \geq m_\epsilon + z\right) \leq c z e^{-2\sqrt{2\pi}z}. \tag{8.15}$$

Proof of Lemma 8.2. We consider the upper bound. By (8.4) it suffices to show that

$$\mathbb{P}\left(\sqrt{\sup_y \bar{\mu}_\tau(y, g^{1/2}(y)\epsilon(1+d_2\epsilon))} \geq m_\epsilon + z\right) \leq c z e^{-2\sqrt{2\pi}z}. \tag{8.16}$$

Since for ϵ in our range

$$(m_\epsilon + z)^2 = \frac{1}{\pi} t_L \left(\sqrt{2\pi}\ z + O(1) \right),$$

(compare (4.7)), (8.16) follows from Lemma 4.4 once we verify the condition that $|\epsilon_y - \epsilon_{y'}| \leq C\, d(y, y')/L$, where now $\epsilon_y = g^{1/2}(y)\epsilon(1+d_2\epsilon))$. This follows easily since g is smooth and we can assume that $\frac{4}{5} \leq g^{1/2}(y) \leq 1$. We also point out that for $\frac{1}{12}h_L \leq \epsilon \leq \frac{1}{3}h_L$ and L large we have $\frac{1}{20}h_L \leq \epsilon_y \leq h_{L+1}$.

The lower bound is similar. □

We note that $\frac{1}{3}h_{L+1} = \frac{1}{3e}h_L \geq \frac{1}{12}h_L$, so all ϵ are covered by Lemma 8.2.

Lemma 8.2 is the analog of Theorem 1.2, but where now θ is the first hitting time of ∂D_*, see (8.2). Theorem 1.2 then follows by Brownian scaling. To spell this out for later use, let θ_a be the first hitting time of $\partial B_e(0, a)$ and set

$$\mu_a(x, \epsilon) = \frac{1}{\pi\epsilon^2} \int_0^{\theta_a} 1_{\{B(x,\epsilon)\}}(W_t)\, dt. \tag{8.17}$$

Then it follows from Brownian scaling that for any $a, b > 0$,

$$\{\mu_a(x, \epsilon_x);\, x, \epsilon_x\} \stackrel{\text{law}}{=} \{\mu_{ba}(bx, b\epsilon_x);\, x, \epsilon_x\}. \tag{8.18}$$

The left tail and then Theorem 1.1 can be proven similarly.

8.1 From r^* small to any $r^* < \pi$

We first note the following extension of Lemma 8.2.

Lemma 8.3. *We can find $0 < c, c', z_0 < \infty$ such that for L large and all $\frac{1}{12}h_L \leq \epsilon_y \leq \frac{1}{3}h_L$ with $|\epsilon_y - \epsilon_{y'}| \leq C|y - y'|/L$ and $z_0 \leq z \leq \log L$,*

$$c' z e^{-2z} \leq \mathbb{P}\left(\sup_y \mu_\theta(y, \epsilon_y) \geq \frac{1}{\pi} t_L\, (z) \right) \leq c z e^{-2z}. \tag{8.19}$$

This follows as in the proof of Lemma 8.2, once we observe that in Lemma 8.1 we can allow the ϵ to depend on x.

It follows from (8.18) that for any fixed $a > 0$, Lemma 8.3 holds with θ replaced by θ_a.

We now show that Theorem 1.4 holds for any $0 < r^* < \pi$. This is done by using Lemma 8.1. That is, with $a = 2\tan(r^*/2)$ we have that for some $-\infty < d_1, d_2, d_3, d_4 < \infty$, all $x \in D_a$ and all ϵ sufficiently small

$$\bar{\mu}_\tau(x, \epsilon) \leq (1 + d_1\ \epsilon)\ \mu_{\theta_a}(x, g^{-1/2}(x)\epsilon(1 + d_2\epsilon)), \tag{8.20}$$

and

$$\bar{\mu}_\tau(x, \epsilon) \geq (1 + d_3\ \epsilon)\ \mu_{\theta_a}(x, g^{-1/2}(x)\epsilon(1 + d_4\epsilon)), \tag{8.21}$$

Theorem 1.4 then follows from Lemma 8.3 just as Lemma 8.2 followed from Lemma 4.4. Theorem 1.3 can be proven similarly.

9 Appendix I: barrier estimates

In what follows, we use the notation $H_{y,\delta} = [y, y + \delta]$ from [8]. The following is a variant of [8, Theorem 1.1], which can be proven similarly. We set

$$f_{a,b}\,(l; L) = a + (b - a)\frac{l}{L}. \tag{9.1}$$

Theorem 9.1. *a) For all fixed $\delta > 0, C \geq 0$, $\eta > 1$ and $\varepsilon \in \left(0, \frac{1}{2}\right)$ we have, uniformly in $\sqrt{2} \leq x, y \leq \eta L$ such that $x^2/2 \in \mathbb{N}$, any $0 \leq x \leq a$, $0 \leq y \leq b$, that*

$$P^{\mathrm{GW}}_{x^2/2}\left(\sqrt{2T_l} \leq f_{a,b}(l;L) + Cl_L^{\frac{1}{2}-\varepsilon}, l = 1, \ldots, L-1, \sqrt{2T_L} \in H_{y,\delta}\right) \leq c\frac{(1+a-x)(1+b-y)}{L}\sqrt{\frac{x}{yL}}e^{-\frac{(x-y)^2}{2L}}. \quad (9.2)$$

b) Let Tube$_{C,\tilde{C}}(l;L) = [f_{x,y}(l;L) - \tilde{C}l_L^{\frac{1}{2}+\varepsilon}, f_{a,b}(l;L) - Cl_L^{\frac{1}{2}-\varepsilon}]$. If, in addition to the conditions in part a), we also have $(1+a-x)(1+b-y) \leq \eta L$, $\max(xy, |y-x|) \geq L/\eta$ and $[y, y+\delta] \cap \sqrt{2\mathbb{Z}} \neq \emptyset$, and Tube$_{C,\tilde{C}}(l;L) \cap \sqrt{2\mathbb{N}} \neq \emptyset$ for all l=1,...,L-1, then

$$P^{\mathrm{GW}}_{x^2/2}\left(\sqrt{2T_l} \in \mathit{Tube}_{C,\tilde{C}}(l;L), l = 1, \ldots, L-1, \sqrt{2T_L} \in H_{y,\delta}\right) \geq c\frac{(1+a-x)(1+b-y)}{L} \times \left(\sqrt{\frac{x}{yL}} \wedge 1\right)e^{-\frac{(x-y)^2}{2L}}, \quad (9.3)$$

and the estimate is uniform in such x, y, a, b and all L.

Similar results hold if we delete the barrier condition on some fixed finite interval.

For the last statement, we simply note that following the proof of [8, Lemma 2.3] we can show that the analogue of [8, Theorem 1.1] holds where we skip some fixed finite interval.

Recall that

$$\rho_L = 2 - \frac{\log L}{L}, \qquad \alpha_{z,\pm}(l) = \alpha(l, L, z) = \rho_L l + z \pm l_L^{1/4}. \quad (9.4)$$

Lemma 9.2. *Let $m = k_y + 1 \leq \log L$. For any $k \geq L - (\log L)^4$, $0 \leq j \leq \alpha_{z,+}(k)/2$ and $z \leq \log L$*

$$\mathbb{P}\left[\sqrt{2T^\tau_{y,l}} \leq \alpha_{z,+}(l),\ l = m, \ldots, k-1;\ \sqrt{2T^\tau_{y,k}} \in I_{\alpha_{z,+}(k)-j}\right] \quad (9.5)$$
$$\leq ce^{-2k-2z-2k_L^{1/4}+2j} \times m^2\left(1 + z + m + k_L^{1/4}\right)(1+j).$$

Proof of Lemma (9.2). Using the Markov property, the probability in (9.5) is bounded by

$$\sum_{s=0}^{\alpha_{z,+}(m)} \mathbb{P}\left[\sqrt{2T^\tau_{y,m}} \in I_s\right] \times \sup_{x \in I_s} \mathbb{P}\left[\sqrt{2T^{m,x^2/2}_{y,l}} \leq \alpha_{z,+}(l),\ l = m+1, \ldots, k-1;\ \sqrt{2T^{m,x^2/2}_{y,k}} \in I_{\alpha_{z,+}(k)-j}\right], \quad (9.6)$$

and using the fact that $T^\tau_{y,m} \leq T^{m,0}_{y,m}$ and (2.8), we see that (9.6) is bounded by

$$\sum_{s=0}^{\alpha_{z,+}(m)} e^{-s^2/2m} \sup_{x \in I_s} \mathbb{P}\left[\sqrt{2T^{m,x^2/2}_{y,l}} \leq \alpha_{z,+}(l),\ l = m+1, \ldots, k-1;\ \sqrt{2T^{m,x^2/2}_{y,k}} \in I_{\alpha_{z,+}(k)-j}\right]. \quad (9.7)$$

Recall that $\alpha_{z,+}(l) = \rho_L l + z + l_L^{1/4}$. Using this we can write the last probability as

$$K_{1,s} := \mathbb{P}\left[\sqrt{2T^{m,x^2/2}_{y,l}} \leq \rho_L l + z + l_L^{1/4} \text{ for } l = m+1, \ldots, k-1;\ \sqrt{2T^{m,x^2/2}_{y,k}} \in I_{\rho_L k + z + k_L^{1/4} - j}\right]. \quad (9.8)$$

Using the fact that for all $1 \leq l \leq k-1$

$$l_L^{1/4} \leq l_k^{1/4} + k_L^{1/4}, \tag{9.9}$$

see [7], it follows that

$$\begin{aligned} K_{1,s} \leq P^{\mathrm{GW}}_{x^2/2} \Big[& \sqrt{2T_l} \leq \rho_L (m+l) + z + (m+l)_k^{1/4} + k_L^{1/4}, \\ & \text{for } l = 1, \ldots, k-m-1; \sqrt{2T_{k-m}} \in I_{\rho_L k + z + k_L^{1/4} - j} \Big]. \end{aligned} \tag{9.10}$$

Thus using (9.2), with $a = \rho_L m + z + (m_k^{1/4} + k_L^{1/4})$ and $b = \rho_L k + z + k_L^{1/4}$, $y = \rho_L k + z + k_L^{1/4} - j$,

$$K_{1,s} \leq c \frac{(1+a-x)(1+j)}{k-m} \sqrt{\frac{x}{y(k-m)}} e^{-\frac{\left(\rho_L k + z + k_L^{1/4} - j - x\right)^2}{2(k-m)}}.$$

We have

$$\begin{aligned} e^{-\frac{\left(\rho_L k + z + k_L^{1/4} - j - x\right)^2}{2(k-m)}} &\leq ce^{-\frac{(\rho_L k)^2}{2k(1-m/k)}} e^{-2\left(z + k_L^{1/4} - j - x\right)} \\ &\leq ce^{2k\frac{\log L}{L}} e^{-2(k+m) - 2z - 2k_L^{1/4} + 2j + 2x}, \end{aligned}$$

$\frac{1}{k-m} \sqrt{\frac{x}{y(k-m)}}\, e^{2k\frac{\log L}{L}} \asymp \sqrt{x}$, and by assumption,

$$a - x \leq c \left(k_L^{1/4} + m + z\right).$$

Hence we can bound (9.7) by

$$c\left(1 + k_L^{1/4} + m + z\right)(1+j)\, e^{-2k - 2z - 2k_L^{1/4} + 2j} \sum_{s=0}^{\alpha_{z,+}(m)} \sqrt{s} e^{-(s-2m)^2/2m}.$$

Our Lemma follows. ☐

Lemma 9.3. *For all L sufficiently large, and all $0 \leq z \leq \log L$,*

$$\begin{aligned} & \mathbb{P}\left[\sqrt{2T^1_{y,l}} \leq \alpha_{z,-}(l) \ \text{ for } l = 1, \ldots, L-1;\ \sqrt{2T^1_{y,L}} \geq \rho_L + z\right] \\ & \leq \mathbb{P}\left[\sqrt{2T^1_{y,l}} \leq \rho_L l + z \ \text{ for } l = 1, \ldots, L-1;\ \sqrt{2T^1_{y,L}} \geq \rho_L L + z\right] \\ & \qquad \leq c(1+z) e^{-2L - 2z - z^2/4L}. \end{aligned} \tag{9.11}$$

and

$$\begin{aligned} & \mathbb{P}\left[\sqrt{2T^1_{y,l}} \leq \alpha_{z,-}(l) \ \text{ for } l = 1, \ldots, L-1;\ \sqrt{2T^1_{y,L}} \in I_{\rho_L L + z}\right] \\ & \qquad \geq c(1+z) e^{-2L - 2z - z^2/4L}. \end{aligned} \tag{9.12}$$

Similar results hold if we delete the barrier condition on some fixed finite interval.

Proof of Lemma 9.3. The first inequality in (9.11) is obvious. Theorem 9.1 requires that $y \leq b$ which we will not have if we go all the way to L. Instead, using the Markov property at $l = L-1$ and (3.22) we bound

$$\begin{aligned} & \mathbb{P}\left[\sqrt{2T^1_{y,l}} \leq \rho_L l + z \text{ for } l = 1, \ldots, L-1;\ \sqrt{2T^1_{y,L}} \geq \rho_L + z\right] \\ & \leq c \sum_{j=1}^{\rho_L(L-1)+z} \mathbb{P}\Big[\sqrt{2T^1_{y,l}} \leq \rho_L l + z \text{ for } l = 1, \ldots, L-2; \\ & \qquad \sqrt{2T^1_{y,L-1}} \in I_{\rho_L(L-1)+z-j}\Big]\, e^{-j^2/2}. \end{aligned} \tag{9.13}$$

If $j \geq L/2$, then $e^{-j^2/2} \leq e^{-L^2/8}$ so we get a bound much smaller than (9.11). Thus we need only bound the sum over $1 \leq j \leq L/2$.

It follows from (9.2), with $a = z$, $b = \rho_L(L-1)+z$, $y = \rho_L(L-1)+z-j$ that the last probability is bounded by

$$c\frac{(1+z)\,(1+j)}{L}\sqrt{\frac{1}{L^2}}e^{-(\rho_L(L-1)+z-j)^2/2(L-1)} \\ \leq c(1+z)\,(1+j)\,e^{-2L-2(z-j)-(z-j)^2/4L},$$

and our upper bound follows after summing over j.

The lower bound follows similarly using (9.3). The last statement in our Lemma comes from the last statement in Theorem 9.1. □

Lemma 9.4. *If $k \geq L/2,\ 0 \leq z \leq \log L$ and L is sufficiently large, then uniformly in $0 \leq p \leq k$,*

$$\mathbb{P}\left[\sqrt{2\mathcal{T}^1_{y,l}} \leq \rho_L l + z \text{ for } l = 1,\ldots,k-1;\ \sqrt{2\mathcal{T}^1_{y,k}} \in I_{\rho_L k+z-p}\right] \\ \leq C(1+z)\,(1+p)\,e^{-2k-2(z-p)-(z-p)^2/4k}. \tag{9.14}$$

Proof of Lemma 9.4. Using Theorem 9.1 with $a = z$, $y = \rho_L k + z - p, b = \rho_L k + z$ this is bounded by

$$c\frac{(1+z)\,(1+p)}{k}\sqrt{\frac{1}{k^2}}e^{-(\rho_L k+z-p)^2/2k} \\ \leq C\frac{(1+z)\,(1+p)}{k^2}e^{2\log(L)k/L}e^{-2k-2(z-p)-(z-p)^2/4k}. \tag{9.15}$$

(9.14) follows since by the convexity of $\log$ we have $e^{2\log(L)k/L} \leq e^{2\log(k)} = k^2$. □

Lemma 9.5. *If $k \leq \log^5 L,\ 0 \leq z \leq \log L$ and L is sufficiently large, then uniformly in $m = \rho_L k + z - t$,*

$$\mathbb{P}\left[\sqrt{2\mathcal{T}^{k,m^2/2}_{y,l}} \leq \rho_L l + z \text{ for } l = k+1,\ldots,L-1;\ \sqrt{2\mathcal{T}^{k,m^2}_{y,L}} \geq \rho_L L + z\right] \\ \leq c(1+t)m^{1/2}e^{-2(L-k)-2t-t^2/4(L-k)}. \tag{9.16}$$

Proof of Lemma 9.5. As before, we can bound the probability by

$$\sum_{j=0}^{\rho_L(L-1)+z} \mathbb{P}\left[\sqrt{2\mathcal{T}^{k,m^2/2}_{y,l}} \leq \rho_L l + z \text{ for } l = k+1,\ldots,L-2; \right. \\ \left. \sqrt{2T^{k,m^2/2}_{y,L-1}} \in I_{\rho_L(L-1)+z-j}\right] e^{-j^2/2}. \tag{9.17}$$

Also, as before, we need only bound the sum over $1 \leq j \leq L/2$.

It follows from (9.2), with the x of that estimate given by $m = a-t$ and $a = \rho_L k+z, b = \rho_L(L-1)+z, y = \rho_L(L-1)+z-j$ that the last probability is bounded by

$$c\frac{(1+t)\,(1+j)}{L-k}\sqrt{\frac{m}{L(L-k)}}e^{-\frac{(\rho_L(L-k-1)+t-j)^2}{2(L-k-1)}}, \tag{9.18}$$

and

$$e^{-\frac{(\rho_L(L-k-1)+t-j)^2}{2(L-k-1)}} \\ \leq ce^{-\frac{(\rho_L(L-k-1))^2}{2(L-k-1)}-2(t-j)-(t-j)^2/2(L-k)} \\ \leq Ce^{2\log(L)(L-k)/L}e^{-2(L-k)-2(t-j)-(t-j)^2/2(L-k)}. \tag{9.19}$$

This gives

$$\mathbb{P}\left[\sqrt{2\mathcal{T}_{y,l}^{k,m^2/2}} \le \rho_L l + z \text{ for } l = k+1,\ldots,L-1;\ \sqrt{2\mathcal{T}_{y,L}^{k,m^2/2}} \ge \rho_L L + z\right]$$
$$\le C \sum_{j=0}^{\rho_L(L-1)+z} \frac{(1+t)\,(1+j)}{L-k}\sqrt{\frac{m}{L(L-k)}}L^{\frac{2(L-k)}{L}}$$
$$e^{-2(L-k)-2(t-j)-(t-j)^2/2(L-k)}e^{-j^2/2}.$$

(9.16) follows since by the convexity of $\log$, $e^{2\log(L)(L-k)/L} \le e^{2\log((L-k))} = (L-k)^2$. □

The following Lemma states the barrier estimates needed for the proof of the left tail estimates in Theorem 1.3. For notation see Section 4.3. The proof of this Lemma is similar to the proofs of Lemmas 9.3-9.5.

Lemma 9.6. *For all L sufficiently large, and all $|z| \le \log L$,*

$$\mathbb{P}\left[\sqrt{2T_{y,l}^1} \le \widehat{\gamma}_{z,-}(l) \text{ for } l = 1,\ldots,L-1;\ \sqrt{2T_{y,L}^1} \ge \rho_L L + z\right]$$
$$\le \mathbb{P}\left[\sqrt{2T_{y,l}^1} \le \widehat{\beta}_z(l) \text{ for } l = 1,\ldots,L-1;\ \sqrt{2T_{y,L}^1} \ge \rho_L L + z\right]$$
$$\le ce^{-2L-2z-z^2/4L}. \tag{9.20}$$

and

$$\mathbb{P}\left[\sqrt{2T_{y,l}^1} \le \widehat{\gamma}_{z,-}(l) \text{ for } l = 1,\ldots,L-1;\ \sqrt{2T_{y,L}^1} \in I_{\rho_L L+z}\right]$$
$$\ge ce^{-2L-2z-z^2/4L}. \tag{9.21}$$

Similar results hold if we delete the barrier condition on some fixed finite interval. If $k \ge L/2,\ |z| \le \log L$, and L is sufficiently large, then uniformly in $p \le k$,

$$\mathbb{P}\left[\sqrt{2\mathcal{T}_{y,l}^1} \le \widehat{\beta}_z(l) \text{ for } l = 1,\ldots,k-1;\ \sqrt{2\mathcal{T}_{y,k}^1} \in I_{\widehat{\beta}_z(k)-p}\right] \tag{9.22}$$
$$\le C\,(1+p)\,e^{-2k-2(z-p)-(z-p)^2/4k}.$$

If $k \le \log^5 L,\ |z| \le \log L$, and L is sufficiently large, then uniformly in $m = \widehat{\beta}_z(k) - t$,

$$\mathbb{P}\left[\sqrt{2\mathcal{T}_{y,l}^{k,m^2}} \le \widehat{\beta}_z(l) \text{ for } l = k+1,\ldots,L-1;\ \sqrt{2\mathcal{T}_{y,L}^{k,m^2}} \ge \rho_L + z\right] \tag{9.23}$$
$$\le c(1+t)m^{1/2}e^{-2(L-k)-2t-t^2/4(L-k)}.$$

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