

Zero distribution of random Bernoulli polynomial mappings*

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Abstract

In this note, we study asymptotic zero distribution of multivariable full system of random polynomials with independent Bernoulli coefficients. We prove that with overwhelming probability their simultaneous zeros sets are discrete and the associated normalized empirical measure of zeros asymptotic to the Haar measure on the unit torus.

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1 Introduction

A random Kac polynomial on the complex plane is of the form

$$f_d(z) = \sum_{j=0}^d a_j z^j \tag{1.1}$$

where the coefficients a_j are independent copies of the (real or complex) standard Gaussian. A classical result due to Kac, Hammersley and Shepp & Vanderbei [21, 17, 25] asserts that almost surely the normalized empirical measure of zeros $\delta_{Z(f_d)} := \frac{1}{d} \sum_{f_d(\zeta)=0} \delta_\zeta$, converges to normalized arc length measure on $S^1 := \{|z| = 1\}$ as $d \rightarrow \infty$. Asymptotic zero distribution of Kac polynomials with independent identically distributed (i.i.d.) discrete random coefficients have also been studied extensively (see e.g. [24, 15]). More recently, Ibragimov and Zaporozhets [20] proved that the empirical measure of zeros $\delta_{Z(f_d)}$ almost surely converges to the normalized arc length measure if and only if the moment condition $\mathbb{E}[\log(1 + |a_i|)] < \infty$ holds. This property can be considered as a global universality property of the zeros of random polynomials (see also [29] for a local version).

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Building upon the work of Shiffman and Zelditch [28], equilibrium distribution of random systems of polynomials with Gaussian coefficients was obtained by Bloom & Shiffman [9] and Shiffman [26]. More recently, these results were generalized for i.i.d. random coefficients with bounded density [1, 2]. We refer the reader to the survey [4] and references therein for the state of the art. On the other hand, asymptotic zero distribution of random polynomial mappings with discrete random coefficients remained open (cf. [3, 8, 5]). In this note, we study asymptotic zero distribution of multivariable full system of random polynomials with independent Bernoulli coefficients.

1.1 Statement of the results

A random Bernoulli polynomial is of the form

$$f_{d,i}(\mathbf{x}) = \sum_{|J| \leq d} \alpha_{i,J} \mathbf{x}^J \in \mathbb{C}[x_1, \dots, x_n]$$

where $\mathbf{x}^J = x_1^{j_1} \dots x_n^{j_n}$ and $\alpha_{i,J}$ are ± 1 Bernoulli random variables for $i = 1, \dots, n$. Throughout this work, we consider systems $(f_{d,1}, \dots, f_{d,n})$ of random Bernoulli polynomials with independent coefficients. We write $\mathbf{f}_d = (f_{d,1}, \dots, f_{d,n})$ for short. We denote the collection of all systems of polynomials in n variables and of degree d by $Poly_{n,d}$ that is endowed with the product probability measure $Prob_d$.

Theorem 1.1. *Let $\mathbf{f}_d = (f_{d,1}, \dots, f_{d,n})$ be a system of random polynomials with independent ± 1 valued Bernoulli coefficients. Then there exists a dimensional constant $K = K(n) > 0$ and an exceptional set $\mathcal{E}_{n,d} \subset Poly_{n,d}$ such that $Prob_d(\mathcal{E}_{n,d}) \leq K/d$ and for all $\mathbf{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d}$ the simultaneous zeros $Z(\mathbf{f}_d)$ of the system \mathbf{f}_d are isolated with $\#Z(\mathbf{f}_d) = d^n$.*

For a system $\mathbf{f}_d \in Poly_{n,d}$, if the simultaneous zeros $Z(\mathbf{f}_d)$ are isolated we denote the corresponding normalized empirical measure by $\delta_{Z(\mathbf{f}_d)}$. That is $\delta_{Z(\mathbf{f}_d)}$ is a probability measure supported on the isolated zeros with equal weight on each zero. We also let ν_{Haar} denote the Haar measure on $(S^1)^n$ of total mass 1. As an application of Theorem 1.1 together with a deterministic equidistribution result [14, Theorem 1.7], we obtain asymptotic zero distribution of random Bernoulli polynomial mappings:

Corollary 1.2. *Let $\mathbf{f}_d = (f_{d,1}, \dots, f_{d,n})$ be system of random polynomials with independent ± 1 valued Bernoulli coefficients and $\mathcal{E}_{n,d} \subset Poly_{n,d}$ be as in Theorem 1.1. Then for each sequence $\mathbf{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d}$ we have*

$$\lim_{d \rightarrow \infty} \delta_{Z(\mathbf{f}_d)} = \nu_{Haar}.$$

in the weak topology. In particular, $\delta_{Z(\mathbf{f}_d)} \rightarrow \nu_{Haar}$ in probability $Prob_d$ as $d \rightarrow \infty$.

Finally, we consider the measure valued random variables

$$\tilde{Z}(\mathbf{f}_d) := \begin{cases} \sum_{\xi_i \in Z(\mathbf{f}_d)} \delta(\xi_i) & \text{for } \mathbf{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d} \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

and define the expected zero measure by

$$\langle \mathbb{E}[\tilde{Z}(\mathbf{f}_d)], \varphi \rangle = \int_{Poly_{n,d} \setminus \mathcal{E}_{n,d}} \sum_{\xi_i \in Z(\mathbf{f}_d)} \varphi(\xi_i) dProb_d(\mathbf{f}_d) \tag{1.3}$$

where φ is a continuous function with compact support in \mathbb{C}^n and $\mathcal{E}_{n,d}$ denote the exceptional set given by Theorem 1.1.

Theorem 1.3. Let $\mathbf{f}_d = (f_{d,1}, \dots, f_{d,n})$ be a system of random polynomials with independent ± 1 valued Bernoulli coefficients. Then

$$\lim_{d \rightarrow \infty} d^{-n} \mathbb{E}[\tilde{Z}(\mathbf{f}_d)] = \nu_{\text{Haar}}$$

in the weak topology.

The outline of this work as follows. In §2, we review some basic properties of resultants. In particular, we recall multi-polynomial resultant and sparse resultant for polynomial systems [16, 11] as well as directional resultant [13]. In §3, we prove the main result Theorem 1.1. Finally, in §4 we prove Theorem 1.3.

2 Preliminaries

In this section, we review some basic results in algebraic geometry and discrepancy theory related to our results. More precisely, we discuss the multi-homogenous (classical) resultant and the sparse eliminant as well as the relation of these two notions. For a detailed account of the subject and proofs we refer the reader to [16, 11]. We also discuss the sparse resultant introduced by D’Andrea and Sombra, and corresponding directional sparse resultants [14, 13].

2.1 Lattice points, polytopes

For a nonempty subset $P \subset \mathbb{R}^n$, we denote its convex hull in \mathbb{R}^n by $\text{conv}(P)$. For two nonempty convex sets Q_1, Q_2 , their Minkowski sum is defined as

$$Q_1 + Q_2 := \{q_1 + q_2 : q_1 \in Q_1, q_2 \in Q_2\}$$

and for $\lambda \in \mathbb{R}$, the scaled polytope is of the form

$$\lambda Q := \{\lambda q : q \in Q\}.$$

It is well known that $\text{Vol}_n(d_1 Q_1 + \dots + d_n Q_n)$ is a homogenous polynomial of degree n in the variables $d_1, \dots, d_n \in \mathbb{Z}_+$ where Vol_n denotes the normalized volume of the subsets in \mathbb{R}^n with respect to the Lebesgue measure. The coefficient of the monomial $d_1 \dots d_n$ is called the *mixed volume* of Q_1, \dots, Q_n and denoted by $MV(Q_1, \dots, Q_n)$. One can use the polarization formula to compute the mixed volume of the convex sets Q_1, \dots, Q_n . Namely,

$$MV_n(Q_1, \dots, Q_n) = \sum_{k=1}^n \sum_{1 \leq j_1 \leq \dots \leq j_k \leq n} (-1)^{n-k} \text{Vol}_n(Q_{j_1} + \dots + Q_{j_k}).$$

In particular, if $Q = Q_1 = \dots = Q_n$ then

$$MV_n(Q) := MV_n(Q, \dots, Q) = n! \text{Vol}_n(Q).$$

For a convex set $Q \subset \mathbb{R}^n$ its *support function* $s_Q : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$s_Q(\mathbf{v}) := \inf_{\mathbf{q} \in Q} \langle \mathbf{q}, \mathbf{v} \rangle \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ represents the Euclidean inner product of \mathbb{R}^n . Given a vector $\mathbf{v} \in \mathbb{R}^n$ the equation

$$\langle \mathbf{q}, \mathbf{v} \rangle = s_Q(\mathbf{v})$$

defines *supporting hyperplane* of Q and \mathbf{v} is called an *inward pointing normal*. The intersection of Q with the supporting hyperplane in the direction $\mathbf{v} \in \mathbb{R}^n$ is denoted by

$$Q^{\mathbf{v}} := \{\mathbf{q} \in Q : \langle \mathbf{q}, \mathbf{v} \rangle = s_Q(\mathbf{v})\}. \tag{2.2}$$

The set $Q^{\mathbf{v}}$ is called the *face* of Q determined by \mathbf{v} . If $Q^{\mathbf{v}}$ has codimension 1, it is called a *facet* of Q .

2.2 Resultant of polynomial systems

2.2.1 Multipolynomial resultant

We consider homogenous polynomials of degree $d_i \geq 0$ of the form

$$\mathbf{F}_i(t_0, \dots, t_n) = \sum_{|J|=d_i} u_{i,J} \mathbf{t}^J$$

for $i = 0, \dots, n$ where J is a multi-index (j_0, \dots, j_n) and $\mathbf{t}^J := t_0^{j_0} \dots t_n^{j_n}$ is the monomial of degree $|J| = \sum_{i=0}^n j_i$. The set of such polynomials form an affine space by identifying $\sum_{|J|=d_i} u_{i,J} \mathbf{t}^J$ with the point $\mathbf{u}_i := (u_{i,J})_{|J|=d_i} \in \mathbb{C}^{N(d_i)}$, where $N(d_i) = \binom{n+d_i-1}{n-1}$. Letting $N := \sum_{i=0}^n N(d_i)$, recall that the *incidence variety* is defined by

$$\mathcal{W} = \{(\mathbf{u}, \mathbf{t}) \in \mathbb{C}^N \times \mathbb{P}(\mathbb{C}^n) : F_0(\mathbf{u}_0, \mathbf{t}) = \dots = F_n(\mathbf{u}_n, \mathbf{t}) = 0\}.$$

We also let $\pi : \mathbb{C}^N \times \mathbb{P}(\mathbb{C}^n) \rightarrow \mathbb{C}^N$ be the projection onto first coordinate where $\mathbb{P}(\mathbb{C}^n)$ denotes the complex projective space. Then by Projective Extension Theorem (see e.g. [11]) the image $\pi(\mathcal{W})$ forms a variety in the affine space \mathbb{C}^N .

Definition 2.1. *The multipolynomial resultant Res_{d_0, \dots, d_n} is defined as the irreducible unique (up to a sign) polynomial in $\mathbb{Z}[\mathbf{u}_0, \dots, \mathbf{u}_n]$ which is the defining equation of the variety $\pi(\mathcal{W})$. The resultant of the homogeneous polynomials F_0, \dots, F_n is the evaluation of Res_{d_0, \dots, d_n} at the coefficients of F_0, \dots, F_n and it is denoted by $Res_{d_0, \dots, d_n}(F_0, \dots, F_n)$.*

Note that if $d_0 = \dots = d_n = 1$, then the evaluation of multipolynomial resultant Res_{d_0, \dots, d_n} at the coefficients of F_0, \dots, F_n is the determinant of the coefficient matrix.

Theorem 2.2 ([16], [11]). *Let $F_0, \dots, F_n \in \mathbb{C}[t_0, \dots, t_n]$ be homogenous polynomials of positive total degrees d_0, \dots, d_n . Then the system $F_0 = \dots = F_n = 0$ has a solution in the complex projective space \mathbb{P}^n if and only if $Res_{d_0, \dots, d_n}(F_0, \dots, F_n) = 0$.*

Theorem 2.2 gives a characterization to determine the existence of nontrivial solutions for the systems of homogenous polynomials based on the coefficients of the polynomials in the system. However, not all the systems of equations are homogenous, and in the power series expansions not all the monomial terms appear. Hence, we need to introduce a more general version of the multi-homogenous resultant.

2.2.2 Sparse eliminant

Following [16], we will recall the definition of sparse resultant. Let A_0, \dots, A_n be a collection of non-empty finite subsets of \mathbb{Z}^n , and let $\mathbf{u}_i := \{u_{i,J}\}_{J \in A_i}$ be a group of $\#A_i$ variables, $i = 0, \dots, n$ and set $\mathbf{u} = \{\mathbf{u}_0, \dots, \mathbf{u}_n\}$. For each i , the general Laurent polynomial f_i with support $A_i := \text{supp}(f_i)$ is given by

$$f_i(\mathbf{u}_i, \mathbf{x}) = \sum_{J \in A_i} u_{i,J} \mathbf{x}^J \in \mathbb{C}[\mathbf{u}][x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

We let $\mathcal{A} = (A_0, \dots, A_n)$ and consider the incidence variety in this setting defined by

$$W_{\mathcal{A}} = \left\{ (\mathbf{u}, \mathbf{x}) \in \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{N_i}) \times (\mathbb{C}^*)^n : f_0(\mathbf{u}_1, \mathbf{x}) = \dots = f_n(\mathbf{u}_n, \mathbf{x}) = 0 \right\} \quad (2.3)$$

where $N_i = \#A_i$. Next, we consider the canonical projection on the first coordinate

$$\pi_{\mathcal{A}} : \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{N_i}) \times (\mathbb{C}^*)^n \rightarrow \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{N_i})$$

and let $\overline{\pi_{\mathcal{A}}(W_{\mathcal{A}})}$ denote the Zariski closure of $W_{\mathcal{A}}$ under the projection $\pi_{\mathcal{A}}$.

Definition 2.3. The sparse eliminant, denoted by $\text{Res}_{\mathcal{A}}$, is defined as follows: if the variety $\overline{\pi_{\mathcal{A}}(W_{\mathcal{A}})}$ has codimension 1, then the sparse eliminant is the unique (up to sign) irreducible polynomial in $\mathbb{Z}[u]$ which is the defining equation of $\overline{\pi_{\mathcal{A}}(W_{\mathcal{A}})}$. If $\text{codim}(\overline{\pi_{\mathcal{A}}(W_{\mathcal{A}})}) \geq 2$, then $\text{Res}_{\mathcal{A}}$ is defined to be the constant polynomial 1. The expression

$$\text{Res}_{\mathcal{A}}(f_0, \dots, f_n)$$

is the evaluation of $\text{Res}_{\mathcal{A}}$ at the coefficients of f_0, \dots, f_n .

Example 2.4. For $A_0 = \{0\}, A_1 = \{0, 1\} \subset \mathbb{Z}$, we have that $\text{Res}_{\mathcal{A}}(\mathbf{u}) = \pm u_{00}$ where $\mathcal{A} = (A_0, A_1)$.

The classical resultant $\text{Res}_{d_0, \dots, d_n}$ is the special case of the sparse eliminant $\text{Res}_{\mathcal{A}}$. Indeed, by letting A_i be the set of all integer points in the d_i -simplex, i.e., $A_i = d_i \Sigma_n \cap \mathbb{Z}^n$ and Σ_n be the standard unit simplex

$$d_i \Sigma_n := \left\{ (a_0, \dots, a_n) \in \mathbb{R}^{n+1} : a_j \geq 0 \text{ and } \sum_j a_j \leq d_i \right\}$$

one recovers $\text{Res}_{\mathcal{A}} = \text{Res}_{d_0, \dots, d_n}$ up to a sign. Indeed, following [11] and [16] for simplicity we let all the sparse polynomials f_0, \dots, f_n have the same support $A_i = d \Sigma_n \cap \mathbb{Z}^n$ for some positive integer d and consider the system

$$\begin{cases} f_0 = u_{01} \mathbf{x}^{\alpha_1} + \dots + u_{0d} \mathbf{x}^{\alpha_n} = 0 \\ \vdots \\ f_n = u_{n1} \mathbf{x}^{\alpha_1} + \dots + u_{nd} \mathbf{x}^{\alpha_n} = 0 \end{cases} \quad (2.4)$$

We also let t_0, \dots, t_n be the homogenous coordinates which are related to x_1, \dots, x_n by $x_i = t_i/t_0$. Then we define the homogenous polynomials

$$F_i(t_0, \dots, t_n) = t_0^d f_i(t_1/t_0, \dots, t_n/t_0) = t_0^d f_i(x_1, \dots, x_n), \quad (2.5)$$

for $0 \leq i \leq n$. This gives $n+1$ homogenous polynomials of total degree d in the variables t_0, \dots, t_n and this procedure is independent of the choice of homogeneous coordinates.

Proposition 2.5 ([11]). Let $A_i := d \Sigma_n \cap \mathbb{Z}^n$ for each $i = 1, \dots, n$ and consider the systems of polynomials \mathbf{F} and \mathbf{f} as above. Then

$$\text{Res}_{\mathcal{A}_d}(f_0, \dots, f_n) = \pm \text{Res}_{d, \dots, d}(F_0, \dots, F_n),$$

where $\mathcal{A}_d := (A_1, \dots, A_n)$.

Using the above proposition, we can give a version of Theorem 2.2 as follows.

Corollary 2.6. Let $\mathbf{f} = (f_1, \dots, f_n)$ be a system of polynomials with $A_i = d \Sigma_n \cap \mathbb{Z}^n$ for $i = 1, \dots, n$. Assume that the system $\mathbf{F} = (F_0, \dots, F_n)$ consists the homogenizations of f_i according to process in (2.5) and denote the set of simultaneous nonzero solutions of \mathbf{F} by $Z(\mathbf{F})$. Suppose that $Z(\mathbf{F}) \cap H^\infty(t_0) = \emptyset$ where $H^\infty(t_0)$ is the hyperplane at infinity for $t_0 = 0$. Then the system of polynomials $\mathbf{f} = 0$ has no solution if and only if $\text{Res}_{\mathcal{A}_d}(f_0, \dots, f_n) \neq 0$ where $\mathcal{A}_d := (A_1, \dots, A_n)$.

Proof. If $\text{Res}_{\mathcal{A}_d}(f_0, \dots, f_n) \neq 0$, then by definition of the sparse resultant the system

$$f_0(x) = \dots = f_n(x) = 0$$

has no solution. Conversely, letting F_i be the homogenization of f_i as in (2.5) with the corresponding variable $\mathbf{t} = (t_0, \dots, t_n)$, i.e. $F_i(\mathbf{t}) = t_0^d f_i(\mathbf{x})$. If the system of polynomials $\mathbf{f} = 0$ has no solution then $F_i(\mathbf{t}) = 0$ for $i = 1, \dots, n$ if and only if $t_0 = 0$ which contradicts our assumption. Hence, by Theorem 2.2 we have

$$\pm \text{Res}_{\mathcal{A}_d}(f_0, \dots, f_n) = \text{Res}_{d_0, \dots, d_n}(F_0, \dots, F_n) \neq 0. \quad \square$$

2.2.3 Sparse resultant

In spite of being a generalization of the multipolynomial resultant and involving considerable large amount of the system of polynomials, the sparse eliminant does not satisfy some essential properties such as additivity property and Poisson formula which are essential in many applications. More recently, D'Andrea and Sombra [13] introduced the following version which has the desired features:

Definition 2.7. *The sparse resultant, denoted by $\mathcal{R}es_{\mathcal{A}}$, is defined as any primitive polynomial in $\mathbb{Z}[u]$ that is the defining equation of the direct image of $\mathcal{W}_{\mathcal{A}}$ where*

$$(\pi_{\mathcal{A}})_*(W_{\mathcal{A}}) = \deg(\pi_{\mathcal{A}}|_{W_{\mathcal{A}}})\overline{\pi_{\mathcal{A}}(W_{\mathcal{A}})}$$

if this variety has codimension one, and otherwise we set $\mathcal{R}es_{\mathcal{A}} \equiv 1$. The expression

$$\mathcal{R}es_{\mathcal{A}}(f_0, \dots, f_n)$$

is the evaluation of $\mathcal{R}es_{\mathcal{A}}$ at the coefficients of f_0, \dots, f_n .

According to this definition, the sparse resultant is not irreducible but it is a power of the irreducible sparse eliminant, i.e.,

$$\mathcal{R}es_{\mathcal{A}} = \pm \text{Res}_{\mathcal{A}}^{\deg(\pi_{\mathcal{A}}|_{W_{\mathcal{A}}})}$$

where $\deg(\pi_{\mathcal{A}}|_{W_{\mathcal{A}}})$ is the degree of the projection $\pi_{\mathcal{A}}$. We also remark that $\mathcal{R}es_{\mathcal{A}} \neq 1$ whenever $\text{Res}_{\mathcal{A}} \neq 1$.

Example 2.8. Let $A_0 = A_1 = A_2 = \{(0, 0), (2, 0), (0, 2)\}$. Then $\text{Res}_{\mathcal{A}} = \det(u_{i,j})$ and $\mathcal{R}es_{\mathcal{A}} = \pm[\det(u_{i,j})]^4$.

For the detailed account of the subject we refer the reader to the manuscripts [13] and [14].

2.2.4 Directional resultant

For a finite subset $A \subset \mathbb{Z}^n$ and a non-zero vector $v \in \mathbb{Z}^n$ we denote

$$\mathcal{A}^v := \{J \in A : \langle J, v \rangle = s_Q(v)\}$$

where $Q = \text{conv}(A)$ and $s_Q(v)$ as in the equation (2.1). For a Laurent polynomial $f(x) = \sum_{J \in A} u_J x^J$ with support $\text{supp}(f) = A$ we also define the directed polynomial

$$f^v(x) := \sum_{J \in \mathcal{A}^v} u_J x^J.$$

Definition 2.9. *Let $A_1, \dots, A_n \subset \mathbb{Z}^n$ be a family of n non-empty finite subsets, $v \in \mathbb{Z}^n \setminus \{0\}$, and $v^\perp \subset \mathbb{R}^n$ the orthogonal subspace. Then there exists $b_{i,v} \in \mathbb{Z}^n$ such that*

$$A_i^v - b_{i,v} \subset \mathbb{Z}^n \cap v^\perp$$

for $i = 1, \dots, n$. The resultant of A_1, \dots, A_n in the direction of v , denoted by $\mathcal{R}es_{\mathcal{A}^v}$ is defined as the sparse resultant of the family of the finite subsets $A_i^v - b_{i,v}$ for $i \in \{1, \dots, n\}$.

Given a collection $f_i \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials with support $\text{supp}(f_i) \subset A_i$ for $i = 1, \dots, n$ we write $f_i^v = x^{b_{i,v}} g_{i,v}$ where $g_{i,v} \in \mathbb{C}[\mathbb{Z}^n \cap v^\perp] \simeq \mathbb{C}[y_1^{\pm 1}, \dots, y_{n-1}^{\pm 1}]$ is a Laurent polynomial with $\text{supp}(g_{i,v}) \subset A_i^v - b_{i,v}$. The expression

$$\mathcal{R}es_{\mathcal{A}^v}(f_1^v, \dots, f_n^v)$$

is defined as the evaluation of the resultant $\mathcal{R}es_{\mathcal{A}^v}$ at the coefficients of the $g_{i,v}$.

We remark that the definition of directional resultant is independent of the choice of the vector $\mathbf{b}_{i,v}$ (see [13, Proposition 3.3]). Moreover, the directional resultant $\mathcal{R}es_{A^v} \neq 1$ only if the direction vector v is an inward pointing normal to a facet of the Minkowski sum $\sum_{i=1}^n conv(A_i)$ (cf. [13, Proposition 3.8]). Therefore, for a family of subsets $A_1, \dots, A_n \subset \mathbb{Z}^n$ there are only finitely many directions $v \in \mathbb{Z}^n \setminus \{0\}$ for which the directional resultant can vanish.

Example 2.10. Let $f(x) = a_0 + \dots + a_n x^n \in \mathbb{C}[x]$ be a polynomial of degree n . Then the nontrivial directional resultants are

$$\mathcal{R}es_A(f^v) = \begin{cases} \pm a_0 & \text{if } v = 1, \\ \pm a_n & \text{if } v = -1 \end{cases}$$

for the polytope $conv(A) = [0, n] \subset \mathbb{R}$.

In the last part of this section we review Bernstein’s Theorem about the number of the common solutions for Laurent polynomial systems and its relation to the directional resultant. The classical Bézout’s Theorem states that for n polynomials $g_1, \dots, g_n \in \mathbb{C}[x_1, \dots, x_n]$ of (positive) degrees d_1, \dots, d_n the system

$$g_1(x_1, \dots, x_n) = \dots = g_n(x_1, \dots, x_n) = 0$$

has either infinite number of solutions or the number of the number of complex roots cannot exceed $d_1 \dots d_n$. Moreover, if the solutions in the hyperplane at infinity are counted with multiplicity, the exact number of solutions in the complex projective space \mathbb{P}^n is $d_1 \dots d_n$ (see e.g. [11]). A generalization of this result to the context of Laurent polynomials was obtained by Bernstein [6] (see also Kushnirenko [23]). More precisely, we have the following:

Theorem 2.11 ([6]). *Let $\mathbf{f} = (f_1, \dots, f_n)$ be a system of Laurent polynomials with support $supp(f_i) = A_i \subset \mathbb{Z}^n$ for $i = 1, \dots, n$. If for any nonzero vector $v \in \mathbb{Z}^n$ the directed system $\mathbf{f}^v = (f_1^v, \dots, f_n^v)$ has no common zeros in $(\mathbb{C}^*)^n$ then the set of solutions of the system $\mathbf{f} = 0$ are isolated and the exact number of the solutions is $\#Z(\mathbf{f}) = MV_n(Q_1, \dots, Q_n)$ where $Q_i = conv(A_i)$ for $i = 1, \dots, n$.*

In particular, for a system of Laurent polynomials $\mathbf{f} = (f_1, \dots, f_n)$ if the directional resultant $\mathcal{R}es_{A^v}(f_1^v, \dots, f_n^v) \neq 0$ for every direction $v \in \mathbb{Z}^n \setminus \{0\}$ then simultaneous solutions of the system \mathbf{f} are isolated. This condition on the directional resultant holds for a generic (i.e. all except for some algebraic subset) choice of \mathbf{f} in the space of coefficients. In the next section, we prove a probabilistic version of this result for polynomial systems with Bernoulli coefficients.

3 Equidistribution of zeros

3.1 Random polynomial systems

First, we recall a theorem of Kozma and Zeitouni [22] asserts that overdetermined random Bernoulli polynomial systems have no common zeros with overwhelming probability:

Theorem 3.1. *Let $f_1, \dots, f_{n+1} \in \mathbb{Z}[x_1, \dots, x_n]$ be $n + 1$ independent random Bernoulli polynomials of degree d and*

$$\mathcal{P}(d, n) := Prob_d\{\exists \mathbf{x} \in \mathbb{C}^n : f_i(\mathbf{x}) = 0 \text{ for } i = 1 \dots, n + 1\}$$

denote the probability that the system $f_1(\mathbf{x}) = \dots = f_{n+1}(\mathbf{x}) = 0$ has a common solution. Then there exists a dimensional constant $K = K(n) < \infty$ such that

$$\mathcal{P}(d, n) \leq K/d$$

for all $d \in \mathbb{Z}_+$.

Next, we prove our main result:

Proof of Theorem 1.1. Let $f_{d,i}$ be a random Bernoulli polynomial of the form

$$f_{d,i}(\mathbf{x}) = \sum_{|J| \leq d} \alpha_{i,J} \mathbf{x}^J \in \mathbb{Z}[x_1, \dots, x_n], \tag{3.1}$$

where $\{\alpha_{i,J}\}$ is a family of independent Bernoulli random variables for $i = 1, \dots, n$.

We investigate the directional resultants of the system \mathbf{f} for all nonzero primitive direction vectors $\mathbf{v} \in \mathbb{Z}^n$. By [13, Proposition 3.8] it is enough to check the inward normals to the Minkowski sum of the supports $nd\Sigma_n$ which has $n + 1$ facets with $n + 1$ inward normals given by $\mathbf{v}_m := \mathbf{e}_m$ for $m = 1, \dots, n$ and $\mathbf{v}_{n+1} := -\sum_{m=1}^n \mathbf{e}_m$ where $\{\mathbf{e}_m\}_{m=1}^n$ is the standard basis of \mathbb{R}^n .

For $\mathbf{v}_m = \mathbf{e}_m$ the intersection of the support with the supporting hyperplane in the direction \mathbf{e}_m is of the form

$$\mathcal{A}^{\mathbf{v}_m} = \left\{ (j_1, \dots, j_n) \in d\Sigma_n \cap \mathbb{Z}^n : j_m = 0 \text{ and } \sum_{l=1}^n j_l \leq d \right\} \tag{3.2}$$

$m = 1, \dots, n$. Hence, the polynomials $f_i^{\mathbf{v}_m}$ can be written as

$$f_i^{\mathbf{v}_m} := \sum_{J \in \mathcal{A}^{\mathbf{v}_m}} \alpha_{i,J} \mathbf{x}^J \tag{3.3}$$

for $i = 1, \dots, n$. Note that polynomials $f_i^{\mathbf{v}_m}$ depend on $n - 1$ variables. As in the Definition 2.9, we choose the vector $\mathbf{b}_{i,\mathbf{v}_m} = \mathbf{0}$ so that $\mathcal{A}^{\mathbf{v}_m} - \mathbf{b}_{i,\mathbf{v}_m} \subset \mathbb{Z}^n \cap \mathbf{v}_m^\perp$ and we may take $g_{i,\mathbf{v}_m} := f_i^{\mathbf{v}_m}$ for each $i = 1, \dots, n$.

Recall that for two univariate polynomials $h_1, h_2 \in \mathbb{C}[x]$, their resultant $\text{Res}(h_1, h_2)$ is zero if and only if h_1 and h_2 have a common solution in \mathbb{C} . Therefore, if $n = 2$ the necessary and sufficient condition for g_{1,\mathbf{v}_m} and g_{2,\mathbf{v}_m} have zero resultant is that they have a common zero. Theorem 3.1 implies that there exists a constant K_m which is independent of d so that the aforementioned event has probability at most K_m/d .

On the other hand, when $n > 2$, we perform the homogenization process to each $(n - 1)$ variable polynomial g_{i,\mathbf{v}_m} for $i = 1, \dots, n$ as described in equation (2.5). We obtain the n variable homogenous polynomials G_{i,\mathbf{v}_m} of the form

$$G_{i,\mathbf{v}_m}(t, \mathbf{x}) = \sum_{J \in \mathcal{A}^{\mathbf{v}_m}} \alpha_{i,J} t^{d-|J|} \mathbf{x}^J. \tag{3.4}$$

In order to compare the sparse resultant of the polynomials g_{i,\mathbf{v}_m} and the multipolynomial resultant of the homogeneous polynomials G_{i,\mathbf{v}_m} , we check the conditions of Corollary 2.6. Let $Z(\mathbf{G})$ be the set of nontrivial solutions of the system $\mathbf{G} = (G_{1,\mathbf{v}_m}, \dots, G_{n,\mathbf{v}_m})$ and suppose that \mathbf{G} has a solution $\boldsymbol{\xi} = (t, \xi_2, \dots, \xi_n)$ in the hyperplane at infinity $H^\infty(t)$. Evaluating these homogeneous polynomials at $t = 0$, we obtain the top degree homogeneous part of the polynomials g_{i,\mathbf{v}_m} for $i = 1, \dots, n$. Since $\boldsymbol{\xi} \in H^\infty(t)$, it has a nonzero coordinate ξ_k for some $k \in \{2, \dots, n\}$. For simplicity, let us assume $k = 2$ and define the new variables $z_i := \xi_{i+2}/\xi_2$ for $i = 1, \dots, n - 2$. Applying this change of variables, we obtain

$$\tilde{G}_{i,\mathbf{v}_m}(z_1, \dots, z_{n-2}) = \sum_{|J| \leq d} \alpha_{i,J} \mathbf{z}^{\varphi(J)} \tag{3.5}$$

where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-2}$ with $\varphi(j_1, \dots, j_n) = (j_3, \dots, j_n)$. This gives n random Bernoulli polynomials of degree d in $n - 2$ variables. Hence by Theorem 3.1, there exists a

positive constant C_i , depending only the dimension n such that the probability that the overdetermined system of Bernoulli polynomials $\tilde{G}_{i,v_m}(z_1, \dots, z_{n-2})$ have a common solution is less than C_i/d . We infer that the system of homogenized polynomials G_{i,v_m} has no common zero at hyperplane at infinity $H^\infty(t)$ except a set that has probability at most C_i/d . Then by Corollary 2.6, outside of a set of small probability, the system of polynomials consisting g_{i,v_m} has a common solution if and only if the directional resultant $\mathcal{R}es_{\mathcal{A}^{v_m}}(f_1^{v_m}, \dots, f_n^{v_m}) = 0$. Now, since the system of Bernoulli polynomials g_{i,v_m} contains n polynomials in $n - 1$ variables, by Theorem 3.1, there is a dimensional constant \tilde{C}_i so that the probability that this system has common solution is at most \tilde{C}_i/d . Hence outside of a set that has probability $K_i/d := C_i/d + \tilde{C}_i/d$, the directional resultant $\mathcal{R}es_{\mathcal{A}^{v_m}}(f_1^{v_m}, \dots, f_n^{v_m}) \neq 0$ for all v_m for $m = 1, \dots, n$.

Next, we consider the inward normal vector $v_{n+1} = -\sum_{m=1}^n e_m$ and we find the minimal weighted set in this direction as $\mathcal{A}^{v_{n+1}} = \{J \in d\Sigma_n \cap \mathbb{Z}^n : |J| = d\}$. Hence, the directed polynomials in this case are of the form

$$f_i^{v_{n+1}}(\mathbf{x}) = \sum_{|J|=d} \alpha_{i,J} \mathbf{x}^J \tag{3.6}$$

In this case $\mathcal{A}^{v_{n+1}}$ is not a subspace of $\mathbb{Z}^n \cap v_{n+1}^\perp$, hence we need to translate it by subtracting a suitable vector $\mathbf{b}_{i,v_{n+1}}$. For Laurent polynomial systems, the sparse resultant is invariant under translations of supports (see [13], Proposition 3.3). Since the polynomials $f_{d,i}$ are not Laurent, we need to determine the effects of this translations. Consider the system of Bernoulli polynomials \mathbf{f}_d and set of its simultaneous zeros $Z(\mathbf{f}_d)$. For a solution $\mathbf{x} = (x_1, \dots, x_n) \in Z(\mathbf{f}_d)$ and assume that $x_1 = 0$. In order to examine the incidence of this case, we evaluate the system \mathbf{f}_d at $x_1 = 0$ and we obtain a new system of n Bernoulli polynomials with $n - 1$ variables. By Theorem 3.1, there exists a constant C_1 which is independent of d such that this system has a common solution with probability at most C_1/d . Therefore the probability of the event that $x_1 = 0$ is less than C_1/d . Hence, there is no harm of translation of supports outside of a set that has probability at most C/d , where $C := \sum_{i=1}^n C_i$. Now, choosing the vector $\mathbf{b}_{i,v_{n+1}} = (d, 0, \dots, 0)$ so that $\mathcal{A}^{v_{n+1}} - \mathbf{b}_{i,v_{n+1}} \subset \mathbb{Z}^n \cap v_{n+1}^\perp$, we obtain the polynomials of the form

$$g_{i,v_{n+1}}(\mathbf{x}) = \sum_{J \in \mathcal{A}^{v_{n+1}} - \mathbf{b}_{i,v_{n+1}}} \alpha_{i,J} \mathbf{x}^{w(J)} \tag{3.7}$$

with $w : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying $(j_1, j_2, \dots, j_n) \mapsto (-d + j_1, j_2, \dots, j_n)$. We substitute the new variables $y_i := x_{i+1}/x_1$ into $g_{i,v_{n+1}}$ for $i = 1, \dots, n - 1$ and obtain

$$g_{i,v_{n+1}}(\mathbf{y}) = \sum_{|J| \leq d} \alpha_{i,J} \mathbf{y}^{\sigma(J)} \tag{3.8}$$

for $\mathbf{y} \in \mathbb{C}^{n-1}$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\sigma(j_1, j_2, \dots, j_n) = (0, j_2, \dots, j_n)$. The system containing the polynomials $g_{i,v_{n+1}}(\mathbf{y})$, $i = 1, \dots, n$ contains n random Bernoulli polynomials with $n - 1$ random variable as in the cases $v_m = e_m$. By applying the same argument, we can show that $\mathcal{R}es_{\mathcal{A}^{v_{n+1}}}(f_1^{v_{n+1}}, \dots, f_n^{v_{n+1}}) \neq 0$ outside of a set that has probability at most K_{i+1}/d .

Now, we define the exceptional set $\mathcal{E}_{n,d}$ as a subset of $Pol_{n,d}$ which contains the systems \mathbf{f}_d that has a zero directional resultant for some nonzero primitive vector \mathbf{v} or the systems \mathbf{f}_d have a common solution $\mathbf{x} \in \mathbb{C}^n$ with $x_i = 0$ for some $i = 1, \dots, n$. More precisely, letting

$$\begin{aligned} \mathcal{E}_{n,d} &:= \{ \mathbf{f}_d \in Pol_{n,d} : \exists \mathbf{v} \in \mathbb{Z}^n \setminus \{0\} \ni \mathcal{R}es_{\mathcal{A}^{\mathbf{v}}} \mathbf{f}_d^{\mathbf{v}} = 0 \} \\ &\cup \{ \mathbf{f}_d \in Pol_{n,d} : \exists \mathbf{x} \in Z(\mathbf{f}_d) \ni \prod x_i = 0 \} \end{aligned} \tag{3.9}$$

we see that there exists a positive constant K which is independent of d such that

$$Prob_d(\mathcal{E}_{n,d}) \leq \frac{K}{d}$$

where $K := \sum_{i=1}^{n+1} K_i + C$. □

Next, we recall a deterministic equidistribution results for the solutions of systems of integer coefficient polynomials [14]. For a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, the supremum norm of f on the unit torus is defined as

$$\|f\|_{\text{sup}} := \sup_{|w_1|=\dots=|w_n|=1} |f(w_1, \dots, w_n)|.$$

Let ν_{Haar} be the Haar measure on \mathbb{C}^n with support $(S^1)^n$ and of total mass 1. Assume that $\mathbf{f} \in Poly_{n,d}$ be a polynomial mapping such that the set of simultaneous zeros $Z(\mathbf{f})$ is a discrete set. We denote by $\delta_{Z(\mathbf{f})}$ the discrete probability measure on \mathbb{C}^n associated to the $Z(\mathbf{f})$ by $\delta_{Z(\mathbf{f})}$. The following result gives the asymptotic distribution of the zeros of such a system \mathbf{f} if the coefficients are integer:

Theorem 3.2 ([14]). *Let $\mathbf{f} = (f_1, \dots, f_n)$ be a polynomial mapping with $f_i \in \mathbb{Z}[x_1, \dots, x_n]$ of degree $d \geq 1$ for each $i = 1, \dots, n$. Assume that $Res_{A^v}(f_1^v, \dots, f_n^v) \neq 0$ for all $v \in \mathbb{Z}^n \setminus \{0\}$ and $\log \|f_i\|_{\text{sup}} = o(d)$. Then*

$$\lim_{d \rightarrow \infty} \delta_{Z(\mathbf{f})} = \nu_{\text{Haar}}$$

in the weak topology.

Proof of Corollary 1.2. Consider the system of Bernoulli polynomials $\mathbf{f}_d = (f_{d,1}, \dots, f_{d,n})$. Since all the coefficients are 1 or -1 , by triangle inequality

$$\|f_{d,i}\|_{\text{sup}} = \sup_{|w_1|=\dots=|w_n|=1} |f_{d,i}(w_1, \dots, w_n)| \leq \binom{n+d}{d} = d^n + O(d^{n-1}) \tag{3.10}$$

where $\binom{n+d}{d}$ is the dimension of space of polynomials $Poly_{n,d}$. This in turn implies that $\log \|f_{d,i}\|_{\text{sup}} = o(d)$. Moreover, by Theorem 1.1 for each $\mathbf{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d}$ we have

$$Res_{A^v}(f_1^v, \dots, f_n^v) \neq 0$$

for all $v \in \mathbb{Z}^n \setminus \{0\}$. Hence, by Theorem 3.2

$$\lim_{d \rightarrow \infty} \delta_{Z(\mathbf{f}_d)} = \nu_{\text{Haar}}$$

in the weak topology. In particular, $\delta_{Z(\mathbf{f}_d)} \rightarrow \nu_{\text{Haar}}$ in probability since $Prob_d(\mathcal{E}_{n,d}) \rightarrow 0$ as $d \rightarrow \infty$. □

4 Expected zero distribution

In this section, we introduce radial and angle discrepancies for random Bernoulli polynomial mappings in order to study asymptotics of expected zero measures. We adapt these concepts from [14] and refer the reader to the manuscript [14] and references therein for a detailed account of the preliminary results this section.

Let Z be a 0-dimensional effective cycle in \mathbb{C}^n that is there is a non-empty finite collection of points $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ and $m_\xi \in \mathbb{N}$, called the multiplicity of ξ , such that $Z = \sum_{\xi} m_\xi [\xi]$. The degree of Z is defined by $\deg(Z) = \sum_{\xi} m_\xi$ which is a positive number.

Definition 4.1 ([14]). Let Z be a 0-dimensional effective cycle in \mathbb{C}^n . For each $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ such that $-\pi \leq \alpha_j < \beta_j \leq \pi$, $j = 1, \dots, n$ we consider the cycle

$$Z_{\alpha, \beta} := \sum_{\{\xi \in Z: \alpha_j < \arg(\xi_j) \leq \beta_j\}} m_{\xi}[\xi].$$

The angle discrepancy of Z is defined as

$$\Delta_{\text{ang}}(Z) := \sup_{\alpha, \beta} \left| \frac{\text{deg}(Z_{\alpha, \beta})}{\text{deg}(Z)} - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} \right|.$$

For $0 < \varepsilon < 1$ we consider the cycle

$$Z_{\varepsilon} := \sum_{\{\xi \in Z: 1 - \varepsilon < |\xi_j| < (1 - \varepsilon)^{-1}\}} m_{\xi}[\xi].$$

The radius discrepancy of Z with respect to ε is defined as

$$\Delta_{\text{rad}}(Z, \varepsilon) := 1 - \frac{\text{deg}(Z_{\varepsilon})}{\text{deg}(Z)}.$$

Note that $0 < \Delta_{\text{ang}}(Z) \leq 1$ and $0 \leq \Delta_{\text{rad}}(Z, \varepsilon) \leq 1$. Observe that the angle discrepancy and the radial discrepancy are generalizations of their one dimensional versions defined in [15, 18].

Let $A_1, \dots, A_n \subset \mathbb{Z}^n$ be a collection of finite sets and let $Q_i = \text{conv}(A_i)$ for each $i = 1, \dots, n$. Throughout this section we assume that $D := MV_n(Q_1, \dots, Q_n) \geq 1$. For a vector $w \in S^{n-1}$ in the unit sphere in \mathbb{R}^n , let w^{\perp} be its orthogonal subspace and $\pi_{w^{\perp}} : \mathbb{R}^n \rightarrow w^{\perp}$ be the corresponding orthogonal projection. We let $MV_{w^{\perp}}$ denote the mixed volume of the convex bodies in w^{\perp} induced by the Euclidean measure on w^{\perp} . We also denote

$$D_{w,i} = MV_{w^{\perp}}(\pi_w(Q_1), \dots, \pi_w(Q_{i-1}), \pi_w(Q_{i+1}), \dots, \pi_w(Q_n)).$$

Let $f = (f_1, \dots, f_n)$ be a mapping such that the coordinates f_i are Laurent polynomials with $\text{supp}(f_i) = A_i$ for $i = 1, \dots, n$. Following [14], we define the Erdős-Turán size of f by

$$\eta(f) := \frac{1}{D} \sup_{w \in S^{n-1}} \log \left(\frac{\prod_{i=1}^n \|f\|_{\text{sup}}^{D_{w,i}}}{\prod_v |\mathcal{R}es_{A^v}(f_1^v, \dots, f_n^v)|^{\frac{|\langle v, w \rangle|}{2}}} \right) \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^n and the product in the denominator is taken over all non-zero primitive vectors $v \in \mathbb{Z}^n$. We remark that the Erdős-Turán size of a polynomial mapping f coincides with the bound in the Erdős-Turán Theorem [15] for univariate polynomials.

The next result gives an upper bound for the Erdős-Turán size of polynomial systems f with integer coefficients.

Proposition 4.2 ([14, Proposition 3.15]). Let A_1, \dots, A_n be a non-empty finite subsets of \mathbb{Z}^n and set $Q_i = \text{conv}(A_i)$ with $MV_n(Q_1, \dots, Q_n) \geq 1$. Let $d_i \in \mathbb{Z}_{\geq 1}$ and $b_i \in \mathbb{Z}^n$ so that $d_i \Sigma_n + b_i$, $i = 1, \dots, n$. Suppose that $f_1, \dots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with $\text{supp}(f_i) \subseteq A_i$ and such that $\mathcal{R}es_{A_1^v, \dots, A_n^v}(f_{d,1}^v, \dots, f_{d,n}^v) \neq 0$ for all $v \in \mathbb{Z}^n \setminus \{0\}$. Then

$$\eta(f) \leq \frac{1}{D} \left((n + \sqrt{n}) \left(\prod_{i=1}^n d_i \right) \sum_{i=1}^n \frac{\log \|f_i\|_{\text{sup}}}{d_i} \right).$$

The following theorem gives bounds for angle discrepancy and radius discrepancy of $Z(\mathbf{f})$ in terms of the Erdős-Turán size of \mathbf{f} . For one dimensional version see for instance [15] and [18].

Theorem 4.3 ([14]). *Let A_1, \dots, A_n be a non-empty finite subsets of \mathbb{Z}^n such that*

$$MV_n(Q_1, \dots, Q_n) \geq 1$$

with $Q_i = \text{conv}(A_i)$ for $n \geq 2$. Let $f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with $\text{supp}(f_i) \subseteq A_i$ and such that $\text{Res}_{\mathcal{A}^v}(f_{d,1}^v, \dots, f_{d,n}^v) \neq 0$ for all $v \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Then

$$\Delta_{\text{ang}}(Z(\mathbf{f})) \leq 66n2^n(18 + \log^+(\eta(\mathbf{f})^{-1}))^{\frac{2}{3}(n-1)}\eta(\mathbf{f})^{\frac{1}{3}}. \tag{4.2}$$

Moreover, for $0 < \varepsilon < 1$,

$$\Delta_{\text{rad}}(Z(\mathbf{f}), \varepsilon) \leq \frac{2n}{\varepsilon}\eta(\mathbf{f}). \tag{4.3}$$

For a random Bernoulli polynomial mapping \mathbf{f}_d we let $Z(\mathbf{f}_d)$ be the set of simultaneous zeros of \mathbf{f}_d . We define the angle discrepancy $\Delta_{\text{ang}}(Z(\mathbf{f}))$ and the radius discrepancy $\Delta_{\text{rad}}(Z(\mathbf{f}), \varepsilon)$ as above whenever $Z(\mathbf{f}_d)$ is a discrete set of points. Otherwise, we set $\Delta_{\text{rad}}(Z(\mathbf{f}), \varepsilon) = \Delta_{\text{ang}}(Z(\mathbf{f})) = 1$. Note that as our probability space $(\text{Poly}_{n,d}, \text{Prob}_d)$ is discrete, measurability of these random variables is not an issue in this setting. Next, we estimate the asymptotic expected discrepancies:

Proposition 4.4. *Let $\mathbf{f}_d = (f_{d,1}, \dots, f_{d,n})$ be a random Bernoulli polynomial mapping of degree $d \geq 1$. Then*

$$\lim_{d \rightarrow \infty} \mathbb{E}[\Delta_{\text{ang}}(Z(\mathbf{f}_d))] = 0 \quad \text{and} \quad \lim_{d \rightarrow \infty} \mathbb{E}[\Delta_{\text{rad}}(Z(\mathbf{f}_d))] = 0. \tag{4.4}$$

Proof. We adapt the argument in [[14], Theorem 4.9] to our setting. Consider the expected value of the angular discrepancy which is

$$\mathbb{E}[\Delta_{\text{ang}}(Z(\mathbf{f}_d))] = \int_{\text{Poly}_{n,d}} \Delta_{\text{ang}}(Z(\mathbf{f}_d)) d\text{Prob}_d(\mathbf{f}_d). \tag{4.5}$$

Let $\mathcal{E}_{n,d}$ be the exceptional set which contains all the systems in $\text{Poly}_{n,d}$ with zero directional resultants for some nonzero primitive vector $v \in \mathbb{Z}^n$ as described in the proof of Theorem 1.1. Since $0 < \Delta_{\text{ang}}(Z(\mathbf{f}_d)) \leq 1$ there exist constants K_1 which is independent of d such that

$$0 \leq \int_{\mathcal{E}_{n,d}} \Delta_{\text{ang}}(Z(\mathbf{f}_d)) d\text{Prob}(\mathbf{f}_d) \leq \text{Prob}_d\{\mathcal{E}_{n,d}\} \leq K_1 d^{-1}. \tag{4.6}$$

Hence,

$$\int_{\mathcal{E}_{n,d}} \Delta_{\text{ang}}(Z(\mathbf{f}_d)) d\text{Prob}_d(\mathbf{f}_d) \rightarrow 0$$

as $d \rightarrow \infty$.

Let $\mathbf{f}_d \in \text{Poly}_{n,d} \setminus \mathcal{E}_{n,d}$, then by Proposition 4.2

$$\eta(\mathbf{f}_d) \leq \frac{1}{d^n} \left(d^{n-1}(n + \sqrt{n}) \sum_{i=1}^n \log \|f_{d,i}\|_{\text{sup}} \right) \tag{4.7}$$

$$\leq \frac{1}{d^n} \left(d^{n-1}(n + \sqrt{n}) \sum_{i=1}^n \log(d + 1) \right) \tag{4.8}$$

$$\leq K_2 \frac{\log d}{d} \tag{4.9}$$

for a constant K_2 which is independent of d . On the other hand, by Theorem 4.3 for $\mathbf{f}_d \in \text{Poly}_{n,d} \setminus \mathcal{E}_{n,d}$ there exists constants K_3, K_4, K_5 and K_6 such that

$$\Delta_{\text{ang}}(Z(\mathbf{f}_d)) \leq K_3 \eta(\mathbf{f}_d)^{\frac{1}{3}} \log \left(\frac{K_4}{\eta(\mathbf{f}_d)} \right)^{\frac{2}{3}(n-1)} \tag{4.10}$$

$$\leq K_5 \left(\frac{\log d}{d} \right)^{\frac{1}{3}} \log \left(\frac{d}{\log d} \right)^{\frac{2}{3}(n-1)} \leq K_6 \frac{\log d^{\frac{2n}{3}-\frac{1}{3}}}{d^{\frac{1}{3}}}, \tag{4.11}$$

since the function $t^{\frac{1}{3}} \log(\frac{a}{t})^{\frac{n-1}{3}}$ is increasing for small values of $t > 0$. Combining the equations (4.9) and (4.11), we deduce that $\lim_{d \rightarrow \infty} \mathbb{E}[\Delta_{\text{ang}}(Z(\mathbf{f}_d))] = 0$.

The proof of the second assertion is analogous and we omit it. □

Proof of Theorem 1.3. We adapt the argument in [14, Theorem 1.8] to our setting. Let us denote $\nu_d := \frac{\mathbb{E}[\tilde{Z}(\mathbf{f}_d)]}{d^n}$, where $\mathbb{E}[\tilde{Z}(\mathbf{f}_d)]$ is the expected zero measure and ν_{Haar} be the Haar probability on $(S^1)^n$. We need to show that for each continuous function φ with compact support in \mathbb{C}^n we have $\int \varphi d\nu_d \rightarrow \int \varphi d\nu_{\text{Haar}}$ as $d \rightarrow \infty$. To this end, it is enough to prove the claim for characteristic functions φ_U of the open sets

$$U := \{(z_1, \dots, z_n) \in \mathbb{C}^n : r_{1,j} < |z_j| < r_{2,j} \text{ and } \alpha_j < \arg(z_j) < \beta_j\} \tag{4.12}$$

where $0 \leq r_{1,j} < r_{2,j} \leq \infty, r_{i,j} \neq 1$ for $i = 1, 2$ and $-\pi < \alpha_j < \beta_j \leq \pi$.

First, we consider the case when $U \cap (S^1)^n = \emptyset$. Then one can find an $0 < \varepsilon < 1$ such that U is disjoint from the set

$$\{(\xi_1, \dots, \xi_n) \in \mathbb{C}^n : 1 - \varepsilon < |\xi_j| < (1 - \varepsilon)^{-1} \text{ for all } j\}. \tag{4.13}$$

Let $\mathcal{E}_{n,d}$ be the exceptional set as in the proof of Theorem 1.1. If $\mathbf{f}_d \in \text{Poly}_{n,d} \setminus \mathcal{E}_{n,d}$ then $Z(\mathbf{f}_d)$ is discrete and

$$\#\{U \cap Z(\mathbf{f}_d)\} \leq \deg(Z(\mathbf{f}_d)) \Delta_{\text{rad}}(\mathbf{f}_d, \varepsilon) \leq d^n \Delta_{\text{rad}}(\mathbf{f}_d, \varepsilon).$$

On the other hand, if $\mathbf{f}_d \in \mathcal{E}_{n,d}$ then by definition $\deg(\tilde{Z}(\mathbf{f}_d)|_U) = 0$. Hence,

$$\nu_d(U) \leq \mathbb{E}[\Delta_{\text{rad}}(\tilde{Z}(\mathbf{f}_d, \varepsilon))]$$

and by Proposition 4.4,

$$\lim_{d \rightarrow \infty} \int_{\text{Poly}_{n,d}} \varphi_U d\nu_d = 0 = \nu_{\text{Haar}}(U).$$

If $U \cap (S^1)^n \neq \emptyset$ let

$$\tilde{U} = \{\mathbf{z} : \alpha_j \leq \arg(z_j) \leq \beta_j \text{ for all } j\}. \tag{4.14}$$

Then we have

$$\nu_d(U) - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} = \left(\nu_d(\tilde{U}) - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} \right) - \nu_d(\tilde{U} \setminus U).$$

By Theorem 1.1 we have

$$\begin{aligned} \left| \nu_d(\tilde{U}) - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} \right| &\leq \int_{\text{Poly}_{n,d} \setminus \mathcal{E}_{n,d}} \left| \frac{\deg(Z(\mathbf{f}_d)_{\alpha, \beta}}{d^n} - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} \right| d\text{Prob}_d(\mathbf{f}_d) + \frac{Kn}{d} \\ &\leq \int_{\text{Poly}_{n,d} \setminus \mathcal{E}_{n,d}} \Delta_{\text{ang}}(Z(\mathbf{f}_d)) d\text{Prob}_d(\mathbf{f}_d) + \frac{Kn}{d}. \end{aligned} \tag{4.15}$$

Note that the set $\tilde{U} \setminus U$ is a union of a finite number of subsets U_m of the form (4.12) such that $U_m \cap (S^1)^n = \emptyset$ for all m , we have $\lim_{d \rightarrow \infty} \nu_d(U_m) = 0$ by previous case and hence $\lim_{d \rightarrow \infty} \nu_d(\tilde{U} \setminus U) = 0$. Therefore, by Proposition 4.4 and (4.15),

$$\lim_{d \rightarrow \infty} \nu_d(U) = \lim_{d \rightarrow \infty} (\tilde{U}) = \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} = \nu_{Haar}(U)$$

which completes the proof. \square

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Random Bernoulli polynomial mappings

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