Electronic Journal of Probability

Electron. J. Probab. **28** (2023), article no. 130, 1–17.
ISSN: 1083-6489 https://doi.org/10.1214/23-EJP1032

A criterion and a Cramér–Wold device for quasi-infinite divisibility for discrete multivariate probability laws*

Ivan Alexeev[†‡] Alexey Khartov[§¶]

Abstract

Multivariate discrete probability laws are considered. We show that such laws are quasi-infinitely divisible if and only if their characteristic functions are separated from zero. We generalize the existing results for the univariate discrete laws and for the multivariate laws on $\mathbb{Z}^d$. The Cramér–Wold devices for infinite and quasi-infinite divisibility are proved.

Keywords: multivariate probability laws; characteristic functions; infinitely divisible laws; the Lévy representation; quasi-infinitely divisible laws; Cramér–Wold device.
MSC2020 subject classifications: 60E05; 60E07; 60E10.
Submitted to EJP on March 8, 2023, final version accepted on September 27, 2023.

1 Introduction

Let F be a distribution function of a multivariate probability law on $\mathbb{R}^d$, where $\mathbb{R}$ is the real line, d is a positive integer. Recall that F and the corresponding law are called *infinitely divisible* if for every positive integer n there exists a distribution function F_n such that $F = F_n^{*n}$, where "$*$" denotes the convolution, i.e. F is the n-fold convolution power of F_n. It is known that F is infinitely divisible if and only if its characteristic function

$$f(t) := \int_{\mathbb{R}^d} e^{i\langle t,x\rangle} dF(x), \quad t \in \mathbb{R}^d,$$

*The work of A. A. Khartov was supported by the Ministry of Science and Higher Education of the Russian Federation, agreement 075-15-2022-289 date 06/04/2022. The work of I. A. Alexeev was supported in part by the Möbius Contest Foundation for Young Scientists.

†Institute for Information Transmission Problems of Russian Academy of Sciences, Bolshoy Karetny per. 19, build.1, Moscow. E-mail: vanyalexeev@list.ru

‡St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, 27 Fontanka, Saint-Petersburg.

§Laboratory for Approximation Problems of Probability, Smolensk State University, 4 Przhevalsky st., 214000 Smolensk, Russia. E-mail: alexeykhartov@gmail.com, https://approlab.org

¶Scientific and Educational Center of Mathematics, ITMO University, Kronverksky Pr. 49, 197101, Saint-Petersburg, Russia.

admits the following *Lévy representation* (see [26, Theorem 8.1])

$$f(t) = \exp\Big\{ i\langle t, \gamma\rangle - \tfrac{1}{2}\langle t, Qt\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle t,x\rangle} - 1 - \tfrac{i\langle t,x\rangle}{1+\|x\|^2} \Big) \nu(dx) \Big\}, \quad t \in \mathbb{R}^d, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$, $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^d$, $\gamma \in \mathbb{R}^d$ is a fixed vector, Q is a symmetric nonnegative-definite $d \times d$ matrix, and ν is a measure on $\mathbb{R}^d$ that satisfies the following conditions

$$\nu(\{\bar{0}\}) = 0, \quad \int_{\mathbb{R}^d} \min\{\|x\|^2, 1\} \nu(dx) < \infty.$$

Here and below, we denote by $\bar{0}$ the zero vector of $\mathbb{R}^d$. The vector (γ, Q, ν) is called a *characteristic triplet* and it is uniquely determined by f and hence by F.

The notion of quasi-infinitely divisible distributions on $\mathbb{R}^d$ was introduced in the recent paper by Berger, Kutlu, and Lindner [5]. Following this paper, a distribution function F and the corresponding law are called *quasi-infinitely divisible*, if there exist infinitely divisible distribution functions F_1 and F_2 such that $F_1 = F * F_2$ (in the papers [13, 14, 15], such property is proposed to be called *rational infinite divisibility*). It was proved in [5] that F is quasi-infinitely divisible if and only if the representation (1.1) holds, where ν is a signed finite measure on $\mathbb{R}^d \setminus (-r, r)^d$ for any $r > 0$ that satisfies $\nu(\{\bar{0}\}) = 0$, and

$$\int_{\mathbb{R}^d} \min\{\|x\|^2, 1\} |\nu|(dx) < \infty,$$

where $|\nu|$ denotes the total variation of the measure ν (see [5] for more details). It is seen that the class of quasi-infinitely divisible distributions is a natural generalization of the class of infinitely divisible distributions.

The examples of univariate quasi-infinitely divisible laws can be found in the classical monographs [9, 19], and [20]. The first detailed analysis of these laws on $\mathbb{R}$ was performed in [18], and a lot of results for the univariate case are contained in the works [1, 3, 4, 12, 13], and [14]. The multivariate case was considered in the recent papers [5, 6], and [23]. The authors of these works studied questions concerning supports, moments, continuity, and the weak convergence. The most complete results were obtained for probability laws on the set $\mathbb{Z}^d$, where $\mathbb{Z}$ is the set of integers. In particular, the following important fact was stated in [6].

Theorem 1.1. *Let F be the distribution function of a probability law on $\mathbb{Z}^d$. Let f be its characteristic function. Then F is quasi-infinitely divisible if and only if $f(t) \neq 0$ for all $t \in \mathbb{R}^d$. In that case, f admits the following representation*

$$f(t) = \exp\Big\{ i\langle t, \gamma\rangle + \sum_{k \in \mathbb{Z}^d \setminus \{\bar{0}\}} \lambda_k \big(e^{i\langle t,k\rangle} - 1 \big) \Big\}, \quad t \in \mathbb{R}^d, \tag{1.2}$$

where $\gamma \in \mathbb{Z}^d$, $\lambda_k \in \mathbb{R}$, $k \in \mathbb{Z}^d \setminus \{\bar{0}\}$, and $\sum_{k \in \mathbb{Z}^d \setminus \{\bar{0}\}} |\lambda_k| < \infty$.

It is clear that (1.2) can be rewritten in the form (1.1). Using this theorem, the authors of [6] also proved the Cramér–Wold device for infinite divisibility of $\mathbb{Z}^d$-valued distributions. We formulate the corresponding result in a simplified form omitting equivalent propositions.

Theorem 1.2. *Let ξ be a $\mathbb{Z}^d$-valued random vector with distribution function F. Let F_c denote the distribution function of $\langle c, \xi\rangle$, $c \in \mathbb{R}^d$. The distribution function F is infinitely divisible if and only if for any $c \in \mathbb{R}^d$ the distribution functions F_c is infinitely divisible.*

Recall that the classical Cramér–Wold device is a fact that a probability distribution of a d-dimensional random vector ξ is uniquely determined by distributions of all linear combinations of its components, i.e. by distributions of $\langle c, \xi \rangle$ for all $c \in \mathbb{R}^d$ (see [7]). Statements concerning some property for multivariate random vectors that can be expressed by corresponding statements for its linear combinations are also called as Cramér–Wold devices. So Theorem 1.2 is an interesting particular example of it. The Cramér–Wold device is well known for strict and symmetric stabilities (see [25]): d-dimensional random vector ξ is strictly (symmetrically) stable if and only if random variable $\langle c, \xi \rangle$ is strictly (symmetrically) stable for any $c \in \mathbb{R}^d$. Note that, however, the Cramér–Wold device for infinite divisibility in general does not hold. If a d-dimensional random vector ξ has infinitely divisible distribution, then the distribution of $\langle c, \xi \rangle$ is infinitely divisible too for all $c \in \mathbb{R}^d$, but for $d \geqslant 2$ there exist examples that the converse is not true (see [8] and [11]).

The purpose of this article is to generalize Theorems 1.1 and 1.2 to arbitrary multivariate discrete distribution functions. More precisely, we obtain a criterion of quasi-infinitely divisibility, we get representations, which are similar to (1.2), and we also prove the Cramér–Wold devices for infinite and quasi-infinite divisibility. The corresponding results are formulated in Section 2. The necessary tools, which are also of independent interest, are formulated in Section 3. All of the mentioned results are proved in Section 4.

2 Main results

Let us consider a multivariate discrete probability law with the following distribution function

$$F(x) = \sum_{\substack{k \in \mathbb{N}: \\ x_k \in (-\infty, x]}} p_{x_k}, \quad x \in \mathbb{R}^d, \tag{2.1}$$

where $x_k \in \mathbb{R}^d$, $k \in \mathbb{N}$, are distinct numbers with probability weights $p_{x_k} \geqslant 0$, $k \in \mathbb{N}$ (the set of positive integers), $\sum_{k=1}^{\infty} p_{x_k} = 1$. We denote by $(-\infty, x]$ with $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$ the set $(-\infty, x^{(1)}] \times \dots \times (-\infty, x^{(d)}] \subset \mathbb{R}^d$. Let f be the characteristic function of F, i.e.

$$f(t) := \int_{\mathbb{R}^d} e^{i\langle t, x \rangle} dF(x) = \sum_{k \in \mathbb{N}} p_{x_k} e^{i\langle t, x_k \rangle}, \quad t \in \mathbb{R}^d. \tag{2.2}$$

We will formulate a criterion for the distribution function F to be quasi-infinitely divisible through condition for characteristic function f. For the sharp formulation of the result we need to introduce the set of all finite $\mathbb{Z}$-linear combinations of elements from a set $Y \subset \mathbb{C}^d$ ($\mathbb{C}$ is the set of complex numbers):

$$\langle Y \rangle := \left\{ \sum_{k=1}^{n} z_k y_k \colon n \in \mathbb{N}, \, z_k \in \mathbb{Z}, \, y_k \in Y \right\}. \tag{2.3}$$

So $\langle Y \rangle$ is a module over the ring $\mathbb{Z}$ with the generating set Y. It is easily seen that $Y \subset \langle Y \rangle$, $\bar{0} \in \langle Y \rangle$. If a countable set $Y \neq \varnothing$, then $\langle Y \rangle$ is an infinite countable set.

Theorem 2.1. *Let F be a discrete distribution function of the form* (2.1) *with characteristic function f of the form* (2.2)*. The following statements are equivalent:*

- (a) *F is quasi-infinitely divisible;*
- (b) $\inf_{t \in \mathbb{R}^d} |f(t)| > 0$.

If one of the conditions is satisfied, and hence all, then f admits the following representation

$$f(t) = \exp\left\{ i\langle t, \gamma \rangle + \sum_{u \in \langle X \rangle \setminus \{\bar{0}\}} \lambda_u \left(e^{i\langle t, u \rangle} - 1 \right) \right\}, \quad t \in \mathbb{R}^d, \tag{2.4}$$

where $X := \{x_k \colon p_{x_k} > 0, k \in \mathbb{N}\} \neq \varnothing$, $\gamma \in \langle X \rangle$, $\lambda_u \in \mathbb{R}$ for all $u \in \langle X \rangle \setminus \{\bar{0}\}$, and $\sum_{u \in \langle X \rangle \setminus \{0\}} |\lambda_u| < \infty$.

It is easily seen that (2.4) can be rewritten in the form (1.1). So if characteristic function of multivariate discrete probability law is represented by (2.4), then the corresponding distribution function F is quasi-infinitely divisible. Also observe that, by this theorem and on account of the conditions and uniqueness of the Lévy representation (1.1), the multivariate discrete distribution function F is infinitely divisible if and only if its characteristic function f admits representation (2.4) with the same X and γ, but with $\lambda_u \geqslant 0$ for all $u \in \langle X \rangle \setminus \{\bar{0}\}$, and $\sum_{u \in \langle X \rangle \setminus \{0\}} \lambda_u < \infty$.

Note that Theorem 2.1 generalizes Theorem 1.1. Indeed, for characteristic function f of probability law on $\mathbb{Z}^d$ the condition that $f(t) \neq 0$, $t \in \mathbb{R}^d$, is equivalent to the condition that $\inf_{t \in \mathbb{R}} |f(t)| > 0$. It follows due to the continuity and 2π-periodicity of the function $|f(t)|$, $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, over each t_j. Theorem 2.1 also generalizes the corresponding results from [1] and [13] for the discrete distributions in the univariate case.

We now formulate the Cramér–Wold devices for the infinite and quasi-infinite divisibility of multivariate discrete distribution functions.

Theorem 2.2. *Let ξ be a discrete random vector with distribution function F of the form* (2.1). *Let F_c denote the distribution function of $\langle c, \xi \rangle$, $c \in \mathbb{R}^d$. The distribution function F is* (*quasi-*)*infinitely divisible if and only if for any $c \in \mathbb{R}^d$ the distribution function F_c is* (*quasi-*)*infinitely divisible.*

It is easily seen that Theorem 2.2 generalizes Theorem 1.2. It should be noted that Theorem 2.2 does not contradict with the results from the [8] and [11], because the distributions from the counterexamples contained an absolutely continuous part.

3 Tools

We will get the main result from more general positions. Namely, we will consequently study admission of the Lévy type representations for general almost periodic functions h, which are very similar to f.

Theorem 3.1. *Let $h \colon \mathbb{R}^d \to \mathbb{C}$ be a function of the following form:*

$$h(t) = \sum_{y \in Y} q_y e^{i \langle t, y \rangle}, \quad t \in \mathbb{R}^d,$$

where $Y \subset \mathbb{R}^d$ is a nonempty at most countable set, $q_y \in \mathbb{C}$ for all $y \in Y$, and $0 < \sum_{y \in Y} |q_y| < \infty$. Assume that $h(\bar{0}) = \sum_{y \in Y} q_y = 1$. If $\inf_{t \in \mathbb{R}^d} |h(t)| = \mu > 0$, then h admits the following representation

$$h(t) = \exp\bigg\{ i\langle t, \gamma \rangle + \sum_{u \in \langle Y \rangle \setminus \{\bar{0}\}} \lambda_u \big(e^{i \langle t, u \rangle} - 1 \big) \bigg\}, \quad t \in \mathbb{R}^d, \tag{3.1}$$

where $\gamma \in \langle Y \rangle$, $\lambda_u \in \mathbb{C}$ for all $u \in \langle Y \rangle \setminus \{\bar{0}\}$, and $\sum_{u \in \langle Y \rangle \setminus \{\bar{0}\}} |\lambda_u| < \infty$.

It should be noted that the function h in Theorem 3.1 is an *almost periodic function* on $\mathbb{R}^n$ with the absolutely convergent Fourier series. Recall that (see [16, p. 255] or [21, Definition 1]) a function $h \colon \mathbb{R}^d \to \mathbb{C}$ is called almost periodic if for any sequence $\{t_n\}_{n \in \mathbb{N}}$ from $\mathbb{R}^d$ there exists a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ and a continuous function $\varphi \colon \mathbb{R}^d \to \mathbb{C}$ such that

$$\sup_{t \in \mathbb{R}^d} \big| h(t + t_{n_k}) - \varphi(t) \big| \underset{k \to \infty}{\longrightarrow} 0.$$

The detailed information about almost periodic functions on $\mathbb{R}^d$ can be found in [2, 16, 17, 21], and [22] with a greater generality (for local compact Abelian groups).

We now turn to the following general version of Theorem 2.1.

Theorem 3.2. *Let $h\colon \mathbb{R}^d \to \mathbb{C}$ be a function of the following form*

$$h(t) = \sum_{y\in Y} q_y e^{i\langle t,y\rangle}, \quad t \in \mathbb{R}^d,$$

where $Y \subset \mathbb{R}^d$ is a nonempty at most countable set, $q_y \in \mathbb{C}$ for all $y \in Y$, and $0 < \sum_{y\in Y} |q_y| < \infty$. Suppose that $h(\bar{0}) = \sum_{y\in Y} q_y = 1$. Then the following statements are equivalent:

(i) $\inf_{t\in\mathbb{R}^d} |h(t)| > 0$;

(ii) *there exist a countable set $Z \subset \mathbb{R}^d$ and coefficients $r_z \in \mathbb{C}$, $z \in Z$, $\sum_{z\in Z} |r_z| < \infty$, such that*

$$\frac{1}{h(t)} = \sum_{z\in Z} r_z e^{i\langle t,z\rangle}, \quad t \in \mathbb{R}^d;$$

(iii) *h admits the representation*

$$h(t) = \exp\Big\{ i\langle t,\gamma\rangle + \sum_{u\in\langle Y\rangle\setminus\{\bar{0}\}} \lambda_u \big(e^{i\langle t,u\rangle} - 1\big) \Big\}, \quad t \in \mathbb{R}^d,$$

where $\gamma \in \langle Y\rangle$, $\lambda_u \in \mathbb{C}$ for all $u \in \langle Y\rangle \setminus \{\bar{0}\}$, and $\sum_{u\in\langle Y\rangle\setminus\{0\}} |\lambda_u| < \infty$;

(iv) *h admits the representation*

$$h(t) = \exp\Big\{ i\langle t,\gamma\rangle - \tfrac{1}{2}\langle t,Qt\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle t,u\rangle} - 1 - \tfrac{i\langle t,u\rangle}{1+\|u\|^2}\Big)\nu(du) \Big\}, \quad t \in \mathbb{R}^d, \tag{3.2}$$

where $\gamma \in \mathbb{C}^d$, $Q \in \mathbb{C}^{d\times d}$ is a matrix, ν is a complex measure on $\mathbb{R}^d$ such that

$$\nu(\{\bar{0}\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}^d} \min\{\|x\|^2, 1\} |\nu|(dx) < \infty.$$

4 Proofs

Proof of Theorem 3.1. We will sequentially consider the following cases: 1) $Y = \mathbb{Z}^d$, 2) Y is a finite subset of $\mathbb{R}^d$, 3) Y is at most countable subset of $\mathbb{R}^d$ (the general case). We always assume that $Y \neq \varnothing$. Each subsequent case will be based on the previous one.

1) Suppose that $Y = \mathbb{Z}^d$. It is easy to see that the function h is 2π-periodic in all coordinates, i.e. for any $k = 1, \dots, d$ and $t \in \mathbb{R}^d$ we have $h(t + 2\pi e_k) = h(t)$, where $\{e_1, e_2, \dots, e_d\}$ denotes the canonical basis in $\mathbb{R}^d$. Let us consider the distinguished logarithm $t \mapsto \operatorname{Ln} h(t)$, $t \in \mathbb{R}^d$, which satisfies $\exp\{\operatorname{Ln} h(t)\} = h(t)$, $t \in \mathbb{R}^d$, and it is uniquely defined by continuity with the condition $\operatorname{Ln} h(\bar{0}) = 0$ (see [26, Lemma 7.6]). For any $k = 1, \dots, d$ we have

$$\exp\{\operatorname{Ln} h(t + 2\pi e_k)\} = h(t + 2\pi e_k) = h(t) = \exp\{\operatorname{Ln} h(t)\}, \quad t \in \mathbb{R}^d.$$

So $\operatorname{Ln} h(t + 2\pi e_k) - \operatorname{Ln} h(t) \in 2\pi i\mathbb{Z}$ for any $k = 1, \dots, d$ and $t \in \mathbb{R}^d$. Since $t \mapsto \operatorname{Ln} h(t + 2\pi e_k) - \operatorname{Ln} h(t)$ is a continuous function on $\mathbb{R}^d$, there exist constants $\gamma_1, \dots, \gamma_d \in \mathbb{Z}$ such that

$$\gamma_k = \frac{\operatorname{Ln} h(t + 2\pi e_k) - \operatorname{Ln} h(t)}{2\pi i}, \quad t \in \mathbb{R}^d, \quad k = 1, \dots, d.$$

Let us define the vector $\gamma = (\gamma_1, \dots, \gamma_d)^T \in \mathbb{Z}^d$. So the function $t \mapsto \operatorname{Ln} h(t) - i\langle t,\gamma\rangle$ is 2π-periodic in all coordinates. By [6, Proposition 3.1], one can conclude that

$$\operatorname{Ln} h(t) = i\langle t,\gamma\rangle + \sum_{u\in\mathbb{Z}^d\setminus\{\bar{0}\}} \lambda_u \big(e^{i\langle t,u\rangle} - 1\big), \quad t \in \mathbb{R}^d,$$

where $\lambda_u \in \mathbb{C}$ for all $u \in \mathbb{Z}^d \setminus \{\bar{0}\}$, and $\sum_{u\in\mathbb{Z}^d\setminus\{\bar{0}\}} |\lambda_u| < \infty$. Note that $\langle Y\rangle = \mathbb{Z}^d$ in this case.

2) Assume that $Y = \{y_1, \dots, y_n\}$, where $y_1, \dots, y_n$ are distinct elements from $\mathbb{R}^d$. So we have $h(t) = \sum_{k=1}^n q_{y_k} e^{i\langle t, y_k\rangle}$, $t \in \mathbb{R}^d$. If $n = 1$ then $Y = \{y_1\}$ and $q_{y_1} = 1$. For this case representation (3.1) holds with $\gamma = y_1$ and $\lambda_u = 0$ for all $u \in \langle Y\rangle \setminus \{\bar{0}\}$. We next suppose that $n \geqslant 2$. We set $y_k = (y_k^{(1)}, \dots, y_k^{(d)})$, $k = 1, \dots, n$. Without loss of generality, we can assume that for every $j = 1, \dots, d$ there exist $k = 1, \dots, n$ such that $y_k^{(j)} \neq 0$, since otherwise we can turn to the space $\mathbb{R}^{d'}$ with some $d' < d$. Next, for every $j = 1, \dots, d$ we can choose non-zero $\beta_1^{(j)}, \dots, \beta_{m_j}^{(j)} \in Y^{(j)} = \{y_1^{(j)}, \dots, y_n^{(j)}\} \subset \mathbb{R}$ that constitute a basis in $Y^{(j)}$ over $\mathbb{Q}$, i.e. for any $j \in \{1, \dots, d\}$ and $k \in \{1, \dots, n\}$ there exist uniquely determined values $c_{k,1}^{(j)}, \dots, c_{k,m_j}^{(j)} \in \mathbb{Q}$, such that $y_k^{(j)} = \sum_{l=1}^{m_j} c_{k,l}^{(j)} \beta_l^{(j)}$ (see [16] p. 67–68). Let $\varkappa^{(j)}$ be the minimal positive integer such that $\tilde{c}_{k,l}^{(j)} := \varkappa^{(j)} c_{k,l}^{(j)} \in \mathbb{Z}$ for any j, k, l. We set $\tilde{\beta}_l^{(j)} := \beta_l^{(j)}/\varkappa^{(j)}$ and we have $y_k^{(j)} = \sum_{l=1}^{m_j} \tilde{c}_{k,l}^{(j)} \tilde{\beta}_l^{(j)}$ for any $j \in \{1, \dots, d\}$ and $k \in \{1, \dots, n\}$.

So it is easy to check that

$$\langle Y\rangle \subset \Big\{ \sum_{l=1}^{m_1} z_l^{(1)} \tilde{\beta}_l^{(1)} :\ z_l^{(1)} \in \mathbb{Z} \Big\} \times \dots \times \Big\{ \sum_{l=1}^{m_d} z_l^{(d)} \tilde{\beta}_l^{(d)} :\ z_l^{(d)} \in \mathbb{Z} \Big\}. \tag{4.1}$$

Note that for every $j = 1, \dots, d$ the values $\tilde{\beta}_1^{(j)}, \dots, \tilde{\beta}_{m_j}^{(j)}$ are linearly independent over $\mathbb{Z}$, i.e. the equation $l_1 \tilde{\beta}_1^{(j)} + \dots + l_{m_j} \tilde{\beta}_{m_j}^{(j)} = 0$ holds with $l_1, \dots, l_{m_j} \in \mathbb{Z}$ if and only if $l_1 = \dots = l_{m_j} = 0$.

We now consider the function

$$\varphi\big(t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_1^{(d)}, \dots, t_{m_d}^{(d)}\big) := \sum_{k=1}^n q_{y_k} \exp\Big\{ i \sum_{j=1}^d \sum_{l=1}^{m_j} \tilde{c}_{k,l}^{(j)} \tilde{\beta}_l^{(j)} t_l^{(j)} \Big\}, \tag{4.2}$$

where $t_l^{(j)} \in \mathbb{R}$, $l = 1, \dots, m_j$, and $j = 1, \dots, d$. If for any such j and l we set $t_l^{(j)} := t^{(j)} \in \mathbb{R}$, then

$$\varphi\big(t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_1^{(d)}, \dots, t_{m_d}^{(d)}\big) = h(t), \tag{4.3}$$

where $t = (t^{(1)}, \dots, t^{(d)})$. We set $M := m_1 + \dots + m_d$. Let us fix an arbitrary $\varepsilon > 0$. Since the function φ is uniformly continuous, there exists $\delta_\varepsilon > 0$ such that for any $\tilde{t}_1$ and $\tilde{t}_2$ from $\mathbb{R}^M$ satisfying $\|\tilde{t}_1 - \tilde{t}_2\| < \delta_e$ we have $\big|\varphi(\tilde{t}_1) - \varphi(\tilde{t}_2)\big| < \varepsilon$. Let us arbitrarily fix the vector $t := \big(t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_1^{(d)}, \dots, t_{m_d}^{(d)}\big) \in \mathbb{R}^M$. We set $b_j := \min\big\{|\tilde{\beta}_1^{(j)}|, \dots, |\tilde{\beta}_{m_j}^{(j)}|\big\} > 0$ for every $j = 1, \dots, d$. Since for every j the values $\tilde{\beta}_1^{(j)}, \dots, \tilde{\beta}_{m_j}^{(j)}$ are linearly independent over $\mathbb{Z}$, then, by the Kronecker theorem (see [17, p.37]), we conclude that the inequalities

$$\big|\tilde{\beta}_l^{(j)} s^{(j)} - t_l^{(j)} - 2\pi n_l^{(j)}\big| < \tfrac{\delta_\varepsilon b_j}{\sqrt{m_j d}}, \quad l = 1, \dots, m_j,$$

have a common solution $s^{(j)} \in \mathbb{R}$ for some $n_l^{(j)} \in \mathbb{Z}$. We fix these numbers and we conclude that

$$\left| s^{(j)} - \frac{t_l^{(j)} + 2\pi n_l^{(j)}}{\tilde{\beta}_l^{(j)}} \right| < \frac{\delta_\varepsilon}{\sqrt{m_j d}}, \quad l = 1, \dots, m_j,$$

and

$$\sum_{j=1}^d \sum_{l=1}^{m_j} \left| s^{(j)} - \frac{t_l^{(j)} + 2\pi n_l^{(j)}}{\tilde{\beta}_l^{(j)}} \right|^2 < \delta_\varepsilon^2.$$

The latter inequality means that $\|s - \tilde{t}\,\| < \delta_\varepsilon$, where

$$s := \big(s^{(1)}, \dots, s^{(1)}, \dots, s^{(d)}, \dots, s^{(d)}\big) \in \mathbb{R}^M,$$
$$\tilde{t} := \left(\tfrac{t_1^{(1)} + 2\pi n_1^{(1)}}{\tilde{\beta}_1^{(1)}}, \dots, \tfrac{t_{m_1}^{(1)} + 2\pi n_{m_1}^{(1)}}{\tilde{\beta}_{m_1}^{(1)}}, \dots, \tfrac{t_1^{(d)} + 2\pi n_1^{(d)}}{\tilde{\beta}_1^{(d)}}, \dots, \tfrac{t_{m_d}^{(d)} + 2\pi n_{m_d}^{(d)}}{\tilde{\beta}_{m_d}^{(d)}} \right) \in \mathbb{R}^M,$$

in the vector s: $s^{(1)}$ repeats m_1 times, $s^{(2)}$ repeats m_2 times, ..., $s^{(d)}$ repeats m_d times. Therefore $\big|\varphi(s) - \varphi(\tilde{t})\big| < \varepsilon$. It is easily seen from (4.2) that

$$\begin{aligned}\varphi(\tilde{t}) &= \varphi\left(\tfrac{t_1^{(1)}+2\pi n_1^{(1)}}{\tilde{\beta}_1^{(1)}}, \dots, \tfrac{t_{m_1}^{(1)}+2\pi n_{m_1}^{(1)}}{\tilde{\beta}_{m_1}^{(1)}}, \dots, \tfrac{t_1^{(d)}+2\pi n_1^{(d)}}{\tilde{\beta}_1^{(d)}}, \dots, \tfrac{t_{m_d}^{(d)}+2\pi n_{m_d}^{(d)}}{\tilde{\beta}_{m_d}^{(d)}}\right)\\ &= \varphi\left(\tfrac{t_1^{(1)}}{\tilde{\beta}_1^{(1)}}, \dots, \tfrac{t_{m_1}^{(1)}}{\tilde{\beta}_{m_1}^{(1)}}, \dots, \tfrac{t_1^{(d)}}{\tilde{\beta}_1^{(d)}}, \dots, \tfrac{t_{m_d}^{(d)}}{\tilde{\beta}_{m_d}^{(d)}}\right)\\ &=: \tilde{\varphi}(t),\end{aligned}$$

i.e.

$$\tilde{\varphi}(t) = \sum_{k=1}^{n} q_{y_k} \exp\bigg\{ i \sum_{j=1}^{d} \sum_{l=1}^{m_j} \tilde{c}_{k,l}^{(j)} t_l^{(j)} \bigg\}; \tag{4.4}$$

since t was fixed arbitrarily, we consider $\tilde{\varphi}$ as a function from $\mathbb{R}^M$ to $\mathbb{C}$. So we have that $\big|\varphi(s) - \tilde{\varphi}(t)\big| < \varepsilon$. Thus, due to (4.3), we get that for any $\varepsilon > 0$ and $t \in \mathbb{R}^M$ there exists $s' = \big(s^{(1)}, \dots, s^{(d)}\big) \in \mathbb{R}^d$ such that $\big|h(s') - \tilde{\varphi}(t)\big| < \varepsilon$. According to the assumption $\inf_{s\in\mathbb{R}^d} |h(s)| > 0$, we conclude that $\inf_{t\in\mathbb{R}^M} |\tilde{\varphi}(t)| > 0$.

We now apply the previous part **1)** to the function (4.4) (it is valid, because there are $\tilde{c}_{k,l}^{(j)} \in \mathbb{Z}$ in (4.4)). So we have the following representation:

$$\begin{aligned}\operatorname{Ln} \tilde{\varphi}(t) &= \operatorname{Ln} \tilde{\varphi}\big(t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_1^{(d)}, \dots, t_{m_d}^{(d)}\big)\\ &= i \sum_{j=1}^{d} \sum_{l=1}^{m_j} \gamma_l^{(j)} t_l^{(j)} + \sum_{z\in\mathbb{Z}^M\setminus\{\bar{0}\}} \lambda_z \bigg(\exp\bigg\{ i \sum_{j=1}^{d} \sum_{l=1}^{m_j} z_l^{(j)} t_l^{(j)} \bigg\} - 1 \bigg),\end{aligned}$$

where $z = \big(z_1^{(1)}, \dots, z_{m_1}^{(1)}, \dots, z_1^{(d)}, \dots, z_{m_d}^{(d)}\big) \in \mathbb{Z}^M \setminus \{\bar{0}\}$, $\gamma_l^{(j)} \in \mathbb{Z}$, $\lambda_z \in \mathbb{C}$ for all $z \in \mathbb{Z}^M \setminus \{\bar{0}\}$, and $\sum_{z\in\mathbb{Z}^M\setminus\{\bar{0}\}} |\lambda_z| < \infty$. From the above, we get

$$\begin{aligned}\operatorname{Ln} \varphi\big(t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_1^{(d)}, \dots, t_{m_d}^{(d)}\big) &= i \sum_{j=1}^{d} \sum_{l=1}^{m_j} \gamma_l^{(j)} \tilde{\beta}_l^{(j)} t_l^{(j)}\\ &\quad + \sum_{z\in\mathbb{Z}^M\setminus\{\bar{0}\}} \lambda_z \bigg(\exp\bigg\{ i \sum_{j=1}^{d} \sum_{l=1}^{m_j} z_l^{(j)} \tilde{\beta}_l^{(j)} t_l^{(j)} \bigg\} - 1 \bigg).\end{aligned}$$

Due to (4.3), for every $t = (t^{(1)}, \dots, t^{(d)})$ we have

$$\operatorname{Ln} h(t) = i \sum_{j=1}^{d} \bigg(\sum_{l=1}^{m_j} \gamma_l^{(j)} \tilde{\beta}_l^{(j)} \bigg) t^{(j)} + \sum_{z\in\mathbb{Z}^M\setminus\{\bar{0}\}} \lambda_z \bigg(\exp\bigg\{ i \sum_{j=1}^{d} \bigg(\sum_{l=1}^{m_j} z_l^{(j)} \tilde{\beta}_l^{(j)} \bigg) t^{(j)} \bigg\} - 1 \bigg).$$

For every $j = 1, \dots, d$ we set $\gamma^{(j)} := \sum_{l=1}^{m_j} \gamma_l^{(j)} \tilde{\beta}_l^{(j)}$, $\gamma := (\gamma^{(1)}, \dots, \gamma^{(d)})$, and $u_z^{(j)} := \sum_{l=1}^{m_j} z_l^{(j)} \tilde{\beta}_l^{(j)}$, $u_z = (u_z^{(1)}, \dots, u_z^{(d)})$. By the well known theorem on the argument of an almost periodic function and its corollaries (see [24] and [16] p. 128–135), γ and all u_z with $\lambda_z \neq 0$ belong to $\langle Y \rangle$. Setting $\lambda_{u_z} := \lambda_z$ for every $z \in \mathbb{Z}^M \setminus \{\bar{0}\}$, we can deal only with λ_u, $u \in \langle Y \rangle \setminus \{\bar{0}\}$ (u determines the corresponding vector z uniquely, because $\beta_l^{(j)}$ constitute a basis, see (4.1) and comments above). Thus we come to the representation (3.1) for h.

3) We now turn to the general case: Y is at most countable subset of $\mathbb{R}^d$. Without loss of generality we can set $Y := \{y_1, y_2, \dots\}$ with distinct $y_k \in \mathbb{R}^d$. So $A := \sum_{k=1}^{\infty} |q_{y_k}| < \infty$ and $h(t) := \sum_{k=1}^{\infty} q_{y_k} e^{i\langle t, y_k\rangle}$, $t \in \mathbb{R}^d$. We approximate h by the following functions:

$$h_n(t) := \sum_{k=1}^{n} q_{n,y_k} e^{i\langle t, y_k\rangle}, \quad t \in \mathbb{R}^d, \quad n \in \mathbb{N},$$

where

$$q_{n,y_k} := \frac{q_{y_k}}{\sum_{m=1}^{n} q_{y_m}}, \quad k = 1, \dots, n, \quad n \in \mathbb{N}.$$

Since $\sum_{k=1}^{\infty} q_{y_k} = 1$, we have $\left|\sum_{m=1}^{n} q_{y_m}\right| \geqslant \frac{1}{2}$ for all $n \geqslant n_0$ with a positive integer n_0. Let us estimate the approximation error for every $n \geqslant n_0$:

$$\begin{aligned}\sup_{t\in\mathbb{R}^d} |h(t) - h_n(t)| &= \sup_{t\in\mathbb{R}^d} \left| \sum_{k=1}^{n} (q_{y_k} - q_{n,y_k}) e^{i\langle t, y_k\rangle} + \sum_{k=n+1}^{\infty} q_{y_k} e^{i\langle t, y_k\rangle} \right| \\ &\leqslant \sum_{k=1}^{n} |q_{y_k} - q_{n,y_k}| + \sum_{k=n+1}^{\infty} |q_{y_k}|.\end{aligned}$$

Due to $\sum_{m=1}^{\infty} q_{y_m} = 1$, we have

$$\begin{aligned}\sum_{k=1}^{n} |q_{y_k} - q_{n,y_k}| &= \left| 1 - \frac{1}{\sum_{m=1}^{n} q_{y_m}} \right| \cdot \sum_{k=1}^{n} |q_{y_k}| \\ &= \left| \frac{\sum_{m=n+1}^{\infty} q_{y_m}}{\sum_{m=1}^{n} q_{y_m}} \right| \cdot \sum_{k=1}^{n} |q_{y_k}| \leqslant 2A \sum_{m=n+1}^{\infty} |q_{y_m}|.\end{aligned}$$

We used $\sum_{k=1}^{\infty} |q_{y_k}| = A$ and $\left|\sum_{m=1}^{n} q_{y_m}\right| \geqslant \frac{1}{2}$ for the last inequality. Thus we obtain

$$\sup_{t\in\mathbb{R}^d} |h(t) - h_n(t)| \leqslant (2A+1) \sum_{m=n+1}^{\infty} |q_{y_m}|, \quad n \geqslant n_0.$$

Since $\sum_{k=1}^{n} |q_{y_k}| < \infty$, we have that $\sup_{t\in\mathbb{R}^d} |h(t) - h_n(t)| \to 0$, $n \to \infty$. Hence for any fixed $\varepsilon \in (0, \frac{1}{4})$ there exists a positive integer $n_\varepsilon \geqslant n_0$ such that for every $n \geqslant n_\varepsilon$ we have

$$\sup_{t\in\mathbb{R}^d} \big| h(t) - h_n(t) \big| \leqslant \varepsilon\mu, \tag{4.5}$$

where we set $\mu := \inf_{t\in\mathbb{R}^d} |h(t)| > 0$. So for every $n \geqslant n_\varepsilon$

$$\inf_{t\in\mathbb{R}^d} \big| h_n(t) \big| \geqslant \inf_{t\in\mathbb{R}^d} \big| h(t) \big| - \sup_{t\in\mathbb{R}^d} \big| h(t) - h_n(t) \big| \geqslant (1-\varepsilon)\mu. \tag{4.6}$$

We now fix $n \geqslant n_\varepsilon$ and we represent $h(t) = h_n(t) \cdot R_n(t)$ with $R_n(t) := h(t)/h_n(t)$, $t \in \mathbb{R}^d$. Since h, h_n, R_n are continuous functions without zeroes on $\mathbb{R}^d$ and they equal 1 at $t = \bar{0}$, we can proceed to the distinguished logarithms:

$$\operatorname{Ln} h(t) = \operatorname{Ln} h_n(t) + \operatorname{Ln} R_n(t), \quad t \in \mathbb{R}^d. \tag{4.7}$$

Let us consider the function $\operatorname{Ln} h_n$. By the result of part **2)**, we have

$$\operatorname{Ln} h_n(t) = i\langle t, \gamma_n\rangle + \sum_{u\in\langle Y_n\rangle\setminus\{\bar{0}\}} \lambda_{n,u} \big(e^{i\langle t,u\rangle} - 1\big), \quad t \in \mathbb{R}^d,$$

with a set $Y_n := \big\{y_k \colon\, q_{y_k} \neq 0,\, k = 1, \dots, n\big\}$, and numbers $\gamma_n \in \langle Y_n\rangle$, $\lambda_{n,u} \in \mathbb{C}$ for all $u \in \langle Y_n\rangle \setminus \{\bar{0}\}$, $\sum_{u\in\langle Y_n\rangle\setminus\{\bar{0}\}} |\lambda_{n,u}| < \infty$. Setting $\lambda_{n,\bar{0}} := -\sum_{u\in\langle Y_n\rangle\setminus\{\bar{0}\}} \lambda_{n,u} \in \mathbb{C}$, we represent $\operatorname{Ln} h_n$ in the following form

$$\operatorname{Ln} h_n(t) = i\langle t, \gamma_n\rangle + \sum_{u\in\langle Y_n\rangle} \lambda_{n,u} e^{i\langle t,u\rangle}, \quad t \in \mathbb{R}^d.$$

Observe that $Y_n \subset Y$, and hence $\langle Y_n\rangle \subset \langle Y\rangle$. So we can write

$$\operatorname{Ln} h_n(t) = i\langle t, \gamma_n\rangle + \sum_{u\in\langle Y\rangle} \lambda_{n,u} e^{i\langle t,u\rangle}, \quad t \in \mathbb{R}^d, \tag{4.8}$$

where for every $u \in \langle Y\rangle \setminus \langle Y_n\rangle$ we define $\lambda_{n,u} := 0$ for the case $\langle Y\rangle \setminus \langle Y_n\rangle \neq \varnothing$.

We next consider the function $\operatorname{Ln} R_n$. Observe that

$$\operatorname{Ln} R_n(t) = \ln\biggl(1 + \frac{h(t) - h_n(t)}{h_n(t)}\biggr), \quad t \in \mathbb{R}^d, \tag{4.9}$$

where the latter is the principal value of the logarithm. Indeed, due to (4.5) and (4.6),

$$\sup_{t\in\mathbb{R}^d}\biggl|\frac{h(t) - h_n(t)}{h_n(t)}\biggr| \leqslant \frac{\sup_{t\in\mathbb{R}^d}|h(t) - h_n(t)|}{\inf_{t\in\mathbb{R}^d}|h_n(t)|} \leqslant \frac{\varepsilon}{1-\varepsilon} < 1, \tag{4.10}$$

and the function in the right-hand side of (4.9) is continuous and it equals 0 at $t = \bar{0}$. Therefore we get the decomposition

$$\operatorname{Ln} R_n(t) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}\biggl(\frac{h(t) - h_n(t)}{h_n(t)}\biggr)^m, \quad t \in \mathbb{R}^d,$$

which yields the estimate

$$\sup_{t\in\mathbb{R}}|\operatorname{Ln} R_n(t)| \leqslant \sum_{m=1}^{\infty}\frac{1}{m}\sup_{t\in\mathbb{R}^d}\biggl|\frac{h(t) - f_n(t)}{f_n(t)}\biggr|^m \leqslant \sum_{m=1}^{\infty}\frac{1}{m}\biggl(\frac{\varepsilon}{1-\varepsilon}\biggr)^m.$$

Since $\varepsilon \in (0, \frac{1}{4})$, we have

$$\sum_{m=1}^{\infty}\frac{1}{m}\biggl(\frac{\varepsilon}{1-\varepsilon}\biggr)^m \leqslant \sum_{m=1}^{\infty}\biggl(\frac{\varepsilon}{1-\varepsilon}\biggr)^m = \frac{\frac{\varepsilon}{1-\varepsilon}}{1-\frac{\varepsilon}{1-\varepsilon}} = \frac{\varepsilon}{1-2\varepsilon} < 2\varepsilon.$$

Thus we obtain

$$\sup_{t\in\mathbb{R}^d}|\operatorname{Ln} R_n(t)| < 2\varepsilon. \tag{4.11}$$

Let us consider the function $(h - h_n)/h_n$ from (4.9). It is clear that $h - h_n$ is an almost periodic function with the absolutely convergent Fourier series. Due to [2, Theorem 3.2] the function $1/h_n$ is also an almost periodic with the absolutely convergent Fourier series. Since the function $z \mapsto \ln(1+z)$, $z \in \mathbb{C}$, is analytic on the unit disk, due to (4.10), [2, Theorem 3.2], and [10, Corollary 5.15], we get that $\operatorname{Ln} R_n$ is an almost periodic function with the absolutely convergent Fourier series:

$$\operatorname{Ln} R_n(t) = \sum_{u\in\Delta_n}\beta_{n,u}e^{i\langle t,u\rangle}, \quad t \in \mathbb{R}^d, \tag{4.12}$$

where Δ_n is at most countable set of vectors from $\mathbb{R}^d$, $\beta_{n,u} \in \mathbb{C}$ for $u \in \Delta_n$, and $\sum_{u\in\Delta_n}|\beta_{n,u}| < \infty$.

We now return to the function $\operatorname{Ln} h$ and (4.7). The formulas (4.8) and (4.12) yield

$$\operatorname{Ln} h(t) = i\langle t, \gamma_n\rangle + \sum_{u\in\langle Y\rangle}\lambda_{n,u}e^{i\langle t,u\rangle} + \sum_{u\in\Delta_n}\beta_{n,u}e^{i\langle t,u\rangle}, \quad t \in \mathbb{R}^d.$$

This formula is valid for every $n \geqslant n_\varepsilon$. Let us fix the vector $\mathbf{e}$ on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d\colon \|x\| = 1\}$. Since $\sum_{u\in\langle Y\rangle}|\lambda_{n,u}| < \infty$ and $\sum_{u\in\Delta_n}|\beta_{n,u}| < \infty$, $n \geqslant n_\varepsilon$, it is easy to see that

$$\lim_{T\to\infty}\frac{\operatorname{Ln} h(T\mathbf{e})}{iT} = \langle\gamma_n, \mathbf{e}\rangle, \quad n \geqslant n_\varepsilon.$$

Since the vector $\mathbf{e}$ is choosen arbitrarily from $\mathbb{S}^{d-1}$, γ_n are equal for $n \geqslant n_\varepsilon$, and we set $\gamma := \gamma_n$. Due to $\gamma_n \in \langle Y_n\rangle \subset \langle Y\rangle$, we have $\gamma \in \langle Y\rangle$. Thus for every $n \geqslant n_\varepsilon$ we obtain

$$\operatorname{Ln} h(t) = i\langle t, \gamma\rangle + \sum_{u\in\langle Y\rangle}\lambda_{n,u}e^{i\langle t,u\rangle} + \sum_{u\in\Delta_n}\beta_{n,u}e^{i\langle t,u\rangle}, \quad t \in \mathbb{R}^d.$$

Due to the uniqueness theorem for the Fourier coefficients (see [2, Lemma 3.1]), one can conclude that

$$\operatorname{Ln} h(t) = i\langle t,\gamma\rangle + \sum_{u\in\langle Y\rangle} \lambda_u e^{i\langle t,u\rangle} + \sum_{u\in Z} \lambda_u e^{i\langle t,u\rangle}, \quad t\in\mathbb{R}^d,$$

where Z is at most countable subset of $\mathbb{R}^d$ such that $\langle Y\rangle\cap Z=\varnothing$, $\lambda_u\in\mathbb{C}$ for all $u\in\langle Y\rangle\cup Z$, $\sum_{u\in\langle Y\rangle\cup Z}|\lambda_u|<\infty$. So for every $n\geqslant n_\varepsilon$ the following estimate is true:

$$\sum_{u\in Z}|\lambda_u|^2 \leqslant \sum_{u\in\Delta_n}|\beta_{n,u}|^2.$$

Using the Parseval identity (see [16, Ch. VI, §4] or [21, Theorem 28]) and (4.11), we get

$$\sum_{u\in Z}|\lambda_u|^2 \leqslant \lim_{T\to\infty}\frac{1}{(2T)^d}\int\limits_{[-T,T]^d}\big|\operatorname{Ln} R_n(t)\big|^2 dt < (2\varepsilon)^2, \quad n\geqslant n_\varepsilon.$$

Since $\varepsilon>0$ can be chosen arbitrarily small, we conclude that $Z=\varnothing$ or $Z\neq\varnothing$, but $\lambda_u=0$ for all $u\in Z$. Thus

$$\operatorname{Ln} h(t) = i\langle t,\gamma\rangle + \sum_{u\in\langle Y\rangle} \lambda_u e^{i\langle t,u\rangle}, \quad t\in\mathbb{R}^d,$$

with $\gamma\in\langle Y\rangle$, $\lambda_u\in\mathbb{C}$ for all $u\in\langle Y\rangle$, and $\sum_{u\in\langle Y\rangle}|\lambda_u|<\infty$. According to $\big(\operatorname{Ln} h(t) - i\langle t,\gamma\rangle\big)\big|_{t=\bar{0}}=0$, we get $\lambda_{\bar{0}} = -\sum_{u\in\langle Y\rangle\setminus\{\bar{0}\}}\lambda_u$ and we come to the required representation (3.1). ☐

We now return to the proof of the Theorem 3.2.

Proof of Theorem 3.2. The proof will be carried out in the following sequence: $(ii)\xrightarrow{\mathbf{I}}(i)\xrightarrow{\mathbf{II}}(iii)\xrightarrow{\mathbf{III}}(iv)\xrightarrow{\mathbf{IV}}(i)\xrightarrow{\mathbf{V}}(ii)$.

I. Due to (ii), we have

$$\sup_{t\in\mathbb{R}^d}\left|\frac{1}{h(t)}\right| \leqslant \sum_{z\in Z}|r_z| = \frac{1}{\mu} < \infty.$$

It follows that

$$\inf_{t\in\mathbb{R}^d}|h(t)| = \frac{1}{\sup_{t\in\mathbb{R}^d}|1/h(t)|} = \mu > 0.$$

II. This implication directly follows from Theorem 3.1.

III. It is clear that (iii) yields (iv) with zero matrix Q and the signed measure

$$\nu(B) = \sum_{u\in B\cap\langle Y\rangle\setminus\{\bar{0}\}}\lambda_u \quad \text{for every Borel set } B.$$

IV. Let us assume the contrary, i.e. h has the representation (3.2) and $\inf_{t\in\mathbb{R}^d}|h(t)|=0$. Since $e^z\neq 0$ for all $z\in\mathbb{C}$, then $h(t)\neq 0$ for all $t\in\mathbb{R}^d$. Hence it is sufficient to focus on the case, where h has the representation (3.2), $h(t)\neq 0$ for all $t\in\mathbb{R}^d$, and $\inf_{t\in\mathbb{R}^d}|h(t)|=0$.

Due to (3.2), for every fixed $\tau\in\mathbb{R}^d$ we have the following representation

$$\frac{h(t+\tau)h(t-\tau)}{h^2(t)} = \exp\bigg\{-\tfrac{1}{2}\langle\tau,Q\tau\rangle + 2\int\limits_{\mathbb{R}^d\setminus\{\bar{0}\}} e^{i\langle t,u\rangle}\big(\cos\big(\langle\tau,u\rangle\big)-1\big)\nu(du)\bigg\}, \quad t\in\mathbb{R}^d.$$

It follows that for any $t \in \mathbb{R}^d$

$$\left|\frac{h(t+\tau)h(t-\tau)}{h^2(t)}\right| \leqslant \exp\left\{\left(\tfrac{1}{2}\|Q\| + \int\limits_{0<\|u\|<1} \|u\|^2 \big|\nu\big|(du)\right)\|\tau\|^2 + 4\int\limits_{\|u\|>1} \big|\nu\big|(du)\right\}.$$

Hence for every $\tau \in \mathbb{R}^d$ there exists C_τ such that

$$\sup_{t\in\mathbb{R}^d}\left|\frac{h(t+\tau)h(t-\tau)}{h^2(t)}\right| \leqslant C_\tau.$$

Let $(t_n)_{n\in\mathbb{N}}$, $t_n \in \mathbb{R}^d$, be a sequence such that $h(t_n)$ tends to 0 as $n \to \infty$. If there exists $R > 0$ such that $\|t_n\| < R$ for every $n \in \mathbb{N}$, then there exists subsequence $(n_k)_{k\in\mathbb{N}}$ satisfying $t_{n_k} \to t_* \in \mathbb{R}^d$ as $k \to \infty$. Since h is a continuous function, $h(t_*) = 0$ that contradicts with the (iv). It follows that $\|t_n\| \to \infty$ as $n \to \infty$. Since h is an almost periodic function, the sequence $(h(\cdot + t_n))_{n\in\mathbb{N}}$ is dense in the set of continuous functions, i.e. there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a continuous function φ such that

$$\sup_{\tau\in\mathbb{R}^d}\big|h(t_{n_k}+\tau) - \varphi(\tau)\big| \underset{k\to\infty}{\longrightarrow} 0.$$

It is obvious that $|\varphi(\tau)| \leqslant C := \sup_{t\in\mathbb{R}^d}|h(t)| < \infty$ for all $\tau \in \mathbb{R}^d$. Then

$$\begin{aligned}
\Delta_k &:= \sup_{\tau\in\mathbb{R}^d}\big|h(t_{n_k}+\tau)h(t_{n_k}-\tau) - \varphi(\tau)\varphi(-\tau)\big| \\
&\leqslant \sup_{\tau\in\mathbb{R}^d}\big|h(t_{n_k}-\tau)\big|\cdot\big|h(t_{n_k}+\tau)-\varphi(\tau)\big| + \sup_{\tau\in\mathbb{R}^d}\big|h(t_{n_k}-\tau)-\varphi(-\tau)\big|\cdot|\varphi(\tau)| \\
&\leqslant 2C\sup_{\tau\in\mathbb{R}^d}\big|h(t_{n_k}+\tau)-\varphi(\tau)\big| \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Let us assume that $\varphi(\tau)\varphi(-\tau) = 0$ for all $\tau \in \mathbb{R}^d$. It follows that

$$\sup_{\tau\in\mathbb{R}^d}\big|h(t_{n_k}+\tau)h(t_{n_k}-\tau)\big| \underset{k\to\infty}{\longrightarrow} 0.$$

So for any fixed $s \in \mathbb{R}^d$

$$h(t_{n_k}+\tau)h(t_{n_k}-\tau)\Big|_{\tau=-t_{n_k}-s} = h(-s)h(2t_{n_k}+s) \underset{k\to\infty}{\longrightarrow} 0.$$

Since $h(s) \neq 0$ for every $s \in \mathbb{R}^d$, we have

$$h(2t_{n_k}+s) \underset{k\to\infty}{\longrightarrow} 0. \tag{4.13}$$

Next, it is easy to see that the function $h(2t_{n_k} + \cdot\,)$ is almost periodic. It means that there exists a subsequence $(n_{k_m})_{m\in\mathbb{N}}$ such that a sequence $\big(h(2t_{n_{k_m}} + \cdot\,)\big)_{m\in\mathbb{N}}$ has a uniform limit. From (4.13) one can conclude that

$$\sup_{s\in\mathbb{R}^d}\big|h(2t_{n_{k_m}}+s)\big| \underset{m\to\infty}{\longrightarrow} 0.$$

Applying this with $s = -2t_{n_{k_m}}$, we come to a contradiction with $h(0) = 1$. Therefore the assumption $\inf_{t\in\mathbb{R}^d}|h(t)| = 0$ is false, i.e. (i) follows from (iv).

V. If (i) holds, then (ii) follows directly from [2, Theorem 3.2]. □

Proof of Theorem 2.1. The implication $(a) \to (b)$ directly follows from the implication $(iv) \to (i)$ of Theorem 3.2. The converse $(b) \to (a)$ holds due to $(i) \to (iv)$ of Theorem 3.2 with applying [5, Theorem 2.7] (so $\gamma \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$, ν is real-valued measure). The representation (2.4) holds due to (iii) of Theorem 3.2 and [5, Theorem 2.7] (so $\gamma \in \mathbb{R}^d$ and $\lambda_u \in \mathbb{R}$). □

Proof of Theorem 2.2. Necessity. Due to Theorem 2.1 and comments below, it is easily seen using formula (2.4) that if the distribution function F of a discrete random vector ξ is (quasi-)infinitely divisible, then for any $c \in \mathbb{R}^d$ distribution functions F_c of the random variables $\langle c, \xi \rangle$ are (quasi-)infinitely divisible, respectively (there is the case $d = 1$ for F_c).

Sufficiency. Let us consider a discrete random vector ξ with distribution function (2.1) and characteristic function (2.2). We write the latter in the expanded form:

$$f(t^{(1)}, \dots, t^{(d)}) = \sum_{k=1}^{\infty} p_{x_k} \exp\Big\{ i \sum_{j=1}^{d} t^{(j)} x_k^{(j)} \Big\},$$

where $x_k = (x_k^{(1)}, \dots, x_k^{(d)}) \in \mathbb{R}^d$ and $t^{(1)}, \dots, t^{(d)} \in \mathbb{R}$.

We now assume that the distribution functions F_c of $\langle c, \xi \rangle$ are quasi-infinitely divisible for any $c = (c^{(1)}, \dots, c^{(d)}) \in \mathbb{R}^d$. Let f_c denote the corresponding characteristic functions. It is easily seen that

$$f_c(t) = f(c^{(1)} t, \dots, c^{(d)} t), \quad t \in \mathbb{R}.$$

Applying Theorem 2.1 to F_c (here the case $d = 1$), we conclude that there exists a constant $\mu_c > 0$ such that

$$\big| f(c^{(1)} t, \dots, c^{(d)} t) \big| \geqslant \mu_c \quad \text{for all} \quad t \in \mathbb{R}. \tag{4.14}$$

In order to prove the quasi-infinite divisibility of F, according to Theorem 2.1, it is sufficient to show that for some $\mu > 0$

$$\big| f(t^{(1)}, \dots, t^{(d)}) \big| \geqslant \mu \quad \text{for all} \quad t^{(1)}, \dots, t^{(d)} \in \mathbb{R}. \tag{4.15}$$

We set $X^{(j)} := \{x_k^{(j)} : p_{x_k} > 0, k \in \mathbb{N}\} \subset \mathbb{R}$, $j = 1, \dots, d$. Let us suppose that $X^{(j)} \neq \{0\}$ for every $j = 1, \dots, d$, i.e. for every j there exists $k \in \mathbb{N}$ such that $x_k^{(j)} \neq 0$. Therefore for every $j = 1, \dots, d$ one can choose non-zero $\beta_l^{(j)} \in X^{(j)}$, $l \in \mathcal{I}^{(j)}$ (here $\mathcal{I}^{(j)}$ is at most countable index set) such that for every $k \in \mathbb{N}$ and for some numbers $z_{k,l}^{(j)} \in \mathbb{Q}$ we have

$$x_k^{(j)} = \sum_{l \in \mathcal{I}^{(j)}} z_{k,l}^{(j)} \beta_l^{(j)}, \tag{4.16}$$

where only finite number of $z_{k,l}^{(j)}$ are non-zero (see [16] p. 67–68). Note that the numbers $\beta_l^{(j)}$ can be chosen as linearly independent over $\mathbb{Q}$, that is the equation $z_1 \beta_{l_1}^{(j)} + \dots + z_n \beta_{l_n}^{(j)} = 0$ holds with $z_1, \dots, z_n \in \mathbb{Q}$, and distinct $l_1, \dots, l_n \in \mathcal{I}^{(j)}$, $n \in \mathbb{N}$, if and only if $z_1 = \dots = z_n = 0$. It follows that the numbers $z_{k,l}^{(j)}$ are uniquely determined for $x_k^{(j)}$ in (4.16). We observe that for every $j = 1, \dots, d$

$$\langle X^{(j)} \rangle_r = \Big\{ z_1 \beta_{l_1}^{(j)} + \dots + z_n \beta_{l_n}^{(j)} :\ z_1, \dots, z_n \in \mathbb{Q},\, l_1, \dots, l_n \in \mathcal{I}^{(j)},\, n \in \mathbb{N} \Big\}, \tag{4.17}$$

where $\langle X^{(j)} \rangle_r$ is the module over the ring $\mathbb{Q}$ with the generating set $X^{(j)}$ (see definition (2.3) for the one-dimesional case with $z_k \in \mathbb{Q}$).

We now propose the procedure of choosing of the numbers $c^{(1)}, \dots, c^{(d)} \in \mathbb{R}$ such that the elements of the union system $\big\{ c^{(1)} \beta_l^{(1)} : l \in \mathcal{I}^{(1)} \big\} \cup \dots \cup \big\{ c^{(d)} \beta_l^{(d)} : l \in \mathcal{I}^{(d)} \big\}$ are linearly independent over $\mathbb{Q}$. We first fix any $c^{(1)} \in \mathbb{R} \setminus \{0\}$. For every $v \in \langle X^{(2)} \rangle_r \setminus \{0\}$ we define

$$D_v^{(2)} := \big\{ c \in \mathbb{R} : cv \in \langle c^{(1)} X^{(1)} \rangle_r \big\}.$$

Here and below, for any set $X \subset \mathbb{R}$ we denote by cX the set $\{cx : x \in X\}$ with $c \in \mathbb{R}$. Observe that every set $D_v^{(2)}$ is countable. Then the set

$$D^{(2)} := \bigcup_{v \in \langle X^{(2)} \rangle_r \setminus \{0\}} D_v^{(2)}$$

is countable too. Hence the set $C^{(2)} := \mathbb{R} \setminus D^{(2)}$ is not empty. We choose any $c^{(2)} \in C^{(2)}$. Observe that

$$C^{(2)} = \mathbb{R} \setminus \bigcup_{v \in \langle X^{(2)} \rangle_r \setminus \{0\}} D_v^{(2)} = \bigcap_{v \in \langle X^{(2)} \rangle_r \setminus \{0\}} \mathbb{R} \setminus D_v^{(2)}.$$

This means that for any $v \in \langle X^{(2)} \rangle_r \setminus \{0\}$ the quantity $c^{(2)}v$ can not be a finite linear combination of elements $c^{(1)}\beta_l^{(1)}$, $l \in \mathcal{I}^{(1)}$, with rational coefficients. Let $v = z_1\beta_{l_1}^{(2)} + \cdots + z_n\beta_{l_n}^{(2)}$ with some $z_1, \ldots, z_n \in \mathbb{Q}$. $l_1, \ldots, l_n \in \mathcal{I}^{(2)}$, and $n \in \mathbb{N}$. Since $c^{(2)}v = z_1(c^{(2)}\beta_{l_1}^{(2)}) + \cdots + z_n(c^{(2)}\beta_{l_n}^{(2)})$, by the above argument, the elements in the union system $\{c^{(1)}\beta_l^{(1)} : l \in \mathcal{I}^{(1)}\} \cup \{c^{(2)}\beta_l^{(2)} : l \in \mathcal{I}^{(2)}\}$ are linear independent over $\mathbb{Q}$. We next consider the set of all finite linear combitations of $\{c^{(1)}\beta_l^{(1)} : l \in \mathcal{I}^{(1)}\} \cup \{c^{(2)}\beta_l^{(2)}\colon l \in \mathcal{I}^{(2)}\}$ with rational coefficients. It is the set $\langle c^{(1)}X^{(1)} \cup c^{(2)}X^{(2)} \rangle_r$. For every $v \in \langle X^{(3)} \rangle_r \setminus \{0\}$ we define

$$D_v^{(3)} := \{c \in \mathbb{R} : cv \in \langle c^{(1)}X^{(1)} \cup c^{(2)}X^{(2)} \rangle_r\}.$$

Every $D_v^{(3)}$ is countable. Hence the set

$$D^{(3)} := \bigcup_{v \in \langle X^{(3)} \rangle_r \setminus \{0\}} D_v^{(3)}$$

is countable too. Since the set $C^{(3)} := \mathbb{R} \setminus D^{(3)}$ is not empty, we choose any $c^{(3)} \in C^{(3)}$. Observe that

$$C^{(3)} = \mathbb{R} \setminus \bigcup_{v \in \langle X^{(3)} \rangle_r \setminus \{0\}} D_v^{(3)} = \bigcap_{v \in \langle X^{(3)} \rangle_r \setminus \{0\}} \mathbb{R} \setminus D_v^{(3)}.$$

Hence for any $v \in \langle X^{(3)} \rangle_r \setminus \{0\}$ the quantity $c^{(3)}v$ can not be a finite linear combination of elements of $\{c^{(1)}\beta_l^{(1)} : l \in \mathcal{I}^{(1)}\} \cup \{c^{(2)}\beta_l^{(2)} : l \in \mathcal{I}^{(2)}\}$ with rational coefficients. This implies that the elements in the union system $\{c^{(1)}\beta_l^{(1)} : l \in \mathcal{I}^{(1)}\} \cup \{c^{(2)}\beta_l^{(2)} : l \in \mathcal{I}^{(2)}\} \cup \{c^{(3)}\beta_l^{(3)} : l \in \mathcal{I}^{(3)}\}$ are linear independent over $\mathbb{Q}$. We next proceed analogously and thus we obtain that the elements of the union system $\{c^{(1)}\beta_l^{(1)} : l \in \mathcal{I}^{(1)}\} \cup \cdots \cup \{c^{(d)}\beta_l^{(d)}\colon l \in \mathcal{I}^{(d)}\}$ are linearly independent over $\mathbb{Q}$ as required.

We now prove (4.15). Suppose, contrary to our claim, that (4.15) is false, i.e. for any $\varepsilon > 0$ there exist $t_\varepsilon^{(1)}, \ldots, t_\varepsilon^{(d)} \in \mathbb{R}$ such that $|f(t_\varepsilon^{(1)}, \ldots, t_\varepsilon^{(d)})| \leqslant \varepsilon$. So we fix $\varepsilon > 0$ and such $t_\varepsilon^{(j)}$, $j = 1, \ldots, d$. We first find $N_\varepsilon \in \mathbb{N}$ such that $\sum_{k=N_\varepsilon+1}^\infty p_{x_k} \leqslant \varepsilon$ (see (2.1) and (2.2), $\sum_{k=1}^\infty p_{x_k} = 1$, $p_{x_k} \geqslant 0$). Then

$$\sup_{t^{(1)}, \ldots, t^{(d)} \in \mathbb{R}} \Big| \sum_{k=N_\varepsilon+1}^\infty p_{x_k} \exp\Big\{ i \sum_{j=1}^d x_k^{(j)} t^{(j)} \Big\} \Big| \leqslant \sum_{k=N_\varepsilon+1}^\infty p_{x_k} \leqslant \varepsilon. \tag{4.18}$$

Hence we get

$$\begin{aligned} \big|f(t_\varepsilon^{(1)}, \ldots, t_\varepsilon^{(d)})\big| &= \Big| \sum_{k \in \mathbb{N}} p_{x_k} \exp\Big\{ i \sum_{j=1}^d x_k^{(j)} t_\varepsilon^{(j)} \Big\} \Big| \\ &\geqslant \Big| \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\Big\{ i \sum_{j=1}^d x_k^{(j)} t_\varepsilon^{(j)} \Big\} \Big| - \Big| \sum_{k=N_\varepsilon+1}^\infty p_{x_k} \exp\Big\{ i \sum_{j=1}^d x_k^{(j)} t_\varepsilon^{(j)} \Big\} \Big| \end{aligned}$$

$$\geqslant \Big|\sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\Big\{ i\sum_{j=1}^{d} x_k^{(j)} t_\varepsilon^{(j)}\Big\}\Big| - \varepsilon.$$

Due to representations (4.16), we write:

$$\begin{aligned}\sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\Big\{ i\sum_{j=1}^{d} x_k^{(j)} t_\varepsilon^{(j)}\Big\} &= \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\Big\{ i\sum_{j=1}^{d}\Big(\sum_{l\in\mathcal{I}^{(j)}} z_{k,l}^{(j)}\beta_l^{(j)}\Big) t_\varepsilon^{(j)}\Big\}\\ &= \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\Big\{ i\sum_{j=1}^{d}\sum_{l\in\mathcal{I}^{(j)}} z_{k,l}^{(j)}\big(\beta_l^{(j)} t_\varepsilon^{(j)}\big)\Big\}. \qquad (4.19)\end{aligned}$$

Let us fix $c^{(1)},\dots,c^{(d)}\in\mathbb{R}$ such that the elements of the union system $\{c^{(1)}\beta_l^{(1)} : l\in\mathcal{I}^{(1)}\}\cup\dots\cup\{c^{(d)}\beta_l^{(d)} : l\in\mathcal{I}^{(d)}\}$ are linearly independent over $\mathbb{Q}$. Let $\varkappa^{(j)}$ be the minimal positive integer such that $\varkappa^{(j)} z_{k,l}^{(j)}\in\mathbb{Z}$ for any $j\in\{1,\dots,d\}$, $k\in\{1,\dots,N_\varepsilon\}$, $l\in\mathcal{I}^{(j)}$. By the Kronecker theorem (see [17, p.37]), for any $\delta>0$ we can find t'_δ such that all following inequalities hold with some integers $n_l^{(j)}$:

$$\Big| c^{(j)}\beta_l^{(j)} t'_\delta - \frac{\beta_l^{(j)} t_\varepsilon^{(j)}}{\varkappa^{(j)}} - 2\pi n_l^{(j)}\Big| < \delta, \quad l\in\mathcal{I}_\varepsilon^{(j)}, \quad j=1,\dots,d, \qquad (4.20)$$

where $\mathcal{I}_\varepsilon^{(j)}$ is the set of all $l\in\mathcal{I}^{(j)}$ such that $z_{k,l}^{(j)}\neq 0$ for some $k=1,\dots,N_\varepsilon$. Since only finite number of $z_{k,l}^{(j)}$ are non-zero in (4.16), the set $\mathcal{I}_\varepsilon^{(j)}$ is finite and the system (4.20) has only finite number of inequalities. Let us choose $\delta=\delta_\varepsilon$ such that

$$\delta_\varepsilon\cdot\max_{k=1,\dots,N_\varepsilon}\Big\{\sum_{j=1}^{d}\sum_{l\in\mathcal{I}_\varepsilon^{(j)}}\varkappa^{(j)}|z_{k,l}^{(j)}|\Big\}\leqslant\varepsilon. \qquad (4.21)$$

Observe that

$$\begin{aligned}\Delta_\varepsilon &:= \Big|\sum_{k=1}^{N_\varepsilon} p_{x_k}\exp\Big\{ i\sum_{j=1}^{d}\sum_{l\in\mathcal{I}_\varepsilon^{(j)}}\varkappa^{(j)} z_{k,l}^{(j)} c^{(j)}\beta_l^{(j)} t'_{\delta_\varepsilon}\Big\}\\ &\qquad - \sum_{k=1}^{N_\varepsilon} p_{x_k}\exp\Big\{ i\sum_{j=1}^{d}\sum_{l\in\mathcal{I}_\varepsilon^{(j)}} z_{k,l}^{(j)}\beta_l^{(j)} t_\varepsilon^{(j)}\Big\}\Big|\\ &\leqslant \sum_{k=1}^{N_\varepsilon} p_{x_k}\Big|\exp\Big\{ i\sum_{j=1}^{d}\sum_{l\in\mathcal{I}_\varepsilon^{(j)}}\varkappa^{(j)} z_{k,l}^{(j)}\Big(c^{(j)}\beta_l^{(j)} t'_{\delta_\varepsilon} - \frac{\beta_l^{(j)} t_\varepsilon^{(j)}}{\varkappa^{(j)}}\Big)\Big\} - 1\Big|\\ &= \sum_{k=1}^{N_\varepsilon} p_{x_k}\Big|\exp\Big\{ i\sum_{j=1}^{d}\sum_{l\in\mathcal{I}_\varepsilon^{(j)}}\varkappa^{(j)} z_{k,l}^{(j)}\Big(c^{(j)}\beta_l^{(j)} t'_{\delta_\varepsilon} - \frac{\beta_l^{(j)} t_\varepsilon^{(j)}}{\varkappa^{(j)}} - 2\pi n_l^{(j)}\Big)\Big\} - 1\Big|.\end{aligned}$$

The last equality holds because all $\varkappa^{(j)} z_{k,l}^{(j)}$ and $n_l^{(j)}$ are integers. Next, using the well known inequality $|e^{iy}-1|\leqslant|y|$, $y\in\mathbb{R}$, and applying (4.20) and (4.21), we obtain

$$\begin{aligned}\Delta_\varepsilon &\leqslant \sum_{k=1}^{N_\varepsilon}\Big(p_{x_k}\sum_{j=1}^{d}\sum_{l\in\mathcal{I}_\varepsilon^{(j)}}\Big(\varkappa^{(j)}|z_{k,l}^{(j)}|\cdot\Big| c^{(j)}\beta_l^{(j)} t'_{\delta_\varepsilon} - \frac{\beta_l^{(j)} t_\varepsilon^{(j)}}{\varkappa^{(j)}} - 2\pi n_l^{(j)}\Big|\Big)\Big)\\ &\leqslant \max_{k=1,\dots,N_\varepsilon}\Big\{\sum_{j=1}^{d}\sum_{l\in\mathcal{I}_\varepsilon^{(j)}}\varkappa^{(j)}|z_{k,l}^{(j)}|\cdot\delta_\varepsilon\Big\}\cdot\sum_{k=1}^{N_\varepsilon} p_{x_k}\end{aligned}$$

$$\leqslant \delta_\varepsilon \cdot \max_{k=1,\ldots,N_\varepsilon} \left\{ \sum_{j=1}^{d} \sum_{l\in\mathcal{I}_\varepsilon^{(j)}} \varkappa^{(j)} |z_{k,l}^{(j)}| \right\} \leqslant \varepsilon.$$

Returning to (4.19), we have

$$\left| \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\left\{ i \sum_{j=1}^{d} \sum_{l\in\mathcal{I}^{(j)}} z_{k,l}^{(j)} \beta_l^{(j)} t_\varepsilon^{(j)} \right\} \right| \geqslant \left| \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\left\{ i \sum_{j=1}^{d} \sum_{l\in\mathcal{I}^{(j)}} \varkappa^{(j)} z_{k,l}^{(j)} c^{(j)} \beta_l^{(j)} t'_{\delta_\varepsilon} \right\} \right| - \varepsilon.$$

Note that we write $\mathcal{I}^{(j)}$ instead of $\mathcal{I}_\varepsilon^{(j)}$ here. This is obviously possible by the definition of $\mathcal{I}_\varepsilon^{(j)}$. Thus we get

$$\left| f(t_\varepsilon^{(1)}, \ldots, t_\varepsilon^{(d)}) \right| \geqslant \left| \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\left\{ i \sum_{j=1}^{d} \sum_{l\in\mathcal{I}^{(j)}} \varkappa^{(j)} z_{k,l}^{(j)} c^{(j)} \beta_l^{(j)} t'_{\delta_\varepsilon} \right\} \right| - 2\varepsilon.$$

According to (4.16), we next write

$$\sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\left\{ i \sum_{j=1}^{d} \sum_{l\in\mathcal{I}^{(j)}} \varkappa^{(j)} z_{k,l}^{(j)} c^{(j)} \beta_l^{(j)} t'_{\delta_\varepsilon} \right\} = \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\left\{ i \sum_{j=1}^{d} \varkappa^{(j)} c^{(j)} x_k^{(j)} t'_{\delta_\varepsilon} \right\}.$$

Due to (4.18), we get

$$\begin{aligned} \left| \sum_{k=1}^{N_\varepsilon} p_{x_k} \exp\left\{ i \sum_{j=1}^{d} \varkappa^{(j)} c^{(j)} x_k^{(j)} t'_{\delta_\varepsilon} \right\} \right| &\geqslant \left| \sum_{k=1}^{\infty} p_{x_k} \exp\left\{ i \sum_{j=1}^{d} \varkappa^{(j)} c^{(j)} x_k^{(j)} t'_{\delta_\varepsilon} \right\} \right| - \varepsilon \\ &= \left| f\left(\varkappa^{(1)} c^{(1)} t'_{\delta_\varepsilon}, \ldots, \varkappa^{(d)} c^{(d)} t'_{\delta_\varepsilon} \right) \right| - \varepsilon. \end{aligned}$$

So we have

$$\varepsilon \geqslant |f(t_\varepsilon^{(1)}, \ldots, t_\varepsilon^{(d)})| \geqslant \left| f\left(\varkappa^{(1)} c^{(1)} t'_{\delta_\varepsilon}, \ldots, \varkappa^{(d)} c^{(d)} t'_{\delta_\varepsilon} \right) \right| - 3\varepsilon.$$

Thus for any $\varepsilon > 0$ we found t'_{δ_ε} such that

$$\left| f\left(\varkappa^{(1)} c^{(1)} t'_{\delta_\varepsilon}, \ldots, \varkappa^{(d)} c^{(d)} t'_{\delta_\varepsilon} \right) \right| \leqslant 4\varepsilon.$$

This obviously contradicts to the assumption (4.14). So (4.15) holds.

We have proved the Cramér–Wold device for the quasi-infinite divisibility. Let us now consider the case of infinite divisibility. Let the distribution functions F_c of random variables $\langle c, \xi \rangle$ be infinitely divisible for any $c \in \mathbb{R}^d$. Then they are also quasi-infinitely divisible. From what has already been proved, the distribution function F of the random vector ξ is also quasi-infinitly divisible and, by Theorem 2.1, its characteristic function f admits the representation

$$f(t) = \exp\left\{ i\langle t, \gamma \rangle + \sum_{u\in\langle X \rangle \setminus \{\bar{0}\}} \lambda_u \left(e^{i\langle t, u \rangle} - 1 \right) \right\}, \quad t \in \mathbb{R}^d,$$

where $\gamma \in \langle X \rangle$, $\lambda_u \in \mathbb{R}$ for all $u \in \langle X \rangle \setminus \{\bar{0}\}$, and $\sum_{u\in\langle X \rangle \setminus \{0\}} |\lambda_u| < \infty$. It remains to show that $\lambda_u \geqslant 0$ for all $u \in \langle X \rangle \setminus \{\bar{0}\}$. Let us write the characteristic function f_c of F_c for any $c \in \mathbb{R}^d$:

$$f_c(t) = \exp\left\{ it\langle c, \gamma \rangle + \sum_{u\in\langle X \rangle \setminus \{\bar{0}\}} \lambda_u \left(e^{it\langle c, u \rangle} - 1 \right) \right\}, \quad t \in \mathbb{R}.$$

Let us fix $c = (c^{(1)}, \ldots, c^{(d)}) \in \mathbb{R}^d$ such that the elements of the union system $\{c^{(1)}\beta_l^{(1)} : l \in \mathcal{I}^{(1)}\} \cup \cdots \cup \{c^{(d)}\beta_l^{(d)} : l \in \mathcal{I}^{(d)}\}$ are linearly independent over $\mathbb{Q}$. On account of (4.16), (4.17), and that $\langle X \rangle \subset \langle X^{(1)} \rangle_r \times \cdots \times \langle X^{(d)} \rangle_r$, we have $\langle c, u_1 \rangle \neq \langle c, u_2 \rangle$ for any distinct $u_1, u_2 \in \langle X \rangle \setminus \{\bar{0}\}$. Since F_c is infinitely divisible (by assumption), we conclude that $\lambda_u \geqslant 0$ for all $u \in \langle X \rangle \setminus \{\bar{0}\}$. □

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