

Regularisation by fractional noise for one-dimensional differential equations with distributional drift

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Abstract

We study existence and uniqueness of solutions to the equation $dX_t = b(X_t)dt + dB_t$, where b is a distribution in some Besov space and B is a fractional Brownian motion with Hurst parameter $H \leq 1/2$. First, the equation is understood as a nonlinear Young equation. This involves a nonlinear Young integral constructed in the space of functions with finite p -variation, which is well suited when b is a measure. Depending on H , a condition on the Besov regularity of b is given so that solutions to the equation exist. The construction is deterministic, and B can be replaced by a deterministic path w with a sufficiently smooth local time. Using this construction we prove the existence of weak solutions (in the probabilistic sense).

We also prove that solutions coincide with limits of strong solutions obtained by regularisation of b . This is used to establish pathwise uniqueness and existence of a strong solution. In particular when b is a finite measure, weak solutions exist for $H < \sqrt{2} - 1$, while pathwise uniqueness and strong existence hold when $H \leq 1/4$. The proofs involve fine properties of the local time of the fractional Brownian motion, as well as new regularising properties of this process which are established using the stochastic sewing Lemma.

Keywords: regularisation by noise; fractional Brownian motion; local time; skew processes.

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1 Introduction

We are interested in the well-posedness of the one-dimensional equation

$$dX_t = b(X_t)dt + dB_t, \tag{1.1}$$

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when b is a distribution in some Besov space and $(B_t)_{t \geq 0}$ is a fractional Brownian motion of Hurst parameter H . We will explain in which sense this equation can be solved when b is a genuine distribution. It is noteworthy that, even when b is a function, this equation can be well-posed while the corresponding equation without noise is not. This effect is often called regularisation by noise. We refer to [15] for a thorough presentation of this phenomenon, in particular on PDE models of fluid mechanics. This equation encompasses at least two classes of equations which have frequently been studied in the literature.

First, when B is the standard Brownian motion, there is an extensive literature which we will not attempt to describe thoroughly. Let us mention the early work of Veretennikov [44] for bounded measurable drifts, then the more general $L^p - L^q$ criterion of Krylov and Röckner [31] for which the authors proved strong existence and uniqueness (both works allowing for time inhomogeneous drifts in dimension $d \geq 1$). Flandoli, Russo and Wolf [17] developed a weak well-posedness theory while Bass and Chen [5] proved existence and uniqueness of strong solutions with the drift being the distributional derivative of a γ -Hölder function for $\gamma > 1/2$. Then Davie [11] provided conditions for path-by-path existence and uniqueness, which is a stronger form of uniqueness, for time inhomogeneous bounded measurable drift. Using rough path methods, Delarue and Diel [12] proved weak existence and uniqueness in dimension 1 when the drift is the distributional derivative of a γ -Hölder function for $\gamma > 1/3$. In higher dimension, Flandoli, Issoglio and Russo [16] identified a class of SDEs with distributional drifts in Bessel spaces such that there exists a solution that is unique in law. In addition, when the drift is random, another well-posedness result is given by Duboscq and Réveillac [14]. We also point out the work [29], with extensions in [33] on an SDE involving the local time at 0 of the solution, which formally corresponds to a drift $b = a\delta_0$, for some $a \in [-1, 1]$ and δ_0 being the Dirac distribution. This setting corresponds to the so-called skew Brownian motion, see [34] for more details and various constructions.

This leads to a second class of interesting problems, namely solving Equation (1.1) when b is a distribution and B is a fractional Brownian motion with sufficiently small Hurst parameter H . A first attempt in this direction seems to be due to Nualart and Ouknine [35], who proved existence and uniqueness for some non-Lipschitz drifts. When $b = a\delta_0$ with $a \in \mathbb{R}$, the well-posedness of this equation was established for $H < 1/4$ by Catellier and Gubinelli [8] (who also consider multidimensional drifts in negative Hölder spaces) and independently for $H < 1/6$ in [1, 3] with extensions to dimension $d \geq 1$ in the three papers [1, 3, 8]. The solution is generally referred to as skew fractional Brownian motion. We observe a gap between the one-dimensional Brownian case ($H = 1/2$), with well posedness for $|a| \leq 1$ proven in [33], and the aforementioned result for fractional Brownian motion with H smaller than $1/4$. The intent of this paper is to partially close this gap. Note also that the case $a = 1$ corresponds to reflection above 0 in the Brownian case. The well-posedness of reflected equations was established even for multiplicative rough noises in case X is one-dimensional [13, 38], while uniqueness might fail as soon as the dimension is greater than 2 (see [23]).

Finally, let us mention that regularisation by noise was also investigated for other types of noise, for instance α -stable noises [9], regular noises [25] and other classes of rough processes [27, 28]. Recently, the regularisation phenomenon was studied for SDEs with multiplicative noise (whether fractional Brownian motion [10], or more general rough paths [7]).

In this paper, the drift b is in some Besov space $\mathcal{B}_{p,\infty}^\beta$ (denoted \mathcal{B}_p^β hereafter). The solutions we consider are stochastic processes of the form

$$X_t = X_0 + K_t + B_t, \tag{1.2}$$

where K_t is the limit in probability of $\int_0^t b^n(X_s) ds$ for any sequence (b^n) of smooth approximations of b (in line with [2, 5]). Roughly, when such a solution exists and X and B are adapted to the filtration of the underlying probability space, we call it a weak solution. When X is adapted to the natural filtration of B , it is called a strong solution. We refer to Definition 2.4 for more details.

Our first main result, Theorem 2.5, gives conditions on β , p and H that ensure the existence of a weak solution to (1.1) when b is a measure. In particular when $b = a\delta_0$, for $a \in \mathbb{R}$, we obtain the existence of weak solutions to (1.1) for any $H < \sqrt{2} - 1$. In the standard Brownian case, Theorem 2.5 provides weak solutions when the drift is in $\mathcal{B}_1^{\frac{1}{4}+}$. This space contains functions which, to the best of our knowledge, are not covered by the existing literature (see Remark 2.7).

To prove Theorem 2.5, we consider another approach to study Equation (1.1) which is via nonlinear Young integrals as introduced in [8], extending Young's theory of integration [46]. Consider the more general equation

$$dX_t = b(X_t) dt + dZ_t, \tag{1.3}$$

where Z is a continuous stochastic process. The idea is to define path-by-path solutions to (1.3), that is, to solve this equation for a fixed realisation of the noise $(Z_t(\omega))_t$. In order to do this one rewrites the equation as a random ODE: $dY_t = b(Y_t + Z_t) dt$, with $Y_t = X_t - Z_t$ and studies the regularity of the averaging operator $T_t^Z b : y \mapsto \int_0^t b(y + Z_s) ds$. In some interesting cases, $T_t^Z b$ is more regular than b itself, which permits to have solutions of the form $Y_t = Y_0 + \int_0^t T_{ds}^Z b(Y_s)$, where the integral is a so-called nonlinear Young integral. We refer to [20] for a review of nonlinear Young integrals in the Hölder setting and also to the recent work [21].

In Theorem 2.15 we give conditions such that solutions w.r.t. the probabilistic approach via approximation of the drift and w.r.t. the approach via nonlinear Young integral theory are equivalent (i.e. weak solutions coincide). Therefore, in order to show existence of a weak solution to Equation (1.1), it is sufficient to construct a solution to the corresponding nonlinear Young integral equation that is additionally adapted to a small enough filtration.

To do so, we construct nonlinear Young integrals in the p -variation setting (instead of Hölder). In our one-dimensional setting this allows us to exploit the nonnegativity of b to get existence of solutions to (1.3) under some milder conditions on the regularity of b . To obtain the regularity of the averaging operator, we proceed similarly to Harang and Perkowski [28] by rewriting it as a convolution between b and the local time of Z . We are then able to deduce the Hölder regularity of the averaging operator from the Besov regularity of the local time of Z (see Lemma 4.4). These two ingredients permit to construct path-by-path solutions by convergence of the Euler scheme associated to the equation, see Theorem 2.14. Alternatively, if the local time of Z has some probabilistic properties, as in the case $Z = B$, we are able to show that the averaging operator has a certain tightness property, see Lemma 4.11. This permits to prove that the sequence of (random) Euler schemes which approximate the nonlinear Young solution is compact in the space of continuous adapted stochastic processes, from which adaptedness of X is deduced.

Our second main result, Theorem 2.10, states pathwise uniqueness of weak solutions in a class of processes which have some Hölder regularity. The main condition is that b is in \mathcal{B}_p^β with β and p satisfying $\beta - \frac{1}{p} \geq 1 - \frac{1}{2H}$. As with the Yamada-Watanabe Theorem, weak existence and pathwise uniqueness also lead here, under the same conditions on β , p and H , to the existence of a strong solution. Moreover, when b is a measure, any strong solution is proven to have sufficient Hölder regularity, thus ensuring pathwise uniqueness. For instance, we get strong existence and pathwise uniqueness to (1.1)

when $b = a\delta_0$ for $a \in \mathbb{R}$ and any $H \leq 1/4$. This extends the previously known condition $H < 1/4$ from [8, 21] to $H = 1/4$.

To prove this theorem, we follow the recent approach of Athreya et al. [2]. In this paper, the authors proved the strong well-posedness of the one-dimensional stochastic heat equation with Besov drift by a tightness-stability argument. The main regularity estimates are obtained via the recent and powerful stochastic sewing Lemma introduced by Lê [32]. To adapt this argument to our setting, we control the Hölder norm of the conditional expectation of $x \mapsto \int_s^t f(x + B_r) dr$ in terms of the Besov norm of f (see Lemma 7.1). One difficulty that arises here is the non-Markovian nature of B , which we could compensate by using a slightly adapted version of the local nondeterminism property of the fractional Brownian motion.

In the development of the proof of Theorem 2.10, we obtain that weak solutions to (1.1) are limits of strong solutions to (1.1) with b replaced by a smooth bounded drift b^n , where the sequence (b^n) converges to b in Besov norm. This result is detailed in Theorem 2.8 and can be of independent interest in view of numerical applications.

Structure of the paper In Sect. 2, the main definitions and results are stated. We also present the organisation of the proofs in the paper. In Sect. 3, we develop the construction of nonlinear Young integrals in p -variation (Theorem 3.1) and use it to find solutions to nonlinear Young integral equations with nonnegative (or nonpositive) drifts, see Theorem 3.4. Then in Sect. 4, we prove successively Theorem 2.15 (relation between different notions of solution) and Theorem 2.14 (existence of path-by-path solutions). Then we conclude with the proof of Theorem 2.5 about the existence of weak solutions. The regularity of weak solutions is studied in Sect. 5. The uniqueness part of Theorem 2.10 is proven in Sect. 6. The tightness-stability argument which leads to the existence of strong solutions is in Sect. 7.

Besides, we recall some useful results on Besov spaces in Appendix A. The local nondeterminism property of the fBm is stated and proven in Appendix B.2, jointly with the proof of the important regularity estimates of the conditional expectation of the fBm (Lemma 5.1). Finally, we recall the stochastic sewing Lemma in Appendix C and use it to derive several Hölder bounds on the integrals of fBm.

1.1 Notations and definitions

Various notations Throughout the paper, we use the following notations and conventions:

- Constants C might vary from line to line.
- For $p \in [1, \infty]$, $p' \in [1, \infty]$ is such that $1/p + 1/p' = 1$.
- For topological spaces X, Y we denote the set of continuous functions from X to Y by $\mathcal{C}_X(Y)$.
- For a Banach space E , the ball of radius $R > 0$ is denoted by $D_R := \{x \in E : \|x\| \leq R\}$.
- Let $s < t$ be two real numbers and $\Pi = (s = t_0 < t_1 < \dots < t_n = t)$ be a partition of $[s, t]$, we denote $|\Pi| = \sup_{i=1, \dots, n} (t_i - t_{i-1})$ the mesh of Π .
- For $s, t \in \mathbb{R}$ with $s \leq t$, we denote $\Delta_{[s,t]} := \{(u, v) : s \leq u \leq v \leq t\}$.
- For any function f defined on $[s, t]$, we denote $f_{u,v} := f_v - f_u$ for $(u, v) \in \Delta_{[s,t]}$.
- For any function g defined on $\Delta_{[s,t]}$ and $s \leq r \leq u \leq v \leq t$, we denote $\delta g_{r,u,v} := g_{r,v} - g_{r,u} - g_{u,v}$.
- For a probability space Ω and $p \in [1, \infty]$, the norm on $L^p(\Omega)$ is denoted by $\|\cdot\|_{L^p}$.
- We denote by $(B_t)_{t \geq 0}$ a fractional Brownian motion with Hurst parameter $H \leq 1/2$.
- The filtration $(\mathcal{F}_t)_{t \geq 0}$ is denoted by \mathbb{F} .
- The filtration generated by a process $(Z_t)_{t \geq 0}$ is denoted by \mathbb{F}^Z .

- All filtrations are assumed to fulfill the usual conditions.
- Let \mathbb{F} be a filtration. We call $(W_t)_{t \geq 0}$ a \mathbb{F} -Brownian motion if $(W_t)_{t \geq 0}$ is a Brownian motion adapted to \mathbb{F} and for $0 \leq s \leq t$, $W_t - W_s$ is independent of \mathcal{F}_s . For such a filtration, the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_s]$ is denoted by $\mathbb{E}^s[\cdot]$.

Gaussian semigroup For any $t > 0$ and $x \in \mathbb{R}$, let $g_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$. For a tempered distribution ϕ on \mathbb{R} , let

$$G_t \phi(x) := (g_t * \phi)(x). \tag{1.4}$$

The occupation time formula Let $T > 0$, $w : [0, T] \rightarrow \mathbb{R}$ be a measurable function and let λ denote the Lebesgue measure on \mathbb{R} . For $A \in \mathcal{B}([0, T])$, let μ_A be the occupation measure defined by $\mu_A(\Lambda) := \lambda(\{t \in A : w_t \in \Lambda\})$ for $\Lambda \in \mathcal{B}(\mathbb{R})$. If $\mu_{[0, T]} \ll \lambda$ then there exists a measurable map $\ell : \mathcal{B}([0, T]) \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that for $A \in \mathcal{B}([0, T])$, $\mu_A(dx) = \ell(A, x)\lambda(dx)$. For any bounded measurable function f , the occupation time formula reads (see [24] for more details)

$$\int_A f(w_r) dr = \int_{\mathbb{R}} f(x) \ell(A, x) dx. \tag{1.5}$$

We define a local time $L : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ by $L_t(x) := \ell([0, t], x)$. By (1.5), it comes that for any bounded measurable f and $t \in [0, T]$,

$$\int_0^t f(w_r) dr = \int_{\mathbb{R}} f(x) L_t(x) dx. \tag{1.6}$$

Note that if $w : [0, T] \rightarrow K \subset \mathbb{R}$ for some compact K , then $L_t(\cdot)$ vanishes on K^c for all $t \in [0, T]$.

Finite variations spaces Let $p \in [1, \infty)$ and $(F, \|\cdot\|_F)$ be a Banach space. Define the p -variation seminorm of a function $x : [s, t] \rightarrow F$ as

$$[x]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(F)} := \sup_{\{t_i\}} \left(\sum_{i=0}^{N-1} \|x_{t_i, t_{i+1}}\|_F^p \right)^{\frac{1}{p}},$$

where the supremum runs over all partitions $\{t_i\}_{i=0}^N$, $N \in \mathbb{N}$, of $[s, t]$. We denote by $\mathcal{C}_{[s,t]}^{p\text{-var}}(F)$ the set of such continuous functions with finite p -variation.

If F is just a Fréchet space, we say that a function mapping from $[s, t]$ to F has finite p -variation if its p -variation is finite with respect to any continuous seminorm and we also use the notation $\mathcal{C}_{[s,t]}^{p\text{-var}}(F)$. If $F = \mathbb{R}$, we use the alleviate notations $\mathcal{C}_{[s,t]}^{p\text{-var}}$ and $[x]_{\mathcal{C}_{[s,t]}^{p\text{-var}}}$.

A continuous function $\varkappa : \Delta_{[s,t]} \rightarrow [0, \infty)$ is a control function if, for $s \leq r \leq u \leq v \leq t$,

$$\varkappa(r, u) + \varkappa(u, v) \leq \varkappa(r, v), \tag{1.7}$$

and $\varkappa(r, r) = 0$ for all $r \in [s, t]$. A typical example of a control function is $[x]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(F)}^p$ (see [18, Prop. 5.8]).

Besov and Hölder spaces

Definition 1.1. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we denote the nonhomogeneous Besov space with these parameters by $\mathcal{B}_{p,q}^s$. For a precise definition see Appendix A.

Besides, for a bounded open interval $\mathcal{I} \subset \mathbb{R}$, we denote by $\mathcal{B}_{p,q}^s(\mathcal{I})$ the space of all distributions u on \mathcal{I} for which there exists $v \in \mathcal{B}_{p,q}^s$ such that $u = v|_{\mathcal{I}}$ (see [43, Def. 1, p. 192]).

If $q = \infty$, we write \mathcal{B}_p^s instead of $\mathcal{B}_{p,\infty}^s$. We have the following important embeddings between Besov spaces.

Remark 1.2. Let $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. From [4, Prop. 2.71], the space \mathcal{B}_{p_1,q_1}^s continuously embeds into $\mathcal{B}_{p_2,q_2}^{s-(p_1^{-1}-p_2^{-1})}$, which we write as $\mathcal{B}_{p_1,q_1}^s \hookrightarrow \mathcal{B}_{p_2,q_2}^{s-(p_1^{-1}-p_2^{-1})}$.

Remark 1.3. Let \mathcal{I} be a bounded open interval. Let $p_1, p_2, q_1, q_2 \in [1, \infty]$ and $-\infty < s_2 < s_1 < \infty$. If $s_1 - 1/p_1 > s_2 - 1/p_2$, then from [43, Th. p.196] we have $\mathcal{B}_{p_1,q_1}^{s_1}(\mathcal{I}) \hookrightarrow \mathcal{B}_{p_2,q_2}^{s_2}(\mathcal{I})$.

For $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $p = q = \infty$, Besov spaces coincide with Hölder spaces (see [4, p. 99]). We now give a definition of Hölder spaces in space domain for $s \in (0, 1]$.

Definition 1.4. Let E, F be Banach spaces, $U \subset E$ and $\beta \in (0, 1]$.

- We denote the supremum norm of $f \in \mathcal{C}_U(F)$ by $\|f\|_{\mathcal{C}_U(F)} = \sup_{x \in U} \|f(x)\|_F$. When U and F are clear from the context, we might also denote $\|f\|_\infty = \|f\|_{\mathcal{C}_U(F)}$.
- The Hölder space $\mathcal{C}_U^\beta(F)$ is the collection of all $f \in \mathcal{C}_U(F)$ such that $\|f\|_{\mathcal{C}_U^\beta(F)}$ is finite, where

$$\|f\|_{\mathcal{C}_U^\beta(F)} := [f]_{\mathcal{C}_U^\beta(F)} + \|f\|_{\mathcal{C}_U(F)} \quad \text{with} \quad [f]_{\mathcal{C}_U^\beta(F)} := \sup_{x \neq y \in U} \frac{\|f(x) - f(y)\|_F}{\|x - y\|_E^\beta}.$$

If $U = E$, we alleviate the notations and write $\|f\|_{\mathcal{C}^\beta(F)}$ and $[f]_{\mathcal{C}^\beta(F)}$. Similarly, if $F = \mathbb{R}$ or if F is clear from the context, we write $\|f\|_{\mathcal{C}_U^\beta}$ and $[f]_{\mathcal{C}_U^\beta}$. Finally, if $U = F = \mathbb{R}$ we just write \mathcal{C}^β .

- The space $\mathcal{C}_{E,loc}^\beta(F)$ of locally Hölder continuous functions is the collection of all $f \in \mathcal{C}_E(F)$ such that $\|f\|_{\mathcal{C}_{D_R}^\beta(F)}$ is finite for all $R > 0$.

Remark 1.5. In some results, we assume that the drift is a (nonnegative) measure in some \mathcal{B}_p^β . This is actually equivalent to the assumption that b is a nonnegative distribution in \mathcal{B}_p^β . Indeed, thanks to [42, Exercise 22.5], any nonnegative distribution is given by a Radon measure (i.e. a locally finite, complete measure fulfilling regularity conditions). Hence, it is sufficient to consider Radon measures lying in Besov spaces instead of considering general nonnegative distributions. Throughout the paper all measures are assumed to be Radon.

2 Main results

2.1 Definitions of solution

We define here weak and strong solutions to (1.1).

Definition 2.1. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$. We say that $(f_n)_{n \in \mathbb{N}}$ converges to f in $\mathcal{B}_p^{\beta-}$ as $n \rightarrow \infty$ if $\sup_{n \in \mathbb{N}} \|f_n\|_{\mathcal{B}_p^\beta} < \infty$ and

$$\forall \beta' < \beta, \quad \lim_{n \rightarrow \infty} \|f_n - f\|_{\mathcal{B}_p^{\beta'}} = 0.$$

Remark 2.2. For any $f \in \mathcal{B}_p^\beta$, there exists a sequence $(f_n)_n$ of bounded smooth functions converging to f in the sense of Definition 2.1: e.g. $f_n := G_{\frac{1}{n}} f$, where G is the Gaussian semigroup introduced in (1.4). This can be seen to hold true by applying Lemmas A.4 and A.6.

We recall here a link between fractional Brownian motion and Brownian motion. For each $H \in (0, \frac{1}{2})$, there exist operators \mathcal{A} and $\tilde{\mathcal{A}}$, where both can be given in terms of

fractional integrals and derivatives (see (B.1) and [36, Th. 11]), such that

$$\text{if } B \text{ is a fractional Brownian motion, } W = \mathcal{A}(B) \text{ is a Brownian motion,} \quad (2.1)$$

$$\text{if } W \text{ is a Brownian motion, } B = \bar{\mathcal{A}}(W) \text{ is a fractional Brownian motion.} \quad (2.2)$$

Besides, B and W generate the same filtration.

We give here the definition of a \mathbb{F} -fractional Brownian motion, for a given filtration \mathbb{F} .

Definition 2.3. Let \mathbb{F} be a filtration. We say that B is a \mathbb{F} -fractional Brownian motion if $W = \mathcal{A}(B)$ is a \mathbb{F} -Brownian motion.

Definition 2.4. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$, $T > 0$ and $X_0 \in \mathbb{R}$.

- Weak solution: We call a couple $((X_t)_{t \in [0, T]}, (B_t)_{t \in [0, T]})$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a weak solution to (1.1) on $[0, T]$, with initial condition X_0 , if

- B is a \mathbb{F} -fBm;
- X is adapted to \mathbb{F} ;
- there exists a process $(K_t)_{t \in [0, T]}$ such that, a.s.,

$$X_t = X_0 + K_t + B_t \text{ for all } t \in [0, T]; \quad (2.3)$$

- for every sequence $(b^n)_{n \in \mathbb{N}}$ of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$, we have that

$$\sup_{t \in [0, T]} \left| \int_0^t b^n(X_r) dr - K_t \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (2.4)$$

If the couple is clear from the context, we simply say that $(X_t)_{t \in [0, T]}$ is a weak solution.

- Pathwise uniqueness: As in the classical literature on SDEs, we say that pathwise uniqueness holds if for any two solutions (X, B) and (Y, B) defined on the same filtered probability space with the same fBm B and same initial condition $X_0 \in \mathbb{R}$, X and Y are indistinguishable.
- Strong solution: A weak solution (X, B) such that X is \mathbb{F}^B -adapted is called a strong solution.

2.2 Existence and uniqueness results

Theorem 2.5. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$ and $H \in (0, \frac{1}{2}]$. Let $b \in \mathcal{B}_p^\beta$ be a measure. Assume that one of the following conditions holds:

- (i) $H \geq \frac{1}{3}$ and $\beta > 1 + \frac{H}{2} - \frac{1}{2H}$;
- (ii) $H < \frac{1}{3}$ and $\beta > 2H - 1$;
- (iii) $p \in [2, \infty]$ and $\beta > -\frac{1}{2H} + 1$.

Then,

- (a) there exists a weak solution X to (1.1) such that the convergence in (2.4) holds a.s.
- (b) Additionally, $X - B \in \mathcal{C}_{[0, T]}^\kappa(L^m)$ for any $\kappa \in (0, 1 + H(\beta - \frac{1}{p}) \wedge 0] \setminus \{1\}$ and $m \geq 2$.

Corollary 2.6. For any finite measure b , there exists a weak solution to (1.1) for $H < \sqrt{2} - 1$. If $b = a\delta_0$, for some $a \in \mathbb{R}$, we call it an a -skew fractional Brownian motion.

Remark 2.7. In the Brownian motion case ($H = 1/2$), we obtain existence of a weak solution in new cases. For instance, consider for $\varepsilon > 0$

$$b(x) = |x|^{-3/4+\varepsilon} a(x),$$

where a is a smooth, compactly supported, nonnegative function equal to 1 on $[-1, 1]$. We have that $b \in \mathcal{B}_1^{\frac{1}{4}+}$ (see [4, Prop. 2.21] for similar computations), and the space $\mathcal{B}_1^{\frac{1}{4}+}$ is covered by Assumption (i) of Theorem 2.5. Since b is neither in $\bigcup_{p \geq 2} L^p(\mathbb{R})$ nor in $\mathcal{C}^{-\frac{2}{3}+}$, we cannot directly apply results from [31] or [12].

The following theorem gives, as Theorem 2.5 does, conditions on the drift and the Hurst parameter such that (1.1) has a weak solution. Note that this time, there is no nonnegativity assumption on b . Even in the case of considering b to be a measure, none of the two theorems is stronger than the other. However, if b is a finite measure, Theorem 2.5 allows for a wider range of Hurst parameters to get existence of weak solutions to (1.1). Moreover, even though Theorem 2.5(iii) is fully covered by (2.5), it still adds value as it gives a.s. convergence in (2.4).

Theorem 2.8. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$ and $X_0 \in \mathbb{R}$. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. Let X^n be the unique strong solution to (1.1) with drift b^n . Assume

$$\beta - \frac{1}{p} > -\frac{1}{2H} + \frac{1}{2}. \tag{2.5}$$

Then, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(X^{n_k})_{k \in \mathbb{N}}$ converges in law w.r.t. $\|\cdot\|_\infty$ to a process X which is a weak solution to (1.1) with drift b . Furthermore, $X - B \in \mathcal{C}_{[0,T]}^\kappa(L^m)$ for any $\kappa \in (0, 1 + H(\beta - \frac{1}{p}) \wedge 0] \setminus \{1\}$ and $m \geq 2$.

Remark 2.9. Without loss of generality, the previous theorem can directly be formulated for Hölder spaces by fixing $p = \infty$, using the embedding from Remark 1.2. However, this is not the case for Theorem 2.10 below. Hence we keep working in general Besov spaces for a better comparison of the results.

Under slightly stronger assumptions than (2.5), the following theorem states strong existence and pathwise uniqueness. In particular, it implies that under this stronger condition, convergence in probability of the approximation scheme in Theorem 2.8 holds without passing to a subsequence.

Theorem 2.10. Let $H < 1/2$, $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$ and $X_0 \in \mathbb{R}$. Assume

$$\beta > -\frac{1}{2H} + 1 \text{ and } \beta - \frac{1}{p} \geq -\frac{1}{2H} + 1. \tag{2.6}$$

Then,

- (a) there exists a strong solution X to (1.1) such that $X - B \in \mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^m)$ for any $m \geq 2$;
- (b) pathwise uniqueness holds in the class of all solutions X such that $X - B \in \mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^2)$;
- (c) for any sequence $(b^n)_{n \in \mathbb{N}}$ of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$, the corresponding sequence of strong solutions $(X^n)_{n \in \mathbb{N}}$ to (1.1) with drift b^n converges in probability to the unique strong solution for which $X - B \in \mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^2)$. In particular X is independent of the chosen sequence of approximations.
- (d) if b is a measure, there exists a pathwise unique strong solution to (1.1).

Remark 2.11. • For b being a finite measure (hence in \mathcal{B}_1^0), Theorem 2.10 gives existence of a unique strong solution to (1.1) for $H \leq 1/4$ and Theorem 2.8 gives existence of a weak solution to (1.1) for $H < 1/3$.

Such a finite measure b is also in \mathcal{B}_∞^{-1} . In this space, the existence of a unique strong solution was shown in [22] for $H < 1/4$ (elaborating on the path-by-path result of [8]). Hence in this case, Theorem 2.10 extends this result to $H = 1/4$.

- In the Brownian motion case, Theorem 2.8 gives existence of weak solutions for $b \in \mathcal{B}_p^\beta$ when $\beta - 1/p > -1/2$. In this regime strong existence and pathwise uniqueness are already known by [5].
- Note that $H = 1/2$ is excluded from Theorem 2.10. Strong existence and uniqueness result are already known under our assumptions (see [31]).

The following diagrams display for which Besov-valued distributions b we have well-posedness for Equation (1.1). The black-hatched area and the turquoise area correspond to the result obtained in Theorem 2.5. The graphics visualize that the weak solution constructed in Theorem 2.5 is a solution that, in some cases, does not arise from the weak solution constructed in Theorem 2.8.

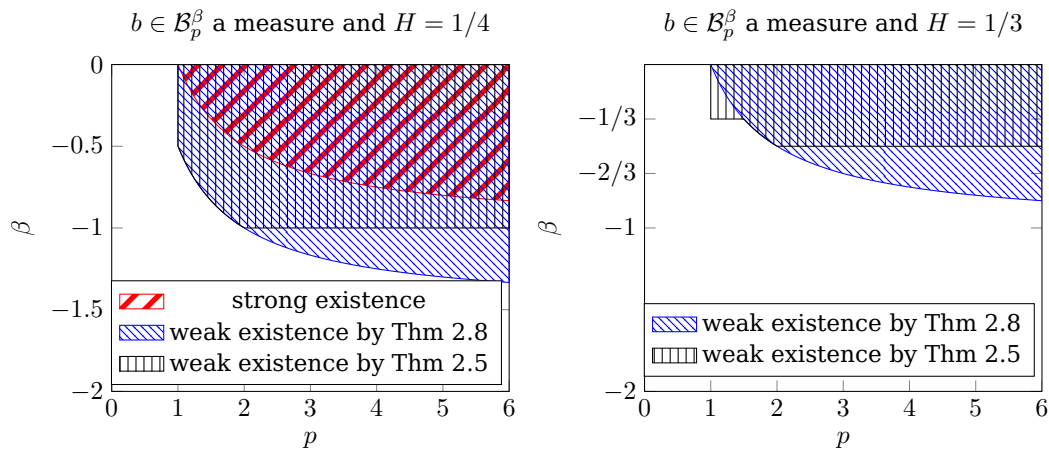


Figure 1: Existence (and uniqueness) for b a measure and H fixed.

2.3 Reformulation as a nonlinear Young equation

Let $w \in \mathcal{C}_{[0,T]}$. Rewriting Equation (1.3) with $Z \equiv w$ and $\tilde{X}_t = X_t - w_t$, we formally obtain

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t b(\tilde{X}_r + w_r) dr. \tag{2.7}$$

For a bounded measurable function b , we define the averaging operator T^w by

$$T_t^w b(x) := \int_0^t b(x + w_r) dr, \text{ for } (t, x) \in [0, T] \times \mathbb{R}. \tag{2.8}$$

Assuming that w has a local time L and using the occupation time formula (1.6), it comes that

$$T_{s,t}^w b(x) = \int_s^t b(x + w_r) dr = \int_{\mathbb{R}} b(x + z) L_{s,t}(z) dz = b * \check{L}_{s,t}(x), \forall x \in \mathbb{R}, \tag{2.9}$$

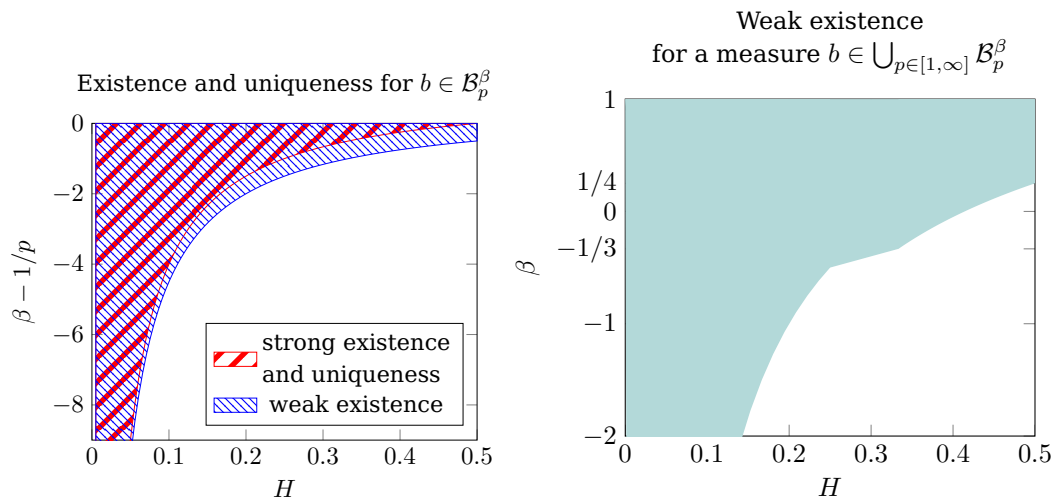


Figure 2: Existence (and uniqueness) for variable H .

where $\check{L}_{s,t}(x) = L_{s,t}(-x)$. This operator and its connection to the local time was already considered in [8]. In view of the expression of $T^w b$ as a convolution, one can expect that for w with a sufficiently regular local time, the definition of $T^w b$ will extend to b with lower regularity (we will consider suitable Besov spaces, see Sect. 4). This idea was exploited in [28], in the case of noises with infinitely differentiable local times.

Using (2.7), we get that for b continuous and bounded, $\tilde{X} \in \mathcal{C}_{[0,T]}$ and a sequence of partitions $\{t_i^n\}_{i=1}^{N_n}$ of $[0, t]$ with mesh size converging to 0,

$$\begin{aligned} \int_0^t b(\tilde{X}_r + w_r) dr &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n-1} \int_{t_i^n}^{t_{i+1}^n} b(\tilde{X}_{t_i^n} + w_r) dr \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n-1} T_{t_i^n, t_{i+1}^n}^w b(\tilde{X}_{t_i^n}) \\ &= \int_0^t T_{dr}^w b(\tilde{X}_r), \end{aligned} \tag{2.10}$$

where the final equality is only formal at this point. We give a rigorous definition of this integral in Sect. 3 and call it a nonlinear Young integral. This also suggests to rewrite Equation (1.3) as a nonlinear Young integral equation. Thus we give another definition of a solution to (1.3) and (1.1). Combining the theory of nonlinear Young integrals from Sect. 3 and the extension of the averaging operator applied to distributions (see Sect. 4) will give sense to the following definition.

Definition 2.12. Let Z be a stochastic process. We call $X : \Omega \rightarrow \mathcal{C}_{[0,T]}$ a path-by-path solution to (1.3) if there exists a null-set \mathcal{N} , $\eta \in (0, 1]$ and $p, q \geq 1$ such that $1/p + \eta/q > 1$ and for any $\omega \notin \mathcal{N}$, $T^{Z(\omega)} b \in \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)$, $X(\omega) - Z(\omega) \in \mathcal{C}_{[0,T]}^{q\text{-var}}$ and

$$X_t(\omega) = X_0 + \int_0^t T_{dr}^{Z(\omega)} b(X_r(\omega) - Z_r(\omega)) + Z_t(\omega), \text{ for all } t \in [0, T]. \tag{2.11}$$

The assumption involving $\eta \in (0, 1]$ and $p, q \geq 1$ is simply the (sufficient) condition formulated in Theorem 3.1 that ensures the existence of the nonlinear Young integral. Note that in the above definition no measurability or adaptedness of X is required.

Theorem 2.14 gives conditions on the local time of the process Z such that there exists a path-by-path solution to (1.3). These conditions will be needed multiple times throughout the paper, hence we give them here.

Assumption 2.13. Let $\gamma, \eta \in (0, 1), \beta \in \mathbb{R}$ and $p \in [1, \infty]$. Let $b \in \mathcal{B}_p^\beta$ and $w \in \mathcal{C}_{[0,T]}$ with local time L .

(I) There exists $\tilde{p} \in [1, \infty]$ with $1/p + 1/\tilde{p} > 1$ and $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta+1/p+1/\tilde{p}-1})$;

(II) There exists $\tilde{p} \in [1, \infty]$ with $1/p + 1/\tilde{p} \leq 1$ and $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta})$.

Theorem 2.14. Let $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$. Let $b \in \mathcal{B}_p^\beta$ be a measure with $\beta \in \mathbb{R}, p \in [1, \infty]$. Let $Z : \Omega \rightarrow \mathcal{C}_{[0,T]}$ be a continuous stochastic process with a local time $L^Z : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$. Assume that there exists a null-set \mathcal{N} such that for any $\omega \notin \mathcal{N}, L^Z(\omega)$ satisfies (I) or (II). Then there exists a path-by-path solution to Equation (1.3).

Recall that Theorem 2.14 does not imply existence of a measurable/adapted solution. However in the case of fBm, using properties of its local time, additionally adaptedness of the path-by-path solution can be proven (see Theorem 2.5).

The following theorem provides a comparison between solutions constructed by approximation with a smooth drift and solutions in the nonlinear Young sense. More precisely, it shows that being a solution to (1.3) (i.e. for a noise Z) in the sense of Definition 2.12 implies being a solution in the sense of Definition 2.4. Under some regularity restrictions, the reverse implication holds as well. We rephrase (2.3) and (2.4) (without specifying the mode of convergence yet) for a noise Z instead of B :

$$X_t = X_0 + K_t + Z_t \text{ for all } t \in [0, T]; \tag{2.12}$$

$$\sup_{t \in [0, T]} \left| \int_0^t b^n(X_r) dr - K_t \right| \xrightarrow[n \rightarrow \infty]{} 0 \tag{2.13}$$

for every sequence $(b^n)_{n \in \mathbb{N}}$ of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$.

Theorem 2.15. Let $\gamma, \eta \in (0, 1)$ and $q \geq 1$ with $\gamma + \eta/q > 1, X_0 \in \mathbb{R}, p \in [1, \infty], \beta \in \mathbb{R}, b \in \mathcal{B}_p^\beta$ and $X : \Omega \rightarrow \mathcal{C}_{[0,T]}$. Let $Z : \Omega \rightarrow \mathcal{C}_{[0,T]}$ be a continuous stochastic process and $L^Z : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ its local time. Assume that there exists a null-set \mathcal{N} such that for any $\omega \notin \mathcal{N}, L^Z(\omega)$ satisfies (I) or (II).

(a) Assume that X is a path-by-path solution to (1.3) and that $X - Z \in \mathcal{C}_{[0,T]}^{q\text{-var}}$. Then for any sequence of smooth bounded functions $(b^n)_{n \in \mathbb{N}}$ that converges to b in $\mathcal{B}_p^{\beta-}, X \cdot(\omega) - Z \cdot(\omega) - X_0 = K \cdot(\omega) = \int_0^\cdot T_{dr}^{Z(\omega)} b(X_r(\omega) - Z_r(\omega))$ is the uniform limit of $\int_0^\cdot b^n(X_r(\omega)) dr$ for all $\omega \notin \mathcal{N}$ (i.e. (2.13) holds on the set of full measure \mathcal{N}^c).

(b) Assume that there exists a process $K : \Omega \rightarrow \mathcal{C}_{[0,T]}$ such that (2.12) and (2.13) hold, where the convergence in (2.13) is in probability. Assume further that a.s., $X(\omega) - Z(\omega) \in \mathcal{C}_{[0,T]}^{q\text{-var}}$. Then X is a path-by-path solution to (1.3).

Remark 2.16. In particular, statement (a) implies that if X is an \mathbb{F} -adapted path-by-path solution, it is a weak solution. Moreover, for the weak solution constructed in Theorem 2.5, the convergence in (2.4) holds on a set of full measure instead of convergence in probability.

2.4 Organisation of the proofs

In Sect. 3, we construct nonlinear Young integrals in p -variation via a classical sewing argument. In particular, we establish existence of solutions to a nonlinear Young

integral equation with monotone drift under milder regularity constraints than in the non-monotone case.

Then we rigorously rewrite Equation (1.1) as a nonlinear Young integral equation. The existence of a solution to the more general Equation (1.3), when Z has a sufficiently regular local time, is stated in Theorem 2.14 and proven in Sect. 4.2. We then establish that a solution to (1.1) in this nonlinear Young sense is also a solution in the sense of Definition 2.4 (see Theorem 2.15, which is proven in Sect. 4.2). For instance, when applied to the case of a fractional Brownian noise (after investigating the regularity properties of its local time), these theorems imply the existence of path-by-path solutions. However, in Theorem 2.5(a) one wants to prove the existence of solutions that are adapted to the filtration \mathbb{F} of the underlying filtered probability space. Theorems 2.14 and 2.15 cannot be applied directly and a modified approach is developed using the tightness of the averaging operator of the fBm (Lemma 4.11) and the continuity of the operator \mathcal{A} transforming a fBm to a Bm, see (2.1) and Lemma B.1. These arguments are given in Sect. 4.4 and lead to the proof of Theorem 2.5(a).

In Sect. 5, we use some new regularity estimates on conditional expectations of the fBm (Lemma 5.1) and the stochastic sewing Lemma with random control (see Lemma C.2) to establish that any weak solution X satisfies $X - B \in \mathcal{C}_{[0,T]}^\kappa(L^m)$ for any $\kappa \in (0, 1 + H(\beta - \frac{1}{p}) \wedge 0] \setminus \{1\}$ and $m \geq 2$ when b is a measure in \mathcal{B}_p^β . This proves Theorem 2.5(b) and Theorem 2.10(d).

In Sect. 6, in order to establish pathwise uniqueness of weak solutions to (1.1) (see Proposition 6.1), we adapt an approach developed recently for the stochastic heat equation with distributional drift, see [2]. This requires several regularity estimates on solutions which are derived from the crucial regularity Lemma C.3 and the stochastic sewing Lemma with critical exponent (Theorem 4.5 in [2]). The proof of Lemma C.3 relies on the stochastic sewing Lemma and the aforementioned regularity estimates on conditional expectations of the fBm (Lemma 5.1).

Theorem 2.8 is proven in Sect. 7 by an approximation of the drift with smooth bounded functions. The corresponding sequence of strong solutions will be shown to be tight and furthermore reveal a stability property, such that we can identify the limit as a solution to (1.1), where continuity of the operator linking fBm to Brownian motion (see Lemma B.1) is needed to prove adaptedness. This works thanks to *a priori* regularity estimates of solutions, see Lemma 7.2 and Lemma 7.3.

Strong existence will follow by a Yamada-Watanabe type argument, using Gyöngy-Krylov's lemma [26, Lem. 1.1] and requires the uniqueness result Proposition 6.1. The proofs of Theorems 2.8 and 2.10 are then completed in Sect. 7.3.

3 Nonlinear Young integrals and nonlinear Young equations in $\mathcal{C}^{p\text{-var}}$

3.1 Construction of nonlinear Young integrals and properties

Throughout this subsection, E and F denote arbitrary Banach spaces. Theorem 3.1 provides conditions for the existence of a nonlinear Young integral in terms of p -variations (rather than Hölder continuity as in [8, Th. 2.4] and [20]).

Theorem 3.1. *Let $\eta \in (0, 1]$ and $p, q \in [1, \infty)$ such that $\theta := 1/p + \eta/q > 1$. Let $A \in \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}_{E,loc}^\eta(F))$ and $x \in \mathcal{C}_{[0,T]}^{q\text{-var}}(E)$. Then for $(s, t) \in \Delta_{[0,T]}$ and any sequence of partitions $(\Pi_n)_n$ of $[s, t]$ with $\lim_n |\Pi_n| = 0$, the sum $\sum_{\Pi_n} A_{t_i, t_{i+1}}(x_{t_i})$ converges. Besides, the limit is independent of the sequence of partitions $(\Pi_n)_n$. We denote it by $\int_s^t A_{dr}(x_r)$ and call it the nonlinear Young integral with respect to A and x .*

In addition, there exists $C(\theta) > 0$ such that for $(s, t) \in \Delta_{[0,T]}$ and $R > 0$ with $\|x\|_\infty \leq R$,

one has

$$\left\| \int_s^t A_{dr}(x_r) - A_{s,t}(x_s) \right\|_F \leq C(\theta) [A]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} [x]_{\mathcal{C}_{[s,t]}^{q\text{-var}}}^\eta. \tag{3.1}$$

Proof. We apply the sewing lemma [19, Theorem 2.2 and Remark 2.3] formulated with controls. Let $0 \leq u \leq v \leq w \leq T$. Then, for $R > 0$ such that $\|x\|_\infty \leq R$,

$$\begin{aligned} |A_{u,w}(x_u) - A_{u,v}(x_u) - A_{v,w}(x_v)| &= |A_{v,w}(x_u) - A_{v,w}(x_v)| \\ &\leq [A_{v,w}]_{\mathcal{C}_{D_R}^\eta} [x]_{\mathcal{C}_{[u,w]}^{q\text{-var}}}^\eta \\ &\leq [A]_{\mathcal{C}_{[u,w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} [x]_{\mathcal{C}_{[u,w]}^{q\text{-var}}}^\eta, \end{aligned}$$

which gives the result as the last expression in the above defines a control raised to the power θ by [18, Exercise 1.9]. \square

As a corollary we obtain the following result.

Corollary 3.2. *Let $T > 0$. Let $p, q \in [1, \infty)$ and $\eta \in (0, 1]$ with $\theta := 1/p + \eta/q > 1$. Let $x \in \mathcal{C}_{[0,T]}^{q\text{-var}}$ and $A \in \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)$. If a sequence A^n converges to A in $\mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)$, then*

$$\sup_{t \in [0, T]} \left| \int_0^t (A^n - A)_{dr}(x_r) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. By (3.1), we have

$$\begin{aligned} \left| \int_0^t (A^n - A)_{dr}(x_r) \right| &\leq \left| \int_0^t (A^n - A)_{dr}(x_r) - (A^n - A)_{0,t}(x_0) \right| + |(A^n - A)_{0,t}(x_0)| \\ &\leq C(\theta) [A^n - A]_{\mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)} [x]_{\mathcal{C}_{[0,T]}^{q\text{-var}}}^\eta + [A^n - A]_{\mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)}. \end{aligned}$$

Taking the supremum over all $t \in [0, T]$ and letting n go to ∞ gives the result. \square

The next lemma establishes the sensitivity in the “ x ” variable of the nonlinear Young integral. See [20, Th. 2.7] for its counterpart in the Hölder setting.

Lemma 3.3. *Let $\eta \in (0, 1]$ and $p, q \in [1, \infty)$ with $\theta := 1/p + \eta/q > 1$. Let $A \in \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}_{E,loc}^\eta(F))$, $x, y \in \mathcal{C}_{[0,T]}^{q\text{-var}}(E)$. We denote $R = \text{Max}(\|x\|_\infty, \|y\|_\infty)$. Then, for $\delta \in (q(1 - \frac{1}{p}), \eta)$ and $(s, t) \in \Delta_{[0,T]}$, one has*

$$\begin{aligned} \left\| \int_s^t A_{dr}(x_r) - \int_s^t A_{dr}(y_r) \right\|_F &\leq C(\theta) [A]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} \left([x]_{\mathcal{C}_{[s,t]}^{q\text{-var}}} + [y]_{\mathcal{C}_{[s,t]}^{q\text{-var}}} \right)^\delta \|x - y\|_\infty^{\eta - \delta} \\ &\quad + \|A_{s,t}\|_{\mathcal{C}_{D_R}^\eta} \|x - y\|_\infty^\eta. \end{aligned} \tag{3.2}$$

Proof. For $(u, w) \in \Delta_{[0,T]}$, let

$$\Gamma_{u,w} := (A_{u,w}(x_u) - A_{u,w}(y_u)).$$

Again, we aim to apply the sewing lemma. For any $z \in \mathcal{C}_{[0,T]}^{q\text{-var}}(E)$, let $\varkappa_z(u, w)^\theta := [A]_{\mathcal{C}_{[u,w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} [z]_{\mathcal{C}_{[u,w]}^{q\text{-var}}}^\eta$. Furthermore, let

$$\hat{\varkappa}(u, w)^{\delta/q+p^{-1}} := (\varkappa_x(u, w) + \varkappa_y(u, w))^{\frac{\delta}{\eta}\theta} \left([A]_{\mathcal{C}_{[u,w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} \right)^{1 - \frac{\delta}{\eta}}.$$

By [18, Exercise 1.9] and by definition of a control, we get that $\varkappa_x, \varkappa_y, \varkappa_x + \varkappa_y$ and $\hat{\varkappa}$ are control functions.

Note that, for $v \in [u, w]$,

$$\begin{aligned} \|\Gamma_{u,w} - \Gamma_{u,v} - \Gamma_{v,w}\|_F &\leq \|A_{v,w}(x_v) - A_{v,w}(x_u)\|_F + \|A_{v,w}(y_v) - A_{v,w}(y_u)\|_F \\ &\leq [A]_{C_{[v,w]}^{p\text{-var}}(C_{D_R}^\eta)}([x]_{C_{[u,v]}^\eta}^{q\text{-var}} + [y]_{C_{[u,v]}^\eta}^{q\text{-var}}) \\ &\leq (\varkappa_x(u, w) + \varkappa_y(u, w))^\theta. \end{aligned}$$

Furthermore, we also have that

$$\begin{aligned} \|\Gamma_{u,w} - \Gamma_{u,v} - \Gamma_{v,w}\|_F &\leq \|A_{v,w}(x_v) - A_{v,w}(y_v)\|_F + \|A_{v,w}(x_u) - A_{v,w}(y_u)\|_F \\ &\leq 2[A]_{C_{[v,w]}^{p\text{-var}}(C_{D_R}^\eta)}\|x - y\|_\infty^\eta. \end{aligned}$$

It clearly holds true that $a \wedge b \leq a^\xi b^{1-\xi}$ for $a, b \geq 0$ and $\xi \in [0, 1]$. Hence, for $\xi = \delta/\eta \in (0, 1)$, it comes

$$\|\Gamma_{u,w} - \Gamma_{u,v} - \Gamma_{v,w}\|_F \leq C\hat{\varkappa}(u, w)^{\frac{\delta}{q} + p - 1} \|x - y\|_\infty^{\eta(1 - \frac{\delta}{\eta})}.$$

Applying the sewing lemma and the inequality

$$\|\Gamma_{u,w}\|_F \leq \|A_{u,w}\|_{C_{D_R}^\eta} \|x - y\|_\infty^\eta$$

gives (3.2). □

3.2 Solving nonlinear Young equations

We are now ready to state a result of existence of solutions to nonlinear Young integral equations with a positive drift. From now on we will work with real-valued functions and vector fields. The proof resembles the one in the Hölder setting, see [20, Th. 3.2]. However, there is one crucial step where the p -variation permits to take into account the nonnegative drift and which then allows to assume milder regularity conditions than in the Hölder setting.

Theorem 3.4. *Let $\eta \in (0, 1]$ and $p \geq 1$ with $1/p + \eta > 1$. Let $A \in C_{[0,T]}^{p\text{-var}}(C^\eta)$ with $A_{s,t}(y) \geq 0$ for all $y \in \mathbb{R}$ and all $(s, t) \in \Delta_{[0,T]}$. Then for any $x_0 \in \mathbb{R}$, there exists a solution $x \in C_{[0,T]}^{1\text{-var}}$ to the nonlinear Young equation*

$$x_t = x_0 + \int_0^t A_{dr}(x_r), \quad \forall t \in [0, T]. \tag{3.3}$$

Proof. Without loss of generality, let $T = 1$. Let $\theta := 1/p + \eta$. For $n \in \mathbb{N}$ and $0 \leq k \leq n$, let $t_k^n := k/n$, $\bar{x}_0^n := x_0$ and define recursively

$$\bar{x}_{k+1}^n = \bar{x}_k^n + A_{t_k^n, t_{k+1}^n}(\bar{x}_k^n).$$

We embed $(\bar{x}_k^n)_{k=0}^n$ into $C_{[0,1]}$ by setting

$$x_t^n = x_0 + \sum_{0 \leq k \leq \lfloor nt \rfloor} A_{t_k^n, t \wedge t_{k+1}^n}(\bar{x}_k^n),$$

which we write as follows

$$x_t^n = x_0 + \int_0^t A_{dr}(x_r^n) + \sum_{0 \leq k \leq \lfloor nt \rfloor} \left(\int_{t_k^n}^{t \wedge t_{k+1}^n} A_{dr}(x_{t_k^n}^n) - A_{dr}(x_r^n) \right). \tag{3.4}$$

Denote the sum in (3.4) by ψ_t^n . Let us introduce the control function

$$\varkappa^n(s, t) := \left([A]_{C_{[s,t]}^{p\text{-var}}(C^\eta)} [x^n]_{C_{[s,t]}^{1\text{-var}}}^\eta \right)^{\frac{1}{\theta}}.$$

Then using (3.1) and the superadditivity property (1.7) of a control,

$$\begin{aligned}
 |\psi_t^n| &= \left| \sum_{0 \leq k \leq \lfloor nt \rfloor} \left(\int_{t_k^n}^{t \wedge t_{k+1}^n} A_{dr}(x_{t_k^n}^n) - A_{dr}(x_r^n) \right) \right| \\
 &\leq C(\theta) \sum_{0 \leq k \leq \lfloor nt \rfloor} \varkappa^n(t_k^n, t \wedge t_{k+1}^n)^\theta \\
 &\leq C(\theta) \varkappa^n(0, t) \max_{0 \leq k \leq \lfloor nt \rfloor} \{ \varkappa^n(t_k^n, t \wedge t_{k+1}^n) \}^{\theta-1}.
 \end{aligned} \tag{3.5}$$

Let $0 \leq s \leq u \leq T = 1$. We note $i = \lfloor ns \rfloor$ and $i + j = \lfloor nu \rfloor$. We assume that $j > 0$ as the case $j = 0$ is simpler. Then

$$\begin{aligned}
 |\psi_{s,u}^n| &\leq \left| \int_s^{t_{i+1}^n} A_{dr}(x_{t_i^n}^n) - A_{dr}(x_r^n) \right| + \sum_{k=1}^j \left| \int_{t_{i+k}^n}^{u \wedge t_{i+k+1}^n} A_{dr}(x_{t_{i+k}^n}^n) - A_{dr}(x_r^n) \right| \\
 &\leq \left| \int_s^{t_{i+1}^n} A_{dr}(x_s^n) - A_{dr}(x_r^n) \right| + |A_{s,t_{i+1}^n}(x_s^n) - A_{s,t_{i+1}^n}(x_{t_i^n}^n)| \\
 &\quad + \sum_{k=1}^j \left| \int_{t_{i+k}^n}^{u \wedge t_{i+k+1}^n} A_{dr}(x_{t_{i+k}^n}^n) - A_{dr}(x_r^n) \right|,
 \end{aligned}$$

so by the estimate (3.1) it comes

$$|\psi_{s,u}^n| \leq C(\theta) \left(\varkappa^n(s, u)^\theta + [A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}(\mathcal{C}^\eta)} [x^n]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{1\text{-var}}}^\eta \right). \tag{3.6}$$

We now look for a bound on the 1-variation norm of x^n . Due to the non-negativity assumption on A , x^n is non-decreasing. Hence

$$[x^n]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{1\text{-var}}} = |x_{t_i^n}^n - x_{t_{i+1}^n}^n| = |A_{t_i^n, t_{i+1}^n}(x_{t_i^n}^n)| \leq [A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}(\mathcal{C}^\eta)}.$$

Then in view of (3.4), (3.6) and applying (3.1) to $\int_s^u A_{dr}(x_r^n)$, we get

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq C(\theta) \left((|A_{s,u}(x_s^n)| + \varkappa^n(s, u)^\theta) + \left(\varkappa^n(s, u)^\theta + [A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}(\mathcal{C}^\eta)}^{1+\eta} \right) \right).$$

Let now $\varepsilon > 0$ and choose n such that $[A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}(\mathcal{C}^\eta)} < \varepsilon^{1/(1+\eta)}$ for all $i = 0, \dots, n - 1$. Then one obtains

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq 2C(\theta) \left([A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} + [A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} [x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} + \varepsilon \right).$$

Using that $a^\eta \leq 1 + a$ for $a \geq 0$ and $\eta \in [0, 1]$, it follows that

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq 2C(\theta) \left([A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} (1 + [x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}}) + \varepsilon \right).$$

Hence, for $s \leq u$ such that $2C(\theta)[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} < 1$, this leads to

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq 2C(\theta) \frac{[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} + \varepsilon}{1 - 2C(\theta)[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)}}, \tag{3.7}$$

which gives both uniform boundedness and equicontinuity. In view of Equation (3.5), the uniform boundedness of $[x^n]_{\mathcal{C}_{[0,t]}^{1\text{-var}}}$ combined with the continuity of $[A]_{\mathcal{C}_{[0,t]}^{p\text{-var}}(\mathcal{C}^\eta)}$ give

that ψ^n converges to 0 uniformly on $[0, 1]$. By the Arzelà-Ascoli Theorem, we deduce that x^n converges uniformly along a subsequence to some non-decreasing $x \in \mathcal{C}_{[0,1]}$. Without loss of generality, we still denote by x^n this subsequence. Then using the uniform boundedness of $[x^n]_{\mathcal{C}_{[0,1]}^{1\text{-var}}}$ and the uniform convergence of x^n to x , one gets from Lemma 3.3 that for any t ,

$$\int_0^t A_{dr}(x_r^n) \xrightarrow{n \rightarrow \infty} \int_0^t A_{dr}(x_r).$$

Hence, passing to the limit in (3.4), we finally obtain that x solves (3.3). □

Remark 3.5. Theorem 3.4 extends to mappings A which are only locally Hölder in space, giving existence possibly up to a blow-up time. Its proof can be done using typical localisation arguments.

Since we are interested in differential equations perturbed by noise, it is natural to look for an extension of Theorem 3.4 in case A is random, and to look for a measurable solution. Therefore we conclude this section with an extension of Theorem 3.4 for random A . We omit the proof as it is similar to the first part of the proof of Theorem 2.5 which is presented in Sect. 4.4 (i.e. showing tightness and then using Skorokhod’s representation Theorem to pass to an almost sure limit).

Corollary 3.6. *Let $\eta \in (0, 1]$ and $p \geq 1$ with $1/p + \eta > 1$. Let $A : \Omega \rightarrow \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)$ be a random variable such that almost surely, $A_{s,t}(y) \geq 0$ for all $y \in \mathbb{R}$ and $(s, t) \in \Delta_{[0,T]}$. Furthermore, assume that for any $\lambda > 0$,*

$$\lim_{\delta \rightarrow 0} \mathbb{P}(\Omega_{\delta,\lambda}) = 1, \text{ where } \Omega_{\delta,\lambda} := \{\omega : \sup_{|t-s| < \delta} [A]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(\mathcal{C}^\eta)} < \lambda\}. \tag{3.8}$$

Then for any $Y_0 \in \mathbb{R}$, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{E}}, \tilde{\mathbb{P}})$, a measurable map \tilde{A} which satisfies $\text{Law}(\tilde{A}) = \text{Law}(A)$ and a measurable map $Y : \tilde{\Omega} \rightarrow \mathcal{C}_{[0,T]}^{1\text{-var}}$ such that almost surely,

$$Y_t = Y_0 + \int_0^t \tilde{A}_{dr}(Y_r), \forall t \in [0, T].$$

4 Existence of weak solutions with nonnegative drift

We prove Theorem 2.5, Corollary 2.6, Theorem 2.14 and Theorem 2.15 in this section. First, we extend the averaging operator defined in (2.8) to distributions b in Besov spaces. Obtaining some Hölder regularity properties for this object allows to prove Theorem 2.15, which states roughly that solutions to (1.1) in the sense of Definition 2.4 are equivalent to solutions in the sense of Definition 2.12. Hence, we then only consider solutions in the sense of Definition 2.12.

The remaining subsections are dedicated to the proof of Theorem 2.5, using some fine results on the joint regularity in time and space of the local time of fractional Brownian motion and the results on nonlinear Young equations from Sect. 3.

4.1 Definition and properties of the averaging operator

In this section we give the definition of the averaging operator $T^w b$ for distributional b and $w \in \mathcal{C}_{[0,T]}$, extending the construction of Sect. 2.3. Note that this was already done in a very general setup in [21, Sect. 3.1]. We take a less general approach by directly mollifying b , which is in line with the definition in [21] by Lemma 3.9 therein.

Definition 4.1. *Let $w \in \mathcal{C}_{[0,T]}$. Let $\beta \in \mathbb{R}$ and $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$. The averaging operator is defined by*

$$T_t^w b(x) := \lim_{n \rightarrow \infty} T_t^w b^n(x),$$

if the limit exists for any sequence (b^n) of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ and is independent of the choice of the sequence.

Lemma 4.2. Consider $\beta, \tilde{\beta} \in \mathbb{R}$ with $\beta - \tilde{\beta} \in (0, 1]$, $\gamma \in (0, 1]$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$ and $w \in \mathcal{C}_{[0,T]}$. Assume that w has a local time $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{p'}^{-\tilde{\beta}})$. For the above choice of b and w , $T^w b$ is well-defined in $\mathcal{C}^\gamma(\mathcal{C}^\rho)$ for any $\rho \in (0, \beta - \tilde{\beta})$.

Remark 4.3. Recall from [4, Prop. 2.76] that for $\eta \in \mathbb{R}$, $p, p', q, q' \in [1, \infty]$ with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, there is a continuous bilinear functional $\langle \cdot, \cdot \rangle : \mathcal{B}_{p,q}^\eta \times \mathcal{B}_{p',q'}^{-\eta} \rightarrow \mathbb{R}$ extending the L^2 inner product.

Proof of Lemma 4.2. Let $\varepsilon = \beta - \tilde{\beta}$. After an embedding of Besov spaces (see Remark 1.2), we know that $b \in \mathcal{B}_p^{\beta-\varepsilon/2}$ and $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{p',1}^{-\tilde{\beta}-\varepsilon/2})$. Let $(b^n)_{n \in \mathbb{N}}$ be any sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. By the convolutional representation (2.9) and Remark 4.3, we have that

$$T_t^w b^n(x) = \langle b^n, L_t(\cdot - x) \rangle.$$

Using the continuity of the bilinear functional $\langle \cdot, \cdot \rangle$, we get, for $n, m \in \mathbb{N}$,

$$\|T^w b^n(\cdot) - T^w b^m(\cdot)\|_\infty \leq C \|L\|_{\mathcal{C}^\gamma(\mathcal{B}_{p',1}^{-\tilde{\beta}-\varepsilon/2})} \|b^n - b^m\|_{\mathcal{B}_p^{\beta-\varepsilon/2}}.$$

Therefore $T^w b^n$ forms a Cauchy sequence and is uniformly convergent. Hence, $T^w b$ is well defined and is easily seen to be independent of the approximating sequence.

Let $\rho \in (0, \beta - \tilde{\beta})$. In order to show that $T^w b$ actually has the required regularity, we have to check that

$$\begin{aligned} \|T^w b\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^\rho)} &= \sup_{s \neq t, x \neq y} \frac{|T_{s,t}^w b(y) - T_{s,t}^w b(x)|}{|t - s|^\gamma |x - y|^\rho} + \sup_{s \neq t} \sup_x \frac{|T_{s,t}^w b(x)|}{|t - s|^\gamma} \\ &\quad + \sup_t \sup_{x \neq y} \frac{|T_t^w b(y) - T_t^w b(x)|}{|x - y|^\rho} + \sup_t \sup_x |T_t^w b(x)| \end{aligned} \tag{4.1}$$

is finite. Fix $s \neq t$ and $x \neq y$. For any $n \in \mathbb{N}$,

$$|T_{s,t}^w b(x) - T_{s,t}^w b(y)| \leq 2 \|T_{s,t}^w b(\cdot) - T_{s,t}^w b^n(\cdot)\|_\infty + |T_{s,t}^w b^n(x) - T_{s,t}^w b^n(y)|.$$

Choosing $n = n(s, t, x, y)$ large enough, we have

$$\|T_{s,t}^w b(\cdot) - T_{s,t}^w b^n(\cdot)\|_\infty \leq |t - s|^\gamma |x - y|^{\beta - \tilde{\beta}}.$$

Moreover, using continuity of the bilinear form, an embedding of Besov spaces and Lemma A.5(b), we have, for $\varepsilon = \beta - \tilde{\beta} - \rho > 0$,

$$\begin{aligned} |T_{s,t}^w b^n(x) - T_{s,t}^w b^n(y)| &\leq C \|b^n\|_{\mathcal{B}_p^{\beta-\varepsilon/2}} \|L_{s,t}(\cdot - x) - L_{s,t}(\cdot - y)\|_{\mathcal{B}_{p'}^{-\beta+\varepsilon}} \\ &\leq C |t - s|^\gamma |x - y|^\rho \|b\|_{\mathcal{B}_p^\beta} \|L\|_{\mathcal{C}^\gamma(\mathcal{B}_{p'}^{-\tilde{\beta}})}. \end{aligned}$$

As the other terms in (4.1) can be controlled similarly, the result follows. \square

Lemma 4.4. Let $p, \tilde{p} \in [1, \infty]$, $\beta \in \mathbb{R}$, $\eta, \gamma \in (0, 1)$ and $b \in \mathcal{B}_p^\beta$. Assume that the local time L of w satisfies Assumption 2.13 (I) or (II). Then, for any $\varepsilon \in (0, \eta)$, $T^w b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})$. Moreover, for any bounded open interval \mathcal{I} that contains the (compact) support of L , there exists a constant $C_{\mathcal{I}}$ such that:

- If (I) holds, then $[T^w b]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} [L]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta+1/p+1/\tilde{p}-1})}$ and $\|T^w b\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} \|L\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta+1/p+1/\tilde{p}-1})}$;

- If (II) holds, then $[T^w b]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} [L]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_p^{-\beta+\eta})}$ and $\|T^w b\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} \|L\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_p^{-\beta+\eta})}$.

Remark 4.5. In particular Lemma 4.4 shows the following: Assume (I) or (II) holds for $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$. Then $T^w b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}})$ for $\tilde{\eta} \in (0, \eta)$ with $\gamma + \tilde{\eta} > 1$.

Proof. The proof follows along the same lines as Lemma 4.2, making use of several Besov embeddings, including $\mathcal{B}_p^{-\beta+\eta+1/p+1/\bar{p}-1}(\mathcal{I}) \hookrightarrow \mathcal{B}_{p',1}^{-\beta+\eta-\varepsilon}(\mathcal{I})$ (see Remark 1.3). \square

4.2 Path-by-path solutions: Existence and comparison of solutions

In this section we prove Theorem 2.14 on the existence of path-by-path solutions to Equation (1.3) and Theorem 2.15 on the comparison of solutions.

Proof of Theorem 2.14. The idea of the proof is to identify a set of full measure on which $T^Z b$ is sufficiently regular and has nonnegative increments. Then, for ω in this set, we can apply the (deterministic) theory of nonlinear Young integral equations developed in Sect. 3. In particular, for any such ω , we use Theorem 3.4 to pick a solution. As the proof of Theorem 3.4 is non-constructive (it relies on Arzelà-Ascoli’s Theorem), the axiom of choice is needed to pick a solution simultaneously for all such ω . For ω outside of this full-measure set, we can define the solution to be identically 0. Note that this construction is indeed done in a path-by-path sense so that the solution solves (1.3) on a set of full measure. However, *a priori* there is no reason why the constructed solution should be adapted.

Let $0 \leq s < t$. Provided that

$$T^Z b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}}) \text{ for } \gamma, \tilde{\eta} \in (0, 1) \text{ with } \gamma + \tilde{\eta} > 1 \text{ and} \tag{4.2}$$

$$T_{s,t}^Z b(x) \geq 0 \text{ for all } x \in \mathbb{R} \tag{4.3}$$

hold a.s., using $\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}}) \subset \mathcal{C}_{[0,T]}^{1/\gamma\text{-var}}(\mathcal{C}^{\tilde{\eta}})$, Theorem 3.4 will give a solution.

Under the assumptions of Theorem 2.14, Lemma 4.4 ensures that (4.2) holds. To see that (4.3) holds, let $\varepsilon \in (0, \eta)$ and $\delta := \eta - \varepsilon$. Both under Assumption 2.13 (I) or (II), we have that $L_{s,t}^Z \in \mathcal{B}_{p',1}^{-\beta+\delta}$ and $L_{s,t}^Z \in \mathcal{B}_{p'}^{-\beta+2\delta/3}$ by the Besov embeddings in Remark 1.2 and Remark 1.3. By Remark 2.2, the sequence of nonnegative function $\phi_n^x := G_{1/n} L_{s,t}^Z(\cdot - x)$ converges to $L_{s,t}^Z(\cdot - x)$ in $\mathcal{B}_{p'}^{(-\beta+2\delta/3)-}$. Hence

$$\lim_n \|L_{s,t}^Z(\cdot - x) - \phi_n^x\|_{\mathcal{B}_{p',1}^{-\beta}} \leq C \lim_n \|L_{s,t}^Z(\cdot - x) - \phi_n^x\|_{\mathcal{B}_{p'}^{-\beta+\delta/3}} = 0. \tag{4.4}$$

Then we get, for any sequence of nonnegative smooth bounded functions b^m converging to b in $\mathcal{B}_p^{\beta-}$,

$$\begin{aligned} T_{s,t}^Z b(x) &= \lim_{m \rightarrow \infty} T_{s,t}^Z b^m(x) \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle b^m, \phi_n^x \rangle, \end{aligned}$$

where the first line holds true by definition and the second line by the continuity of $\langle \cdot, \cdot \rangle$ and (4.4). The inequality (4.3) now follows from the fact that $\langle b^m, \phi_n^x \rangle \geq 0$, for any n, m and x . \square

Proof of Theorem 2.15. (a): Let $\varepsilon \in (0, \eta)$ such that $\gamma + (\eta - \varepsilon)/q > 1$. As Assumption 2.13 (I) or (II) is fulfilled, we know by Lemma 4.4 that $T^Z b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})$ on \mathcal{N}^c

(which we do not mention in the rest of the proof, although all equalities and convergences happen on this set). Then, by the definition of the nonlinear Young integral of $T^Z b^n$ in Theorem 3.1, we get that Equality (2.10) does hold for b^n :

$$\int_0^t b^n(X_r)dr = \int_0^t T_{dr}^Z b^n(X_r - Z_r). \tag{4.5}$$

By Lemma 4.4, we get that $T^Z b^n$ converges to $T^Z b$ in $C_{[0,T]}^\gamma(C^{n-\varepsilon})$. Hence, by Corollary 3.2, we obtain that

$$\sup_{t \in [0,T]} \left| \int_0^t b^n(X_r)dr - \int_0^t T_{dr}^Z b(X_r - Z_r) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Therefore the convergence in the statement holds for $K_t := \int_0^t T_{dr}^Z b(X_r - Z_r)$.

(b): By (2.13), we know that for some subsequence $(n_k)_{k \in \mathbb{N}}$,

$$\sup_{t \in [0,T]} \left| \int_0^t b^{n_k}(X_r)dr - K_t \right| \longrightarrow 0 \text{ a.s.}$$

We have again that (4.5) holds true and we deduce that

$$\sup_{t \in [0,T]} \left| \int_0^t T_{dr}^Z b^{n_k}(X_r - Z_r) - K_t \right| \longrightarrow 0 \text{ a.s.}$$

By Lemma 4.4 and Corollary 3.2, we get that almost surely, $K_t = \int_0^t T_{dr}^Z b(X_r - Z_r)$ for all $t \in [0, T]$. Equation (2.11) now follows from (2.12). \square

4.3 Joint regularity of the local time of the fractional Brownian motion

In the rest of this section, L denotes the local time of a one-dimensional fBm B and T^B denotes the averaging operator associated to it, as constructed in Sect. 4.1.

First, we recall Theorem 3.1 from [30]. This result is stated in [30] for a compact hypercube with side length equal to one. By dilatation, the result also holds for an arbitrary large hyperrectangle.

Theorem 4.6 (Th. 3.1 in [30]). *Consider the rectangle $\mathcal{R} = [m_x, M_x] \times [m_y, M_y]$ for some $m_x < M_x$ and $m_y < M_y$. Let $Y : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous stochastic process. Suppose that for $\gamma, \beta > 0, \alpha > 1$ and all $(x_1, y_1), (x_2, y_2) \in \mathcal{R}$,*

$$\mathbb{E} [|Y(x_2, y_2) - Y(x_2, y_1) - Y(x_1, y_2) + Y(x_1, y_1)|^\alpha] \leq K |x_2 - x_1|^{1+\gamma} |y_2 - y_1|^{1+\beta}.$$

Then for every $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 \alpha < \gamma$ and $0 < \varepsilon_2 \alpha < \beta$, there exists a random variable ρ with $\mathbb{E}[\rho^\alpha] \leq K$ and a constant $C = C(\mathcal{R}) > 0$ such that almost surely,

$$|Y(x_2, y_2) - Y(x_2, y_1) - Y(x_1, y_2) + Y(x_1, y_1)| \leq C \rho |x_2 - x_1|^{\gamma/\alpha - \varepsilon_1} |y_2 - y_1|^{\beta/\alpha - \varepsilon_2},$$

for all $(x_1, y_1), (x_2, y_2) \in \mathcal{R}$.

Remark 4.7. Let $T > 0$ and $n \in \mathbb{N}$. By Lemma 8.12 in [45], for any $n \geq 1$ there exists a constant $C > 0$ such that for any $(x, y) \in \mathbb{R}^2, (s, t) \in \Delta_{[0,T]}$ and $0 < \bar{\beta} < (1/(2H) - 1/2) \wedge 1$, the local time L of a one-dimensional fBm fulfills

$$\mathbb{E} [|L_{s,t}(y) - L_{s,t}(x)|^n] \leq C |t - s|^{n(1-H(1+\bar{\beta}))} |x - y|^{n\bar{\beta}}. \tag{4.6}$$

Lemma 4.8. *Let $0 < \bar{\beta} < (1/(2H) - 1/2) \wedge 1$ and $0 < \gamma < 1 - H(1 + \bar{\beta})$. Then, almost surely, $L \in C_{[0,T]}^\gamma(C^{\bar{\beta}})$. Additionally for any $n \in \mathbb{N}$ there exists a random variable ρ with $\mathbb{E}[\rho^n] < \infty$ such that for any $M > 0, (s, t) \in \Delta_{[0,T]}$ and x, y with $|x|, |y| \leq M$, there exists a constant $C_M > 0$ such that*

$$|L_{s,t}(x) - L_{s,t}(y)| \leq C_M \rho |t - s|^\gamma |x - y|^{\bar{\beta}}. \tag{4.7}$$

Proof. Note that by [24, Th. 26.1] we can assume L to be jointly continuous in (t, x) . Then choosing n large enough in (4.6) and by Theorem 4.6, there exists a random variable ρ with finite n -th moment such that, a.s., for $(s, t) \in \Delta_{[0, T]}$ and x, y with $|x|, |y| \leq M$,

$$|L_{s,t}(x) - L_{s,t}(y)| \leq C_M \rho |t - s|^\gamma |x - y|^{\tilde{\beta}}.$$

As L is a.s. compactly supported, after exhausting $[0, T] \times \mathbb{R}$ with an increasing sequence of compacts $[-M_i, M_i]$, it follows that $L \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{\tilde{\beta}})$ a.s. \square

Note that Lemma 4.8 does not give differentiability in space for the local time of a fBm, even in case of a small Hurst parameter. It is also possible to get time-space Sobolev regularity of L by following the methodology of [28], and in particular to obtain differentiability for small enough H . This is used in proving Theorem 2.5 under Assumption (iii).

Remark 4.9. With the same parameters as in the previous lemma, if $b \in \mathcal{B}_p^\beta$ for $\beta \in (-\tilde{\beta}, 1 - \tilde{\beta})$ and $p \in [1, \infty]$, then Lemma 4.4 and Lemma A.7 imply that $T^B b \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{\tilde{\beta} + \beta - \varepsilon})$ almost surely, for any $\varepsilon \in (0, \tilde{\beta} + \beta)$.

4.4 Existence of a weak solution

We first show in Proposition 4.10 that the assumptions of Theorem 2.5 imply that either Assumption (I) or Assumption (II) holds. In particular, we get existence of a path-by-path solution by Theorem 2.14. Then we observe, by “randomizing” the Euler scheme, that it is actually possible to construct a weak solution (i.e. adapted solution).

Proposition 4.10. *Let β, p, b be as in Theorem 2.5.*

- (a) *Assume that (i) or (ii) from Theorem 2.5 holds. Then, a.s., L satisfies Assumption 2.13 (II) for $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$, $0 < -\beta + \eta < 1$ and $\tilde{p} = \infty$.*
- (b) *Assume that (iii) from Theorem 2.5 holds. Then, a.s., L satisfies Assumption 2.13 (II) for $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$, $\eta < (\frac{1}{2H} + \beta - 1/2)$ and $\tilde{p} = 2$.*

Hence, by Theorem 2.14, whenever one of the conditions (i)–(iii) from Theorem 2.5 holds, there exists a path-by-path solution to (1.3).

Proof of Proposition 4.10. It is used multiple times throughout the proof that we can also consider b to be in $\mathcal{B}_p^{\tilde{\beta}}$ for any $\tilde{\beta} < \beta$ by an embedding.

(a): Assume that (i) holds. W.l.o.g. assume that $\beta \in (1 + \frac{H}{2} - \frac{1}{2H}, \frac{3}{2} - \frac{1}{2H})$. In view of Lemma 4.8 and the assumption $H \geq \frac{1}{3}$, $L \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{-\beta + \eta})$ for $\eta \in \mathbb{R}$ such that $0 < -\beta + \eta < \frac{1}{2H} - \frac{1}{2}$ and $0 < \gamma < 1 - H(1 - \beta + \eta)$. Thus Assumption (II) is fulfilled for $\tilde{p} = \infty$. Choose $\eta = \beta + \frac{1}{2H} - \frac{1}{2} - \varepsilon$ and $\gamma = 1 - H(\frac{1}{2} - \varepsilon + \frac{1}{2H}) - \varepsilon$. Then for small enough ε one gets $\gamma \in (0, 1)$, $\eta \in (0 \vee \beta, 1)$ and $\gamma + \eta > 1$.

Assume that (ii) holds. W.l.o.g. assume that $\beta \in (2H - 1, 0)$. In view of Lemma 4.8 and the assumption $H < \frac{1}{3}$, we have that $L \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{-\beta + \eta})$ for $\eta \in \mathbb{R}$ such that $0 < -\beta + \eta < 1$ and $0 < \gamma < 1 - H(1 - \beta + \eta)$. Thus Assumption (II) is fulfilled for $\tilde{p} = \infty$. Choose $\eta = \beta + 1 - \varepsilon$ and $\gamma = 1 - H(2 - \varepsilon) - \varepsilon$. Then again, one gets $\gamma \in (0, 1)$, $\eta \in (0 \vee \beta, 1)$ and $\gamma + \eta > 1$ for small enough ε .

(b): Assume that (iii) holds. By Theorem 3.1 in [28], we know that $L \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{B}_{2,2}^\lambda)$ almost surely for $\lambda < \frac{1}{2H} - 1/2$ and $0 < \gamma < 1 - H(\lambda + 1/2)$.¹ Hence, after a Besov space embedding (see Remark 1.3), Assumption (II) is fulfilled for $\tilde{p} = 2$, $0 < \eta < (\frac{1}{2H} - 1/2 + \beta) \wedge 1$ and $0 < \gamma < (1 - H(\eta - \beta + 1/2)) \wedge 1$. The assumption $\beta > 1 - \frac{1}{2H}$ ensures that we can choose η and γ such that $\eta + \gamma > 1$. \square

¹Actually in [28] they use a Bessel space instead of $\mathcal{B}_{2,2}^\lambda$, but by Proposition 2.(iii) on page 47 and the Theorem on page 88 in [43], this is equivalent.

To construct weak solutions, we will proceed with an approximation by an Euler scheme, similarly to the proof of Theorem 3.4. Although this time, we need the following lemma, which gives tightness and is a crucial step in showing adaptedness by an argument using Skorokhod’s representation Theorem.

Lemma 4.11. *Assume that one of the assumptions (i)–(iii) in Theorem 2.5 holds. Then there exist $\gamma, \tilde{\eta} \in (0, 1)$ with $\gamma + \tilde{\eta} > 1$ such that for any $\lambda > 0$,*

$$\lim_{\delta \rightarrow 0} \mathbb{P}(\Omega_{\delta, \lambda}) = 1, \text{ where } \Omega_{\delta, \lambda} := \left\{ \omega : \sup_{|t-s| < \delta} [T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma\text{-var}(\mathcal{C}^{\tilde{\eta}})}} < \lambda \right\}.$$

Proof. In view of Proposition 4.10, L satisfies either Assumption 2.13 (I) or (II) for some $\gamma, \eta \in (0, 1)$ such that $\gamma + \eta > 1$. Hence by Lemma 4.4, there exists $\tilde{\eta} < \eta$ such that $\tilde{\eta} + \gamma > 1$ and $T^B b \in \mathcal{C}_{[0,T]}^{\gamma}(\mathcal{C}^{\tilde{\eta}})$.

For $M > 0$, define $L^M := \mathbb{1}_{\Omega_M} L$, where $\Omega_M = \{\omega : \sup_{t \in [0,T]} |B_t| \leq M\}$. Then,

$$\mathbb{P}(\Omega_{\delta, \lambda}^c) \leq \mathbb{P}\left(\left\{ \omega : \delta^{\gamma} \sup_{|t-s| < \delta} [T^B b]_{\mathcal{C}_{[s,t]}^{\gamma}(\mathcal{C}^{\tilde{\eta}})} \geq \lambda \right\} \cap \Omega_M\right) + \mathbb{P}(\Omega_M^c), \tag{4.8}$$

as $[T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma\text{-var}(\mathcal{C}^{\tilde{\eta}})}} \leq |t-s|^{\gamma} [T^B b]_{\mathcal{C}_{[s,t]}^{\gamma}(\mathcal{C}^{\tilde{\eta}})}$.

We will distinguish two cases, depending on whether (i) or (ii) is satisfied (first case), or (iii) is satisfied (second case).

First case. Assume that (i) or (ii) in Theorem 2.5 holds true. Then by Proposition 4.10(a), $L \in \mathcal{C}_{[0,T]}^{\gamma}(\mathcal{C}^{-\beta+\eta})$ with $0 < -\beta + \eta < 1$ and by Lemma 4.8 $\mathbb{E}[[L^M]_{\mathcal{C}_{[s,t]}^{\gamma}(\mathcal{C}^{-\beta+\eta})}^m]$ is finite for any $m \geq 1$. Moreover, by Lemma 4.4, we have on Ω_M that $[T^B b]_{\mathcal{C}_{[0,T]}^{\gamma}(\mathcal{C}^{\tilde{\eta}})} \leq C_M [L]_{\mathcal{C}_{[0,T]}^{\gamma}(\mathcal{C}^{-\beta+\eta})}$, where C_M depends on M (but not on the realisation ω). Hence, we can bound the right hand side of (4.8) from above by

$$\begin{aligned} \forall m \geq 1, \quad \mathbb{P}\left(\left\{ \omega : \delta^{\gamma} \sup_{|t-s| < \delta} C_M [L^M]_{\mathcal{C}_{[s,t]}^{\gamma}(\mathcal{C}^{-\beta+\eta})} \geq \lambda \right\}\right) + \mathbb{P}(\Omega_M^c) \\ \leq C_M^m \lambda^{-m} \delta^{m\gamma} \mathbb{E}[[L^M]_{\mathcal{C}_{[0,T]}^{\gamma}(\mathcal{C}^{-\beta+\eta})}^m] + \mathbb{P}(\Omega_M^c). \\ \leq C(m, M) \lambda^{-m} \delta^{m\gamma} + \mathbb{P}(\Omega_M^c). \end{aligned} \tag{4.9}$$

Let $\varepsilon > 0$. By Fernique’s Theorem, $\|B\|_{\infty}$ has exponential moments and we can choose M such that $\mathbb{P}(\Omega_M^c) < \varepsilon/2$. Then δ can be chosen such that the other term in (4.9) can also be controlled by $\varepsilon/2$.

Second case. Now assume that (iii) in Theorem 2.5 holds. By Proposition 4.10(b), L satisfies Assumption 2.13 (II) with $\gamma < (1 - H(-\beta + \eta + 1/2)) \wedge 1$, $\tilde{p} = 2$ and $\eta < (\frac{1}{2H} - 1/2 + \beta) \wedge 1$. In view of Lemma 4.4 and an embedding of Besov spaces (see Remark 1.2), we have that on Ω_M ,

$$[T^B b]_{\mathcal{C}_{[0,T]}^{\gamma}(\mathcal{C}^{\tilde{\eta}})} \leq C_M [L]_{\mathcal{C}_{[0,T]}^{\gamma}(\mathcal{B}_{2,2}^{-\beta+\eta})} \tag{4.10}$$

for some constant $C_M > 0$ that depends on M but not on the realisation ω . Next, we use the chain of inequalities on page 12 in [28] setting λ' therein equal to $-\beta + \eta + 1/2 + \varepsilon$. To check the condition $\lambda' < 1/(2H)$ appearing in [28], note that $-\beta + \eta + 1/2 + \varepsilon < 1/(2H)$ for ε small enough. Hence, we get for any $m \geq 2$ and some $\tilde{\varepsilon} > 0$,

$$\mathbb{E}[\|L_{u,v}^M\|_{\mathcal{B}_{2,2}^{-\beta+\eta}}^m] \leq C |u-v|^{m(1-H(-\beta+\eta+1/2+\varepsilon))} \leq C |u-v|^{m(\gamma+\tilde{\varepsilon})}.$$

An application of Kolmogorov’s continuity Theorem for Banach-valued stochastic processes (see [41, Th. 4.3.2]) gives that $\mathbb{E}[[L^M]_{\mathcal{C}_{[0,T]}^{\gamma}(\mathcal{B}_{2,2}^{-\beta+\eta})}^m]$ is finite. Using (4.10), we deduce as in the first case (see (4.9)) that

$$\mathbb{P}(\Omega_{\delta, \lambda}^c) \leq C(m, M) \lambda^{-m} \delta^{m\gamma} + \mathbb{P}(\Omega_M^c),$$

for a constant $C(m, M)$ only depending on m and M . We can make the right-hand side arbitrarily small by choosing M large enough and δ small enough. \square

Proof of Theorem 2.5. Proof of (a): By Proposition 4.10, Assumption 2.13 (I) or (II) is fulfilled and thus there are some $\gamma \in (0, 1)$ and $\tilde{p} \in [1, \infty]$ such that $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^\xi)$ a.s., for some ξ , with its precise value given in Assumption 2.13 (I) or (II). Besides it follows by Remark 4.5 that $T^B b \in \mathcal{C}^\gamma(\mathcal{C}^{\tilde{\eta}})$ for some $\tilde{\eta} \in (0, 1)$ such that $\gamma + \tilde{\eta} > 1$.

W.l.o.g. let $T = 1$. First, we use the Euler scheme as in the proof of Theorem 3.4 in order to construct a measurable solution. Let X^n be the random counterpart of x^n in the proof of Theorem 3.4 for $A = T^B b$, $p = 1/\gamma$ and $\theta := \gamma + \tilde{\eta}$. Let $\nu > 0$. The computations done in the proof of Theorem 3.4 hold for almost every $\omega \in \Omega$ until Equation (3.7). Choose $\lambda \in (0, 1)$ such that $2\lambda C(\theta) < 1$ for $C(\theta)$ as in (3.7). Let δ such that $\mathbb{P}(\Omega_{\delta,\lambda}) \geq 1 - \nu$, which is possible by Lemma 4.11. Then one can choose $N = N(\delta)$ large enough so that for any $n \geq N$ and any $\omega \in \Omega_{\delta,\lambda}$,

$$\begin{aligned} [X^n]_{\mathcal{C}_{[s,t]}^{1-\text{var}}} &\leq 2C(\theta) \frac{[T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma\text{-var}}(\mathcal{C}^{\tilde{\eta}})} + \lambda}{1 - 2C(\theta)[T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma\text{-var}}(\mathcal{C}^{\tilde{\eta}})}} \\ &\leq 2C(\theta) \frac{2\lambda}{1 - 2C(\theta)\lambda}, \quad \forall (s, t) \in \Delta_{[0,1]} \text{ with } |t - s| < \delta. \end{aligned}$$

It follows that we can choose M sufficiently large so that for any $n \geq N$,

$$\mathbb{P}(\|X^n\|_\infty > M) \leq \mathbb{P}(\Omega_{\delta,\lambda} \cap \{\|X^n\|_\infty > M\}) + \mathbb{P}(\Omega_{\delta,\lambda}^c) < \nu.$$

Therefore the sequence $(\text{Law}(X^n))_{n \in \mathbb{N}}$ is tight in the space of continuous functions. Hence, along some subsequence that we do not relabel, (X^n, B, L) converges in law in $\mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^\xi)$ to some (X, \hat{B}, \hat{L}) . By Skorokhod's representation Theorem, there exists a sequence $(Y^n, B^n, L^{B^n})_{n \in \mathbb{N}}$ with $\text{Law}(Y^n, B^n, L^{B^n}) = \text{Law}(X^n, B, L)$ for all $n \in \mathbb{N}$, such that $(Y^n, B^n, L^{B^n})_{n \in \mathbb{N}}$ converges a.s. to some (Y, \hat{B}, \hat{L}) . To get that \hat{L} is the local time of \hat{B} , observe that for any bounded measurable function f ,

$$\begin{aligned} 1 &= \mathbb{P}\left(\left\{\omega : \forall t \in [0, T], \int_0^t f(B_r) dr = \int_{\mathbb{R}} f(x) L_t(x) dx\right\}\right) \\ &= \hat{\mathbb{P}}\left(\left\{\omega : \forall t \in [0, T], \int_0^t f(\hat{B}_r) dr = \int_{\mathbb{R}} f(x) \hat{L}_t(x) dx\right\}\right). \end{aligned} \quad (4.11)$$

As the local time of a fBm is characterised by the occupation time formula, we deduce that \hat{L} is the local time of \hat{B} . By (3.5), we have that for $T^B b(x) = \langle b, L^B(\cdot - x) \rangle$,

$$\left| X_t^n - X_0 - \int_0^t T_{dr}^B b(X_r^n) \right| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0.$$

Hence for $T^{B^n} b(x) = \langle b, L^{B^n}(\cdot - x) \rangle$,

$$\left| Y_t^n - X_0 - \int_0^t T_{dr}^{B^n} b(Y_r^n) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

and therefore a.s. along a subsequence, which we do not relabel. Hence, using that $\hat{L} \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^\xi)$, we get by Remark 4.5 and Lemma 3.3 that for $T^{\hat{B}} b(x) = \langle b, \hat{L}(\cdot - x) \rangle$,

$$Y_t = X_0 + \int_0^t T_{dr}^{\hat{B}} b(Y_r), \text{ for all } t \in [0, T] \text{ a.s.}$$

In order for Y to be a weak solution, it remains to show that Y is adapted to a filtration $\hat{\mathbb{F}}$ such that \hat{B} is an $\hat{\mathbb{F}}$ -fBm. First note that by construction X^n is adapted to \mathbb{F}^B . Hence, Y^n is \mathbb{F}^{B^n} measurable as

$$\text{Law}(Y^n, B^n, L^{B^n}) = \text{Law}(X^n, B, L).$$

Therefore B^n is an $\hat{\mathbb{F}}^n$ -fBm for $\hat{\mathcal{F}}_t^n := \sigma(Y_s^n, B_s^n, s \in [0, t])$. By definition this implies that, for $s < t$, $W_t^n - W_s^n = \mathcal{A}(B^n)_t - \mathcal{A}(B^n)_s$ is independent of $\hat{\mathcal{F}}_s^n$. After passing to the limit and using Lemma B.1 we infer that $W_t - W_s = \mathcal{A}(\hat{B})_t - \mathcal{A}(\hat{B})_s$ is independent of $\hat{\mathcal{F}}_s := \sigma(Y_r, \hat{B}_r, r \in [0, s])$. Hence, $\hat{B} = \bar{\mathcal{A}}(W)$ is an $\hat{\mathbb{F}}$ -fBm and therefore (Y, \hat{B}) is a weak solution as Y is clearly $\hat{\mathbb{F}}$ -adapted.

Proof of (b): this proof will be presented at the end of Sect. 5 as it follows from the stochastic sewing arguments developed in the next section. \square

Proof of Corollary 2.6. Any finite measure lies in \mathcal{B}_1^0 by similar computations as in Proposition 2.39 in [4]. Hence, if b is a measure, we can choose $\beta = 0$ in Theorem 2.5. For $H < 1/3$ condition (i) therein is clearly fulfilled and condition (ii) is fulfilled if $H^2 + 2H - 1 < 0$, giving existence of a weak solution for $H < \sqrt{2} - 1$. \square

5 Regularity of weak solutions

We first state Lemma 5.1 which establishes various regularity estimates on the conditional expectation of fractional Brownian motion. It is an extension to the fBm of [2, Lemma C.4] (which was for standard Brownian motion). It is used several times in the remainder of the paper and its proof is postponed to Appendix B.2. In particular, the proof of Lemma 5.1(d) relies on a variant of local nondeterminism of the fBm, see Lemma B.2. Note that Lemma 5.1(a) was already stated and proven in Proposition 3.6(iii) of [6].

Lemma 5.1. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and B be an \mathbb{F} -fBm. Let $\gamma < 0$ and $p \in [1, \infty]$. Let $d \in \mathbb{N}$, $(t_1, t_2) \in \Delta_{[0, T]}$ and $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded measurable function and Ξ be an \mathcal{F}_{t_1} -measurable \mathbb{R}^d -valued random variable. Assume that $\|f(\cdot, \Xi)\|_{C^1} < \infty$ almost surely. Then there exists a constant $C > 0$ such that*

$$(a) \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)] = \int_{\mathbb{R}} g_{\sigma_{t_1, t_2}^2}(x) f(\mathbb{E}^{t_1}[B_{t_2}] - x, \Xi) dx, \text{ also written } \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)] = G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi), \text{ where } g \text{ is the Gaussian density and } G \text{ is the Gaussian semigroup introduced in (1.4) and } \sigma_{t_1, t_2}^2 := \text{Var}(B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}]);$$

$$(b) |\mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]| \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} (t_2 - t_1)^{H(\gamma - 1/p)};$$

$$(c) \|f(B_{t_2}, \Xi) - \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]\|_{L^1} \leq C \| \|f(\cdot, \Xi)\|_{C^1} \|_{L^2} (t_2 - t_1)^H.$$

Furthermore, for $n \in [1, p]$, there exists a constant $C > 0$ such that for any \tilde{t} in the interval (t_1, t_2) ,

$$(d) \|\mathbb{E}^{\tilde{t}}[f(B_{t_2}, \Xi)]\|_{L^n} \leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t_2 - \tilde{t})^{H\gamma} (\tilde{t} - t_1)^{-\frac{1}{2p}} (t_2 - t_1)^{\frac{1-2H}{2p}}.$$

Remark 5.2. In this section we assume that $b \in \mathcal{B}_p^\beta$ for $p \in [1, \infty]$ and $\beta \in \mathbb{R}$ with $\beta - 1/p > -1/(2H)$. Note that this condition allows negative values of β for any $H < \frac{1}{2}$. In the proofs of this section, we consider the case $\beta < 0$ and $p \geq m$ for some $m \geq 2$. Indeed, it is always possible to come back to these cases in the following way: If $\beta \geq 0$, $p \in [1, \infty]$ and $m \geq 2$, there exist $\tilde{\beta} < 0$, $\tilde{p} \geq m$ fulfilling $\tilde{\beta} - 1/\tilde{p} > -1/(2H)$ such that $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}$. This can be seen using the embeddings $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_p^{\beta - (\frac{1}{p} - \frac{1}{\tilde{p}})}$ (see Remark 1.2) and $\mathcal{B}_p^s \hookrightarrow \mathcal{B}_{\tilde{p}}^s$ for $s > \tilde{s}$.

The following proposition ensures smoothness of $X - B$ for any solution X of (1.1).

Proposition 5.3. *Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$ with $0 > \beta - 1/p > -1/(2H)$. Suppose that $b \in \mathcal{B}_p^\beta$ is a measure. Then every weak solution X to (1.1) fulfills $X - B \in \mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^m)$, for any $m \geq 2$.*

Thus if X is a solution to (1.1) with b a finite measure, we deduce that $X - B \in \mathcal{C}_{[0,T]}^{1-H}(L^m)$, for any $m \geq 2$. It will imply Theorem 2.5(b) and is the main step in the proof of Theorem 2.10(d).

Proof of Proposition 5.3. Let X be a weak solution to (1.1) and $m \geq 2$. W.l.o.g. we assume that $X_0 = 0$. We choose a sequence $(b^k)_{k \in \mathbb{N}}$ of smooth nonnegative bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ with $\|b^k\|_{\mathcal{B}_p^\beta} \leq \|b\|_{\mathcal{B}_p^\beta}$ for all k . For $k \in \mathbb{N}$ and $t \in [0, T]$, let

$$K_t^k := \int_0^t b^k(X_r) dr = \int_0^t b^k(B_r + K_r) dr.$$

By Definition 2.4 of a weak solution, we know that K^k converges in probability to K with respect to $\|\cdot\|_\infty$ on $[0, T]$. Hence it also converges almost surely, up to passing to a subsequence (without loss of generality, we do not relabel K^k and assume it converges a.s.). By nonnegativity of b^k , K^k is monotone for all k , and therefore K is monotone as well. For any $(s, t) \in \Delta_{[0,T]}$, let

$$A_{s,t}^k := \int_s^t b^k(B_r + K_s) dr.$$

We now want to apply the stochastic sewing Lemma with random controls (see Lemma C.4) for $K^k = \mathcal{A}^k$. In order to see that all conditions of this theorem are fulfilled (i.e. (C.2) and (C.4)), we will show the following for a constant C that is independent of k and for any $u \in (s, t)$,

- (i) $\|A_{s,t}^k\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}$;
- (ii) $|\mathbb{E}^u[\delta A_{s,u,t}^k]| \leq C \|b\|_{\mathcal{B}_p^\beta} |K_u - K_s| (t-u)^{H(\beta-1/p-1)+1}$;
- (iii) Identifying K^k with the limit of $\sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n}^k$ along any sequence of partitions $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of $[0, t]$ with mesh converging to 0.

Notice that (i) gives (C.2) for $n = m$ and that $1 + H(\beta - 1/p) > 1/2$. Furthermore, (ii) gives condition (C.4) for $\alpha_1 = H(\beta - 1/p - 1) + 1 > 0$, $\beta_1 = 1$ and $\lambda(s, t) := K_t - K_s$, which is a random control function by monotonicity of K .

Assume, for the moment, that (i)–(ii)–(iii) hold true. Then by Lemma C.2, there exists a process D^k such that

$$|K_t^k - K_s^k - A_{s,t}^k| \leq C \|b\|_{\mathcal{B}_p^\beta} (K_t - K_s) (t-s)^{H(\beta-1/p-1)+1} + D_{s,t}^k, \tag{5.1}$$

with $\|D_{s,t}^k\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}$. Hence, by (i),

$$\|K_t^k - K_s^k\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} \|K_t - K_s\|_{L^m} (t-s)^{H(\beta-1/p-1)+1} + C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}.$$

Hence, after letting k go to ∞ in the previous inequality, we obtain that for $(s, t) \in \Delta_{[0,T]}$ such that $C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{H(\beta-1/p-1)+1} < 1/2$,

$$\|K_t - K_s\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}.$$

After covering $[0, T]$ with a finite number of small enough intervals, we obtain the result.

Let us now verify (i)–(ii)–(iii).

Proof of (i): By Lemma C.3 applied to $\Xi = K_s$, $f(z, x) = b^k(z + x)$ and $n = m$, we have

$$\|A_{s,t}^k\|_{L^m} \leq C \| \|b^k(\cdot + K_s)\|_{\mathcal{B}_p^\beta} \|_{L^m} (t - s)^{1+H(\beta-1/p)}.$$

Using Lemma A.5(a), we thus get

$$\begin{aligned} \|A_{s,t}^k\|_{L^m} &\leq C \|b^k\|_{\mathcal{B}_p^\beta} (t - s)^{1+H(\beta-1/p)} \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} (t - s)^{1+H(\beta-1/p)}. \end{aligned} \tag{5.2}$$

Proof of (ii): By Lemma 5.1(a) applied to $\Xi = (K_s, K_u)$ and $f(z, (x_1, x_2)) = b^k(z + x_1) - b^k(z + x_2)$, we get

$$|\mathbb{E}^u[\delta A_{s,u,t}^k]| = \left| \int_u^t \mathbb{E}^u[b^k(B_r + K_s) - b^k(B_r + K_u)] dr \right| \leq \int_u^t \|G_{\sigma_{u,r}^2} b^k\|_{\mathcal{C}^1} |K_u - K_s| dr,$$

where we recall $\sigma_{u,r}^2 = \text{Var}(B_r - \mathbb{E}^u[B_r])$. From (B.4), we have that $\sigma_{u,r}^2 = C(u - r)^{2H}$. Apply now Lemma A.4(d), which is possible as $\beta - 1/p < 0$, and then use that $\beta - 1/p > -1/2H$ to ensure integrability. This gives

$$\begin{aligned} |\mathbb{E}^u[\delta A_{s,u,t}^k]| &\leq C \int_u^t |r - u|^{H(\beta-1/p-1)} \|b^k\|_{\mathcal{B}_p^\beta} |K_u - K_s| dr \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} |K_u - K_s| (t - u)^{H(\beta-1/p-1)+1}. \end{aligned}$$

Proof of (iii): For a sequence $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of partitions of $[0, t]$ with mesh size going to 0, we have

$$\begin{aligned} \left| K_t^k - \sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n}^k \right| &\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} |b^k(B_r + K_r) - b^k(B_r + K_{t_i^n})| dr \\ &\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} \|b^k\|_{\mathcal{C}^1} |K_r - K_{t_i^n}| dr \\ &\leq \sum_i \|b^k\|_{\mathcal{C}^1} (t_{i+1}^n - t_i^n) |K_{t_{i+1}^n} - K_{t_i^n}| \\ &\leq \|b^k\|_{\mathcal{C}^1} |\Pi_n| (K_t - K_0) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \end{aligned} \quad \square$$

We conclude this section with the proof of Theorem 2.5(b).

Proof of Theorem 2.5(b). It is a direct consequence of Proposition 5.3. More precisely, in order to apply Proposition 5.3, we check that $\beta - 1/p > -1/(2H)$ holds in all cases (i)–(iii). In case (i) holds, the assumption $\beta - 1 > -1/(2H) + H/2$ and $p \geq 1$ yields $\beta - 1/p > -1/(2H)$. In case (ii) holds, then $\beta - 1/p > 2H - 2$ and $2H - 2 > -1/(2H)$ for $H < 1/3$. Finally, in case (iii) the result follows from $p \geq 1$. \square

6 Uniqueness

In this section we state and prove Proposition 6.1, which gives pathwise uniqueness to (1.1) among all weak solutions X fulfilling $X - B \in \mathcal{C}_{[0,T]}^{1/2+H}(L^2)$. This is a crucial step towards the proof of the uniqueness part in Theorem 2.10. Note that we do not assume anymore that b is a measure. The scheme of proof of Proposition 6.1, which is briefly described in Sect. 2.4, is inspired by the proof of Proposition 2.1 in [11] and closely follows the steps of the proof of Proposition 3.6 in [2]. More precisely, we establish that if two solutions X and Y are such that $X - B$ and $Y - B$ are $(\frac{1}{2} + H)$ -Hölder continuous,

then their difference Z satisfies inequality (6.9), and since any continuous function which satisfies such an inequality must be 0, the uniqueness follows. The most technical part is to establish (6.9). This relies on variations of the stochastic sewing Lemma (Sects. 6.1, 6.2 and Appendix C).

Even when it is not explicitly written, we assume in the whole section that $H < 1/2$.

In the rest of the paper we need the following definition. Let $0 \leq s \leq t$. Let $\psi : [s, t] \times \Omega \rightarrow \mathbb{R}$ be a stochastic process. For $\alpha \in (0, 1]$ and $m, n \in [1, \infty]$, define

$$[\psi]_{C_{[s,t]}^\alpha(L^{m,n})} := \sup_{(u,v) \in \Delta_{[s,t]}} \frac{\|\mathbb{E}^u[|\psi_v - \psi_u|^m]^{\frac{1}{m}}\|_{L^n}}{(v-u)^\alpha}. \tag{6.1}$$

where the conditional expectation is taken w.r.t. the filtration the space is equipped with. By the tower property and Jensen’s inequality for conditional expectation, we know that, for $1 \leq m \leq n \leq \infty$,

$$[\psi]_{C_{[s,t]}^\alpha(L^m)} = [\psi]_{C_{[s,t]}^\alpha(L^{m,m})} \leq [\psi]_{C_{[s,t]}^\alpha(L^{m,n})} \leq [\psi]_{C_{[s,t]}^\alpha(L^n)}. \tag{6.2}$$

Proposition 6.1. *Let $H < 1/2$. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$ such that (2.6) holds. Let $b \in \mathcal{B}_p^\beta$, and $(X_t)_{t \in [0,T]}$ and $(Y_t)_{t \in [0,T]}$ be two weak solutions to (1.1) in the sense of Definition 2.4 with the same initial condition X_0 , both being defined on the same probability space and adapted to the same filtration \mathbb{F} . Suppose that $(X - B)$ and $(Y - B)$ are in $C_{[0,T]}^{1/2+H}(L^2)$. Further assume that*

$$\|X - B\|_{C_{[0,T]}^{1/2+H}(L^{2,\infty})} < \infty \quad a.s. \tag{6.3}$$

Then X and Y are indistinguishable.

Remark 6.2. In the above proposition, note that condition (2.6) allows negative values of β for any $H < \frac{1}{2}$. In the proofs in this section, we will thus only consider the case $\beta < 0$ and $p \geq 2$. Indeed, it is always possible to come back to these cases in the following way: If $\beta \geq 0$ and $p \in [1, \infty]$, there exist $\tilde{\beta} < 0$, $\tilde{p} \geq 2$ fulfilling (2.6) such that $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}$. This can be seen using the embeddings $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_p^{\beta - (\frac{1}{p} - \frac{1}{\tilde{p}})}$ (see Remark 1.2) and $\mathcal{B}_p^s \hookrightarrow \mathcal{B}_{\tilde{p}}^{\tilde{s}}$ for $s > \tilde{s}$.

6.1 Uniqueness: Proof of Proposition 6.1

For the proof of Proposition 6.1, which is detailed at the end of this subsection, we will use Lemma 6.3, Lemma 6.4 and Lemma 6.5. From now on, assume w.l.o.g. that $X_0 = 0$. Let $K := X - B$ and $\tilde{K} := Y - B$. Let $Z := X - Y = K - \tilde{K}$.

Lemma 6.3. *Let the assumptions of Proposition 6.1 hold. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ with $\sup_n \|b^n\|_{\mathcal{B}_p^\beta} \leq \|b\|_{\mathcal{B}_p^\beta}$. Then for any $(s, t) \in \Delta_{[0,T]}$, there exist random variables $\tilde{T}_{s,t}^K$ and $\tilde{T}_{s,t}^{\tilde{K}}$ such that*

$$\int_s^t b^n(B_r + K_s) dr \xrightarrow[n \rightarrow \infty]{L^2} \tilde{T}_{s,t}^K \quad \text{and} \quad \int_s^t b^n(B_r + \tilde{K}_s) dr \xrightarrow[n \rightarrow \infty]{L^2} \tilde{T}_{s,t}^{\tilde{K}}. \tag{6.4}$$

Moreover, there exists $C > 0$ such that, for any $(s, t) \in \Delta_{[0,T]}$, we have that

$$\|\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} \|Z_s\|_{L^2} (t-s)^{1/2} \quad \text{and} \tag{6.5}$$

$$\|\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1/2+H}. \tag{6.6}$$

The proof of this lemma is moved to Sect. 6.2.

For $(s, t) \in \Delta_{[0,T]}$, let

$$R_{s,t} := (K_t - K_s - \tilde{T}_{s,t}^K) - (\tilde{K}_t - \tilde{K}_s - \tilde{T}_{s,t}^{\tilde{K}}).$$

It is now necessary to estimate the regularity of this remainder term. The proof of the following lemma, which heavily relies on stochastic sewing, is moved to Sect. 6.2.

Lemma 6.4. *Let the assumptions of Proposition 6.1 hold. There exists $\delta > 0$ such that for any $\alpha \in (1/2, 1)$, there exists $C > 0$ for which the following holds: for all $(u, v) \in \Delta_{[0, T]}$, we have*

$$\|Z_v - Z_u\|_{L^2} \leq C \|R\|_{C_{[u, v]}^\alpha(L^2)} (v - u)^{1/2 + \delta} \tag{6.7}$$

$$+ C \|Z\|_{C_{[u, v]}(L^2)} |\log \|Z\|_{C_{[u, v]}(L^2)}| (v - u) + C \|Z\|_{C_{[u, v]}(L^2)} (v - u)^{1/2},$$

$$\|R_{u, v}\|_{L^2} \leq C \left(\|Z\|_{C_{[u, v]}(L^2)} + \|R\|_{C_{[u, v]}^\alpha(L^2)} \right) (v - u)^{1/2 + \delta} \tag{6.8}$$

$$+ C \|Z\|_{C_{[u, v]}(L^2)} |\log \|Z\|_{C_{[u, v]}(L^2)}| (v - u).$$

One can now deduce a regularity estimate on Z alone.

Lemma 6.5. *Let the assumptions of Proposition 6.1 hold. Let $\tilde{T} \in (0, T]$ such that $\|Z\|_{C_{[0, \tilde{T}]}(L^2)} \leq 1/e$. Then there exist $C > 0$ and $l > 0$ such that for any $(s, t) \in \Delta_{[0, \tilde{T}]}$ with $t - s < l$, we have*

$$\|Z_t - Z_s\|_{L^2} \leq C \|Z\|_{C_{[s, t]}(L^2)} (t - s)^{1/2} + C \|Z\|_{C_{[s, t]}(L^2)} |\log \|Z\|_{C_{[s, t]}(L^2)}| (t - s). \tag{6.9}$$

Proof. Consider δ as in Lemma 6.4 and assume w.l.o.g. that $\delta < 2H$. By assumption on X and Y and by (6.6) we know that $\|R\|_{C_{[0, T]}^{1/2+H}(L^2)} < \infty$. Let $(s, t) \in \Delta_{[0, \tilde{T}]}$. After dividing both sides of Equation (6.8) by $(v - u)^{1/2 + \delta/2}$ and taking the supremum over all $(u, v) \in \Delta_{[s, t]}$, we obtain for $\alpha = 1/2 + \delta/2$ that

$$\|R\|_{C_{[s, t]}^\alpha(L^2)} \leq C \left(\|Z\|_{C_{[s, t]}(L^2)} + \|R\|_{C_{[s, t]}^\alpha(L^2)} \right) (t - s)^{\delta/2} \tag{6.10}$$

$$+ C \|Z\|_{C_{[s, t]}(L^2)} |\log \|Z\|_{C_{[s, t]}(L^2)}| (t - s)^{1/2 - \delta/2}.$$

In the above we used that

$$\sup_{(u, v) \in \Delta_{[s, t]}} \|Z\|_{C_{[u, v]}(L^2)} |\log \|Z\|_{C_{[u, v]}(L^2)}| \leq \|Z\|_{C_{[s, t]}(L^2)} |\log \|Z\|_{C_{[s, t]}(L^2)}|$$

as $f(x) = x \log(1/x)$ is increasing on $[0, 1/e]$ and $\|Z\|_{C_{[s, t]}(L^2)} \leq 1/e$ by assumption. Using $\delta < 2H$, we have that $\|R\|_{C_{[0, T]}^\alpha(L^2)} \leq C \|R\|_{C_{[0, T]}^{1/2+H}(L^2)} < \infty$. Let $l \in (0, \tilde{T})$ be such that $C l^{\delta/2} < 1/2$ for C as in (6.10). For $t - s < l$, we have

$$\frac{1}{2} \|R\|_{C_{[s, t]}^\alpha(L^2)} \leq C \|Z\|_{C_{[s, t]}(L^2)} (t - s)^{\delta/2}$$

$$+ C \|Z\|_{C_{[s, t]}(L^2)} |\log \|Z\|_{C_{[s, t]}(L^2)}| (t - s)^{1/2 - \delta/2}.$$

Plugging this into (6.7) for $u = s$ and $v = t$ finishes the proof. □

Proof of Proposition 6.1. Let $\tilde{T} := \sup\{t \in [0, T] : \sup_{0 \leq s \leq t} \|Z_s\|_{L^2} \leq 1/e\}$. We show in the next paragraph that Z is indistinguishable from 0 on $[0, \tilde{T}]$. Note that this implies that $\tilde{T} = T$: Indeed, by definition of \tilde{T} and continuity of $Z : [0, T] \rightarrow L^2$, $\|Z_{\tilde{T}}\|_{L^2} = 1/e$ if $\tilde{T} < T$. This would contradict Z being indistinguishable from 0 on $[0, \tilde{T}]$. Hence, we get that Z is actually indistinguishable from 0 on $[0, T]$.

Assume that $\|Z_t\|_{L^2}$ is not identically 0 on $[0, \tilde{T}]$. Choose $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \sup_{t \in [0, \tilde{T}]} \|Z_t\|_{L^2}$. For $k \geq k_0$ let $t_k := \inf\{t : \|Z_t\|_{L^2} \geq 2^{-k}\}$. We have that $\|Z_t\|_{L^2} < 2^{-k}$ for $t < t_k$ and $\|Z_{t_k}\|_{L^2} = 2^{-k}$ as $t \mapsto \|Z_t\|_{L^2}$ is continuous by (6.9). For l as in Lemma 6.5, choose $a \in (0, l)$ such that $C a^{1/2} < 1/4$ for C as in (6.9). As the sequence t_k is strictly

decreasing we can choose M such that $t_k - t_{k+1} < a$ for $k \geq M$. Therefore by (6.9) we have, for $k \geq M$, that

$$2^{-(k+1)} \leq \|Z_{t_k} - Z_{t_{k+1}}\|_{L^2} \leq 2^{-(k+2)} + C 2^{-k} k \log(2) (t_k - t_{k+1})$$

and therefore

$$\frac{1}{4C \log(2)k} \leq t_k - t_{k+1}, \tag{6.11}$$

which leads to a contradiction as the sequence t_k is strictly decreasing and the left hand side of (6.11) is diverging when summing over $k \geq M$. Hence, $\|Z_t\|_{L^2}$ is identically 0 on $[0, \tilde{T}]$. Now using that $Z \in C_{[0, \tilde{T}]}^{1/2+H}(L^2)$, we get that $\mathbb{E}[|Z_t - Z_s|^2] \leq \|Z\|_{C_{[0, \tilde{T}]}^{1/2+H}(L^2)}^2 |t-s|^{1+2H}$ for any $s, t \in [0, \tilde{T}]$. Hence by Kolmogorov's continuity Theorem, Z is a.s. continuous and is therefore indistinguishable from 0. \square

6.2 Intermediate regularity results: Proofs of Lemma 6.3 and Lemma 6.4

The proofs of Lemma 6.3 and Lemma 6.4 rely on several results about the regularity of $\int_s^t f(B_r, \Xi) dr$ in terms of $t - s$ and of the Besov norm of the random mapping $f(\cdot, \Xi)$, for Ξ an \mathcal{F}_s -measurable random variable. These regularity results are stated and proven in Appendix C, and are derived from two main ingredients: the stochastic sewing Lemma (Lemma C.1) and Lemma 5.1.

We now turn to the proofs of Lemmas 6.3 and 6.4.

Proof of Lemma 6.3. Recall that we assumed $\beta < 0$ at the beginning of this section (Remark 6.2). Let $\beta' \in (-\frac{1}{2H}, \beta)$ and $(s, t) \in \Delta_{[0, T]}$. Let $k, l \in \mathbb{N}$. Applying the crucial regularity Lemma C.3 for $\Xi = K_s$ and $f(z, x) = b^k(z + x) - b^l(z + x)$, we get that

$$\begin{aligned} \left\| \int_s^t b^k(B_r + K_s) dr - \int_s^t b^l(B_r + K_s) dr \right\|_{L^2} &\leq C \left\| \|b^k(\cdot + K_s) - b^l(\cdot + K_s)\|_{\mathcal{B}_p^{\beta'}} \right\|_{L^2} \\ &\leq C \|b^k - b^l\|_{\mathcal{B}_p^{\beta'}}, \end{aligned}$$

using Lemma A.5(a) in the last inequality. Hence, $(\int_s^t b^k(B_r + K_s) dr)_{k \in \mathbb{N}}$ is a Cauchy sequence in L^2 , so it converges to some $\tilde{T}_{s,t}^K$. The same holds for the sequence with K being replaced by \tilde{K} . Using Fatou's Lemma and Corollary C.4, for $\lambda \in [0, 1]$ we obtain

$$\begin{aligned} \|\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}\|_{L^2} &\leq \liminf_n \left\| \int_s^t b^n(B_r + K_s) dr - \int_s^t b^n(B_r + \tilde{K}_s) dr \right\|_{L^2} \\ &\leq C \sup_n \|b^n\|_{\mathcal{B}_p^\beta} \|K_s - \tilde{K}_s\|_{L^2}^\lambda (t-s)^{1+H(\beta-\lambda-1/p)} \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} \|K_s - \tilde{K}_s\|_{L^2}^\lambda (t-s)^{1/2+H-\lambda H}, \end{aligned}$$

using that $\beta - 1/p \geq -1/(2H) + 1$. Setting $\lambda = 1$ gives (6.5) and setting $\lambda = 0$ gives (6.6). \square

Proof of Lemma 6.4. Let $(b^k)_{k \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ with $\sup_k \|b^k\|_{\mathcal{B}_p^\beta} \leq \|b\|_{\mathcal{B}_p^\beta}$. Fix $(u, v) \in \Delta_{[0, T]}$. For $(s, t) \in \Delta_{[u, v]}$, let

$$\begin{aligned} A_{s,t}^k &:= \int_s^t (b^k(B_r + K_s) - b^k(B_r + \tilde{K}_s)) dr, \\ \mathcal{A}_t^k &:= \int_u^t (b^k(B_r + K_r) - b^k(B_r + \tilde{K}_r)) dr. \end{aligned} \tag{6.12}$$

Then, for $\theta \in (s, t)$,

$$\delta A_{s,\theta,t}^k := \int_{\theta}^t \left(b^k(B_r + K_s) - b^k(B_r + \tilde{K}_s) - b^k(B_r + K_{\theta}) + b^k(B_r + \tilde{K}_{\theta}) \right) dr. \quad (6.13)$$

We now verify the conditions of the stochastic sewing Lemma with critical exponents, as stated in [2, Th. 4.5]. To show that the conditions of this theorem hold (i.e. (C.1) and (C.2) with $\beta_2 = 0$ and $m = n = 2$ in the current paper, and (4.11) from [2]), we verify that there exists $C > 0$ independent of s, t, u, v and θ such that for any $k \in \mathbb{N}$,

- (i) $\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C(t-s)^{1+H}$;
- (ii) $\|\delta A_{s,\theta,t}^k\|_{L^2} \leq C \left(\|Z\|_{C_{[u,v]}(L^2)} + \|R\|_{C_{[u,v]}^{\alpha}(L^2)} \right) (t-s)^{1/2+\delta}$;
- (iii) $\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C \|Z\|_{C_{[u,v]}(L^2)}(t-s) + C \|R\|_{C_{[u,v]}^{\alpha}(L^2)}(t-s)^{1/2+\alpha}$;
- (iv) Identifying A^k , as given in (6.12), with the limit of $\sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n}^k$ along any sequence of partitions $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of $[u, t]$ with mesh converging to 0.

Assume for the moment that the above properties hold. Then applying Theorem 4.5 in [2], we get that for any $(s, t) \in \Delta_{[u,v]}$,

$$\begin{aligned} & \|\mathcal{A}_t^k - \mathcal{A}_s^k - A_{s,t}^k\|_{L^2} \\ & \leq C \|Z\|_{C_{[u,v]}(L^2)} \left(1 + \left| \log \frac{T^H}{\|Z\|_{C_{[u,v]}(L^2)}} \right| \right) |t-s| \\ & \quad + C \left(\|Z\|_{C_{[u,v]}(L^2)} + \|R\|_{C_{[u,v]}^{\alpha}(L^2)} \right) (t-s)^{1/2+\delta} + C \|R\|_{C_{[u,v]}^{\alpha}(L^2)}(t-s)^{1/2+\alpha} \\ & \leq C \left(\|Z\|_{C_{[u,v]}(L^2)} + \|R\|_{C_{[u,v]}^{\alpha}(L^2)} \right) (t-s)^{1/2+\delta} \\ & \quad + C \|Z\|_{C_{[u,v]}(L^2)} |\log \|Z\|_{C_{[u,v]}(L^2)}| (t-s). \end{aligned} \quad (6.14)$$

By Definition 2.4 and Lemma 6.3, we have the following convergence in probability:

$$\lim_{k \rightarrow \infty} \mathcal{A}_t^k - \mathcal{A}_s^k - A_{s,t}^k = Z_t - Z_s - (\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}) = R_{s,t}.$$

Hence and letting k go to ∞ in (6.14) and using Fatou's Lemma, we get (6.8) by choosing $(s, t) = (u, v)$. Putting together (6.8) and (6.5), we get (6.7).

Proof of (i): Recall that we consider times $0 \leq u \leq s \leq \theta \leq t \leq v \leq T$. Let $f_1 : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} f_1(z, (x_1, x_2, x_3, x_4)) &= b^k(z + x_1) - b^k(z + x_2) + b^k(z + x_3) - b^k(z + x_4), \\ f_2(z, (x_1, x_2)) &= b^k(z + x_1) - b^k(z + x_2), \end{aligned}$$

and consider the \mathcal{F}_{θ} -measurable random vectors Ξ_1 and Ξ_2 given by:

$$\begin{aligned} \Xi_1 &= (K_s, \tilde{K}_s, K_{\theta} + \tilde{K}_s - K_s, K_{\theta}) \\ \Xi_2 &= (\tilde{K}_{\theta}, K_{\theta} + \tilde{K}_s - K_s). \end{aligned}$$

We can rewrite the integrand on the right hand side of (6.13) as $f_1(B_r, \Xi_1) + f_2(B_r, \Xi_2)$. Hence using Lemma 5.1(b), we get that for $\lambda \in [0, 1]$,

$$\begin{aligned} |\mathbb{E}^s \delta A_{s,\theta,t}^k| &= |\mathbb{E}^s \mathbb{E}^{\theta} \delta A_{s,\theta,t}^k| \\ &\leq C \int_{\theta}^t \left\{ (r-\theta)^{H(\beta-1-\lambda-1/p)} \mathbb{E}^s \|f_1(\cdot, \Xi_1)\|_{\mathcal{B}_p^{\beta-1-\lambda}} \right. \\ & \quad \left. + (r-\theta)^{H(\beta-1-1/p)} \mathbb{E}^s \|f_2(\cdot, \Xi_2)\|_{\mathcal{B}_p^{\beta-1}} \right\} dr. \end{aligned} \quad (6.15)$$

Apply Lemma A.5(c) first, then Jensen's inequality for conditional expectation to get that

$$\begin{aligned} \mathbb{E}^s \|f_1(\cdot, \Xi_1)\|_{\mathcal{B}_p^{\beta-1-\lambda}} &\leq C \|b^k\|_{\mathcal{B}_p^\beta} |K_s - \tilde{K}_s|^\lambda \mathbb{E}^s |K_\theta - K_s| \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} |K_s - \tilde{K}_s|^\lambda (\mathbb{E}^s [|K_\theta - K_s|^2])^{1/2} \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} |Z_s|^\lambda [K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} (\theta - s)^{1/2+H}. \end{aligned}$$

Hence,

$$\|\mathbb{E}^s \|f_1(\cdot, \Xi_1)\|_{\mathcal{B}_p^{\beta-1-\lambda}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} \|Z\|_{\mathcal{C}_{[u,v]}^\lambda(L^2)} [K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} (t - s)^{1/2+H}. \quad (6.16)$$

By Lemma A.5(b), we have that $\mathbb{E}^s \|f_2(\cdot, \Xi_2)\|_{\mathcal{B}_p^{\beta-1}} \leq C \|b^k\|_{\mathcal{B}_p^\beta} \mathbb{E}^s |Z_\theta - Z_s|$. Therefore using the contraction property of conditional expectation and (6.2), we get that

$$\|\mathbb{E}^s \|f_2(\cdot, \Xi_2)\|_{\mathcal{B}_p^{\beta-1}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} \|Z_\theta - Z_s\|_{L^2} \quad (6.17)$$

$$\leq C \|b\|_{\mathcal{B}_p^\beta} \left([K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} + [\tilde{K}]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2)} \right) (t - s)^{1/2+H}. \quad (6.18)$$

After putting $\lambda = 0$ in (6.15) and using inequalities (6.16) and (6.18), we get

$$\begin{aligned} |\mathbb{E}^s \delta A_{s,\theta,t}^k| &\leq C \|b\|_{\mathcal{B}_p^\beta} \left([K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} + [\tilde{K}]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2)} \right) (t - s)^{1/2+H} \int_\theta^t (r - \theta)^{H(\beta-1-1/p)} dr. \end{aligned}$$

Now using that $\beta - 1/p \geq -1/(2H) + 1$, we obtain that for some C depending only on T , $\|b\|_{\mathcal{B}_p^\beta}$, $[K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)}$ and $[\tilde{K}]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2)}$,

$$\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C (t - s)^{1+H}.$$

Proof of (ii): We use Corollary C.5 with $f = b^k$, $\gamma = \beta$, $\varepsilon > 0$ small enough to ensure that $0 > \beta > -1/(2H) + 1 + \varepsilon$, $\lambda = \lambda_1 = 1$, $\lambda_2 = \varepsilon$, $\kappa_1 = K_s$, $\kappa_2 = \tilde{K}_s$, $\kappa_3 = K_\theta$ and $\kappa_4 = \tilde{K}_\theta$.

Hence, we get

$$\begin{aligned} \|\delta A_{s,\theta,t}^k\|_{L^2} &\leq C \|b\|_{\mathcal{B}_p^\beta} \|\mathbb{E}^s [|K_\theta - K_s|^2]^{1/2}\|_{L^\infty}^\varepsilon \|K_s - \tilde{K}_s\|_{L^2} (t - \theta)^{1+H(\beta-1/p-1-\varepsilon)} \\ &\quad + C \|b\|_{\mathcal{B}_p^\beta} \|Z_\theta - Z_s\|_{L^2} (t - \theta)^{1+H(\beta-1-1/p)}. \end{aligned} \quad (6.19)$$

We have $\|Z_\theta - Z_s\|_{L^2} \leq \|R_{s,\theta}\|_{L^2} + \|\tilde{T}_{s,\theta}^K - \tilde{T}_{s,\theta}^{\tilde{K}}\|_{L^2}$. Thus by Lemma 6.3,

$$\|Z_\theta - Z_s\|_{L^2} \leq \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} (\theta - s)^\alpha + C \|b\|_{\mathcal{B}_p^\beta} \|Z_s\|_{L^2} (\theta - s)^{1/2}. \quad (6.20)$$

It follows directly from the definitions that

$$\|\mathbb{E}^s [|K_\theta - K_s|^2]^{1/2}\|_{L^\infty} \leq C (\theta - s)^{1/2+H} [K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)}$$

and that $\|K_s - \tilde{K}_s\|_{L^2} \leq \|Z\|_{\mathcal{C}_{[u,v]}(L^2)}$. Plugging the last three inequalities into (6.19) we get that

$$\begin{aligned} \|\delta A_{s,\theta,t}^k\|_{L^2} &\leq C \|b\|_{\mathcal{B}_p^\beta} [K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)}^\varepsilon \|Z\|_{\mathcal{C}_{[u,v]}(L^2)} (t - s)^{1-H+H(\beta-1/p)+\varepsilon/2} \\ &\quad + C \|b\|_{\mathcal{B}_p^\beta} \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} (t - s)^{\alpha+1+H(\beta-1-1/p)} \\ &\quad + C \|b\|_{\mathcal{B}_p^\beta}^2 \|Z\|_{\mathcal{C}_{[u,v]}(L^2)} (t - s)^{3/2-H+H(\beta-1/p)}. \end{aligned}$$

After using that $\beta - 1/p \geq -1/(2H) + 1$, we obtain that for some C depending only on T , $\|b\|_{\mathcal{B}_p^\beta}$ and $[K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2,\infty)}$ and some $\delta > 0$,

$$\|\delta A_{s,\theta,t}^k\|_{L^2} \leq C \left(\|Z\|_{\mathcal{C}_{[u,v]}(L^2)} + \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} \right) (t-s)^{1/2+\delta}.$$

Proof of (iii): We use (6.15) with $\lambda = 1$, (6.16), (6.17) and (6.20) to obtain that

$$\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C \|Z\|_{\mathcal{C}_{[u,v]}(L^2)}(t-s) + C \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)}(t-s)^{1/2+\alpha}.$$

That (iv) holds can be shown by similar arguments as for (iii) in the proof of Proposition 5.3. □

7 Existence of weak and strong solutions

In this section we prove Theorem 2.8 and Theorem 2.10. As in Sect. 6, recall that b is not assumed to be a measure.

In the proofs, we assume that $p \in [m, \infty]$ and $\beta < 0$ for some $m \geq 2$. As explained in Remarks 5.2 and 6.2, it is not a restriction as we can always reduce to this case under our assumptions.

The scheme of proof is the following: we use a sequence of smooth approximations of the drift $(b^n)_{n \in \mathbb{N}}$ that converges in $\mathcal{B}_p^{\beta-}$ to b . We prove that the sequence $(X^n)_{n \in \mathbb{N}}$ of solutions associated to b^n is tight and that each limit point is a weak solution to (1.1). To prove the existence of a strong solution (which is in some sense a Yamada-Watanabe result), we rely on a classical argument of Krylov [26], a new result on the continuity of fractional operators (Lemma B.1) and the aforementioned construction of weak solutions.

On the technical level, we follow the approach of [2] which is based on applications of the stochastic sewing Lemma. However there are noticeable differences due to the non-Markovian character of the fBm. In particular we state immediately below:

- Lemma 7.1, which is a crucial estimate on the regularity of the integrals of the fBm, relying on fine properties of the fBm (notably Lemma 5.1) and stochastic sewing, and which will be used in Sect. 7.1 to establish *a priori* estimates on the solutions;
- Lemma B.1, which will be used in Sect. 7.3 to prove that solutions are adapted.

Lemma 7.1. *Let $(\psi_t)_{t \in [0,T]}$ be a stochastic process adapted to \mathbb{F} . Let $m \in [2, \infty)$, $n \in [m, \infty]$, $p \in [n, \infty]$ and $\gamma < 0$ such that $(\gamma - 1/p)H > -1/2$. Let $\alpha \in (0, 1)$ such that $H(\gamma - 1/p - 1) + \alpha > 0$. There exists a constant $C > 0$ such that for any $f \in \mathcal{C}_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\gamma$ and any $(s, t) \in \Delta_{[0,T]}$ we have*

$$\begin{aligned} \left\| \mathbb{E}^s \left[\left| \int_s^t f(B_r + \psi_r) dr \right|^m \right]^{1/m} \right\|_{L^n} &\leq C \|f\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma-1/p)} \\ &+ C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})} (t-s)^{1+H(\gamma-1-1/p)+\alpha}, \end{aligned} \tag{7.1}$$

where we recall that the seminorm $[\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})}$ is defined in (6.1).

A similar regularisation result was proposed in [6, Lemma 4.7] for functions of positive regularity. There, the deterministic sewing lemma is used instead of the stochastic sewing Lemma, hence the conditional expectation does not appear. Note that due to an embedding argument, it would be no restriction here to work in Hölder spaces, yet we directly work in Besov spaces to stay consistent throughout the paper.

The proof of Lemma 7.1 is postponed to Appendix C.

7.1 A priori estimates

Before starting the main proofs of existence of solutions, we need *a priori* estimates on solutions. These estimates are established for drifts in $C_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\beta$ and therefore the solution is unique and strong. Note that the following lemma looks similar to Proposition 5.3. However we work here within the assumptions of Theorem 2.8, that is with a drift that may not be a measure and $\beta - 1/p > 1/2 - 1/(2H)$ (recall that Proposition 5.3 allows the milder condition $\beta - 1/p > -1/(2H)$).

Lemma 7.2. *Let $m \in [2, \infty)$. There exists $C > 0$, such that, for any $b \in C_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\beta$ with $\beta - 1/p > 1/2 - 1/(2H)$, one has*

$$[X - B]_{C_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq C \|b\|_{\mathcal{B}_p^\beta} \left(1 + \|b\|_{\mathcal{B}_p^\beta}^{\frac{H(1/p-\beta)}{1+H(\beta-1/p)-H}} \right) \leq C \left(1 + \|b\|_{\mathcal{B}_p^\beta}^2 \right), \quad (7.2)$$

where X is the strong solution to (1.1) with drift b , and we recall that the seminorm $[\cdot]_{C_{[s,t]}^\alpha(L^{m,n})}$ was defined in (6.1).

Another difference with Proposition 5.3 is that we obtain here an estimate on the seminorm $[\cdot]_{C_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})}$, which is stronger (see (6.2)) than the seminorm $[\cdot]_{C_{[0,T]}^{1+H(\beta-1/p)}(L^m)}$ used in Proposition 5.3.

Proof. Note that it is sufficient to show the first inequality in (7.2) as the second one follows immediately from $1 + H(\beta - 1/p) - H > H(1/p - \beta) > 0$. Without loss of generality, we assume that $X_0 = 0$ and denote $K = X - B$. Then $[K]_{C_{[0,T]}^\alpha(L^{m,\infty})}$ is finite for any $\alpha \in (0, 1]$ as

$$|K_t - K_s| = \left| \int_s^t b(B_r + K_r) dr \right| \leq \|b\|_\infty |t - s|.$$

We will apply Lemma 7.1 with $n = \infty$, $\alpha = 1 + H(\beta - 1/p)$ and considering $b \in \mathcal{B}_\infty^{\beta-1/p}$ after an embedding. Remark that $\alpha > \alpha - H > 1/2 - H/2 > 0$ and since we assume $\beta < 0$ in the whole section, $\alpha \in (0, 1)$. In addition, $H(\beta - 1/p) > H/2 - 1/2 > -1/2$ and $H(\beta - 1/p - 1) + \alpha > 0$. So, the assumptions of Lemma 7.1 are fulfilled. Then we get for some $C, \tilde{C} > 0$ that

$$\begin{aligned} & \|E^s [|K_t - K_s|^m]^{1/m} \|_{L^\infty} \\ & \leq C \|b\|_{\mathcal{B}_\infty^{\beta-1/p}} \left((t-s)^{1+H(\beta-1/p)} + [K]_{C_{[s,t]}^\alpha(L^{m,\infty})} (t-s)^{1+H(\beta-1/p)+\alpha-H} \right) \\ & \leq \tilde{C} \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)} \left(1 + [K]_{C_{[s,t]}^\alpha(L^{m,\infty})} (t-s)^{\alpha-H} \right), \end{aligned} \quad (7.3)$$

where we used a Besov space embedding in the second line.

Choose $l = (4\tilde{C}\|b\|_{\mathcal{B}_p^\beta})^{1/(H-\alpha)}$ so that $\tilde{C}\|b\|_{\mathcal{B}_p^\beta} l^{\alpha-H} < 1/2$. Let $u \in [0, T]$. After dividing both sides in (7.3) by $(t-s)^{1+H(\beta-1/p)}$ and taking the supremum over $(s, t) \in \Delta_{[u, (u+l) \wedge T]}$ we get

$$[K]_{C_{[u, (u+l) \wedge T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq \left(\tilde{C}\|b\|_{\mathcal{B}_p^\beta} + 1/2[K]_{C_{[u, (u+l) \wedge T]}^\alpha(L^{m,\infty})} \right)$$

and therefore

$$[K]_{C_{[u, (u+l) \wedge T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq 2\tilde{C}\|b\|_{\mathcal{B}_p^\beta}. \quad (7.4)$$

If $l > T$, then (7.2) follows immediately from (7.4). Hence, assume $l \leq T$. In order to obtain (7.2) we will iteratively apply inequality (7.4). Let $(s, t) \in \Delta_{[0,T]}$ be arbitrary. Let $N = \lceil T/l \rceil$ and define the sequence $(s_k)_{k=0}^N$ by $s_k = s + k(t-s)/N$. Using the triangle

inequality, that $\mathcal{F}_s \subset \mathcal{F}_{s_{k-1}}$ for $k \geq 1$, the contraction property of conditional expectation and (7.4) we get

$$\begin{aligned} \|\mathbb{E}^s [|K_t - K_s|^m]^{1/m}\|_{L^\infty} &\leq \sum_{k=1}^N \|\mathbb{E}^s [|K_{s_k} - K_{s_{k-1}}|^m]^{1/m}\|_{L^\infty} \\ &\leq \sum_{k=1}^N \|\mathbb{E}^{s_{k-1}} [|K_{s_k} - K_{s_{k-1}}|^m]^{1/m}\|_{L^\infty} \\ &\leq \tilde{C} \|b\|_{\mathcal{B}_p^\beta} \sum_{k=1}^N (s_k - s_{k-1})^{1+H(\beta-1/p)} \\ &\leq \tilde{C} \|b\|_{\mathcal{B}_p^\beta} \sum_{k=1}^N \left(\frac{t-s}{N}\right)^{1+H(\beta-1/p)} \\ &\leq \tilde{C} \|b\|_{\mathcal{B}_p^\beta} N^{H(1/p-\beta)} (t-s)^{1+H(\beta-1/p)}. \end{aligned}$$

Using $N \leq 1 + \frac{T}{t} \leq 2\frac{T}{t} \leq C \|b\|_{\mathcal{B}_p^\beta}^{-\frac{1}{H-\alpha}}$, we get

$$\|b\|_{\mathcal{B}_p^\beta} N^{H(1/p-\beta)} \leq C \|b\|_{\mathcal{B}_p^\beta} \|b\|_{\mathcal{B}_p^\beta}^{-\frac{H(1/p-\beta)}{H-\alpha}}$$

and therefore (7.2). □

Lemma 7.3. *Let $b, h \in C_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\beta$ where $\beta - 1/p > 1/2 - 1/(2H)$. Let X be the strong solution to (1.1) with drift b . Let $\delta \in (0, 1 + H(\beta - 1/p))$. Then there exists a constant $C > 0$ which is independent of X_0, b and h , and a nonnegative random variable Z which satisfies $\mathbb{E}[Z] \leq C \|h\|_{\mathcal{B}_p^\beta} (1 + \|b\|_{\mathcal{B}_p^\beta}^2)$ such that*

$$\left| \int_s^t h(X_r) dr \right| \leq Z |t - s|^\delta. \tag{7.5}$$

Proof. Let $\Lambda_t^h := \int_0^t h(X_r) dr$. We apply Lemma 7.1 to establish an upper bound for $\|\Lambda_t^h - \Lambda_s^h\|_{L^m}$, with the parameters $n = m, \gamma = \beta (< 0)$ and $\alpha = 1 + H(\beta - 1/p)$. Hence, we get

$$\|\Lambda_t^h - \Lambda_s^h\|_{L^m} \leq C \|h\|_{\mathcal{B}_p^\beta} (1 + [X - B]_{C_{[0,T]}^\alpha(L^m)})(t - s)^{1+H(\beta-1/p)}. \tag{7.6}$$

By (6.2) and Lemma 7.2 we know that

$$[X - B]_{C_{[0,T]}^\alpha(L^m)} \leq [X - B]_{C_{[0,T]}^\alpha(L^{m,\infty})} \leq C (1 + \|b\|_{\mathcal{B}_p^\beta}^2).$$

Since for any m there exists a constant C such that (7.6) holds, the result follows from Kolmogorov’s continuity criterion. □

7.2 Tightness and stability

In Proposition 7.4 we show that if $(b^n)_{n \in \mathbb{N}}$ approximates b , the solution X^n to (1.1) with drift b^n converges weakly (up to taking a subsequence). Proposition 7.6 states that the limit in probability of such a sequence $(X^n)_{n \in \mathbb{N}}$ is a solution to the original equation with drift b .

Proposition 7.4. *Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ where $b \in \mathcal{B}_p^\beta$ for $\beta - 1/p > 1/2 - 1/(2H)$. For $n \in \mathbb{N}$, let X^n be the strong solution to (1.1) with initial condition X_0 and drift b^n . Then there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(X^{n_k}, B)_{k \in \mathbb{N}}$ converges weakly in the space $[C_{[0,T]}]^2$.*

Proof. Let $K_t^n := \int_0^t b^n(X_r^n) dr$. For $M > 0$, let

$$A_M := \{f \in \mathcal{C}_{[0,T]} : f(0) = 0, |f(t) - f(s)| \leq M(t - s)^{1+H(\beta-1/p)}, \forall (s, t) \in \Delta_{[0,T]}\}.$$

By Arzelà-Ascoli's Theorem, A_M is compact in $\mathcal{C}_{[0,T]}$. Applying Lemma 7.2 and Markov's inequality, we get

$$\begin{aligned} \mathbb{P}(K^n \notin A_M) &\leq \mathbb{P}(\exists (s, t) \in \Delta_{[0,T]} : |K_{s,t}^n| > M(t - s)^{1+H(\beta-1/p)}) \\ &\leq C(1 + \sup_{n \in \mathbb{N}} \|b^n\|_{\mathcal{B}_p^\beta}^2) M^{-1}. \end{aligned}$$

Hence, the sequence $(K^n)_{n \in \mathbb{N}}$ is tight in $\mathcal{C}_{[0,T]}$. So $(K^n, B)_{n \in \mathbb{N}}$ is tight in $[\mathcal{C}_{[0,T]}]^2$. Thus by Prokhorov's Theorem, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(K^{n_k}, B)_{k \in \mathbb{N}}$ converges weakly in the space $[\mathcal{C}_{[0,T]}]^2$, and so does $(X^{n_k}, B)_{k \in \mathbb{N}}$. \square

Remark 7.5. The previous proposition can be generalised to any pair of sequences of approximations $(X^{1,n}, X^{2,n})_{n \in \mathbb{N}}$ of strong solution to (1.1) with two sequences $(b_1^n)_{n \in \mathbb{N}}$ and $(b_2^n)_{n \in \mathbb{N}}$ of smooth approximations of b .

Proposition 7.6. Let $(\tilde{b}^k)_{k \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ where $b \in \mathcal{B}_p^\beta$ for $\beta - 1/p > 1/2 - 1/(2H)$. Let \tilde{B}^k have the same law as B . We consider \tilde{X}^k the strong solution to (1.1) for $B = \tilde{B}^k$, initial condition X_0 and drift \tilde{b}^k . We assume that there exist stochastic processes $\tilde{X}, \tilde{B} : [0, T] \rightarrow \mathbb{R}$ such that $(\tilde{X}^k, \tilde{B}^k)_{k \in \mathbb{N}}$ converges to (\tilde{X}, \tilde{B}) on $[\mathcal{C}_{[0,T]}]^2$ in probability. Then \tilde{X} fulfills (2.3) and (2.4) from Definition 2.4 and for any $m \in [2, \infty)$, there exists $C > 0$ such that

$$[\tilde{X} - \tilde{B}]_{\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq C(1 + \sup_{k \in \mathbb{N}} \|\tilde{b}^k\|_{\mathcal{B}_p^\beta}^2) < \infty. \tag{7.7}$$

Proof. We again assume here w.l.o.g. that $X_0 = 0$ and let $\tilde{K} := \tilde{X} - \tilde{B}$, so that (2.3) is automatically verified. Let now $(b^n)_{n \in \mathbb{N}}$ be any sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. In order to verify that \tilde{K} and \tilde{X} fulfill (2.4) from Definition 2.4, we have to show that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,T]} \left| \int_0^t b^n(\tilde{X}_r) dr - \tilde{K}_t \right| = 0 \text{ in probability.} \tag{7.8}$$

By the triangle inequality we have that for $k, n \in \mathbb{N}$ and $t \in [0, T]$,

$$\begin{aligned} \left| \int_0^t b^n(\tilde{X}_r) dr - \tilde{K}_t \right| &\leq \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^k) dr \right| + \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| \\ &\quad + \left| \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr - \tilde{K}_t \right| =: A_1 + A_2 + A_3. \end{aligned} \tag{7.9}$$

Now we will show that all summands on the right hand side of (7.9) converge to 0 uniformly on $[0, T]$ in probability as $n \rightarrow \infty$, choosing $k = k(n)$ accordingly.

First we bound A_1 . Notice that

$$\begin{aligned} \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^k) dr \right| &\leq \|b^n\|_{\mathcal{C}^1} \int_0^t |\tilde{X}_r - \tilde{X}_r^k| dr \\ &\leq \|b^n\|_{\mathcal{C}^1} T \sup_{t \in [0,T]} |\tilde{X}_t - \tilde{X}_t^k|. \end{aligned}$$

For any $\varepsilon > 0$, we can choose an increasing sequence $(k(n))_{n \in \mathbb{N}}$ such that

$$\mathbb{P}\left(\|b^n\|_{\mathcal{C}^1} T \sup_{t \in [0,T]} |\tilde{X}_t - \tilde{X}_t^{k(n)}| > \varepsilon\right) < \frac{1}{n}, \forall n \in \mathbb{N}.$$

Hence, we get that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^{k(n)}) dr \right| = 0 \text{ in probability.}$$

Now, let us bound A_2 . Let $\beta' < \beta$ with $\beta' - 1/p > -1/(2H) + 1/2$. By Lemma 7.3 applied to \tilde{X}^k , $h = b^n - \tilde{b}^k$ and β' instead of β , we know that there exists a random variable $Z_{n,k}$ such that

$$\begin{aligned} \mathbb{E}[Z_{n,k}] &\leq C \|b^n - \tilde{b}^k\|_{\mathcal{B}_p^{\beta'}} (1 + \|\tilde{b}^k\|_{\mathcal{B}_p^{\beta'}}^2) \\ &\leq C (\|b^n - b\|_{\mathcal{B}_p^{\beta'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_p^{\beta'}}) (1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^{\beta'}}^2), \end{aligned} \tag{7.10}$$

for C independent of k, n and that we have

$$\sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| \leq Z_{n,k} (1 + T).$$

Using Markov's inequality and (7.10) we obtain that

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| > \varepsilon \right) &\leq \varepsilon^{-1} \mathbb{E}[Z_{n,k}] (1 + T) \\ &\leq C \varepsilon^{-1} (1 + T) (\|b^n - b\|_{\mathcal{B}_p^{\beta'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_p^{\beta'}}) (1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^{\beta'}}^2). \end{aligned}$$

Choosing $k = k(n)$ as before we get

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^{k(n)}) dr - \int_0^t \tilde{b}^{k(n)}(\tilde{X}_r^{k(n)}) dr \right| = 0 \text{ in probability.}$$

To bound the last summand A_3 , recall that $\tilde{X}_t^k = \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr + \tilde{B}_t^k$. Hence, we get that

$$\sup_{t \in [0, T]} \left| \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr - \tilde{K}_t \right| \leq \sup_{t \in [0, T]} (|\tilde{X}_t^k - \tilde{X}_t| + |\tilde{B}_t^k - \tilde{B}_t|).$$

Since by assumption $(\tilde{X}^k, \tilde{B}^k)_{k \in \mathbb{N}}$ converges to (\tilde{X}, \tilde{B}) on $[\mathcal{C}_{[0, T]}]^2$ in probability, we get that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t \tilde{b}^{k(n)}(\tilde{X}_r^{k(n)}) dr - \tilde{K}_t \right| = 0 \text{ in probability}$$

and therefore (7.8) holds true.

It remains to show that (7.7) holds true. By Lemma 7.2, we know that there exists some $C > 0$ such that for any $(s, t) \in \Delta_{[0, T]}$,

$$\|\mathbb{E}^s [|(\tilde{X}_t^k - \tilde{B}_t^k) - (\tilde{X}_s^k - \tilde{B}_s^k)|^m]^{1/m}\|_{L^\infty} \leq C (1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^\beta}^2) (t - s)^{1+H(\beta-1/p)}. \tag{7.11}$$

Using that $\int_0^t \tilde{b}^k(\tilde{X}_r^k) dr$ converges to K_t on $\mathcal{C}_{[0, T]}$ in probability (by assumption) and that $\sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^\beta}$ is finite, we get (7.7) by applying Fatou's Lemma to (7.11). \square

7.3 Approximation by smooth drifts

Proof of Theorem 2.8. Instead of constructing a single weak solution, we will construct a couple (Y^1, Y^2) . This does not change the nature of the proof, but it will be extremely

useful in the proof of Theorem 2.10 when we will seek to construct a strong solution, via a Gyöngy-Krylov argument.

Let $(b_1^n)_{n \in \mathbb{N}}, (b_2^n)_{n \in \mathbb{N}}$ be sequences of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. By Proposition 7.4 and Remark 7.5, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(X^{1,n_k}, X^{2,n_k}, B)_{k \in \mathbb{N}}$ converges weakly in $[\mathcal{C}_{[0,T]}]^3$. Without loss of generality, we assume that $(X^{1,n}, X^{2,n}, B)_{n \in \mathbb{N}}$ converges weakly. By the Skorokhod representation Theorem, there exists a sequence of random variables $(Y^{1,n}, Y^{2,n}, \hat{B}^n)_{n \in \mathbb{N}}$ defined on a common probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, such that

$$\text{Law}(Y^{1,n}, Y^{2,n}, \hat{B}^n) = \text{Law}(X^{1,n}, X^{2,n}, B), \quad \forall n \in \mathbb{N}, \tag{7.12}$$

and $(Y^{1,n}, Y^{2,n}, \hat{B}^n)$ converges a.s. to some (Y^1, Y^2, \hat{B}) in $[\mathcal{C}_{[0,T]}]^3$. As $X^{i,n}$ solves the SDE (1.1) with drift b_i^n , we know by (7.12) that $Y^{i,n}$ also solves (1.1) with drift b_i^n and \hat{B}^n instead of B . As $X^{i,n}$ is a strong solution, we have that $X^{i,n}$ is adapted to \mathbb{F}^B . Hence by (7.12), we know that $Y^{i,n}$ is adapted to $\mathbb{F}^{\hat{B}^n}$ as the conditional laws of $Y^{i,n}$ and $X^{i,n}$ agree and therefore they are strong solutions to (1.1) with \hat{B}^n instead of B .

By Proposition 7.6, we know that Y^1 and Y^2 fulfill (2.3) and (2.4) from Definition 2.4 with \hat{B} instead of B and they are both adapted with respect to the filtration $\hat{\mathbb{F}}$ defined by $\hat{\mathcal{F}}_t := \sigma(Y_s^1, Y_s^2, \hat{B}_s, s \in [0, t])$.

Following the same arguments as in the proof of Theorem 2.5, after passing to the limit, we know that \hat{B} is an $\hat{\mathbb{F}}$ -fBm and Y^1 and Y^2 are weak solutions adapted to $\hat{\mathbb{F}}$.

Lastly, (7.7) gives that

$$[Y^i - \hat{B}]_{\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})} < \infty, \quad i = 1, 2, \tag{7.13}$$

and thus $Y^1 - \hat{B}$ and $Y^2 - \hat{B}$ belong to $\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^m)$. □

7.4 Existence of strong solutions

In order to prove strong existence, we follow a Yamada-Watanabe argument: this is done here by combining the construction of weak solutions from the previous proof, the uniqueness result Proposition 6.1 and Gyöngy-Krylov's result.

Lemma 7.7. *Let $H < 1/2$, $p \in [1, \infty]$ and β such that $\beta > -1/(2H) + 1$ and $\beta - 1/p \geq -1/(2H) + 1$. Let $b \in \mathcal{B}_p^\beta$, and $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. For $n \in \mathbb{N}$, consider X^n the strong solution to (1.1) with drift b^n and initial condition $X_0 \in \mathbb{R}$. Then there exists an \mathbb{F}^B -adapted process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ such that*

- (i) $\sup_{t \in [0, T]} |X_t^n - X_t| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$;
- (ii) $[X - B]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^{2,\infty})} < \infty$ a.s.

Proof. Let $(X^{\phi(n)})_{n \in \mathbb{N}}$ and $(X^{\psi(n)})_{n \in \mathbb{N}}$ be two arbitrary subsequences of $(X^n)_{n \in \mathbb{N}}$. From the previous proof, we know that there exists a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{P})$, an $\hat{\mathbb{F}}$ -fBm \hat{B} and a pair (Y^1, Y^2) of weak solutions to (1.1) adapted to $\hat{\mathbb{F}}$. By tightness (Proposition 7.4 and Remark 7.5), we also have that there exist subsequences $\tilde{\phi}(n)$ and $\tilde{\psi}(n)$ such that $(X^{\tilde{\phi}(n)}, X^{\tilde{\psi}(n)})$ converges weakly to (Y^1, Y^2) on $[\mathcal{C}_{[0,T]}]^2$ for $n \rightarrow \infty$.

Combining (7.13), (6.2) and that by assumption $1 + H(\beta - 1/p) \geq 1/2 + H$, we get that

$$[Y^i - \hat{B}]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2)} \leq [Y^i - \hat{B}]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^{2,\infty})} < \infty, \quad i = 1, 2.$$

By Proposition 6.1, this gives $Y^1 = Y^2$ a.s. Hence by Lemma 1.1 in [26], we get that there exists X such that X^n converges in probability to X on $\mathcal{C}_{[0,T]}$. Notice that X^n is adapted to \mathbb{F}^B , for any $n \in \mathbb{N}$, as they are strong solutions to (1.1). So X is adapted to the same filtration. Lastly, (ii) follows from (7.7). □

Proof of Theorem 2.10. (a) Let X be the process constructed in Lemma 7.7. Proposition 7.6 yields that X is a strong solution to (1.1) fulfilling (7.7) for any $m \geq 2$. Since $\beta - 1/p \geq 1 - 1/(2H)$, we get that $[X - B]_{\mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^{m,\infty})} < \infty$ for any $m \geq 2$.

(b) Let (Y^1, B) and (Y^2, B) be weak solutions defined on the same probability space, with $Y^1 - B$ and $Y^2 - B$ in $\mathcal{C}_{[0,T]}^{1/2+H}(L^2)$. On this probability space, let X be a strong solution which satisfies (6.3) with the same fBm B . Since X is also a weak solution adapted to \mathbb{F} , it follows from Proposition 6.1 that $X = Y^i$ for $i = 1, 2$.

(c) By Lemma 7.7 the sequence of strong solutions corresponding to an approximation sequence of the drift b converges in probability to a strong solution X to (1.1) such that

$$[X - B]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^{2,\infty})} < \infty. \tag{7.14}$$

The limit X is the same for any approximation sequence as uniqueness in the class of solutions fulfilling (7.14) holds by (b).

(d) Observe that if b is a measure, then we know by applying Proposition 5.3 and using $\beta - 1/p \geq 1 - 1/(2H)$ that any solution X to (1.1) verifies $X - B \in \mathcal{C}_{[0,T]}^{1/2+H}(L^2)$. Hence by the previous uniqueness statement, X is the pathwise unique solution to (1.1). \square

Appendices

A Elementary results on Besov spaces

Definition A.1 (Partition of unity). Let $\chi, \rho \in C^\infty(\mathbb{R})$ be even functions and for $j \geq 0$, $\rho_j(x) = \rho(2^{-j}x)$. We assume that there exists $a, b, c > 0$ with $\text{supp}(\chi) \subset [-c, c]$ and $\text{supp}(\rho) \subset [-b, -a] \cup [a, b]$. Moreover, we have

$$\chi + \sum_{j \geq 0} \rho_j \equiv 1, \tag{A.1}$$

$$\text{supp}(\chi) \cap \text{supp}(\rho_j) = \emptyset, \quad \forall j \geq 1, \tag{A.2}$$

$$\text{supp}(\rho_j) \cap \text{supp}(\rho_i) = \emptyset, \quad \text{if } |i - j| \geq 2. \tag{A.3}$$

Then we call the pair (χ, ρ) a partition of unity.

Existence of a partition of unity is proven in [4, Prop. 2.10]. Let such a partition be fixed. We denote the Schwartz space on \mathbb{R} by \mathcal{S} and the space of tempered distributions by \mathcal{S}' .

Definition A.2 (Littlewood-Paley blocks). Let $f \in \mathcal{S}'$. We define its j -th Littlewood-Paley block by

$$\Delta_j f = \begin{cases} \mathcal{F}^{-1}(\rho_j \mathcal{F}(f)) & \text{for } j \geq 0, \\ \mathcal{F}^{-1}(\chi \mathcal{F}(f)) & \text{for } j = -1, \\ 0 & \text{for } j \leq -2, \end{cases}$$

where \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and its inverse.

Definition A.3. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, let the nonhomogeneous Besov space $\mathcal{B}_{p,q}^s$ be the space of tempered distributions $u \in \mathcal{S}'$ such that

$$\|u\|_{\mathcal{B}_{p,q}^s} := \left(\sum_{j \in \mathbb{Z}} (2^{js} \|\Delta_j u\|_{L^p(\mathbb{R})})^q \right)^{\frac{1}{q}} < \infty.$$

Lemma A.4 and A.5, both taken from [2], are used on a regular basis throughout the paper. For the readers convenience we state them again below.

Lemma A.4. *Let $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$ and the Gaussian semigroup G_t defined by (1.4). Then there exists $C > 0$ such that for any $f \in \mathcal{B}_p^\gamma$,*

- (a) $\|G_t f\|_{L^p(\mathbb{R})} \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{\gamma/2}$ for any $t > 0$, provided that $\gamma < 0$;
- (b) $\lim_{t \rightarrow 0} G_t f = f$ in $\mathcal{B}_p^{\tilde{\gamma}}$ for every $\tilde{\gamma} < \gamma$;
- (c) $\sup_{t > 0} \|G_t f\|_{\mathcal{B}_p^\gamma} \leq \|f\|_{\mathcal{B}_p^\gamma}$;
- (d) $\|G_t f\|_{C^1} \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{(\gamma-1/p-1)/2}$ for all $t > 0$, provided that $\gamma - 1/p < 0$.

Lemma A.5. *Let f be a tempered distribution on \mathbb{R} , $\gamma \in \mathbb{R}$, $p \in [1, \infty]$. Then for any $a, a_1, a_2, a_3 \in \mathbb{R}$, $\alpha, \alpha_1, \alpha_2 \in [0, 1]$, there exists a constant $C > 0$ such that*

- (a) $\|f(a + \cdot)\|_{\mathcal{B}_p^\gamma} = \|f\|_{\mathcal{B}_p^\gamma}$;
- (b) $\|f(a_1 + \cdot) - f(a_2 + \cdot)\|_{\mathcal{B}_p^\gamma} \leq C |a_1 - a_2|^\alpha \|f\|_{\mathcal{B}_p^{\gamma+\alpha}}$;
- (c) $\|f(a_1 + \cdot) - f(a_2 + \cdot) - f(a_3 + \cdot) + f(a_3 + a_2 - a_1 + \cdot)\|_{\mathcal{B}_p^\gamma} \leq C |a_1 - a_2|^{\alpha_1} |a_1 - a_3|^{\alpha_2} \|f\|_{\mathcal{B}_p^{\gamma+\alpha_1+\alpha_2}}$.

Lemma A.6. *Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ and $f \in \mathcal{B}_p^\gamma$ and $t > 0$, the function $G_t f$ defined by (1.4) is smooth and bounded.*

Proof. Smoothness comes from [40, Th 3.13]. We now prove boundedness: First, we consider the case $f \in \mathcal{B}_p^\gamma$ for $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$ such that $\gamma - 1/p < 0$. Using Lemma A.4(a) and the embedding $\mathcal{B}_p^\gamma \hookrightarrow \mathcal{B}_\infty^{\gamma-1/p}$, we have

$$\|G_t f\|_{L^\infty} \leq C_\gamma \|f\|_{\mathcal{B}_\infty^{\gamma-1/p}} t^{\frac{\gamma}{2} - \frac{1}{2p}} \leq C_\gamma \|f\|_{\mathcal{B}_p^\gamma} t^{\frac{\gamma}{2} - \frac{1}{2p}}.$$

So, $G_t f$ is bounded. Now for arbitrary $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$ such that $f \in \mathcal{B}_p^\gamma$, one can choose $\gamma' < \gamma$ such that $\gamma' - 1/p < 0$. We then have $f \in \mathcal{B}_p^{\gamma'}$ since $\mathcal{B}_p^\gamma \subset \mathcal{B}_p^{\gamma'}$. By the first part of the proof $G_t f$ is bounded. \square

The following lemma provides an embedding between Hölder and Besov spaces for functions with compact support. It is useful when we deal with local times.

Lemma A.7. *Let $M \subset \mathbb{R}$ be a compact set, $0 < \beta < \hat{\beta} < 1$ and $p, q \in [1, \infty]$. There exists constants $C, \tilde{C} > 0$ only depending on M such that for any $f \in \mathcal{C}^{\hat{\beta}}$ with support included in M , we have*

$$\|f\|_{\mathcal{B}_{p,q}^\beta} \leq C \|f\|_{\mathcal{B}_\infty^{\hat{\beta}}} \leq \tilde{C} \|f\|_{\mathcal{C}^{\hat{\beta}}}.$$

The first inequality follows by Remark 1.3 and the second inequality by the Proposition on page 14 in [39].

B Properties of fBm

B.1 Operator linking Bm and fBm

In Lemma B.1 we prove a continuity property of the operator \mathcal{A} introduced in (2.1). This is crucial in order to show adaptedness in the proof of 2.5(a) and Theorem 3.4. The explicit form of the operator \mathcal{A} is:

$$\mathcal{A} = \tilde{\Pi}^{H-1/2} \mathcal{I}^{1/2-H} \tilde{\Pi}^{1/2-H}, \tag{B.1}$$

where the operators $\tilde{\Pi}^h$ and \mathcal{I}^h are defined as follows:

$$\begin{aligned} \forall h \in \mathbb{R}, \quad (\tilde{\Pi}^h f)(t) &:= t^h f(t) - h \int_0^t s^{h-1} f(s) ds, \\ \forall h > 0, \quad (\mathcal{I}^h f)(t) &:= \frac{1}{\Gamma(h)} \int_0^t (t-s)^{h-1} f(s) ds. \end{aligned}$$

Lemma B.1. *Let $H < 1/2$. The operator \mathcal{A} defined in (B.1) continuously maps the space $(\mathcal{C}_{[0,T]}, \|\cdot\|_\infty)$ to itself.*

Proof. The operator \mathcal{A} is linear, therefore it is sufficient to show that it is bounded. For $f \in \mathcal{C}_{[0,T]}$ and $t \in [0, T]$, we have by (B.1) that

$$\tilde{\Pi}^{H-1/2} \mathcal{I}^{1/2-H} \tilde{\Pi}^{1/2-H} f(t) = \sum_{i=1}^4 f_i(t),$$

where

$$\begin{aligned} f_1(t) &= t^{H-1/2} \frac{1}{\Gamma(1/2-H)} \int_0^t (t-y)^{-1/2-H} y^{1/2-H} f(y) dy; \\ f_2(t) &= t^{H-1/2} (1/2-H) \frac{-1}{\Gamma(1/2-H)} \int_0^t (t-y)^{-1/2-H} \int_0^y x^{-1/2-H} f(x) dx dy; \\ f_3(t) &= (1/2-H) \frac{1}{\Gamma(1/2-H)} \int_0^t s^{H-3/2} \int_0^s (s-y)^{-1/2-H} y^{1/2-H} f(y) dy ds; \\ f_4(t) &= (1/2-H)^2 \frac{-1}{\Gamma(1/2-H)} \int_0^t s^{H-3/2} \int_0^s (s-y)^{-1/2-H} \int_0^y r^{-1/2-H} f(r) dr dy ds. \end{aligned}$$

We have, after doing a change of variables of the form $\xi = y/t$, uniformly on $[0, T]$, that

$$\begin{aligned} |f_1(t)| &\leq C t^{H-1/2} \|f\|_\infty \int_0^1 t^{-1/2-H} t^{1/2-H} t(1-z)^{-1/2-H} z^{1/2-H} dz \leq C T^{1/2-H} \|f\|_\infty; \\ |f_2(t)| &\leq C t^{H-1/2} \|f\|_\infty \int_0^t (t-y)^{-1/2-H} y^{1/2-H} dy \leq C T^{1/2-H} \|f\|_\infty; \\ |f_3(t)| &\leq C \|f\|_\infty \int_0^t s^{H-3/2} s^{1-2H} ds \leq C T^{1/2-H} \|f\|_\infty; \\ |f_4(t)| &\leq C \|f\|_\infty \int_0^t s^{H-3/2} \int_0^s (s-y)^{-1/2-H} y^{1/2-H} dy ds \leq C T^{1/2-H} \|f\|_\infty. \end{aligned}$$

Hence, the operator is bounded. It remains to prove that $\mathcal{A}(f)$ is a bounded continuous function. For f_1 , we perform the change of variables $z = y/t$ and it comes

$$f_1(t) = t^{1/2-H} \frac{1}{\Gamma(1/2-H)} \int_0^1 (1-z)^{-1/2-H} z^{1/2-H} f(tz) dz.$$

By the dominated convergence theorem, this is a continuous function. We proceed similarly for f_2 . As for $i = 3, 4$, we have

$$f_i(t) = \int_0^t g_i(s) ds$$

for functions g_i that are integrable on $[0, T]$, which implies the continuity of f_3 and f_4 . \square

B.2 Local nondeterminism of fBm and proof of Lemma 5.1

The goal of this section is to prove Lemma 5.1. As an intermediate step, we obtain first the following lemma, which is a local nondeterminism property for the fractional Brownian motion.

Lemma B.2. *Let $(B_t)_{t \geq 0}$ be an \mathbb{F} -fractional Brownian motion with $H \leq 1/2$. There exists $C > 0$ such that for any $0 \leq s < u < t$,*

$$\mathbb{E} [(\mathbb{E}^u[B_t] - \mathbb{E}^s[B_t])^2] \geq C(u - s)(t - s)^{2H-1}.$$

Note that in the above the conditional expectation is taken as usual with respect to \mathbb{F} and not with respect to the filtration generated by B .

Proof. The case $H = 1/2$ is trivial, so we assume $H < 1/2$. Recall that the process $W = \mathcal{A}(B)$, where \mathcal{A} is the operator given in (2.1), is an \mathbb{F} -Brownian motion. Moreover, by Theorem 11 in [36], it satisfies

$$B_t = \int_0^t K_H(t, r) dW_r, \tag{B.2}$$

where, for some $d_H > 0$,

$$K_H(t, r) = d_H \left[\left(\frac{t}{r}\right)^{H-1/2} (t - r)^{H-1/2} + (1/2 - H)r^{1/2-H} \int_r^t z^{H-3/2} (z - r)^{H-1/2} dz \right].$$

Therefore

$$\begin{aligned} \mathbb{E} [(\mathbb{E}^u[B_t] - \mathbb{E}^s[B_t])^2] &= \mathbb{E} \left[\left(\mathbb{E}^u \left[\int_0^t K_H(t, r) dW_r \right] - \mathbb{E}^s \left[\int_0^t K_H(t, r) dW_r \right] \right)^2 \right] \\ &= \mathbb{E} \left[\left(\int_s^u K_H(t, r) dW_r \right)^2 \right] \\ &= \int_s^u K_H(t, r)^2 dr \\ &\geq (u - s) \min_{r \in [s, u]} K_H(t, r)^2. \end{aligned} \tag{B.3}$$

Notice that by the change of variables $\tilde{z} = z/r$,

$$\int_r^t z^{H-3/2} (z - r)^{H-1/2} dz = r^{2H-1} \beta_H(t/r),$$

for $\beta_H(\tau) = \int_1^\tau \tilde{z}^{H-3/2} (\tilde{z} - 1)^{H-1/2} d\tilde{z}$. Hence,

$$K_H(t, r) = d_H \left[\left(\frac{t}{r}\right)^{H-1/2} (t - r)^{H-1/2} + (1/2 - H)r^{H-1/2} \beta_H(t/r) \right].$$

Using that

$$\begin{aligned} \beta_H \left(\frac{t}{r}\right) &= \int_1^{t/r} z^{H-3/2} (z - 1)^{H-1/2} dz \\ &\geq \int_1^{t/r} z^{2H-2} dz \\ &= \frac{1}{1 - 2H} \left(1 - \left(\frac{t}{r}\right)^{2H-1} \right) \end{aligned}$$

and subadditivity of $x \mapsto x^\alpha$ for $\alpha \in (0, 1)$ we get, for $0 < r < t$, that

$$\begin{aligned} \frac{1}{d_H} K_H(t, r) &\geq \left(\frac{t}{r}\right)^{H-1/2} (t-r)^{H-1/2} + \frac{1/2-H}{1-2H} r^{H-1/2} \left(1 - \left(\frac{t}{r}\right)^{2H-1}\right) \\ &\geq (t-r)^{H-1/2} \left(\left(\frac{r}{t}\right)^{1/2-H} + \frac{1}{2}(t-r)^{1/2-H} r^{H-1/2} \left(1 - \left(\frac{r}{t}\right)^{1-2H}\right)\right) \\ &\geq (t-r)^{H-1/2} \left(\left(\frac{r}{t}\right)^{1/2-H} + \frac{1}{2}(t^{1/2-H} - r^{1/2-H}) r^{H-1/2} \left(1 - \left(\frac{r}{t}\right)^{1-2H}\right)\right) \\ &= \frac{1}{2}(t-r)^{H-1/2} \left(\left(\frac{r}{t}\right)^{1/2-H} + \left(\frac{t}{r}\right)^{1/2-H} + \left(\frac{r}{t}\right)^{1-2H} - 1\right) \\ &\geq \frac{1}{2}(t-r)^{H-1/2}, \end{aligned}$$

where the last line holds true as $x + x^{-1} \geq 2$ for $x > 0$. Plugging this into (B.3) we get that

$$\begin{aligned} \mathbb{E} [(\mathbb{E}^u[B_t] - \mathbb{E}^s[B_t])^2] &\geq (u-s) \min_{r \in [s, u]} K_H(t, r)^2 \\ &\geq \frac{d_H^2}{4} (u-s)(t-s)^{2H-1}. \quad \square \end{aligned}$$

Proof of Lemma 5.1. Proof of (a): Notice that $B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}]$ is Gaussian with zero mean and variance σ_{t_1, t_2}^2 . Furthermore, it is independent of \mathcal{F}_{t_1} , which can be seen using the integral representation (see (B.2)). Hence, we have

$$\begin{aligned} \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)] &= \mathbb{E}^{t_1}[f(B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}] + \mathbb{E}^{t_1}[B_{t_2}], \Xi)] \\ &= G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi). \end{aligned}$$

Proof of (b): By the local nondeterminism property of fBm (see Lemma 7.1 in [37]), there exists $C > 0$ such that

$$\sigma_{t_1, t_2}^2 = C(t_2 - t_1)^{2H}. \tag{B.4}$$

To see that (b) holds true we use (a), Lemma A.4(a), the embedding $\mathcal{B}_p^\gamma \hookrightarrow \mathcal{B}_\infty^{\gamma-1/p}$ and (B.4) to get that

$$\|\mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]\| \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_\infty^{\gamma-1/p} \sigma_{t_1, t_2}^{\gamma-1/p}} \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} (t_2 - t_1)^{H(\gamma-1/p)}.$$

Proof of (c): First notice that for fixed $x \in \mathbb{R}$, $t > 0$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ with $\|\phi\|_{C^1} < \infty$,

$$|\phi(x) - G_t \phi(x)| \leq C \|\phi\|_{C^1} \sqrt{t}. \tag{B.5}$$

To see that (c) holds true note that due to (a), (B.5) and Cauchy-Schwarz's inequality we get

$$\begin{aligned} \mathbb{E}[|f(B_{t_2}, \Xi) - \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]|] &= \mathbb{E}[|f(B_{t_2}, \Xi) - G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi)|] \\ &\leq \mathbb{E}[|f(B_{t_2}, \Xi) - f(\mathbb{E}^{t_1}[B_{t_2}], \Xi)|] + \mathbb{E}[|f(\mathbb{E}^{t_1}[B_{t_2}], \Xi) - G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi)|] \\ &\leq \mathbb{E}[\|f(\cdot, \Xi)\|_{C^1} |B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}]|] + C \mathbb{E}[\|f(\cdot, \Xi)\|_{C^1}] \sigma_{t_1, t_2} \\ &\leq C \| \|f(\cdot, \Xi)\|_{C^1} \|_{L^2} (t_2 - t_1)^H, \end{aligned}$$

where the last line holds true by (B.4) and as

$$\|(B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}])\|_{L^2} \leq \|B_{t_2} - B_{t_1}\|_{L^2} + \|B_{t_1} - \mathbb{E}^{t_1}[B_{t_2}]\|_{L^2} \leq 2(t_2 - t_1)^H$$

by Jensen's inequality for conditional expectations.

Proof of (d): In order to prove (d) we first state the following inequality, for some $C > 0$: for any $t_1 < \tilde{t} < t_2$,

$$\hat{\sigma}_{t_1, \tilde{t}, t_2}^2 := \text{Var}(\mathbb{E}^{\tilde{t}}[B_{t_2}] - \mathbb{E}^{t_1}[B_{t_2}]) \geq C(\tilde{t} - t_1)(t_2 - t_1)^{-1+2H}. \tag{B.6}$$

The above inequality holds true by Lemma B.2. Notice that $\mathbb{E}^{\tilde{t}}[B_{t_2}] - \mathbb{E}^{t_1}[B_{t_2}]$ is Gaussian with mean zero and variance $\hat{\sigma}_{t_1, \tilde{t}, t_2}^2$ and it is independent of \mathcal{F}_{t_1} . Using this, (a) and Hölder's inequality for $q = p/n \geq 1$ and $q' = q/(q - 1)$, we get

$$\begin{aligned} \mathbb{E}^{t_1} \left[|\mathbb{E}^{\tilde{t}}[f(B_{t_2}, \Xi)]|^n \right] &= \mathbb{E}^{t_1} \left[|G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} f(\mathbb{E}^{\tilde{t}}[B_{t_2}], \Xi)|^n \right] \\ &= \int g_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2}(z) |G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}] + z, \Xi)|^n dz \\ &\leq \|g_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2}\|_{L^{q'}(\mathbb{R})} \left\| \left(G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}] + \cdot, \Xi) \right)^n \right\|_{L^q(\mathbb{R})} \\ &= \|G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} \delta_0\|_{L^{q'}(\mathbb{R})} \|G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} f(\cdot, \Xi)\|_{L^p(\mathbb{R})}^n. \end{aligned}$$

Using Lemma A.4(a), (B.4), (B.6) and that $\|\delta_0\|_{\mathcal{B}_x^{-1+1/x}} < \infty$ for $x \geq 1$, we get

$$\begin{aligned} \|G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} \delta_0\|_{L^{q'}(\mathbb{R})} \|G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} f(\cdot, \Xi)\|_{L^p(\mathbb{R})}^n &\leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n \|\delta_0\|_{\mathcal{B}_{q'}^{-n/p}} \sigma_{t_1, \tilde{t}, t_2}^{\gamma n} \hat{\sigma}_{t_1, \tilde{t}, t_2}^{-n/p} \\ &\leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n \|\delta_0\|_{\mathcal{B}_{q'}^{-n/p}} (t_2 - \tilde{t})^{H\gamma n} (\tilde{t} - t_1)^{-n/(2p)} (t_2 - t_1)^{n(1-2H)/(2p)}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}^{t_1} \left[|\mathbb{E}^{\tilde{t}}[f(B_{t_2}, \Xi)]|^n \right] &\leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n \|\delta_0\|_{\mathcal{B}_{q'}^{-n/p}} (t_2 - \tilde{t})^{H\gamma n} (\tilde{t} - t_1)^{-n/(2p)} (t_2 - t_1)^{n(1-2H)/(2p)}. \end{aligned} \tag{B.7}$$

After taking expectations in (B.7) and raising both sides to the power $1/n$, we obtain (d). \square

C Stochastic sewing Lemma and regularising properties of the fBm

In this section we will state and prove some results that are crucial throughout the paper. The statements and their proofs are close extensions to fBm of Lemma 6.1, its corollaries and Lemma 5.2 in [2].

First, in the following two lemmas we recall two recent extensions of the stochastic sewing Lemma (see [32, Th. 4.1 and Th. 4.7]). In both statements, let $0 < S < T$, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let $A : \Delta_{[S, T]} \rightarrow L^m$ such that $A_{s, t}$ is \mathcal{F}_t -measurable for $(s, t) \in \Delta_{[S, T]}$.

Lemma C.1 (Stochastic sewing Lemma). *Let $n \in [m, \infty]$. Assume that there exist constants $\Gamma_1, \Gamma_2 \geq 0$, $\varepsilon_1, \varepsilon_2 > 0$, $\alpha_2 \in [0, 1/2)$ such that for every $(s, t) \in \Delta_{[S, T]}$ and $u := (s + t)/2$,*

$$\|\mathbb{E}^s[\delta A_{s, u, t}]\|_{L^n} \leq \Gamma_1 (t - s)^{1+\varepsilon_1}, \tag{C.1}$$

$$\|\mathbb{E}^S[|\delta A_{s, u, t}|^m]^{1/m}\|_{L^n} \leq \Gamma_2 (u - S)^{-\alpha_2} (t - s)^{1/2+\varepsilon_2}. \tag{C.2}$$

Suppose there exists a process $(\mathcal{A}_t)_{t \in [S, T]}$ such that, for any $t \in [S, T]$ and any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[S, t]$ with mesh size going to zero, we have

$$\mathcal{A}_t = \lim_{k \rightarrow \infty} \sum_{i=0}^{N_k} A_{t_i^k, t_{i+1}^k} \text{ in probability.} \tag{C.3}$$

Moreover, there exists a constant $C = C(\varepsilon_1, \varepsilon_2, \alpha_2, m)$ independent of S, T such that for every $(s, t) \in \Delta_{[S, T]}$ we have

$$\|\mathbb{E}^S[\mathcal{A}_t - \mathcal{A}_s - A_{s,t}]^{1/m}\|_{L^n} \leq C\Gamma_2(t-s)^{1/2-\alpha_2+\varepsilon_2} + C\Gamma_1(t-s)^{1+\varepsilon_1}$$

and

$$\|\mathbb{E}^S[\mathcal{A}_t - \mathcal{A}_s - A_{s,t}]\|_{L^n} \leq C\Gamma_1(t-s)^{1+\varepsilon_1}.$$

Lemma C.2 (Stochastic sewing lemma with random controls). *Let λ be a random control. Assume that there exist constants $\Gamma_1, \alpha_1, \beta_1 \geq 0$ with $\alpha_1 + \beta_1 > 1$ such that*

$$|\mathbb{E}^u \delta A_{s,u,t}| \leq \Gamma_1 |t-s|^{\alpha_1} \lambda(s,t)^{\beta_1} \text{ a.s.} \tag{C.4}$$

for all $(s, t) \in \Delta_{[S, T]}$ and $u := (s+t)/2$. Additionally, assume that (C.2) holds for $m = n$ and $\alpha_2 = 0$ and that there exists a process $(\mathcal{A}_t)_{t \in [S, T]}$ such that (C.3) holds. Then there is a map $B: \Delta_{[S, T]} \rightarrow L^m$ and a constant $C > 0$ such that for all $(s, t) \in \Delta_{[S, T]}$,

$$|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}| \leq C\Gamma_1 |t-s|^{\alpha_1} \lambda(s,t)^{\beta_1} + B_{s,t} \text{ a.s. and} \tag{C.5}$$

$$\|B_{s,t}\|_{L^m} \leq C\Gamma_2 |t-s|^{1/2+\varepsilon_2}. \tag{C.6}$$

Lemma C.3. *Let $\gamma \in (-1/(2H), 0)$, $m \in [2, \infty)$, $n \in [m, \infty]$, $p \in [n, \infty]$ and $d \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for any $0 \leq S \leq T$, any \mathcal{F}_S -measurable random variable Ξ in \mathbb{R}^d and any bounded measurable function $f: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ fulfilling*

(i) $\mathbb{E}[\|f(\cdot, \Xi)\|_{C^1}^2] < \infty;$

(ii) $\mathbb{E}[\|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n] < \infty,$

we have for any $t \in [S, T]$ that

$$\left\| \mathbb{E}^S \left[\left| \int_S^t f(B_r, \Xi) dr \right|^m \right]^{1/m} \right\|_{L^n} \leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t-S)^{1+H(\gamma-1/p)}. \tag{C.7}$$

Proof. In order to show (C.7), we will apply Lemma C.1. For $S \leq s \leq t \leq T$, let

$$\mathcal{A}_t := \int_S^t f(B_r, \Xi) dr \quad \text{and} \quad A_{s,t} := \mathbb{E}^s \left[\int_s^t f(B_r, \Xi) dr \right].$$

Notice that we have $\mathbb{E}^s[\delta A_{s,u,t}] = 0$, so (C.1) trivially holds.

In order to establish inequality (C.2), we show that

$$\|\delta A_{s,u,t}\|_{L^n} \leq \Gamma_2(u-S)^{-\alpha_2}(t-s)^{1/2+\varepsilon_2} \tag{C.8}$$

holds true for some $\alpha_2 \in [0, 1/2)$ and $\varepsilon_2 > 0$, which is sufficient by the tower property and conditional Jensen's inequality. For $u = (s+t)/2$ we have by Minkowski's integral inequality, Jensen's inequality for conditional expectation and Lemma 5.1(d) that

$$\begin{aligned} \|\delta A_{s,u,t}\|_{L^n} &\leq \left\| \mathbb{E}^s \left[\int_u^t f(B_r, \Xi) dr \right] \right\|_{L^n} + \left\| \mathbb{E}^u \left[\int_u^t f(B_r, \Xi) dr \right] \right\|_{L^n} \\ &\leq \int_u^t (\|\mathbb{E}^s f(B_r, \Xi)\|_{L^n} + \|\mathbb{E}^u f(B_r, \Xi)\|_{L^n}) dr \\ &\leq 2 \int_u^t \|\mathbb{E}^u f(B_r, \Xi)\|_{L^n} dr \\ &\leq C \int_u^t \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (r-u)^{H\gamma} (u-S)^{-\frac{1}{2p}} (r-S)^{\frac{1-2H}{2p}} dr \\ &\leq C \int_u^t \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (r-u)^{H\gamma} (u-S)^{-\frac{H}{p}} dr \\ &\leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t-u)^{1+H\gamma} (u-S)^{-\frac{H}{p}}, \end{aligned}$$

where the penultimate inequality holds true as $r - S \leq 2(u - S)$. Hence, we have (C.8) for $\varepsilon_2 = 1/2 + H\gamma > 0$ and $\alpha_2 = H/p < 1/2$.

Let $t \in [S, T]$. Let $(\Pi_k)_{k \in \mathbb{N}}$ be a sequence of partitions of $[S, t]$ with mesh size converging to zero. For each k , denote $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$. By Lemma 5.1(c) we have that

$$\begin{aligned} \|\mathcal{A}_t - \sum_i A_{t_i^k, t_{i+1}^k}\|_{L^1} &\leq \sum_i \int_{t_i^k}^{t_{i+1}^k} \|f(B_r, \Xi) - \mathbb{E}^{t_i^k} f(B_r, \Xi)\|_{L^1} dr \\ &\leq C \|f(\cdot, \Xi)\|_{C^1} \|L^2\| (t - S) |\Pi_k|^H \rightarrow 0. \end{aligned}$$

Hence (C.3) holds true.

Applying Lemma C.1, we get

$$\begin{aligned} \|\mathbb{E}^S[|\mathcal{A}_t - \mathcal{A}_S|^m]^{1/m}\|_{L^n} &\leq \|A_{S,t}\|_{L^n} + C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|L^n\| \left(\int_S^t (r - S)^{-\frac{2H}{p}} dr\right)^{\frac{1}{2}} (t - S)^{\frac{1}{2} + H\gamma} \\ &\leq \|A_{S,t}\|_{L^n} + C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|L^n\| (t - S)^{1 + H(\gamma - 1/p)}. \end{aligned}$$

Applying Minkowski's integral inequality and Lemma 5.1(b), we get that

$$\begin{aligned} \|A_{S,t}\|_{L^n} &= \left\| \mathbb{E}^S \int_S^t f(B_r, \Xi) dr \right\|_{L^n} \\ &\leq \int_S^t \|\mathbb{E}^S f(B_r, \Xi)\|_{L^n} dr \\ &\leq C \int_S^t \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|L^n\| (r - S)^{H(\gamma - 1/p)} dr \\ &\leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|L^n\| (t - S)^{1 + H(\gamma - 1/p)}. \end{aligned}$$

Hence

$$\|\mathbb{E}^S[|\mathcal{A}_t - \mathcal{A}_S|^m]^{1/m}\|_{L^n} \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|L^n\| (t - S)^{1 + H(\gamma - 1/p)}. \quad \square$$

Corollary C.4. Let $\gamma \in (-1/(2H), 0)$, $m \in [2, \infty)$ and $p \in [m, \infty]$. Let $\lambda \in [0, 1]$ and assume that $\gamma > -1/(2H) + \lambda$. There exists $C > 0$ such that for any $f \in \mathcal{C}_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\gamma$, any $0 \leq s \leq t \leq T$ and any \mathcal{F}_s -measurable random variables $\kappa_1, \kappa_2 \in L^m$, one has

$$\begin{aligned} \left\| \int_s^t (f(B_r + \kappa_1) - f(B_r + \kappa_2)) dr \right\|_{L^m} &\leq C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^\lambda (t - s)^{1 + H(\gamma - \lambda - 1/p)}. \end{aligned} \quad (C.9)$$

Proof. We aim to apply Lemma C.3 for the function $(z, (x_1, x_2)) \mapsto f(z + x_1) - f(z + x_2)$ with $m = n$ and $\Xi = (\kappa_1, \kappa_2)$. By Lemma A.5(b) and Jensen's inequality, we have that

$$\begin{aligned} \|f(\cdot + \kappa_1) - f(\cdot + \kappa_2)\|_{\mathcal{B}_p^{\gamma - \lambda}} \|L^m\| &\leq C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^\lambda \\ &\leq C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^\lambda. \end{aligned}$$

Moreover

$$\|f(\cdot + \kappa_1) - f(\cdot + \kappa_2)\|_{C^1} \leq 2\|f\|_{C^1} < \infty.$$

Therefore all assumptions of Lemma C.3 are fulfilled and the result follows. □

Corollary C.5. *Let $\gamma \in (-1/(2H), 0)$, $m \in [2, \infty)$ and $p \in [m, \infty]$. Let $\lambda, \lambda_1, \lambda_2 \in (0, 1]$ and assume that $\gamma > -1/(2H) + \lambda$ and $\gamma > -1/(2H) + \lambda_1 + \lambda_2$. There exists a constant $C > 0$ such that for any $f \in C_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\gamma$, any $0 \leq s \leq u \leq t \leq T$, any \mathcal{F}_s -measurable random variables $\kappa_1, \kappa_2 \in L^m$ and any \mathcal{F}_u -measurable random variables $\kappa_3, \kappa_4 \in L^m$, we have that*

$$\begin{aligned} & \left\| \int_u^t (f(B_r + \kappa_1) - f(B_r + \kappa_2) - f(B_r + \kappa_3) + f(B_r + \kappa_4)) dr \right\|_{L^m} \\ & \leq C \|f\|_{\mathcal{B}_p^\gamma} \|\mathbb{E}^s[|\kappa_1 - \kappa_3|^m]^{1/m}\|_{L^\infty}^{\lambda_2} \|\kappa_1 - \kappa_2\|_{L^m}^{\lambda_1} (t-u)^{1+H(\gamma-\lambda_1-\lambda_2-1/p)} \\ & \quad + C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2 - \kappa_3 + \kappa_4\|_{L^m}^{\lambda} (t-u)^{1+H(\gamma-\lambda-1/p)}. \end{aligned} \tag{C.10}$$

Proof. Let $h : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ be defined by

$$h : (z, (x_1, x_2, x_3, x_4)) \mapsto f(z + x_1) - f(z + x_2) - f(z + x_3) + f(z + x_3 + x_2 - x_1).$$

and $g : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ by

$$g : (z, (x_1, x_2, x_3, x_4)) \mapsto f(z + x_4) - f(z + x_3 + x_2 - x_1).$$

Let $\Xi = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. Hence the integrand on the left hand side of (C.10) is $h(B_r, \Xi) + g(B_r, \Xi)$. The proof will consist in applying Lemma C.3 to the integral of h , and Corollary C.4 to the integral of g . By Lemma A.5(c), we have

$$\|h(\cdot, \Xi)\|_{\mathcal{B}_p^{\gamma-\lambda_1-\lambda_2}} \leq \|f\|_{\mathcal{B}_p^\gamma} |\kappa_1 - \kappa_2|^{\lambda_1} |\kappa_1 - \kappa_3|^{\lambda_2}. \tag{C.11}$$

Using that $\kappa_1 - \kappa_2$ is \mathcal{F}_s -measurable, by the tower property, Jensen's inequality for conditional expectation and (C.11), we get that

$$\begin{aligned} \mathbb{E} \|h(\cdot, \Xi)\|_{\mathcal{B}_p^{\gamma-\lambda_1-\lambda_2}}^m & \leq \|f\|_{\mathcal{B}_p^\gamma} \mathbb{E} [|\kappa_1 - \kappa_2|^{m\lambda_1} \mathbb{E}^s[|\kappa_1 - \kappa_3|^{m\lambda_2}]] \\ & \leq \|f\|_{\mathcal{B}_p^\gamma} \mathbb{E} [|\kappa_1 - \kappa_2|^{m\lambda_1}] \|\mathbb{E}^s[|\kappa_1 - \kappa_3|^{m\lambda_2}]\|_{L^\infty} \\ & \leq \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^{m\lambda_1} \|\mathbb{E}^s[|\kappa_1 - \kappa_3|^m]^{1/m}\|_{L^\infty}^{m\lambda_2}. \end{aligned}$$

Furthermore

$$\|h(\cdot, \Xi)\|_{C^1} \leq 4\|f\|_{C^1}.$$

Hence, we get the result by applying Lemma C.3 for $S = u$ to the integral of h and Corollary C.4 to the integral of g . \square

As another consequence of the stochastic sewing Lemma, we finally prove Lemma 7.1.

Proof of Lemma 7.1. We assume that $[\psi]_{C_{[s,t]}^\alpha(L^{m,n})} < \infty$, otherwise (7.1) trivially holds. For $(\tilde{s}, \tilde{t}) \in \Delta_{[s,t]}$, let

$$A_{\tilde{s}, \tilde{t}} := \int_{\tilde{s}}^{\tilde{t}} f(B_r + \psi_{\tilde{s}}) dr \text{ and } \mathcal{A}_t := \int_s^{\tilde{t}} f(B_r + \psi_r) dr. \tag{C.12}$$

In the following, we check the necessary conditions in order to apply Lemma C.1. In order to show that (C.1) and (C.2) hold true with $\alpha_2 = 0$, $\varepsilon_1 = H(\gamma - 1/p - 1) + \alpha > 0$ and $\varepsilon_2 = 1/2 + H(\gamma - 1/p) > 0$, we show that there exists a constant $C > 0$ independent of s, t, \tilde{s} and \tilde{t} such that

$$(i) \quad \|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{C_{[s,t]}^\alpha(L^{m,n})} (\tilde{t} - \tilde{s})^{1+H(\gamma-1-1/p)+\alpha}.$$

- (ii) $\|\mathbb{E}^s [|\delta A_{\tilde{s}, u, \tilde{t}}|^m]^{1/m} \|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}$.
- (iii) If (i) and (ii) are verified, then (C.3) gives the convergence in probability of $\sum_{i=0}^{N_n-1} A_{t_i^k, t_{i+1}^k}$ along any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[s, \tilde{t}]$ with mesh converging to 0. We will show that the limit is the process \mathcal{A} given in (C.12).

Assume for now that (i), (ii) and (iii) hold. Applying Lemma C.1, we obtain that

$$\begin{aligned} \left\| \mathbb{E}^s \left[\left| \int_{\tilde{s}}^{\tilde{t}} f(B_r + \psi_r) dr \right|^m \right]^{1/m} \right\|_{L^n} &\leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)} \\ &\quad + C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}_{[s, \tilde{t}]}(L^{m, n})}^\alpha (\tilde{t} - \tilde{s})^{1+H(\gamma-1-1/p)+\alpha} \\ &\quad + \|\mathbb{E}^s [A_{\tilde{s}, \tilde{t}}^m]^{1/m} \|_{L^n}. \end{aligned}$$

In (C.14) we will see that $\|\mathbb{E}^s [A_{\tilde{s}, \tilde{t}}^m]^{1/m} \|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}$. Then, choosing $(\tilde{s}, \tilde{t}) = (s, t)$ we get (7.1).

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): For $s \leq \tilde{s} \leq u \leq \tilde{t} \leq t$, we have

$$\delta A_{\tilde{s}, u, \tilde{t}} = \int_u^{\tilde{t}} f(B_r + \psi_{\tilde{s}}) - f(B_r + \psi_u) dr.$$

Hence, by the tower property of conditional expectation and Fubini's Theorem, we get

$$|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}| = \left| \mathbb{E}^{\tilde{s}} \int_u^{\tilde{t}} \mathbb{E}^u [f(B_r + \psi_{\tilde{s}}) - f(B_r + \psi_u)] dr \right|.$$

Now using Lemma 5.1(b) with the \mathcal{F}_u -measurable variable $\Xi = (\psi_{\tilde{s}}, \psi_u)$, Lemma A.5(b) for $\alpha = 1$ and again Fubini's Lemma, we obtain that

$$\begin{aligned} \left| \mathbb{E}^{\tilde{s}} \int_u^{\tilde{t}} \mathbb{E}^u [f(B_r + \psi_{\tilde{s}}) - f(B_r + \psi_u)] dr \right| &\leq \mathbb{E}^{\tilde{s}} \int_u^{\tilde{t}} \|f(\cdot + \psi_{\tilde{s}}) - f(\cdot + \psi_u)\|_{\mathcal{B}_p^{\gamma-1}} (r - u)^{H(\gamma-1-1/p)} dr \\ &\leq C \|f\|_{\mathcal{B}_p^\gamma} \int_u^{\tilde{t}} \mathbb{E}^{\tilde{s}} [|\psi_u - \psi_{\tilde{s}}|] (r - u)^{H(\gamma-1-1/p)} dr. \end{aligned}$$

Hence we get

$$\|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} \int_u^{\tilde{t}} \|\mathbb{E}^{\tilde{s}} |\psi_u - \psi_{\tilde{s}}|\|_{L^n} (r - u)^{H(\gamma-1-1/p)} dr. \tag{C.13}$$

By the conditional Jensen's inequality, we have

$$\|\mathbb{E}^{\tilde{s}} [|\psi_u - \psi_{\tilde{s}}|]\|_{L^n} \leq [\psi]_{\mathcal{C}_{[s, \tilde{t}]}^\alpha(L^{m, n})} (u - \tilde{s})^\alpha.$$

Combining this with Equation (C.13), we get

$$\|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}_{[s, \tilde{t}]}^\alpha(L^{m, n})} (\tilde{t} - \tilde{s})^{1+H(\gamma-1-1/p)+\alpha}.$$

Proof of (ii): Note that by Jensen's inequality for conditional expectation, tower property and Lemma C.3 we have that

$$\begin{aligned} \|\mathbb{E}^s [A_{\tilde{s}, \tilde{t}}^m]^{1/m} \|_{L^n} &= \left(\mathbb{E} [(\mathbb{E}^s \mathbb{E}^{\tilde{s}} |A_{\tilde{s}, \tilde{t}}|^m)^{n/m}] \right)^{1/n} \leq \left(\mathbb{E} \mathbb{E}^s [\mathbb{E}^{\tilde{s}} [A_{\tilde{s}, \tilde{t}}^m]^{n/m}] \right)^{1/n} \\ &= \left(\mathbb{E} [\mathbb{E}^{\tilde{s}} [A_{\tilde{s}, \tilde{t}}^m]^{n/m}] \right)^{1/n} \\ &\leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}. \end{aligned} \tag{C.14}$$

After similarly controlling $\|\mathbb{E}^s[|A_{\tilde{s},u}|^m]^{1/m}\|_{L^n}$ and $\|\mathbb{E}^s[|A_{u,\tilde{t}}|^m]^{1/m}\|_{L^n}$, we get

$$\|\mathbb{E}^s[|\delta A_{\tilde{s},u,\tilde{t}}|^m]^{1/m}\|_{L^n} \leq C\|f\|_{\mathcal{B}_p^\gamma}(\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}.$$

The proof of (iii) can be done by similar arguments as for (iii) in the proof of Proposition 5.3. \square

References

- [1] O. Amine, D. R. Baños, and F. Proske, *Regularity properties of the stochastic flow of a skew fractional Brownian motion*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **23** (2020), no. 1, 2050005, 19. MR4107242
- [2] S. Athreya, O. Butkovsky, K. Lê, and L. Mytnik, *Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation*, to appear in *Comm. Pure Appl. Math.* (2023), arXiv:2011.13498
- [3] D. Baños, S. Ortiz-Latorre, A. Pilipenko, and F. Proske, *Strong solutions of stochastic differential equations with generalized drift and multidimensional fractional Brownian initial noise*, *J. Theoret. Probab.* **35** (2022), no. 2, 714–771. MR4414403
- [4] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 343, Springer, Heidelberg, 2011. MR2768550
- [5] R. F. Bass and Z.-Q. Chen, *Stochastic differential equations for Dirichlet processes*, *Probab. Theory Related Fields* **121** (2001), no. 3, 422–446. MR1867429
- [6] O. Butkovsky, K. Dareiotis, and M. Gerencsér, *Approximation of SDEs: a stochastic sewing approach*, *Probab. Theory Related Fields* **181** (2021), no. 4, 975–1034. MR4344136
- [7] R. Catellier and R. Duboscq, *Regularization by noise for rough differential equations driven by gaussian rough paths*, arXiv:2207.04251, 2022.
- [8] R. Catellier and M. Gubinelli, *Averaging along irregular curves and regularisation of ODEs*, *Stochastic Process. Appl.* **126** (2016), no. 8, 2323–2366. MR3505229
- [9] P.-E. Chaudru de Raynal, S. Menozzi, and E. Priola, *Weak well-posedness of multidimensional stable driven SDEs in the critical case*, *Stoch. Dyn.* **20** (2020), no. 6, 2040004, 20. MR4161968
- [10] K. Dareiotis and M. Gerencsér, *Path-by-path regularisation through multiplicative noise in rough, young, and ordinary differential equations*, arXiv:2207.03476, 2022.
- [11] A. M. Davie, *Uniqueness of solutions of stochastic differential equations*, *Int. Math. Res. Not. IMRN* (2007), no. 24, Art. ID rnm124, 26. MR2377011
- [12] F. Delarue and R. Diel, *Rough paths and 1d SDE with a time dependent distributional drift: application to polymers*, *Probab. Theory Related Fields* **165** (2016), no. 1-2, 1–63. MR3500267
- [13] A. Deya, M. Gubinelli, M. Hofmanová, and S. Tindel, *One-dimensional reflected rough differential equations*, *Stochastic Process. Appl.* **129** (2019), no. 9, 3261–3281. MR3985562
- [14] R. Duboscq and A. Réveillac, *Stochastic regularization effects of semi-martingales on random functions*, *J. Math. Pures Appl.* (9) **106** (2016), no. 6, 1141–1173. MR3565418
- [15] F. Flandoli, *Random perturbation of PDEs and fluid dynamic models*, *Lecture Notes in Mathematics*, vol. 2015, Springer, Heidelberg, 2011, Lectures from the 40th Probability Summer School held in Saint-Flour, 2010. MR2796837
- [16] F. Flandoli, E. Issoglio, and F. Russo, *Multidimensional stochastic differential equations with distributional drift*, *Trans. Amer. Math. Soc.* **369** (2017), no. 3, 1665–1688. MR3581216
- [17] F. Flandoli, F. Russo, and J. Wolf, *Some SDEs with distributional drift. I. General calculus*, *Osaka J. Math.* **40** (2003), no. 2, 493–542. MR1988703
- [18] P. K. Friz and N. B. Victoir, *Multidimensional stochastic processes as rough paths*, *Cambridge Studies in Advanced Mathematics*, vol. 120, Cambridge University Press, Cambridge, 2010. MR2604669
- [19] P. K. Friz and H. Zhang, *Differential equations driven by rough paths with jumps*, *J. Differential Equation.* **264** (2018), no. 10, 6226–6301. MR2604669

- [20] L. Galeati, *Nonlinear Young differential equations: a review*, J. Dynam. Differential Equations. **35** (2023), no. 2, 985–1046. MR4594437
- [21] L. Galeati and M. Gubinelli, *Noiseless regularisation by noise*, Rev. Mat. Iberoam. **38** (2022), no. 2, 433–502. MR4404773
- [22] L. Galeati, F. A. Harang, and A. Mayorcas, *Distribution dependent SDEs driven by additive fractional Brownian motion*, Probab. Theory Related Fields. **185** (2023), no. 1-2, 251–309. MR4528970
- [23] P. Gassiat, *Non-uniqueness for reflected rough differential equations*, Ann. Inst. Henri Poincaré Probab. Stat. **57** (2021), no. 3, 1369–1387. MR4291451
- [24] D. Geman and J. Horowitz, *Occupation densities*, Ann. Probab. **8** (1980), no. 1, 1–67. MR556414
- [25] M. Gerencsér, *Regularisation by regular noise*, Stoch. Partial Differ. Equ. Anal. Comput. **11** (2023), no. 2, 714–729. MR4588621
- [26] I. Gyöngy and N. Krylov, *Existence of strong solutions for Itô's stochastic equations via approximations*, Probab. Theory Related Fields **105** (1996), no. 2, 143–158. MR1392450
- [27] F. A. Harang and C. Ling, *Regularity of local times associated with Volterra–Lévy processes and path-wise regularization of stochastic differential equations*, J. Theoret. Probab. **35** (2022), no. 3, 1706–1735. MR4488556
- [28] F. A. Harang and N. Perkowski, *C^∞ -regularization of ODEs perturbed by noise*, Stoch. Dyn. **21** (2021), no. 8, Paper No. 2140010, 29. MR4342752
- [29] J. M. Harrison and L. A. Shepp, *On skew Brownian motion*, Ann. Probab. **9** (1981), no. 2, 309–313. MR606993
- [30] Y. Hu and K. Lê, *A multiparameter Garsia-Rodemich-Rumsey inequality and some applications*, Stochastic Process. Appl. **123** (2013), no. 9, 3359–3377. MR3071383
- [31] N. V. Krylov and M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift*, Probab. Theory Related Fields **131** (2005), no. 2, 154–196. MR2117951
- [32] K. Lê, *A stochastic sewing lemma and applications*, Electron. J. Probab. **25** (2020), Paper No. 38, 55. MR4089788
- [33] J.-F. Le Gall, *One-dimensional stochastic differential equations involving the local times of the unknown process*, Stochastic Analysis and Applications (Swansea, 1983), Lecture Notes in Math., vol. 1095, Springer, Berlin, 1984, pp. 51–82. MR777514
- [34] A. Lejay, *On the constructions of the skew Brownian motion*, Probab. Surv. **3** (2006), 413–466. MR2280299
- [35] D. Nualart and Y. Ouknine, *Regularization of differential equations by fractional noise*, Stochastic Process. Appl. **102** (2002), no. 1, 103–116. MR1934157
- [36] J. Picard, *Representation formulae for the fractional Brownian motion*, Séminaire de Probabilités XLIII, Lecture Notes in Math., vol. 2006, Springer, Berlin, 2011, pp. 3–70. MR2790367
- [37] L. D. Pitt, *Local times for Gaussian vector fields*, Indiana Univ. Math. J. **27** (1978), no. 2, 309–330. MR471055
- [38] A. Richard, E. Tanré, and S. Torres, *Penalisation techniques for one-dimensional reflected rough differential equations*, Bernoulli **26** (2020), no. 4, 2949–2986. MR4140534
- [39] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996. MR1419319
- [40] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971, Princeton Mathematical Series, No. 32. MR0304972
- [41] D. W. Stroock, *Probability theory, an analytic view*, Cambridge University Press, Cambridge, 2011. MR2760872
- [42] F. Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967. MR0225131
- [43] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983. MR781540

- [44] A. J. Veretennikov, *Strong solutions and explicit formulas for solutions of stochastic integral equations*, Mat. Sb. (N.S.) **111(153)** (1980), no. 3, 434–452, 480. MR568986
- [45] Y. Xiao, *Sample path properties of anisotropic Gaussian random fields*, A minicourse on stochastic partial differential equations, Lecture Notes in Math., vol. 1962, Springer, Berlin, 2009, pp. 145–212. MR2508776
- [46] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. **67** (1936), no. 1, 251–282. MR1555421

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