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Sequences of expected record values*

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Abstract

We investigate conditions in order to decide whether a given sequence of real numbers represents expected record values arising from an independent, identically distributed, sequence of real-valued random variables. The main result provides a necessary and sufficient condition, relating any expected record sequence with the Stieltjes moment problem. The results are proved by means of a useful transformation on random variables. Some properties of this mapping, and its inverse, are discussed in detail, and, under mild conditions, an explicit inversion formula for the random variable that admits a given expected record sequence is obtained.

Keywords: characterizations; expected record values; Stieltjes moment problem; transformation of random variables; inversion formula.
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1 Introduction

Let X be a real-valued random variable (r.v.) with distribution function (d.f.) F, and suppose that $X_1, X_2, \ldots$ is an independent, identically distributed (i.i.d.) sequence from F. The usual record times, T_n, and (upper) record values, R_n, corresponding to the i.i.d. sequence $X_1, X_2, \ldots$, are defined by $T_1 = 1$, $R_1 = X_1$, and, inductively, by

$$T_{n+1} = \inf\Big\{m > T_n : X_m > R_n\Big\}, \quad R_{n+1} = X_{T_{n+1}} \quad (n = 1, 2, \ldots). \tag{1.1}$$

It is obvious that (1.1) produces an infinite sequence of records (= record values) if and only if F has not an atom at its upper end-point (if finite). In a similar manner, one can define the so called *weak* (upper) records, W_n, by $\widetilde{T}_1 = 1$, $W_1 = X_1$, and

$$\widetilde{T}_{n+1} = \min\Big\{m > \widetilde{T}_n : X_m \ge W_n\Big\}, \quad W_{n+1} = X_{\widetilde{T}_{n+1}} \quad (n = 1, 2, \ldots); \tag{1.2}$$

clearly, the sequence W_n in (1.2) is non-terminating for every d.f. F.

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These models have been studied extensively in the literature. The interested reader is referred to [1, 3, 25]. Moreover, several characterization results based on the regressions of (weak or ordinary) record values are given in a number of papers, including [22, 24, 17, 32, 2, 6, 21, 29, 5, 37].

Clearly, the record processes (1.1) and (1.2) coincide with probability (w.p.) 1 whenever F is continuous (i.e., free of atoms). In that case, the record process $(R_1, R_2 \ldots)$ has the same distribution as the sequence

$$\Big(F^{-1}(U_1), F^{-1}(U_2), \ldots\Big), \tag{1.3}$$

where $U_1 < U_2 < \cdots$ is the record process from the standard uniform d.f., $U(0,1)$, and $F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \ge u\}$, $0 < u < 1$, is the left-continuous inverse d.f. of F. It should be noted, however, that the records, as defined by (1.3), are neither weak nor ordinary records (when F is arbitrary). To illustrate the situation, consider the case where F is Bernoulli$(1/2)$, in which case

$$F^{-1}(u) = \begin{cases} 0, & 0 < u \le 1/2, \\ 1, & 1/2 < u < 1. \end{cases}$$

The following table provides a realization of the corresponding i.i.d. and record processes.

Table 1:

Random mechanism producing i.i.d. from $U(0,1)$	0.13	0.32	0.01	0.44	0.57	0.52	0.64	0.12	...
Uniform records U_n	0.13	0.32	$*$	0.44	0.57	$*$	0.64	$*$	...
Records $F^{-1}(U_n)$ – see (1.3)	0	0	$*$	0	1	$*$	1	$*$	...
i.i.d. observations from $b(1/2)$	0	0	0	0	1	1	1	0	...
Weak records W_n from the i.i.d. observations – see (1.2)	0	0	0	0	1	1	1	$*$	...
Ordinary records R_n from the i.i.d. observations – see (1.1)	0	$*$	$*$	$*$	1	$*$	$*$	$*$	...

Table 1 shows that $W_2 = F^{-1}(U_2) = 0$ while $R_2 = 1$. Also, $W_4 = 0$ while $F^{-1}(U_4) = 1$ (and R_4 is undefined); thus, $F^{-1}(U_n)$ is neither R_n nor W_n in general.

From now on we shall constantly use the notation R_n for $F^{-1}(U_n)$, where $\{U_n\}_{n=1}^{\infty}$ is the sequence of uniform records – the effect is not essential in applications, where it is customarily assumed that F is absolutely continuous. Of course, the three notions of records coincide (w.p. 1) if and only if $F^{-1}(u)$ is strictly increasing in $u \in (0,1)$, and this is equivalent to the fact that $\mathbb{P}(X = x) = 0$ for all x.

The present work is concentrated on questions of the form

Does a given real sequence $\{\rho_n\}_{n=1}^{\infty}$ represents an expected record sequence (ERS) of some r.v. X?

That is, can we find an r.v. X such that $\mathbb{E}\, R_n = \rho_n$ for all n, where the record process R_n is defined by (1.3)? Moreover, if the answer is in the affirmative, is (the d.f. of) this r.v. unique? How can we re-construct it from its ERS?

One of the central results of the paper reads as follows.

Theorem 1.1. A real sequence $\{\rho_n\}_{n=1}^{\infty}$ is an expected record sequence corresponding to a non-degenerate r.v. X if and only if $\rho_1 < \rho_2$ and

$$\frac{\rho_{n+2} - \rho_{n+1}}{\rho_2 - \rho_1} = \frac{\mathbb{E}\, T^n}{(n+1)!}, \quad n = 0, 1, \ldots, \tag{1.4}$$

for some r.v. T, with $\mathbb{P}(T > 0) = 1$, possessing finite moments of any order.

Characterizations of the parent d.f. through its expected records (under mild additional assumptions like continuity and finite moment of order greater than one) are present in the bibliography for a long time, the most relevant being those given by Kirmani and Beg [15] and Lin [18]; see also [19]. However, these authors do not provide an explicit connection to the Stieltjes moment problem. In the contrary, the corresponding theory for an *expected maxima sequence* (EMS) $\mu_n = \mathbb{E}\max\{X_1, \ldots, X_n\}$, is well-understood from Kadane [13, 14]. Namely, Kadane showed that the real sequence $\{\mu_n\}_{n=1}^{\infty}$ (with $\mu_1 < \mu_2$) represents an EMS (of a non-degenerate, integrable, parent population) if and only if there exists an r.v. T, with $\mathbb{P}(0 < T < 1) = 1$, such that

$$\frac{\mu_{n+2} - \mu_{n+1}}{\mu_2 - \mu_1} = \mathbb{E}\, T^n, \quad n = 0, 1, \ldots. \tag{1.5}$$

The representation (1.5) is closely connected to the Hausdorff moment problem, [8], and improves on Hoeffding's characterization, [11]. The above kind of results enable further applications in the theory of maxima and order statistics, see, e.g., [9, 10, 12, 16]. Moreover, the r.v. T in (1.5), (the d.f. of) which is clearly unique, admits the representation $T = F(V)$ where F is the parent d.f. and V has density $f_V(x) = F(x)(1-F(x)) \Big/ \int_{\mathbb{R}} F(y)(1-F(y))dy$ – cf. [28]. Conversely, the parent d.f. is characterized from the sequence $\{\mu_n\}_{n=1}^{\infty}$, and its location-scale family from T.

In the case of a record process we would like to verify similar results, guaranteeing that the theory of maxima can be suitably adapted to that of records. However, there are essential differences between these two models – see, e.g., [30, 31, 23, 36, 7, 27, 4]; see also Section 2. In this spirit, (1.4) can be viewed as the natural record-analogue of (1.5).

The results presented here are based on a suitable mapping φ on the d.f. of an r.v. Using φ, the location-scale family of any suitable X is transformed to (the d.f. of) a unique positive r.v. T with finite moments of any order. The mapping is one to one and onto (hence, invertible), and several properties of the expected record sequence of X are easily extracted from the behavior of $T = \varphi(X)$. The basic properties of the mapping φ are discussed in Section 3. Using them, we provide a complete description of the class of r.v.'s that are characterized from their expected record sequence – see Theorem 4.1. Moreover, under mild assumptions, an inversion formula for the d.f. of the r.v. that admits a given expected record sequence is obtained; see Theorem 4.2. The main results are presented in Sections 3 and 4.

Through the rest of the article, $X = Y$ for r.v.'s X, Y means that X, Y are identically distributed, and inverse d.f.'s are always taken to be left-continuous, namely, $F^{-1}(u) := \inf\{x : F(x) \geq u\}$, $u \in (0, 1)$.

2 Existence of expectations of records

It is well-known (see, e.g., [3]) that U_n in (1.3) has density

$$f_{U_n}(u) = \frac{L(u)^{n-1} I(0 < u < 1)}{(n-1)!}, \; n = 1, 2, \ldots, \text{ where } L(u) := -\log(1-u), \; 0 < u < 1, \tag{2.1}$$

and I denotes the indicator function. We may use (2.1) to calculate the d.f. F_n of R_n as follows:

$$F_n(x) = \mathbb{P}\left(F^{-1}(U_n) \leq x\right) = \mathbb{P}\left(U_n \leq F(x)\right) = \frac{1}{(n-1)!} \int_0^{F(x)} L(u)^{n-1} du.$$

Substituting $L(u) = y$ in the integral we see that $F_n(x) = \mathbb{P}\left(E_1 + \cdots + E_n \leq L(F(x))\right)$, where $E_1, \ldots, E_n$ are i.i.d. from the standard exponential, Exp(1). From the well-known relationship regarding waiting times for the standard (with intensity one) Poisson process, $\{Y_t, t \ge 0\}$, we have

$$\mathbb{P}\Big(E_1 + \cdots + E_n \le t\Big) = 1 - \mathbb{P}(Y_t \le n-1) = 1 - \mathrm{e}^{-t}\sum_{k=0}^{n-1}\frac{t^k}{k!}, \quad t \ge 0.$$

Therefore, with $t = L(F(x))$, we obtain (cf. [23])

$$F_n(x) = 1 - (1-F(x))\sum_{k=0}^{n-1}\frac{L(F(x))^k}{k!}, \quad x \in \mathbb{R} \quad (n = 1, 2, \ldots). \tag{2.2}$$

In the above sum, the term $L(F(x))^0$ should be treated as 1 for all x; moreover, the product $(1 - F(x))L(F(x))^k$ should be treated as 0 whenever $k \ge 1$ and $F(x) = 1$. Hence, (2.2) yields $F_1(x) = F(x)$ and, e.g.,

$$F_2(x) = \begin{cases} 1 - \Big(1 - F(x)\Big)\Big(1 + L(F(x))\Big), & \text{if } F(x) < 1, \\ 1, & \text{if } F(x) = 1. \end{cases}$$

Since our problem concerns the expectations $\mathbb{E}\,R_n$ for all n, we have to define an appropriate space of r.v.'s (d.f.'s) to work with; that is, to guarantee that these expectations are, all, finite. In view of Proposition 2.1, below, the natural space is the following.

Definition 2.1. A function $H : (0, \infty) \to \mathbb{R}$ belongs to $\mathcal{H}^*$ if it is non-constant, non-decreasing, left continuous, and satisfies

$$\int_0^\infty y^m \mathrm{e}^{-y}\big|H(y)\big|dy < \infty, \quad m = 0, 1, \ldots.$$

Furthermore, $\mathcal{H}_0 := \Big\{H \in \mathcal{H}^* : \int_0^\infty \mathrm{e}^{-y}H(y)dy = 0 \text{ and } \int_0^\infty y\mathrm{e}^{-y}H(y)dy = 1\Big\}$. By definition, the r.v. X belongs to $\mathcal{H}^*$ if its inverse d.f. G can be written as $G(u) = H(-\log(1-u))$, $0 < u < 1$, for some $H \in \mathcal{H}^*$. Similarly, $X_0 \in \mathcal{H}_0$ if its inverse d.f. G_0 can be written as $G_0(u) = H_0(-\log(1-u))$, $0 < u < 1$, for some $H_0 \in \mathcal{H}_0$. Here, identically distributed r.v's are considered as equal. Finally, $\mathcal{H} := \mathcal{H}^* \cup \{\text{the constant functions}\}$. (The notation $\mathcal{H}$ will be used for both functions and r.v.'s, since this simplifies the presentation considerably – the probability space where the r.v.'s are defined is immaterial to our subject.)

Notice that $X \in \mathcal{H}^*$ if and only if X is non-degenerate and the corresponding record process in (1.3) satisfies $\mathbb{E}\,|R_n| < \infty$ for all n. Indeed, we have the following results.

Proposition 2.1. The following statements are equivalent:

(i) $X \in \mathcal{H}$, i.e., $H \in \mathcal{H}$, where $H(y) := F^{-1}(1 - \mathrm{e}^{-y})$, $y > 0$, and F is the d.f. of X.

(ii) $\mathbb{E}\,|R_n| < \infty$ for all $n = 1, 2, \ldots$.

(iii) $\mathbb{E}\,X^- < \infty$ and $\mathbb{E}\,X(\log^+ X)^m < \infty$ for all $m > 0$, where $\log^+ x = \log x$ if $x \ge 1$ and $= 0$ otherwise.

(iv) $\int_0^1 L(u)^m |F^{-1}(u)|du < \infty$, $m = 0, 1, \ldots$.

(v) $\int_0^\infty y^m \mathrm{e}^{-y}|H(y)|dy < \infty$, $m = 0, 1, \ldots$.

If (i)–(v) are satisfied, then the sequence $\rho_n = \mathbb{E}\,R_n$ is given by

$$\rho_n = \int_{-\infty}^\infty (I(x > 0) - F_n(x))dx = \int_0^1 \frac{L(u)^{n-1}}{(n-1)!}F^{-1}(u)du = \int_0^\infty \frac{y^{n-1}}{(n-1)!}\mathrm{e}^{-y}H(y)dy, \tag{2.3}$$

$n = 1, 2, \ldots$, with F_n given by (2.2) and L by (2.1).

Proposition 2.2. For $\alpha \geq 1$ set $L^\alpha = \{X : \mathbb{E}|X|^\alpha < \infty\}$, where identically distributed r.v.'s are considered as equal. Then, $\cup_{\delta>0} L^{1+\delta} \subsetneqq \mathcal{H} \subsetneqq L^1$.

The preceding results are due to Nagaraja, [23], in the particular case where X has a density and/or is non-negative, but his proofs continue to hold in our case too.

It is worth pointed out that every $X \in \mathcal{H}$ admits the equivalent representation $X = H(E)$, where the function H belongs to $\mathcal{H}$ and E is a standard exponential r.v. This says that a left-continuous, non-decreasing function H belongs to $\mathcal{H}$ if and only if $\mathbb{E}|H(S_m)| < \infty$ for all $m \geq 1$, where S_m follows the Erlang distribution with parameters m and 1, i.e., S_m is the sum of m i.i.d. standard exponential r.v.'s. The subspace $\mathcal{H}_0$ consists of those $H \in \mathcal{H}$ for which $\mathbb{E}\, H(S_1) = 0$, $\mathbb{E}\, H(S_2) = 1$.

3 The mapping φ with applications to characterizations

For the investigation of the mapping φ it is necessary to introduce another suitable space as follows.

Definition 3.1. The space $\mathcal{T}$ consists of all r.v.'s T, with $\mathbb{P}(T > 0) = 1$, possessing finite moments of any (positive) order, where identically distributed r.v.'s are considered as equal. We customarily write $F_T \in \mathcal{T}$ in order to denote $T \in \mathcal{T}$, where F_T is the d.f. of T. (The notation $\mathcal{T}$ will be used for both d.f.'s and r.v.'s, since this simplifies the presentation considerably.)

We are now ready to define the mapping φ and its inverse φ'. What we shall prove in the sequel is that, essentially, the spaces $\mathcal{H}_0$ and $\mathcal{T}$ are identified through the restriction of φ on $\mathcal{H}_0$.

Definition 3.2. Set $L(u) = -\log(1-u)$, $0 < u < 1$. For $X \in \mathcal{H}^*$, $T = \varphi(X) \in \mathcal{T}$ is defined to be the r.v. $T = L(F(V))$, where F is the d.f. of X and the r.v. V has density (with respect to Lebesgue measure) given by $f_V(x) = (1-F(x))L(F(x)) \Big/ \int_{\mathbb{R}} (1-F(y))L(F(y))dy$, $x \in \mathbb{R}$, and where $(1 - F(x))L(F(x)) = 0$ if $F(x) = 1$ or 0.

The mapping φ is well-defined because $X \in \mathcal{H}^*$ so that f_V is integrable, strictly positive in the non-empty interval $\{x : 0 < F(x) < 1\}$, and zero otherwise.

Definition 3.3. For any $T \in \mathcal{T}$ with d.f. F_T we define $X_0 = \varphi'(T) \in \mathcal{H}_0$ to be the r.v. with inverse d.f. G_0, for which the function $H_0(y) = G_0(1 - \mathrm{e}^{-y})$, $y > 0$, is given by the formula

$$H_0(y) := \frac{\mathrm{e}^y}{y} F_T(y-) - \int_1^y \frac{x-1}{x^2} \mathrm{e}^x F_T(x) dx - c_T, \quad y > 0. \tag{3.1}$$

In this formula, $F_T(y-) = \mathbb{P}(T < y)$, $\int_1^y dx := \int_y^1 -dx$ if $0 < y < 1$, and c_T is the unique constant (depending only on T) for which $\int_0^\infty \mathrm{e}^{-y} H_0(y) dy = 0$.

We shall prove in Lemma A.7 that

$$c_T = \mathbb{E}\left[\frac{1}{T} I(T > 1)\right] + \mathbb{E}\left[\frac{1 + \mathrm{e}T - \mathrm{e}^T}{T} I(T \leq 1)\right], \tag{3.2}$$

where I denotes an indicator function.

Proposition 3.1. Both transformations φ and φ' are well-defined, with domains $\mathcal{H}^*$ and $\mathcal{T}$, and ranges $\mathcal{T}$ and $\mathcal{H}_0$, respectively. Moreover, if $X_1 \in \mathcal{H}^*$ and $X_2 = c + \lambda X_1$, where $c \in \mathbb{R}$ and $\lambda > 0$, then $X_2 \in \mathcal{H}^*$ and $\varphi(X_1) = \varphi(X_2)$.

Theorem 1.1 holds true, since it is an immediate corollary of the following result.

Theorem 3.1. (i) Given $X \in \mathcal{H}^*$ with expected record sequence $\{\rho_n\}_{n\geq 1}$, the r.v. $T = \varphi(X) \in \mathcal{T}$ satisfies (1.4), where the mapping φ is given by Definition 3.2.
(ii) Given $T \in \mathcal{T}$, the r.v. $X_0 = \varphi^{-1}(T) \in \mathcal{H}_0$ has expected record sequence $\{\rho_n\}_{n\geq 1}$ that satisfies (1.4) with $\rho_1 = 0$, $\rho_2 = 1$, where the mapping $\varphi^{-1} = \varphi'$ is as in Definition 3.3.

Remark 3.1. Given $T \in \mathcal{T}$, $\mu \in \mathbb{R}$, and $\lambda > 0$, we can always construct an r.v. $X \in \mathcal{H}^*$ with ERS $\{\rho_n\}_{n\geq 1}$ satisfying (1.4), with $\rho_1 = \mu$, $\rho_2 - \rho_1 = \lambda$, namely, $X := \mu + \lambda\varphi^{-1}(T)$.

Since the proofs of Theorem 3.1(ii) and of the second part of Proposition 3.1 require a series of auxiliary lemmas (regarding the inversion of φ), we have postponed the relative material to Appendix A.

Proof of Theorem 3.1(i) *and of the first part of Proposition 3.1.* Pick an r.v. $X \in \mathcal{H}^*$ with d.f. F and set $\rho_n = \mathbb{E}\, R_n$ (finite for all n). Then,

$$\rho_n = \int_{-\infty}^{\infty} (I(x > 0) - F_n(x))dx \in \mathbb{R},$$

and from (2.2) we see that

$$\rho_{n+1} - \rho_n = \int_{-\infty}^{\infty} (F_n(x) - F_{n+1}(x))dx = \frac{1}{n!}\int_{\alpha}^{\omega} (1 - F(x))L(F(x))^n dx, \;\; n = 1, 2, \ldots, \tag{3.3}$$

where $\alpha = \inf\{x : F(x) > 0\}$, $\omega = \sup\{x : F(x) < 1\}$. Note that $F_n(x) - F_{n+1}(x) = 0$ for $x \notin [\alpha, \omega)$. Since $\alpha < \omega$ (because F is non-degenerate), the above relation shows that

$$\rho_2 - \rho_1 = \int_{\alpha}^{\omega} (1 - F(x))L(F(x))dx > 0,$$

because $(1 - F(x))L(F(x)) > 0$ for $x \in (\alpha, \omega)$. It follows that the function

$$f_V(x) := \begin{cases} (1 - F(x))L(F(x))/(\rho_2 - \rho_1), & \alpha < x < \omega, \\ 0, & \text{otherwise}, \end{cases}$$

defines a Lebesgue density of an absolutely continuous r.v. V with support (α, ω). Setting $T := -\log(1 - F(V)) = L(F(V))$ we see that $0 < T < \infty$ w.p. 1 (because $\alpha < V < \omega$ so that $0 < F(V) < 1$ w.p. 1). Thus, we can rewrite (3.3) as

$$\frac{\rho_{n+2} - \rho_{n+1}}{\rho_2 - \rho_1} = \frac{1}{(n+1)!}\int_{\mathbb{R}} f_V(x)L(F(x))^n dx = \frac{\mathbb{E}\, L(F(V))^n}{(n+1)!}, \;\; n = 0, 1, \ldots,$$

and (1.4) is proved with $T = L(F(V)) = \varphi(X)$; this also verifies the first counterpart of Proposition 3.1, i.e., that φ has domain $\mathcal{H}^*$ and takes values into $\mathcal{T}$. The fact that $\varphi(X_1) = \varphi(\lambda X_1 + c)$ (when $X_1 \in \mathcal{H}^*$, $c \in \mathbb{R}$, $\lambda > 0$) is trivial. □

Theorem 3.2. The transformation $\varphi : \mathcal{H}_0 \to \mathcal{T}$ of Definition 3.2 is one to one and onto, with inverse $\varphi^{-1} : \mathcal{T} \to \mathcal{H}_0$, where $\varphi^{-1} = \varphi'$ is given by Definition 3.3.

For the proof we shall make use of the following two lemmas.

Lemma 3.1. If $X_1, X_2 \in \mathcal{H}_0$ and $\varphi(X_1) = \varphi(X_2)$ then $X_1 = X_2$.

Proof. Let F_i (resp., F_i^{-1}) be the d.f. (resp., the inverse d.f.) of X_i, and V_i the corresponding r.v. with density $f_i = (1 - F_i)L(F_i)$ ($i = 1, 2$). It is easy to verify that the events $\{L(F_i(V_i)) < t\}$ and $\{V_i < F_i^{-1}(1 - \mathrm{e}^{-t})\}$ are equivalent for all $t > 0$. Setting $T_i = \varphi(X_i)$ and $u = 1 - \mathrm{e}^{-t}$, the assumption $T_1 = T_2$ is equivalent to $\mathbb{P}(T_1 < t) = \mathbb{P}(T_2 < t)$ for all $t > 0$, i.e.,

$$\int_{-\infty}^{F_1^{-1}(u)} (1 - F_1(x))L(F_1(x))dx = \int_{-\infty}^{F_2^{-1}(u)} (1 - F_2(x))L(F_2(x))dx, \;\; 0 < u < 1. \tag{3.4}$$

For every r.v. X with d.f. F and inverse d.f. F^{-1}, the following identity is valid (see, e.g., Lemma 4.1 in [26]):

$$\int_{-\infty}^{F^{-1}(u)} F(x)^m dx = m\int_0^u w^{m-1}\Big(F^{-1}(u) - F^{-1}(w)\Big)dw, \;\; 0 < u < 1. \tag{3.5}$$

Using (3.5) and $\mathbb{E}\,X^- < \infty$, i.e., $F^{-1} \in L^1(0, 1-\delta)$ for any $\delta \in (0,1)$, we obtain

$$\begin{aligned}\int_{-\infty}^{F^{-1}(u)} (1-F(x))L(F(x))dx &= \sum_{m=1}^{\infty} \frac{1}{m} \int_{-\infty}^{F^{-1}(u)} (1-F(x))F(x)^m dx \\ &= \sum_{m=1}^{\infty} \int_0^u w^{m-1}\Big(F^{-1}(u) - F^{-1}(w)\Big)\Big(1-(1+1/m)w\Big)dw \\ &= \int_0^u \Big(F^{-1}(u) - F^{-1}(w)\Big)\Big(1-L(w)\Big)dw. \end{aligned} \tag{3.6}$$

Define $s(w) := 1 - L(w)$ and $S(w) := \int_0^w s(t)dt$, so that $S'(w) = s(w)$, $0 < w < 1$. In view of (3.6), (3.4) reads as

$$h(u) = \frac{1}{S(u)} \int_0^u s(w)h(w)dw, \ \ 0 < u < 1,$$

where $h := F_1^{-1} - F_2^{-1}$. This relation shows that h is absolutely continuous in every compact interval $[x,y] \subseteq (0,1)$, and

$$h'(u) = \frac{s(u)}{S(u)} \left(h(u) - \frac{1}{S(u)} \int_0^u s(w)h(w)dw \right) = 0, \ \ \text{for almost all} \ \ u \in (0,1).$$

Therefore, $h = c$, constant. Finally, from the assumption $X_1, X_2 \in \mathcal{H}_0$, we must have $c = \int_0^1 h(u)du = 0$; hence, $F_1^{-1} = F_2^{-1}$, i.e., $X_1 = X_2$. □

Remark 3.2. The equation (3.6) provides an explicit expression for the d.f. of $T = \varphi(X)$ when $X \in \mathcal{H}_0$, namely,

$$\mathbb{P}(T < t) = \int_0^{1-\mathrm{e}^{-t}} \big(F^{-1}(1-\mathrm{e}^{-t}) - F^{-1}(w)\big)\big(1-L(w)\big)dw, \ \ t > 0.$$

Lemma 3.2. If $T_1, T_2 \in \mathcal{T}$ and $\varphi'(T_1) = \varphi'(T_2)$ then $T_1 = T_2$.

Proof. Write $H_i(y)$ for $G_i(1-\mathrm{e}^{-y})$, $y > 0$, where G_i is the inverse d.f. of $X_i = \varphi'(T_i) \in \mathcal{H}_0$ ($i = 1,2$). From the proof in Appendix A we know that $H_i \in \mathcal{H}_0$. Note that H_i is the function H_0 given by (3.1), on substituting $T = T_i$ ($i = 1,2$). By assumption, $H_1 = H_2$. Thus,

$$F_{T_1}(y-) - F_{T_2}(y-) = y\mathrm{e}^{-y} \int_1^y \frac{x-1}{x^2} \mathrm{e}^x \big(F_{T_1}(x) - F_{T_2}(x)\Big)dx + cy\mathrm{e}^{-y}, \ \ y > 0,$$

where $c = c_{T_1} - c_{T_2}$ (see (3.2)) and F_{T_i} is the d.f. of T_i. Setting $h = F_{T_1} - F_{T_2}$, the above relation implies that h is absolutely continuous in every compact interval $[y_1, y_2] \subseteq (0,\infty)$, so that $h(y-) = h(y)$ for all $y > 0$, yielding

$$w(y) - \int_1^y \frac{x-1}{x} w(x)dx = c, \quad y > 0,$$

where $w(y) := \mathrm{e}^y h(y)/y$. Hence, $0 = w'(y) - (y-1)w(y)/y = \mathrm{e}^y h'(y)/y$ for almost all $y \in (0,\infty)$. Thus, $h'(y) = 0$ and, therefore, $h = c_1$, constant. Finally, taking limits as $y \to \infty$, we conclude that $c_1 = 0$ and $F_{T_1} = F_{T_2}$. □

Proof of Theorem 3.2. In view of Lemmas 3.1, 3.2, and the sufficiency proof of Theorem 1.1 – see Appendix A – it remains to verify that $\varphi'(\varphi(X)) = X$ for every $X \in \mathcal{H}_0$. If this is proved, then $\varphi(\varphi'(T)) = T$ for each $T \in \mathcal{T}$, and thus, $\varphi' = \varphi^{-1}$. To see that $\varphi'(\varphi(X)) = X$ implies $\varphi(\varphi'(T)) = T$, set $T_1 = \varphi(\varphi'(T)) \in \mathcal{T}$. Then, $\varphi'(T_1) = \varphi'\Big(\varphi\big(\varphi'(T)\big)\Big) = \varphi'(T)$, because $\varphi'(T) \in \mathcal{H}_0$. Thus, $T_1 = T$ from the one to one property of φ' – Lemma 3.2. Pick now $X \in \mathcal{H}_0$, and set $H(y) = G(1 - \mathrm{e}^{-y})$, $y > 0$, where G is the distribution inverse of X; set also $T = \varphi(X)$. From Remark 3.2 we have

$$F_T(y-) := \mathbb{P}(T < y) = y\mathrm{e}^{-y}H(y) - \int_0^y (1-w)\mathrm{e}^{-w}H(w)dw, \quad y > 0.$$

Let $X_0 = \varphi'(T)$, assume that G_0 is the distribution inverse of X_0, and set $H_0(y) = G_0(1 - \mathrm{e}^{-y})$, $y > 0$. Applying (3.1) to F_T we find

$$\begin{aligned} H_0(y) = H(y) - \frac{\mathrm{e}^y}{y}\int_0^y (1-w)\mathrm{e}^{-w}H(w)dw - \int_1^y \frac{x-1}{x}H(x)dx \\ + \int_1^y \frac{x-1}{x^2}\mathrm{e}^x \int_0^x (1-w)\mathrm{e}^{-w}H(w)dwdx - c_T, \quad y > 0. \end{aligned}$$

The double integral can be rewritten as

$$\begin{aligned} &\int_1^y \frac{x-1}{x^2}\mathrm{e}^x \int_0^1 (1-w)\mathrm{e}^{-w}H(w)dwdx + \int_1^y \frac{x-1}{x^2}\mathrm{e}^x \int_1^x (1-w)\mathrm{e}^{-w}H(w)dwdx \\ &\quad = \left(\frac{\mathrm{e}^y}{y} - \mathrm{e}\right)\int_0^1 (1-w)\mathrm{e}^{-w}H(w)dw + \int_1^y (1-w)\mathrm{e}^{-w}H(w)\int_w^y \frac{x-1}{x^2}\mathrm{e}^x dxdw \\ &\quad = \left(\frac{\mathrm{e}^y}{y} - \mathrm{e}\right)\int_0^1 (1-w)\mathrm{e}^{-w}H(w)dw + \int_1^y (1-w)\mathrm{e}^{-w}H(w)\left(\frac{\mathrm{e}^y}{y} - \frac{\mathrm{e}^w}{w}\right)dw \\ &\quad = \frac{\mathrm{e}^y}{y}\int_0^y (1-w)\mathrm{e}^{-w}H(w)dw + \int_1^y \frac{w-1}{w}H(w)dw - c_1, \end{aligned}$$

where $c_1 = \mathrm{e}\int_0^1 (1-w)\mathrm{e}^{-w}H(w)dw$, and the change in the order of integration is justified from Tonelli-Fubini, since $w \to (1-w)\mathrm{e}^{-w}H(w) \in L^1(0,\infty)$; see Appendix A. Thus,

$$\begin{aligned} H_0(y) - H(y) = -\frac{\mathrm{e}^y}{y}\int_0^y (1-w)\mathrm{e}^{-w}H(w)dw - \int_1^y \frac{x-1}{x}H(x)dx \\ + \frac{\mathrm{e}^y}{y}\int_0^y (1-w)\mathrm{e}^{-w}H(w)dw + \int_1^y \frac{w-1}{w}H(w)dw + c_2 = c_2, \end{aligned}$$

where $c_2 = -c_T - c_1$. Since $c_2 = \int_0^\infty \mathrm{e}^{-y}(H_0(y) - H(y))dy = 0$ (because $X, X_0 \in \mathcal{H}_0$), the theorem is proved. $\square$

4 Further characterizations and remarks

With the aim of mapping φ, a complete characterization result based on the expected record sequence becomes possible, as follows.

Theorem 4.1. A random variable $X \in \mathcal{H}^*$ is characterized from its expected record sequence if and only if the r.v. $T = \varphi(X) \in \mathcal{T}$ is characterized from its moments, where the mapping φ is given by Definition 3.2.

Proof. Assume first that $X \in \mathcal{H}^*$ is characterized from $\{\rho_n\}_{n=1}^\infty$, where $\rho_n = \mathbb{E}\,R_n$, and set $X_0 = (X - \rho_1)/(\rho_2 - \rho_1) \in \mathcal{H}_0$. If $T = \varphi(X) = \varphi(X_0) \in \mathcal{T}$ is not characterized from its moments, then we can find an r.v. $T_1 \in \mathcal{T}$, $T_1 \neq T$, such that $\mathbb{E}\,T_1^n = \mathbb{E}\,T^n$ for all $n \in \{0, 1, \ldots\}$. Then, from Theorem 3.1(ii), the r.v. $X_1 := \varphi^{-1}(T_1) \in \mathcal{H}_0$ possesses the same expected record sequence as X_0, and, thus, the r.v. $X' := \rho_1 + (\rho_2 - \rho_1)X_1$ has the same ERS as X. Since φ^{-1} is one to one and $T_1 \neq T$, it follows that $X_1 \neq X_0$ and, consequently, $X' \neq X$, which contradicts the assumption that X is characterized from its ERS.

Next, assume that $T=\varphi(X)\in\mathcal{T}$ is characterized from its moments. Suppose, in contrary, that $X\in\mathcal{H}^*$ is not characterized from its ERS. Then, we can find an r.v. $X'\in\mathcal{H}^*$, $X'\neq X$, with the same ERS as X. Obviously, if $\{\rho_n\}_{n=1}^{\infty}$ is the common ERS, both r.v.'s $X_0:=(X-\rho_1)/(\rho_2-\rho_1)$ and $X_0':=(X'-\rho_1)/(\rho_2-\rho_1)$ belong to $\mathcal{H}_0$ and posses the same expected record sequence, while $X_0'\neq X'$. From Theorem 3.1(i) it follows that the r.v.'s T and $T':=\varphi(X_0')=\varphi(X')\in\mathcal{T}$ posses identical moments. However, since $X_0'\neq X_0$ and $X_0,X_0'\in\mathcal{H}_0$, it follows that $T'=\varphi(X_0')\neq\varphi(X_0)=T$, and this contradicts the assumption that T is characterized from its moments. □

Our final result is applicable to most practical situations regarding characterizations (and inversions) in terms of the expected record sequence.

Theorem 4.2. Let $X\in\mathcal{H}^*$ with ERS $\{\rho_n\}_{n\geq 1}$, set $w_n:=\mathbb{E}\,|R_n|$, $T=\varphi(X)$, and define the following generating functions:

$$G_\rho(t):=\sum_{n\geq 1}\rho_n t^{n-1}\qquad\text{[the generating function of }\rho_n\text{'s]}$$
$$G_w(t):=\sum_{n\geq 1}w_n t^{n-1}\qquad\text{[the generating function of }w_n\text{'s]}$$
$$M_T(a):=\mathbb{E}\,\mathrm{e}^{aT}\qquad\text{[the moment generating function of }T\text{].}$$

Then, the following statements are equivalent.

(i) $\mathbb{E}(X^+)^p<\infty$ for some $p>1$.

(ii) $G_\rho(t)$ is finite for t in a neighborhood of zero.

(iii) $G_w(t)<\infty$ for some $t>0$.

(iv) $M_T(a)<\infty$ for some $a>0$.

If (i)–(iv) hold, then we can find $\epsilon_0\in(0,1)$ such that

$$G_\rho(t)=\frac{1}{1-t}\left\{\rho_1+(\rho_2-\rho_1)\int_0^t M_T(s)ds\right\},\quad |t|<\epsilon_0. \tag{4.1}$$

Consequently,

$$M_T(t)=\frac{1}{\rho_2-\rho_1}\,\frac{d}{dt}\Big((1-t)G_\rho(t)\Big),\quad |t|<\epsilon_0. \tag{4.2}$$

Therefore, under assumption (i), X is characterized from the generating function G_ρ of its ERS through $X=\rho_1+(\rho_2-\rho_1)\varphi^{-1}(T)$, where T has moment generating function M_T, given by (4.2), and $\varphi^{-1}=\varphi'$ as in Definition 3.3.

Proof. Without loss of generality assume that $X\in\mathcal{H}_0$ (equivalently, $H\in\mathcal{H}_0$ where $H(y):=F_X^{-1}(1-\mathrm{e}^{-y})$, $y>0$) has ERS $\{\rho_1,\rho_2,\rho_3,\ldots\}=\{0,1,\rho_3,\ldots\}$.

Suppose that (ii) is satisfied. Then, we can find $t_0\in(0,1)$ such that $G_\rho(t_0)=t_0+\rho_3 t_0^2+\cdots<\infty$; note that $\rho_n>0$ for all $n\geq 2$. Moreover,

$$w_n=\mathbb{E}\,|R_n|=\int_0^\infty\frac{y^{n-1}}{(n-1)!}\mathrm{e}^{-y}\big|H(y)\big|dy=\rho_n+2\int_0^m\frac{y^{n-1}}{(n-1)!}\mathrm{e}^{-y}\big|H(y)\big|dy,$$

where $m:=\sup\{y>0:H(y)\leq 0\}\in(0,\infty)$, because H is non-decreasing and $\int_0^\infty\mathrm{e}^{-y}H(y)dy=0$. This verifies that $G_w(t_0)<\infty$, since $y\to\mathrm{e}^{-y}H(y)\in L^1(0,\infty)$, $y\to\mathrm{e}^{t_0y}$ is bounded in $[0,m]$ and

$$G_w(t_0)=G_\rho(t_0)+2\int_0^m\mathrm{e}^{t_0y}\mathrm{e}^{-y}\big|H(y)\big|dy.$$

It follows that $\int_0^\infty \mathrm{e}^{t_0 x}\mathrm{e}^{-y}\big|H(x)\big|dx = G_w(t_0) < \infty$, and thus,

$$\lim_{y\to\infty} \beta(y) := \lim_{y\to\infty} \int_y^\infty \mathrm{e}^{t_0 x}\mathrm{e}^{-x}\big|H(x)\big|dx = 0.$$

Since $H(x)$ is non-decreasing and non-negative in (m,∞), we obtain

$$\beta(y) \ge H(y) \int_y^\infty \mathrm{e}^{-(1-t_0)x} dx = H(y)\frac{\mathrm{e}^{-(1-t_0)y}}{1-t_0}, \ \ y > m,$$

and hence, $H(y) = |H(y)| \le C\mathrm{e}^{(1-t_0)y}$, $y > m$, where $C = (1-t_0)\beta(m) < \infty$. It follows that

$$\int_m^\infty \mathrm{e}^{-y}\big|H(y)\big|^{1+\delta} dy \le C^{1+\delta} \int_m^\infty \exp\Big[-y\Big(1-(1-t_0)(1+\delta)\Big)\Big]dy < \infty$$

for $0 \le \delta < t_0/(1-t_0)$. Since $\mathbb{E}(X^+)^p = \int_m^\infty \mathrm{e}^{-y}|H(y)|^p dy$, (i) is deduced. Fix now $\delta_0 \in (0, t_0/(1-t_0))$ and $a > 0$. Then, in view of (1.4), we have

$$M_T(a) = \sum_{n=0}^\infty \frac{a^n}{n!}\,\mathbb{E}\,T^n = \sum_{n=0}^\infty \frac{a^n}{n!}(n+1)!(\rho_{n+2}-\rho_{n+1}) \le \sum_{n=0}^\infty (n+1)(w_{n+2}+w_{n+1})a^n.$$

The last sum equals to

$$\int_0^\infty \left(\sum_{n=0}^\infty \frac{(y+n+1)(ay)^n}{n!}\right)\mathrm{e}^{-y}\big|H(y)\big|dy = \int_0^\infty (y+1+ay)\mathrm{e}^{ay}\mathrm{e}^{-y}\big|H(y)\big|dy.$$

Since the function $y \to (y+1+ay)\mathrm{e}^{ay}$ is bounded for $y \in [0,m]$, and $y \to \mathrm{e}^{-y}H(y) \in L^1(0,\infty)$, the integral $\int_0^m (y+1+ay)\mathrm{e}^{ay}\mathrm{e}^{-y}|H(y)|dy$ is finite for all a. Furthermore,

$$I := \int_m^\infty (y+1+ay)\mathrm{e}^{ay}\mathrm{e}^{-y}\big|H(y)\big|dy = \int_m^\infty \Big(\mathrm{e}^{-y/(1+\delta_0)}\big|H(y)\big|\Big)\Big((y+1+ay)\mathrm{e}^{-\kappa y}\Big)dy,$$

where $\kappa := \delta_0/(1+\delta_0) - a > 0$ if $a \in (0, \delta_0/(1+\delta_0))$. An application of Hölder's inequality (with $p = 1+\delta_0$, $q = 1 + 1/\delta_0$) to the last integral yields

$$I \le \left(\int_m^\infty \mathrm{e}^{-y}\big|H(y)\big|^{1+\delta_0} dy\right)^{1/(1+\delta_0)} \left(\int_m^\infty (y+1+ay)^{1+1/\delta_0}\mathrm{e}^{-\lambda y} dy\right)^{\delta_0/(1+\delta_0)},$$

where $\lambda := (1+1/\delta_0)\kappa > 0$. Hence, I is finite and, consequently, $M_T(a) < \infty$, for any $a < \delta_0/(1+\delta_0)$.

So far, we have shown that (ii)⇒(iii)⇒(i)⇒(iv). The remaining implication, (iv)⇒(ii), is a by-product of Theorem 1.1. Indeed, if $M_T(a) = \mathbb{E}\,\mathrm{e}^{aT}$ is finite for some $a \in (0,1)$, then (1.4) shows that

$$G_\rho(a) = \sum_{n\ge 2} a^{n-1} \sum_{j=0}^{n-2} \frac{\mathbb{E}\,T^j}{(j+1)!} = \sum_{j\ge 0} \frac{\mathbb{E}\,T^j}{(j+1)!} \sum_{n \ge j+2} a^{n-1} = \frac{1}{1-a}\,\mathbb{E}\Big[\frac{\mathrm{e}^{aT}-1}{T}\Big].$$

Since $(\mathrm{e}^{at}-1)/t < a\mathrm{e}^{at}$ for $t > 0$, (ii) is proved. Finally, from the preceding calculation,

$$G_\rho(t) = \frac{1}{1-t}\,\mathbb{E}\int_0^t \mathrm{e}^{sT} ds = \frac{1}{1-t}\int_0^t \mathbb{E}\,\mathrm{e}^{sT} ds, \quad 0 < t \le a,$$

and (4.1) is deduced.

Obviously, the results extend to a t-interval $-\epsilon_0 < t < \epsilon_0$ by analytic continuation. □

Suppose that for a given (non-degenerate) r.v. X, $\mathbb{E}\,X^- < \infty$ and $\mathbb{E}(X^+)^p < \infty$ for some $p > 1$. According to Theorem 4.2, the transformation $T = \varphi(X)$ of any such r.v. has finite moment generating function at a neighborhood of zero; hence T it characterized from its moments, and we obtain the following result.

Corollary 4.1 (Kirmani and Beg, [15])**.** Every r.v. X with finite absolute moment of order $p > 1$ is characterized from its expected record sequence.

However, we emphasize that the Kirmani-Beg characterization does not extend to $\mathcal{H}^*$:

Example 4.1. There exist different r.v.'s in $\mathcal{H}_0$ with identical expected record sequence. A concrete example leading to absolutely continuous r.v.'s can be constructed by means of the classical example due to Stieltjes, as follows. Let T be the lognormal r.v. with density $f_T(t) = \mathrm{e}^{-(\log t)^2/2}/(t\sqrt{2\pi})$, $t > 0$, and moments $\mathbb{E}\,T^n = \mathrm{e}^{n^2/2}$. Each density in the set $\Big\{f_\lambda(t) := (1 + \lambda \sin(\pi \log t))f_T(t),\ -1 \le \lambda \le 1\Big\}$ admits the same moments as T – see [34] or [35]. Assume that T_λ has density f_λ, and consider the r.v. $X_\lambda = \varphi^{-1}(T_\lambda)$, with distribution inverse given by (see Remark 4.1, below)

$$G_\lambda(u) := \int_1^{-\log(1-u)} \frac{\mathrm{e}^y}{y} f_\lambda(y)dy + \int_0^1 \frac{\mathrm{e}^t - 1}{t} f_\lambda(t)dt - \int_1^\infty \frac{1}{t} f_\lambda(t)dt, \quad 0 < u < 1.$$

Using an obvious notation, it is clear from Theorem 3.1(ii) that $\mathbb{E}\,R_1(X_\lambda) = 0$, $\mathbb{E}\,R_2(X_\lambda) = 1$, and the sequence $\rho_n^{(\lambda)} := \mathbb{E}\,R_n(X_\lambda)$ satisfies (1.4) with T_λ in place of T. Thus, each X_λ, $-1 \le \lambda \le 1$, has the same expected record sequence, namely,

$$\rho_n^{(\lambda)} = \int_0^1 \frac{[-\log(1-u)]^{n-1}}{(n-1)!} G_\lambda(u)du = \sum_{k=0}^{n-2} \frac{\mathrm{e}^{k^2/2}}{(k+1)!}, \quad n = 1, 2, 3, \dots, \quad -1 \le \lambda \le 1,$$

where an empty sum should be treated as zero. Differentiating G_λ, it follows that $G_{\lambda_1} \ne G_{\lambda_2}$ for $\lambda_1 \ne \lambda_2$. Therefore, the function $g := G_{\lambda_1} - G_{\lambda_2}$ belongs to

$$\mathcal{H}(0,1) := \left\{ g : (0,1) \to \mathbb{R} : \int_0^1 |g(u)|[-\log(1-u)]^k du < \infty \quad \text{for} \quad k = 0, 1, \dots \right\},$$

it is non-zero (when $\lambda_1 \ne \lambda_2$) in a set of positive measure, and satisfies

$$\int_0^1 g(u)[-\log(1-u)]^k du = 0, \quad k = 0, 1, \dots. \tag{4.3}$$

It is easily checked that every X_λ admits a density.

In fact, the Kirmani-Beg characterization holds true because the system of functions $\mathcal{L} := \Big\{[-\log(1-u)]^k,\ \ k = 0, 1, \dots\Big\}$ is complete in $\cup_{\delta>0} L^{1+\delta}(0,1)$, see Lemma 3 in Lin [18], while (4.3) implies that $\mathcal{L}$ is not complete in the larger space $\mathcal{H}(0,1)$.

Remark 4.1. In the particular case where $T \in \mathcal{T}$ admits a density f_T, the inversion formula (3.1) simplifies considerably, after an obvious application of Tonelli's theorem. Indeed, the function H_0 in (3.1) is given by

$$H_0(y) = \int_1^y \frac{\mathrm{e}^x}{x} f_T(x)dx + \int_0^1 \frac{\mathrm{e}^t - 1}{t} f_T(t)dt - \int_1^\infty \frac{1}{t} f_T(t)dt, \qquad y > 0. \tag{4.4}$$

Moreover, the r.v. $X_0 = \varphi^{-1}(T)$ has a continuous d.f. if $f_T(y) > 0$ for almost all $y > 0$.

Remark 4.2. The formula (4.4) is unable to describe several continuous r.v.'s in $\mathcal{H}_0$, for which, however, the ordinary record process $\{R_n\}_{n\ge1}$ is well-defined. This is so because any r.v. $T \in \mathcal{T}$ with dense support in $(0,\infty)$ will produce a continuous r.v. $X_0 = \varphi^{-1}(T) \in \mathcal{H}_0$. This observation is a consequence of (A.3), which implies that, for such an r.v. T, H_0 is strictly increasing, and hence, its d.f. is continuous. It is obvious that we can find discrete r.v.'s T with dense support and finite moment generating function at a neighborhood of zero. As a concrete example, set $T = T_1 + T_2$, where T_1 follows a Poisson d.f. with mean 1, $\mathbb{P}(T_2 = r_n) = 2^{-n}$ ($n = 1, 2, \ldots$), with $\{r_1, r_2, \ldots\}$ being an enumeration of the rationals of the interval $(0, 1]$, and assume that T_1, T_2 are independent. Set also $X = \varphi^{-1}(T)$. According to Theorem 4.2, this particular continuous r.v. X is, indeed, characterized from its ERS.

It is well-known that any r.v. T is uniquely determined from its moments, if it admits a finite moment generating function at a neighborhood of zero. On the other hand, it is also known that we can find several r.v.'s $T \in \mathcal{T}$ that are characterized from their moment sequence, although $\mathbb{E}\, \mathrm{e}^{aT} = \infty$ for all $a > 0$. A large family of such r.v.'s is the so called *Hardy class* – see [34] and [20] for more details. Clearly, the corresponding r.v.'s $\varphi^{-1}(T)$ can not be treated by Kirmani-Beg's [15] characterization, which shows that the proposed method, based on the transformation φ, is quite efficient.

A Invertibility of the transformation: construction of $X_0 \in \mathcal{H}_0$ from $T \in \mathcal{T}$

We shall provide a detailed proof of Theorem 3.1(ii), which also verifies the half counterpart of Proposition 3.1, showing that the mapping φ' is well-defined with domain $\mathcal{T}$ and range into $\mathcal{H}_0$, as stated. We notice that the present appendix is self-contained; it does not require any further results from the present article.

Suppose we are given an r.v. $T \in \mathcal{T}$ with d.f. F_T, i.e., $F_T(0) = 0$ and $\mathbb{E}\, T^n < \infty$ for all n. Define

$$H(y) := H_0(y) + c_T - \mathrm{e}F_T(1-), \quad y > 0, \tag{A.1}$$

where H_0 is as in (3.1) and c_T as in (3.2), and rewrite (3.1) as

$$H(y) = \begin{cases} -\mathrm{e}\big[F_T(1-) - F_T(y-)\big] - \int_y^1 \frac{1-x}{x^2}\mathrm{e}^x \big[F_T(x) - F_T(y-)\big]dx, & 0 < y \le 1, \\ \mathrm{e}\big[F_T(y-) - F_T(1-)\big] + \int_1^y \frac{x-1}{x^2}\mathrm{e}^x \big[F_T(y-) - F_T(x)\big]dx, & 1 \le y < \infty. \end{cases} \tag{A.2}$$

From (A.2) we see that $H(y) \le 0$ for $y \le 1$ and ≥ 0 otherwise.

Lemma A.1. H is non-decreasing and left-continuous.

Proof. Left-continuity is obvious. Also, H is non-positive in $(0, 1]$ and non-negative in $[1, \infty)$. Choose now y_1, y_2 with $0 < y_1 < y_2 \le 1$. Then,

$$H(y_2) - H(y_1) = \frac{\mathrm{e}^{y_2}}{y_2}\, \mathbb{P}(y_1 \le T < y_2) + \int_{y_1}^{y_2} \frac{1-x}{x^2} \mathrm{e}^x\, \mathbb{P}(y_1 \le T \le x)dx \ge 0. \tag{A.3}$$

A similar argument applies to the case $1 \le y_1 < y_2 < \infty$. □

Lemma A.2. (i) For all $x \ge 0$,

$$\int_0^x \mathrm{e}^{-y}\Big[F_T(x) - F_T(y)\Big]dy = \int_{(0,x]} (1 - \mathrm{e}^{-t})dF_T(t). \tag{A.4}$$

(ii) The function $y \to \mathrm{e}^{-y}H(y) \in L^1(0, m)$ for any finite $m > 0$.

Proof. Consider the non-negative r.v. $Y := (1-\mathrm{e}^{-T})I(T \le x) = g(T)$, for which it is easily verified that $\mathbb{P}(Y > u) = F_T(x) - F_T(-\log(1-u))$ for $0 \le u < 1 - \mathrm{e}^{-x}$, and $\mathbb{P}(Y > u) = 0$

for $u \ge 1-\mathrm{e}^{-x}$. Then, we can compute the expectation of Y by means of two different integrals, namely,

$$\mathbb{E}\,Y = \int_0^{1-\mathrm{e}^{-x}} \Big[F_T(x) - F_T(-\log(1-u))\Big]du, \quad \mathbb{E}\,g(T) = \int_{(0,x]} (1-\mathrm{e}^{-t})dF_T(t).$$

The integrals above are equal, and the substitution $-\log(1-u) = y$ yields (A.4). Fix now $\delta \in (0,1)$. From (A.2) we have

$$\int_0^\delta \mathrm{e}^{-y}|H(y)|dy \le \mathrm{e} + \int_0^\delta \mathrm{e}^{-y} \int_y^1 \frac{1-x}{x^2}\mathrm{e}^x \Big[F_T(x) - F_T(y)\Big]dxdy = \mathrm{e} + I, \text{ say,}$$

noting that

$$\mathrm{e}^{-y}\frac{1-x}{x^2}\mathrm{e}^x\Big[F_T(x) - F_T(y-)\Big] = \mathrm{e}^{-y}\frac{1-x}{x^2}\mathrm{e}^x\Big[F_T(x) - F_T(y)\Big]$$

for almost all $(y,x) \in (0,\infty)\times(0,1)$. Interchanging the order of integration according to Tonelli's theorem, we get

$$\begin{aligned} I &= \int_0^\delta \frac{1-x}{x^2}\mathrm{e}^x \int_0^x \mathrm{e}^{-y}\Big[F_T(x) - F_T(y)\Big]dydx \\ &\quad + \int_\delta^1 \frac{1-x}{x^2}\mathrm{e}^x \int_0^\delta \mathrm{e}^{-y}\Big[F_T(x) - F_T(y)\Big]dydx = I_1 + I_2. \end{aligned}$$

Obviously, I_2 is finite and it remains to verify that $I_1 < \infty$. In view of (A.4) we obtain

$$I_1 = \int_0^\delta \frac{1-x}{x^2}\mathrm{e}^x \int_{(0,x]} (1-\mathrm{e}^{-t})dF_T(t)dx = \int_{(0,\delta]} (1-\mathrm{e}^{-t}) \int_t^\delta \frac{1-x}{x^2}\mathrm{e}^x dxdF_T(t).$$

The last equation shows that I_1 is finite, because the inner integral is less than e^t/t. Thus,

$$I_1 \le \int_{(0,\delta]} \frac{\mathrm{e}^t - 1}{t} dF_T(t) = \mathbb{E}\left[\frac{\mathrm{e}^T - 1}{T} I(T \le \delta)\right]$$

and the function $T \to (\mathrm{e}^T - 1)I(T \le \delta)/T$ is (non-negative and) bounded. Finally, since $|H|$ is bounded in $[\delta, m]$ (see Lemma A.1), the lemma is proved. □

Lemma A.3. Define

$$a_k(t) := \int_t^\infty u^k \mathrm{e}^{-u}du = k!\mathrm{e}^{-t}\sum_{j=0}^k \frac{t^j}{j!}, \qquad t \ge 0, \ \ k = 0,1,\ldots. \tag{A.5}$$

Then,

$$\int_x^\infty y^k \mathrm{e}^{-y}\Big[F_T(y) - F_T(x)\Big]dy = \int_{(x,\infty)} a_k(t)dF_T(t), \ \ x \ge 0, \ \ k = 0,1,\ldots. \tag{A.6}$$

Proof. a_k is strictly decreasing with $a_k(0) = k!$ and $a_k(\infty) = 0$. Fix $x \ge 0$ and consider the bounded non-negative r.v. $Y := a_k(T)I(T > x)$. Then, $\mathbb{P}(Y > y) = F_T(a_k^{-1}(y)-) - F_T(x)$ for $0 \le y < a_k(x)$, and $\mathbb{P}(Y > y) = 0$ for $y \ge a_k(x)$, where a_k^{-1} is the (usual) inverse function of a_k. Since $F_T(a_k^{-1}(y)-) = F_T(a_k^{-1}(y))$ for almost all $y \in (0,\infty)$, we obtain

$$\mathbb{E}\,Y = \int_0^{a_k(x)} \Big[F_T(a_k^{-1}(y)) - F_T(x)\Big]dy = \int_x^\infty t^k\mathrm{e}^{-t}\Big[F_T(t) - F_T(x)\Big]dt,$$

where we made use of the substitution $t = a_k^{-1}(y)$. On the other hand,

$$\mathbb{E}\,Y = \mathbb{E}\Big[a_k(T)I(T > x)\Big] = \int_{(x,\infty)} a_k(t)dF_T(t),$$

and (A.6) is proved. □

Lemma A.4. $\int_0^\infty y^k \mathrm{e}^{-y}|H(y)|dy < \infty$ for $k = 0, 1, \ldots$.

Proof. Fix $k \in \{0, 1, \ldots\}$, $m \in (1, \infty)$, and write

$$\int_0^\infty y^k \mathrm{e}^{-y}|H(y)|dy = \int_0^m + \int_m^\infty y^k \mathrm{e}^{-y}|H(y)|dy = I_1 + I_2.$$

Lemma A.2(ii) shows that I_1 is finite, and we proceed to verify that I_2 is also finite. Using (A.2) we have

$$I_2 \le \mathrm{e}k! + \int_m^\infty \int_1^y y^k \mathrm{e}^{-y} \frac{x-1}{x^2} \mathrm{e}^x \Big[F_T(y) - F_T(x)\Big] dxdy = \mathrm{e}k! + I_3,$$

noting that

$$y^k \mathrm{e}^{-y} \frac{x-1}{x^2} \mathrm{e}^x \Big[F_T(y-) - F_T(x)\Big] = y^k \mathrm{e}^{-y} \frac{x-1}{x^2} \mathrm{e}^x \Big[F_T(y) - F_T(x)\Big]$$

for almost all $(y, x) \in (m, \infty) \times (1, \infty)$. It remains to show $I_3 < \infty$. Using Tonelli's theorem,

$$\begin{aligned} I_3 &= \int_1^m \frac{x-1}{x^2} \mathrm{e}^x \int_m^\infty y^k \mathrm{e}^{-y} \Big[F_T(y) - F_T(x)\Big] dydx \\ &\quad + \int_m^\infty \frac{x-1}{x^2} \mathrm{e}^x \int_x^\infty y^k \mathrm{e}^{-y} \Big[F_T(y) - F_T(x)\Big] dydx = J_1 + J_2. \end{aligned}$$

Now J_1 is obviously finite, because the inner integral is less that $k!$ and the function $x \to (x-1)\mathrm{e}^x/x^2$ is bounded for $x \in [1, m]$. Applying Lemma A.3 to the inner integral in J_2 we obtain

$$J_2 = \int_m^\infty \frac{x-1}{x^2} \mathrm{e}^x \int_{(x,\infty)} a_k(t) dF_T(t) dx = \int_{(m,\infty)} a_k(t) \int_m^t \frac{x-1}{x^2} \mathrm{e}^x dx dF_T(t).$$

Therefore, since the inner integral is less than e^t/t, and

$$\frac{\mathrm{e}^t}{t} a_k(t) = k! \left(\frac{1}{t} + \sum_{j=0}^{k-1} \frac{t^j}{(j+1)!} \right) \le \frac{k!}{m} + k! \sum_{j=0}^{k-1} \frac{t^j}{(j+1)!}, \quad t > m,$$

see (A.5), we arrive at the inequality $J_2 \le k!/m + k! \sum_{j=0}^{k-1} \mathbb{E}\, T^j/(j+1)!$, and this is finite because T has been assumed to possess finite moments of any order. $\square$

From Lemmas A.1, A.4 we conclude that $H \in \mathcal{H}$, so that $H_0 \in \mathcal{H}$ (since these functions differ by a constant–see (A.1)). We now proceed to show that $H_0 \in \mathcal{H}_0$.

Lemma A.5. For each $k = 1, 2, \ldots$,

$$\int_0^\infty (y^k - k!)\mathrm{e}^{-y} F_T(y) dy = \mathbb{E}\left[a_k(T) - k!\mathrm{e}^{-T}\right], \tag{A.7}$$

where a_k is given by (A.5).

Proof. Substitute $x = 0$ in Lemma A.3 and observe that $a_k(0) = k!$, $a_0(t) = \mathrm{e}^{-t}$ and $F_T(0) = 0$. $\square$

Lemma A.6. For each $k = 1, 2, \ldots$,

$$\int_0^\infty (y^k - k!)\mathrm{e}^{-y} H(y) dy = k! \sum_{j=0}^{k-1} \frac{\mathbb{E}\, T^j}{(j+1)!}. \tag{A.8}$$

Proof. Set $\beta_k := k!^{1/k}$ so that $1 = \beta_1 < \beta_2 < \cdots \to \infty$, as $k \to \infty$, and note that the integral in (A.8) is finite – see Lemma A.4. Clearly, $y^k > k!$ for $y \in (\beta_k, \infty)$ and $y^k < k!$ for $y \in (0, \beta_k)$. We split the integral in (A.8) as follows:

$$\int_0^1 (k! - y^k)\mathrm{e}^{-y}(-H(y))dy - \int_1^{\beta_k} (k! - y^k)\mathrm{e}^{-y}H(y)dy \\ + \int_{\beta_k}^{\infty} (y^k - k!)\mathrm{e}^{-y}H(y)dy = I_1 - I_2 + I_3.$$

Now we compute these three integrals. From (A.2),

$$I_1 = \mathrm{e}\int_0^1 (k! - y^k)\mathrm{e}^{-y}\Big[F_T(1-) - F_T(y)\Big]dy \\ + \int_0^1 (k! - y^k)\mathrm{e}^{-y}\int_y^1 \frac{1-x}{x^2}\mathrm{e}^x\Big[F_T(x) - F_T(y)\Big]dxdy = \mathrm{e}I_{11} + I_{12}.$$

Similarly,

$$I_2 = \mathrm{e}\int_1^{\beta_k} (k! - y^k)\mathrm{e}^{-y}\Big[F_T(y) - F_T(1-)\Big]dy \\ + \int_1^{\beta_k} (k! - y^k)\mathrm{e}^{-y}\int_1^y \frac{x-1}{x^2}\mathrm{e}^x\Big[F_T(y) - F_T(x)\Big]dxdy = \mathrm{e}I_{21} + I_{22},$$

and, finally,

$$I_3 = \mathrm{e}\int_{\beta_k}^{\infty} (y^k - k!)\mathrm{e}^{-y}\Big[F_T(y) - F_T(1-)\Big]dy \\ + \int_{\beta_k}^{\infty} (y^k - k!)\mathrm{e}^{-y}\int_1^y \frac{x-1}{x^2}\mathrm{e}^x\Big[F_T(y) - F_T(x)\Big]dxdy = \mathrm{e}I_{31} + I_{32}.$$

The above calculation shows that

$$\mathrm{e}(I_{11} - I_{21} + I_{31}) = \mathrm{e}F_T(1-)\int_0^{\infty}(k! - y^k)\mathrm{e}^{-y}dy + \mathrm{e}\int_0^{\infty}(y^k - k!)\mathrm{e}^{-y}F_T(y)dy.$$

Similarly,

$$I_{12} - I_{22} + I_{32} = \int_0^1 (k! - y^k)\mathrm{e}^{-y}\int_y^1 \frac{1-x}{x^2}\mathrm{e}^x\Big[F_T(x) - F_T(y)\Big]dxdy \\ + \int_1^{\infty} (y^k - k!)\mathrm{e}^{-y}\int_1^y \frac{x-1}{x^2}\mathrm{e}^x\Big[F_T(y) - F_T(x)\Big]dxdy.$$

Observing that $\int_0^{\infty}(k! - y^k)\mathrm{e}^{-y}dy = 0$, we finally obtain

$$\int_0^{\infty}(y^k - k!)\mathrm{e}^{-y}H(y)dy = \mathrm{e}J_1 + J_2 + J_3, \tag{A.9}$$

where

$$J_1 = \int_0^{\infty}(y^k - k!)\mathrm{e}^{-y}F_T(y)dy,$$
$$J_2 = \int_0^1\int_y^1 (k! - y^k)\mathrm{e}^{-y}\frac{1-x}{x^2}\mathrm{e}^x\Big[F_T(x) - F_T(y)\Big]dxdy,$$
$$J_3 = \int_1^{\infty}\int_1^y (y^k - k!)\mathrm{e}^{-y}\frac{x-1}{x^2}\mathrm{e}^x\Big[F_T(y) - F_T(x)\Big]dxdy.$$

The integrand in J_2 is non-negative, so we can change the order of integration. In order to justify that this is also permitted for J_3, we compute

$$\begin{aligned}
&\int_1^\infty \int_1^y \left|(y^k - k!)\mathrm{e}^{-y}\frac{x-1}{x^2}\mathrm{e}^x\Big[F_T(y) - F_T(x)\Big]\right| dxdy \\
&\le \int_1^\infty \frac{x-1}{x^2}\mathrm{e}^x \int_x^\infty (y^k + k!)\mathrm{e}^{-y}\Big[F_T(y) - F_T(x)\Big] dydx \\
&= \int_1^\infty \frac{x-1}{x^2}\mathrm{e}^x \int_{(x,\infty)} \Big[a_k(t) + k!a_0(t)\Big] dF_T(t)dx && \text{(Lemma A.3)} \\
&= \int_{(1,\infty)} \Big[a_k(t) + k!a_0(t)\Big]\left(\frac{\mathrm{e}^t}{t} - e\right) dF_T(t) && \text{(Tonelli)} \\
&\le \int_{(1,\infty)} \Big[a_k(t) + k!a_0(t)\Big]\frac{\mathrm{e}^t}{t} dF_T(t) < \infty,
\end{aligned}$$

because, for $t > 1$,

$$\Big[a_k(t) + k!a_0(t)\Big]\frac{\mathrm{e}^t}{t} = \frac{2k!}{t} + k!\sum_{j=0}^{k-1}\frac{t^j}{(j+1)!} \le 2k! + k!\sum_{j=0}^{k-1}\frac{t^j}{(j+1)!},$$

and T has finite moments of any order. Thus,

$$\begin{aligned}
J_2 &= \int_0^1 \frac{1-x}{x^2}\mathrm{e}^x \int_0^x (k! - y^k)\mathrm{e}^{-y}\Big[F_T(x) - F_T(y)\Big] dydx, \\
J_3 &= \int_1^\infty \frac{x-1}{x^2}\mathrm{e}^x \int_x^\infty (y^k - k!)\mathrm{e}^{-y}\Big[F_T(y) - F_T(x)\Big] dydx.
\end{aligned}$$

The inner integral in J_3 equals to $\int_{(x,\infty)}\big[a_k(t) - k!\mathrm{e}^{-t}\big]dF_T(t)$; see Lemma A.3. Since $a_k(t) - k!\mathrm{e}^{-t} = k!\mathrm{e}^{-t}\sum_{j=1}^k t^j/j!$, we obtain (after changing the order of integration once again)

$$\begin{aligned}
J_3 &= k!\int_{(1,\infty)} \left(\mathrm{e}^{-t}\sum_{j=1}^k \frac{t^j}{j!}\right)\left(\frac{\mathrm{e}^t}{t} - \mathrm{e}\right) dF_T(t) \\
&= k!\int_{(1,\infty)} \sum_{j=0}^{k-1}\frac{t^j}{(j+1)!} dF_T(t) - \mathrm{e}k!\int_{(1,\infty)} \mathrm{e}^{-t}\sum_{j=1}^k \frac{t^j}{j!} dF_T(t). && \text{(A.10)}
\end{aligned}$$

Next, we make similar calculations for the inner integral in J_2. We have

$$\begin{aligned}
&\int_0^x (k! - y^k)\mathrm{e}^{-y}\Big[F_T(x) - F_T(y)\Big] dy \\
&= \int_0^\infty (y^k - k!)\mathrm{e}^{-y}\Big[F_T(y) - F_T(x)\Big] dy - \int_x^\infty (y^k - k!)\mathrm{e}^{-y}\Big[F_T(y) - F_T(x)\Big] dy \\
&= \int_0^\infty (y^k - k!)\mathrm{e}^{-y} F_T(y) dy - \int_{(x,\infty)} \Big[a_k(t) - k!\mathrm{e}^{-t}\Big] dF_T(t) \\
&= \mathbb{E}\Big[a_k(T) - k!\mathrm{e}^{-T}\Big] - \int_{(x,\infty)} \Big[a_k(t) - k!\mathrm{e}^{-t}\Big] dF_T(t) \\
&= \int_{(0,x]} \Big[a_k(t) - k!\mathrm{e}^{-t}\Big] dF_T(t),
\end{aligned}$$

where we made use of Lemmas A.3 and A.5 and the fact that $\int_0^\infty (y^k - k!)\mathrm{e}^{-y}dy = 0 =$

$a_k(0) - k!\mathrm{e}^{-0}$. Therefore,

$$J_2 = -\mathrm{e}\int_{(0,1]} \Big[a_k(t) - k!\mathrm{e}^{-t}\Big] dF_T(t) + \int_{(0,1]} \frac{\mathrm{e}^t}{t} \Big[a_k(t) - k!\mathrm{e}^{-t}\Big] dF_T(t)$$
$$= -\mathrm{e}k! \int_{(0,1]} \mathrm{e}^{-t} \sum_{j=1}^{k} \frac{t^j}{j!}\, dF_T(t) + k! \int_{(0,1]} \sum_{j=0}^{k-1} \frac{t^j}{(j+1)!} dF_T(t). \tag{A.11}$$

Finally, from Lemma A.5,

$$\mathrm{e}J_1 = \mathrm{e}\int_{(0,\infty)} \Big[a_k(t) - k!\mathrm{e}^{-t}\Big] dF_T(t) = \mathrm{e}k! \int_{(0,\infty)} \mathrm{e}^{-t} \sum_{j=1}^{k} \frac{t^j}{j!} dF_T(t). \tag{A.12}$$

Combining (A.10)–(A.12) we obtain

$$\mathrm{e}J_1 + J_2 + J_3 = k! \int_{(0,1]} \sum_{j=0}^{k-1} \frac{t^j}{(j+1)!} dF_T(t) + k! \int_{(1,\infty)} \sum_{j=0}^{k-1} \frac{t^j}{(j+1)!} dF_T(t),$$

and from (A.9) we conclude (A.8). □

Lemma A.7. $\int_0^\infty \mathrm{e}^{-y} H(y)dy = c_T - \mathrm{e}F_T(1-)$, where c_T is as in (3.2) and H as in (A.1).

Proof. Using (A.2) and applying Lemma A.3 (for $k = 0$), we obtain

$$\int_1^\infty \mathrm{e}^{-y} H(y)dy = \mathbb{P}(T = 1) + \mathrm{e}\int_{(1,\infty)} \mathrm{e}^{-t} dF_T(t)$$
$$+ \int_1^\infty \frac{x-1}{x^2} \mathrm{e}^x \int_{(x,\infty)} \mathrm{e}^{-t} dF_T(t)dx = \mathbb{P}(T = 1) + \mathbb{E}\Big[\frac{1}{T} I(T > 1)\Big].$$

Similarly, from Lemma A.2(i), we get

$$\int_0^1 \mathrm{e}^{-y}(-H(y))dy = -(\mathrm{e} - 1)\,\mathbb{P}(T = 1) + \mathrm{e}\int_{(0,1]} (1 - \mathrm{e}^{-t}) dF_T(t)$$
$$+ \int_0^1 \frac{1-x}{x^2} \mathrm{e}^x \int_{(0,x]} (1 - \mathrm{e}^{-t}) dF_T(t)dx$$
$$= -(\mathrm{e} - 1)\,\mathbb{P}(T = 1) + \mathbb{E}\Big[\frac{\mathrm{e}^T - 1}{T} I(T \le 1)\Big].$$

Subtracting the above equations we deduce the desired result. □

Proof of Theorem 3.1(ii) *and of the second part of Proposition 3.1.* Let E be a standard exponential r.v., and set $X_0 := H_0(E)$, where H_0 is given by (3.1). From Lemmas A.7, A.1, A.4 and A.6 (with $k = 1$), and in view of (2.3), we see that $H_0 \in \mathcal{H}_0$, i.e., $\rho_1 = 0$, $\rho_2 = 1$, where $\{\rho_n\}_{n\ge1}$ is the ERS from X_0. Noting that $X_0 = \varphi'(T)$, see Definition 3.3, we have proven that the mapping φ' is well-defined for all $T \in \mathcal{T}$, and its values are in $\mathcal{H}_0$. Finally, from Lemma A.6,

$$\rho_{n+1} = \rho_{n+1} - \rho_1 = \int_0^\infty \left(\frac{y^n}{n!} - 1\right) \mathrm{e}^{-y} H_0(y)dy = \sum_{j=0}^{n-1} \frac{\mathbb{E}\,T^j}{(j+1)!}, \quad n = 1, 2, \dots$$

(the last equality is justified because $H_0 - H = c$, constant), completing the proof. □

References

[1] Ahsanullah, M. (1995). *Record Statistics.* Nova Science Publishers, N.Y. MR1443904

[2] Aliev, F.A. (1998). Characterizations of distributions through weak records. *J. Appl. Stat. Science*, **8**, 13–16. MR1659333

[3] Arnold, B.C.; Balakrishnan, N.; Nagaraja, H.N. (1998). *Records.* Wiley, N.Y. MR1628157

[4] Barakat, H.M.; Nigm, E.M.; Khaled, O.M. (2019). *Statistical Techniques for Modelling Extreme Value Data and Related Applications.* Cambridge Scholars Publishing, U.K.

[5] Danielak, K.; Dembińska, A. (2007). On characterizing discrete distributions via conditional expectations of weak record values. *Metrika*, **66**(2), 129–138. MR2329028

[6] Dembińska, A.; Wesolowski, J. (2000). Linearity of regression for non-adjacent record values. *J. Statist. Plann. Inference*, **90**, 195–205. MR1795596

[7] Embrechts, P.; Klüppelberg, C; Mikosch, T. (1997). *Modelling Extremal Events.* Springer-Verlag, Berlin. MR1458613

[8] Hausdorff, F. (1921). Summationmethoden und momentfolgen. I. *Math. Zeitchrift,* **9**(1), 74–109. MR1544453

[9] Hill, T.P.; Spruill, M.C. (1994). On the relationship between convergence in distribution and convergence of expected extremes. *Proc. Amer. Math. Soc.*, **121**(4), 1235–1243. MR1195722

[10] Hill, T.P.; Spruill, M.C. (2000). Erratum to “On the relationship between convergence in distribution and convergence of expected extremes”. *Proc. Amer. Math. Soc.*, **128**, 625–626. MR1707517

[11] Hoeffding, W. (1953). On the distribution of the expected values of the order statistics. *Ann. Math. Statist.*, **24**(1), 93–100. MR0054197

[12] Huang, J.S. (1998). Sequences of expectations of maximum-order statistics. *Statist. Probab. Lett.*, **38**, 117–123. MR1627910

[13] Kadane, J.B. (1971). A moment problem for order statistics. *Ann. Math. Statist.*, **42**, 745–751. MR0292241

[14] Kadane, J.B. (1974). A characterization of triangular arrays which are expectations of order statistics. *J. Appl. Probab.*, **11**, 413–416. MR0362700

[15] Kirmani, S.N.U.A.; Beg, M.I. (1984). On characterization of distributions by expected records. *Sankhyā A*, **46**, 463–465. MR0798052

[16] Kolodynski, S. (2000). A note on the sequence of expected extremes. *Statist. Probab. Lett.*, **47**, 295–300. MR1747490

[17] Korwar, R.M. (1984). On characterizing distributions for which the second record value has a linear regression on the first. *Sankhyā B*, **46**, 108–109. MR0773650

[18] Lin, G.D. (1987). On characterizations of distributions via moments of record values. *Probab. Th. Rel. Fields*, **74**, 479–483. MR0876250

[19] Lin, G.D.; Huang, J.S. (1987). A note on the sequence of expectations of maxima and of record values. *Sankhyā A*, **3**, 272–273. MR1055792

[20] Lin, G.D.; Stoyanov, J. (2016). On the moment determinacy of products of non-identically distributed random variables. *Probab. Math. Statist.*, **36**(1), 21–33. MR3529337

[21] Lopez-Blazquez, F.; Wesolowski, J. (2001). Discrete distributions for which the regression of the first record on the second is linear. *Test*, **10**, 121–131. MR1856187

[22] Nagaraja, H.N. (1977). On a characterization based on record values. *Austral. J. Statist.*, **19**, 70–73. MR0458667

[23] Nagaraja, H.N. (1978). On the expected values of record values. *Austral. J. Statist.*, **20**, 176–182. MR0548184

[24] Nagaraja, H.N. (1988). Some characterizations of continuous distributions based on regressions of adjacent order statistics and record values. *Sankhyā A*, **50**, 70–73. MR1056242

[25] Nevzorov, V.B. (2001). *Records: Mathematical Theory.* American Mathematical Society, Providence, RI. MR1791071

[26] Papadatos, N. (2001). Distribution and expectation bounds on order statistics from possibly dependent variates. *Statistics and Probability Letters*, **54**, 21–31. MR1857867

[27] Papadatos, N. (2012). Linear estimation of location and scale parameters using partial maxima. *Metrika*, **75**, 243–270. MR2887565

[28] Papadatos, N. (2017). On sequences of expected maxima and expected ranges. *J. Appl. Probab.*, **54**, 1144–1166. MR3731289

[29] Raqab M.Z. (2002). Characterizations of distributions based on the conditional expectations of record values. *Statistics & Decisions*, **20**, 309–319. MR1945941

[30] Resnick, S.I. (1973). Record values and maxima. *Ann. Probab.*, **1**(4), 650–662. MR0356186

[31] Resnick, S.I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, N.Y. MR0900810

[32] Stepanov, A.V. (1993). A characterization theorem for weak records. *Theory Probab. Appl.*, **38**, 762–764. MR1318007

[33] Stoyanov, J. (2013). *Counterexamples in Probability*. Dover Publications, N.Y. MR3837562

[34] Stoyanov, J.; Lin, G.D. (2013). Hardy's condition in the moment problem for probability distributions. *Theory. Probab. Appl.* (SIAM), **57**(4), 699–708. MR3201676

[35] Stoyanov, J.; Tolmatz, L. (2005). Method for constructing Stieltjes classes for M-indeterminate probability distributions. *Appl. Math. Comput.*, **165**, 669–685. MR2138909

[36] Tryfos, P; Blackmore, R. (1985). Forecasting records. *J. Am. Stat. Assoc.*, **80**, 46–50. MR0786596

[37] Yanev, G.P. (2012). Characterization of exponential distribution via regression of one record value on two non-adjacent record values. *Metrika*, **75**(6), 743–760. MR2956274

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