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On Hadamard powers of random Wishart matrices

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Abstract

A famous result of Horn and Fitzgerald is that the β-th Hadamard power of any $n \times n$ positive semi-definite (p.s.d.) matrix with non-negative entries is p.s.d. for all $\beta \geq n-2$ and is not necessarily p.s.d. for $\beta < n-2$, with $\beta \notin \mathbb{N}$. In this article, we study this question for random Wishart matrix $A_n := X_n X_n^T$, where X_n is $n \times n$ matrix with i.i.d. Gaussian entries. It is shown that applying $x \to |x|^\alpha$ entrywise to A_n, the resulting matrix is p.s.d., with high probability, for $\alpha > 1$ and is not p.s.d., with high probability, for $\alpha < 1$. It is also shown that if X_n are $\lfloor n^s \rfloor \times n$ matrices, for any $s < 1$, then the transition of positivity occurs at the exponent $\alpha = s$.

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1 Introduction

Entrywise exponents of matrices preserving positive semi-definiteness has been a topic of active research (see [4, 8, 10, 7]). They appear naturally in many fields of pure and applied mathematics. For example, in high-dimensional probability, entrywise exponents are applied to covariance matrices to obtain regularized estimators (see [9]). The resulting matrices are further subjected to statistical procedures that require positive semi-definiteness. Therefore it is important to know if Hadamard powers preserve positive semi-definiteness.

An important theorem in this field is the result of Horn and Fitzgerald [3]. Let $\mathcal{P}_n^+$ denote the set of $n \times n$ p.s.d. matrices with non-negative entries. The Schur product theorem gives us that the k-th Hadamard power $A^{\circ k} := [a_{ij}^k]$ of any p.s.d. matrix $A = [a_{ij}] \in \mathcal{P}_n^+$ is again p.s.d. for every positive integer k. Horn and Fitzgerald proved that $n-2$ is the 'critical exponent' for such matrices, i.e., $n-2$ is the least number for which $A^{\circ\alpha} \in \mathcal{P}_n^+$ for every $A \in \mathcal{P}_n^+$ and for every real number $\alpha \geq n-2$. They considered the matrix $A \in \mathcal{P}_n^+$ with (i,j)-th entry $1 + \varepsilon ij$ and showed that if α is not an integer and $0 < \alpha < n-2$, then $A^{\circ\alpha}$ is not p.s.d. for a sufficiently small positive number ε (also see [8, 7]).

We consider a random matrix version of this problem. Let $X := [X_{ij}]$ be an $n \times n$ matrix, where X_{ij} are i.i.d. standard normal random variables. Define $A_n := \frac{XX^T}{n}$ and $|A_n|^{\circ\alpha}$ as the matrix obtained by applying $x \to |x|^\alpha$ function entrywise to A_n. Let $B_{n,\alpha} := |A_n|^{\circ\alpha}$. We are interested in the values of real $\alpha > 0$ for which the matrix $B_{n,\alpha}$ is p.s.d., with high probability.

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Table 1: Table of smallest eigenvalues for varying α and s with $n = 5000$.

s	1	1	1	1	0.8	0.8	0.8	0.8
α	0.98	0.99	1.06	1.07	0.78	0.79	0.81	0.82
$\lambda_{\min}$	-0.288	-0.246	0.016	0.046	-0.076	-0.049	0.017	0.041

In probability and statistical mechanics, phase transitions refer to the phenomenon of abrupt changes in the properties of a system as a parameter approaches a 'critical point'. The phase transition results that we prove are novel and simulations suggest that our results are true even when Gaussians in X are replaced by other i.i.d. random variables.

Simulations show that for large values of n, if $\alpha > 1$ then with high probability, $B_{n,\alpha}$ is p.s.d. and if $\alpha < 1$ then with high probability, $B_{n,\alpha}$ is not p.s.d. (as shown in Table 1).

We prove the theorem that these observations from simulations are indeed true. In fact, we prove a stronger result. Fix any $s \leq 1$ and let $m = \lfloor n^s \rfloor$. Let $X_{m,n} := [X_{ij}]$ be an $m \times n$ matrix, where X_{ij} are i.i.d. standard normal random variables. Define $A_{n,s} := \frac{X_{m,n} X_{m,n}^T}{n}$ and $B_{n,\alpha,s} := |A_{n,s}|^{\circ\alpha}$. Let $\lambda_{\min}(A),\ \lambda_{\max}(A)$ denote the smallest and largest eigenvalue of a symmetric $m \times m$ matrix A. We prove the following main result.

Theorem 1.1. *Let $s \leq 1$. Then there exists $\varepsilon_s = \varepsilon(s) > 0$ such that as $n \to \infty$,*

$$\begin{aligned} \mathbb{P}\big(\lambda_{\min}(B_{n,\alpha,s}) \geq \varepsilon_s\big) \to 1 \qquad & \text{if } \alpha > s, \\ \mathbb{P}\big(\lambda_{\min}(B_{n,\alpha,s}) < 0\big) \to 1 \qquad & \text{if } \alpha < s \end{aligned}$$

Remark 1.2. Simulations show that Theorem 1.1 holds if i.i.d. Gaussians are replaced by other i.i.d. random variables with finite second moment like Uniform$(0,1)$, Exp(1) and even heavy tailed distributions like Cauchy distribution, distributions with densities $f(x) = bx^{-1-b}, \forall x \geq 1$, all with transition of positivity at exponent $\alpha = s$. This suggests that the transition of matrix positivity happens for a large family of distributions. In this direction we prove the below proposition where we show that $B_{n,\alpha,s}$ is p.s.d. for the range of $\alpha > 2s$, when $X_{m,n}$ has sub-Gaussian entries.

Proposition 1.3. *Let $m = \lfloor n^s \rfloor$ for $s \leq 1$ and let the entries of $X_{m,n}$ be i.i.d. sub-Gaussian random variables with mean 0 and unit variance. Fix $\alpha > 2s$ and $\varepsilon > 0$. Define $B_{n,\alpha,s}$ as before. Then as $n \to \infty$*

$$\mathbb{P}\big(\lambda_{\min}(B_{n,\alpha,s}) \leq 1 - \varepsilon\big) \to 0, \tag{1.1}$$

$$\mathbb{P}\big(\lambda_{\max}(B_{n,\alpha,s}) \geq 1 + \varepsilon\big) \to 0. \tag{1.2}$$

Remark 1.4. Although Theorem 1.1 and Proposition 1.3 hold for $m = \Theta(n^s)$, for definiteness we have considered $m = \lfloor n^s \rfloor$. For fixed $a > 0$ and $m = a \times n$, the transition of positivity is at exponent 1. For the critical exponent to be less than 1, we need $m = \Theta(n^s)$ with $s < 1$, which is much smaller than n, unlike in the study of spectrum of Wishart matrices.

A standard way to study the distribution of eigenvalues of a random matrix is to look at the limit of empirical spectral distributions using method of moments. For example, Wigner's proof of semi-circle law for Gaussian ensemble uses this method (for more see [1]). In our case, the entries of the matrix $B_{n,\alpha,s}$ are sums of products of random variables and the entries on the same row or column are correlated. The entrywise absolute fractional power makes this problem intractable, if we try to use method of moments or Stieltjes transforms.

1.1 Outline of the paper

First we prove Proposition 1.3 in Section 2. This is done using Gershgorin's circle theorem and the sub-exponential Bernstein's inequality. Note that this proposition is not needed to prove Theorem 1.1.

The proof of Theorem 1.1 is divided into two parts. In the first part of the proof, we consider the range $\alpha < s$. We use Lemma 3.3 to conclude that the expected empirical spectral distribution (EESD) of $B_{n,\alpha,s}$ has positive weight on negative reals. Using a concentration of measure result, we then show that with high probability, $B_{n,\alpha,s}$ has negative eigenvalues. This is done in Subsection 3.1.

In the second part of the proof, we consider the range $s < \alpha$. We further divide this range by looking at $(\frac{k+1}{k})s < \alpha$, where k is an integer greater than 1 and let $k \to \infty$. For $(\frac{k+1}{k})s < \alpha$, we consider C_m, a different modification of $B_{n,\alpha,s}$, whose EESD has $2k$-th moment converging to 0 to conclude that the probability of $B_{n,\alpha,s}$ having a negative eigenvalue converges to 0. We then let k be arbitrarily large. This is done in Subsection 3.2.

1.2 Notation

We use the following notations in this paper.

1) $m = \lfloor n^s \rfloor$.
2) $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of symmetric matrix A respectively.
3) R_i denotes the i-th row of $X_{m,n}$ ($R_i^T \sim N(0, I_n)$ in Section 3 but not necessarily in Section 2).
4) $\rho_{ij} = \frac{\langle R_i, R_j \rangle}{\|R_i\|\|R_j\|}$.
5) $\ell_\alpha = \mathbb{E}[|Z|^\alpha]$, where Z is a standard normal random variable.
6) $J_n =$ All ones matrix of size $n \times n$ and $I_n = n \times n$ identity matrix.
7) $\mathcal{F}_{i,j} =$ The sigma algebra generated from the i-th row and j-th row of $X_{m,n}$.
8) $\sigma_i = \|R_i\|/\sqrt{n}$.
9) $Y_{ij} = \mathbb{E}[(|\frac{\langle R_i, R_k \rangle}{\sqrt{n}}|^\alpha - m_\alpha)(|\frac{\langle R_k, R_j \rangle}{\sqrt{n}}|^\alpha - m_\alpha) \mid \mathcal{F}_{i,j}]$.

2 Proof of Proposition 1.3

For the rest of this section we fix $s \le 1$ and $\alpha > 2s$. Also we recall that the entries of R_i in $X_{m,n}$ for this section are i.i.d. sub-Gaussian random variables. For ease of notation, we write $B_{n,\alpha,s}$ as B_n. We will use concentration inequalities to show that $B_n(i,i) \in [1-\varepsilon, 1+\varepsilon]$ and $\sum_{i \neq j} |B_n(i,j)| \le \varepsilon$ with high probability. We then apply Gershgorin circle theorem to prove that all eigenvalues of B_n are in $[1-2\varepsilon, 1+2\varepsilon]$ with high probability.

Proof of Proposition 1.3. The diagonal entries of B_n are of the form $(\frac{\langle R_i, R_i \rangle}{n})^\alpha$ and off-diagonal entries are of the form $|\frac{\langle R_i, R_j \rangle}{n}|^\alpha$. Note that all the off-diagonal entries are identically distributed and all the diagonal entries are identically distributed. First we give an upper bound for the probability that $\sum_{i=2}^m (B_n)_{1i} > \varepsilon$.

$$\mathbb{P}\left(\sum_{i=2}^m (B_n)_{1i} > \varepsilon\right) \le m\mathbb{P}\left((B_n)_{12} > \frac{\varepsilon}{m}\right).$$

Note that $(B_n)_{12}$ is a function of sum of n independent sub-exponential random variables (product of independent Gaussians is sub-exponential (Lemma 2.7.7 of [11]). We now recall the Bernstein inequality for sub-exponential random variables from [11].

Theorem 2.1 (Theorem 2.8.1 of [11]). *Let $X_1, X_2, \dots, X_N$ be independent, mean zero, sub-exponential random variables. Then, for every $t \geq 0$, we have*

$$\mathbb{P}\left(\left|\sum_{i=1}^{N} X_i\right| \geq t\right) \leq 2\exp\left[-c\min\left(\frac{t^2}{\sum_{i=1}^{N}\|X_i\|_{\psi_1}^2}, \frac{t}{\max_i \|X_i\|_{\psi_1}}\right)\right]$$

where $c > 0$ is an absolute constant and $\|X\|_{\psi_1}$ is the sub-exponential norm of X.

Bernstein's inequality and the fact that $m = \lfloor n^s \rfloor$ gives us that

$$\mathbb{P}\left((B_n)_{12} > \frac{\varepsilon}{m}\right) = \mathbb{P}\left(\left|\langle R_1, R_2\rangle\right| \geq n\left(\frac{\varepsilon}{m}\right)^{1/\alpha}\right) \leq 2\exp\left(-c_1 n^{1-\frac{2s}{\alpha}}\right)$$

for some constant $c_1 = c_1(\varepsilon)$. This implies that

$$\mathbb{P}\left(\sum_{i=2}^{m}(B_n)_{1i} > \varepsilon\right) \leq 2m\exp\left(-c_1 n^{1-\frac{2s}{\alpha}}\right).$$

Using the identical distribution of off-diagonal entries, we get that

$$\mathbb{P}\left(\bigcup_{i=1}^{m}\left(\sum_{j=1, j\neq i}^{m}(B_n)_{ij} > \varepsilon\right)\right) \leq 2m^2\exp\left(-c_1 n^{1-\frac{2s}{\alpha}}\right). \tag{2.1}$$

For the diagonal entry $(B_n)_{11}$, we have

$$\mathbb{P}\left((B_n)_{11} \leq 1-\varepsilon\right) \leq \mathbb{P}\left(\langle R_1, R_1\rangle - n \leq n\left((1-\varepsilon)^{1/\alpha} - 1\right)\right) \leq 2\exp(-c_2 n),$$

for a constant $c_2 = c_2(\varepsilon, \alpha)$. Here we have used Theorem 2.1 in the last inequality, as $\langle R_1, R_1\rangle - n$ is a sum of n mean 0, i.i.d. sub-exponential random variables and $t = n((1-\varepsilon)^{1/\alpha} - 1)$.

This implies that

$$\mathbb{P}\left(\bigcup_{i=1}^{m}\left((B_n)_{ii} \leq 1-\varepsilon\right)\right) \leq 2m\exp(-c_2 n). \tag{2.2}$$

Similarly

$$\mathbb{P}\left(\bigcup_{i=1}^{m}\left((B_n)_{ii} \geq 1+\varepsilon\right)\right) \leq 2m\exp(-c_2 n). \tag{2.3}$$

Applying Gershgorin circle theorem (Theorem 6.1.1 of [6]) to B_n, using (2.1), (2.2), (2.3), gives us that, with probability at least $1 - 4m^2\exp(-c_3 n^{1-\frac{2s}{\alpha}})$, $\lambda_{\min} \geq 1 - 2\varepsilon$ and $\lambda_{\max} \leq 1 + 2\varepsilon$. Here $c_3 > 0$ depends on ε and α. As $\alpha > 2s$, this completes the proof of Proposition 1.3. □

3 Proof of Theorem 1.1

3.1 Proof of Theorem 1.1 for the range $\alpha < s$

For the proof we define the following matrices. Let $C_{n,\alpha,s} := \frac{B_{n,\alpha,s}}{n^{\frac{s-\alpha}{2}}}$. For ease of notation, we write $C_{n,\alpha,s}$ as C_m. C_m is a $m \times m$ matrix where $m = \lfloor n^s \rfloor$. Define the diagonal matrix D_m, with $D_m(i,i) := C_m(i,i) - \frac{\ell_\alpha}{n^{s/2}}$ and $E_m := C_m - D_m - \frac{\ell_\alpha}{n^{s/2}} J_m$, where ℓ_α, J_m are as defined in Subsection 1.2. We define a few terms here which will be used in the rest of the article. Empirical spectral distribution of a symmetric random matrix A_n is the random probability measure $\mu_{A_n} := \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}$, where λ_is are the eigenvalues of A_n. Expected empirical spectral distribution(EESD) of A_n is the probability measure $\bar{\mu}_{A_n}$ such that $\int_{\mathbb{R}} f d\bar{\mu}_{A_n} = \mathbb{E}[\int_{\mathbb{R}} f \, d\mu_{A_n}]$, for all bounded continuous functions f (for more see [1]).

We use Lemma 3.3 which shows that the limiting distribution of EESDs of E_m has positive weight on the negative reals. We then use a concentration of measure result to prove Lemma 3.1 which immediately implies Theorem 1.1 for the range $\alpha < s$. We then give the proof of Lemma 3.3.

Lemma 3.1. *Fix $\alpha < s$. Then $\mathbb{P}(\lambda_{\min}(C_{n,\alpha,s}) < 0) \to 1$, as $n \to \infty$.*

Proof of Lemma 3.1. We complete the proof of Lemma 3.1 assuming Lemma 3.3 and then provide the proof of Lemma 3.3. For the sake of contradiction assume that $\mathbb{P}(\lambda_{\min}(C_m) < 0)$ does not converge to 1, then by going to a subsequence we may assume that $\exists\, \varepsilon > 0$ such that $\mathbb{P}(\lambda_{\min}(C_m) \geq 0) > \varepsilon$.

Let $\bar{\mu}_{E_m}$ converge weakly to some probability distribution μ (using (ii) of Lemma 3.3 we get the tightness of $\bar{\mu}_{E_m}$). Using Lemma 3.3 and uniform integrability one can see that μ must have mean 0, positive variance (see Remark 3.4). As μ has zero mean and positive variance, $\mu(-\infty, -\omega) \geq \eta$ for some $\eta, \omega > 0$. This gives us that $\bar{\mu}_{E_m}(-\infty, -\omega) > \frac{\eta}{2}$ for large enough n. We would like to say with high probability, empirical spectral distributions of E_m also have positive weight on the negative reals. This would imply the existence of negative eigenvalues, with high probability. Here we make use of the following McDiarmid-type concentration result due to Guntuboyina and Leeb [5]. Let F_{μ_A} denote the cumulative distribution function of μ_A and $F_{\mu_A}(f) = \int_{\mathbb{R}} f d\mu_A$. The Kolmogorov-Smirnov distance between two probability measures μ, μ' is defined as $d_{KS}(\mu, \mu') := \|F_\mu - F_{\mu'}\|_\infty$. Let $V_g([a,b])$ denote the total variation of the function g on an interval $[a,b]$ and $V_g(\mathbb{R}) := \sup_{[a,b]} V_g([a,b])$.

Theorem 3.2 (Theorem 6 of [5])**.** *Let M be a random symmetric $n \times n$ matrix that is a function of m independent random quantities $Y_1, Y_2, \ldots, Y_m$, i.e., $M = M(Y_1, Y_2, \ldots, Y_m)$. Write $M_{(i)}$ for the matrix obtained from M after replacing Y_i by an independent copy, i.e., $M_{(i)} = M(Y_1, \ldots, Y_{i-1}, Y_i^*, Y_{i+1}, \ldots, Y_m)$ where Y_i^* is distributed as Y_i and independent of $Y_1, Y_2 \ldots, Y_m$. For $S = M/\sqrt{m}$ and $S_{(i)} = M_{(i)}/\sqrt{m}$, assume that*

$$\|F_S - F_{S_{(i)}}\|_\infty \leq \frac{r}{n}$$

holds (almost surely) for each $i = 1, 2, \ldots, m$ and for some (fixed) integer r. Finally, assume that $g : \mathbb{R} \to \mathbb{R}$ is of bounded variation on $\mathbb{R}$. For each $\varepsilon > 0$, we then have

$$\mathbb{P}\big(\big|F_S(g) - \mathbb{E}\big[F_S(g)\big]\big| \geq \varepsilon\big) \leq 2\exp\left[-\frac{n^2 2\varepsilon^2}{mr^2 V_g^2(\mathbb{R})}\right].$$

We apply Theorem 3.2 where E_m is the matrix M which is a function of the $\lfloor n^s \rfloor$ rows (independent) of $X_{m,n}$. In order to apply Theorem 3.2, we need to show

$$\|F_{E_m} - F_{E_{m(i)}}\|_\infty \leq \frac{r}{\lfloor n^s \rfloor} \tag{3.1}$$

almost surely. Here $E_{m(i)}$ is the matrix obtained when i-th row of $X_{m,n}$ is replaced by an independent and identical copy. Using the fact that rank$(E_m - E_{m(i)}) \leq 2$ and the standard rank inequality (Lemma 2.5 of [2]) we see that (3.1) holds for $r = 2$.

We can now apply Theorem 3.2 to the matrices E_m, using the function $f = \mathbb{1}_{(-\infty,-\omega)}$. Note that f is of bounded variation and $V_f(\mathbb{R})$ is finite and independent of n. Applying Theorem 3.2, we get

$$\mathbb{P}\big(\big|F_{E_m}(f) - \mathbb{E}\big[F_{E_m}(f)\big]\big| \geq \eta/4\big) \leq 2\exp\big(-c\lfloor n^s \rfloor \eta^2\big) \tag{3.2}$$

for some $c > 0$. As $\bar{\mu}_{E_m}(-\infty, -\omega) > \frac{\eta}{2}$ and using (3.2), we get that, for large enough n

$$\mathbb{P}\big(\mu_{E_m}(-\infty, -\omega) \geq \eta/4\big) \geq 1 - \frac{\varepsilon}{2}.$$

E_m is almost C_m, with diagonals made 0 and then off-diagonals are subtracted by $\ell_\alpha / \lfloor n^s \rfloor$.

Using (2.3), it can be seen that

$$\mathbb{P}\left(\bigcup_{i=1}^{m} \big((C_m)_{ii} \geq n^{\frac{\alpha-s}{2}}(1+\varepsilon)\big)\right) \leq 2m \exp(-c_2 n) \tag{3.3}$$

$$\mathbb{P}\left(\bigcup_{i=1}^{m} \left((D_m)_{ii} \geq n^{\frac{\alpha-s}{2}}\left(1+\varepsilon - \frac{\ell_\alpha}{n^{\alpha/2}}\right)\right)\right) \leq 2m \exp(-c_2 n). \tag{3.4}$$

Weyl's inequality (Theorem $4.3.1$ of [6]) bounds the amount of perturbation of eigenvalues due to perturbation of a matrix. Using Weyl's inequality, along with (3.4) gives that,

$$\mathbb{P}(\mu_{E_m + D_m}\left(-\infty, -\omega + n^{\frac{\alpha-s}{2}}\left(1+\varepsilon - \frac{\ell_\alpha}{n^{\alpha/2}}\right) \geq \eta/4\right)$$
$$\geq 1 - \frac{\varepsilon}{2} - 2m\exp(-c_2 n).$$

As rank$(E_m + D_m - C_m) = 1$ and $\alpha < s$, using rank inequality (Lemma 2.5 of [2]) again, we get that

$$\mathbb{P}(\text{all the eigenvalues of } C_m \text{ are non-negative}) < \frac{\varepsilon}{2} + \frac{1}{n} + 2m\exp(-c_2 n),$$

which contradicts the earlier assumption. This completes the proof of Lemma 3.1. □

Lemma 3.3. *Let $\bar{\mu}_{E_m}$ be the EESD of E_m. Then*

i) *Limit of first moment of $\bar{\mu}_{E_m}$ is 0*
ii) *Limit of second moment of $\bar{\mu}_{E_m}$ is a positive constant*
iii) *The fourth moments of $\bar{\mu}_{E_m}$ are uniformly bounded.*

Remark 3.4. As $\bar{\mu}_{E_m}$ is a tight sequence of measures, any subsequential limit must have mean zero and finite variance.

Proof of Lemma 3.3. **Computation of moments of $\bar{\mu}_{E_m}$:** Before we start the computations, we make a note of the form of entries of E_m.

Diagonal entries: $(E_m)_{ii} = 0$

Off diagonal entries: $(E_m)_{ij} = \frac{1}{n^{s/2}}(|\frac{\langle R_i, R_j\rangle}{\sqrt{n}}|^\alpha - \ell_\alpha)$

We prove limits of first and second moments of $\bar{\mu}_{E_m}$ are 0 and a positive value.

Limit of first moments: One can see that the limit is 0 as

$$\int_{\mathbb{R}} x \, d\bar{\mu}_{E_m}(x) = \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}\big[(E_m)_{ii}\big] = 0.$$

Limit of second moments: $\int_{\mathbb{R}} x^2 \, d\bar{\mu}_{E_m}(x) = \mathbb{E}[\int_{\mathbb{R}} x^2 \, d\mu_{E_m}(x)] = \frac{1}{m}\sum_{i,j} \mathbb{E}[((E_m)_{ij})^2]$. As the off-diagonal entries are identically distributed, it is enough to look at the limit of $\sum_{i=1}^{m} \mathbb{E}[((E_m)_{1i})^2]$.

$$\lim_{n\to\infty} \sum_{i=1}^{m} \mathbb{E}\big[\big((E_m)_{1i}\big)^2\big] = \lim_{n\to\infty}(m-1)\mathbb{E}\big[\big((E_m)_{12}\big)^2\big].$$

Using central limit theorem, uniform bound on $\mathbb{E}[(\frac{\langle R_1,R_2\rangle}{\sqrt{n}})^4]$ and $m=\lfloor n^s\rfloor$, one can see that the limit is $\mathbb{E}[(|Z|^\alpha-\ell_\alpha)^2]$. We now prove that the fourth moments of $\bar{\mu}_{E_m}$ are uniformly bounded.

Uniform bound of fourth moments:

$$\int_{\mathbb{R}} x^4\, d\bar{\mu}_{E_m}(x)=\frac{1}{m}\sum_{i_1i_2i_3i_4}\mathbb{E}\big[(E_m)_{i_1i_2}(E_m)_{i_2i_3}(E_m)_{i_3i_4}(E_m)_{i_4i_1}\big].$$

This is a sum of expectations with each term corresponding to a closed walk of length 4 on the complete graph K_m. It is enough to look at closed walks starting and ending at vertex 1. Such walks can visit $2,3$ or 4 different vertices, including the vertex 1.

$$\int_{\mathbb{R}} x^4\, d\bar{\mu}_{E_m}(x)=\sum_{i\neq 1}\mathbb{E}\big[(E_m)_{1i}^4\big]+\sum_{j,k\neq 1}\mathbb{E}\big[(E_m)_{1j}^2(E_m)_{1k}^2\big]$$
$$+\sum_{i,j\neq 1}\mathbb{E}\big[(E_m)_{1i}^2(E_m)_{ij}^2\big]+\sum_{i,j,k\neq 1}\mathbb{E}\big[(E_m)_{1i}(E_m)_{ij}(E_m)_{jk}(E_m)_{k1}\big]$$

The four terms in the above equation correspond to four different types of walks as shown in above figures.

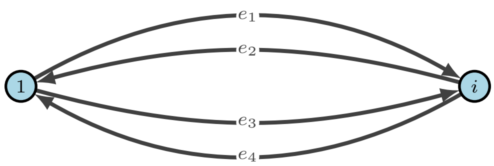

Figure 1: The walk corresponding to $\mathbb{E}[(E_m)_{1i}^4]$.

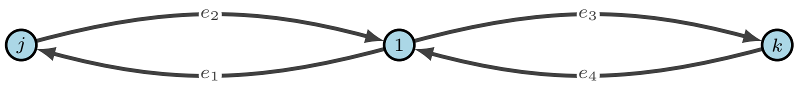

Figure 2: The walk corresponding to $\mathbb{E}[(E_m)_{1j}^2(E_m)_{1k}^2]$.

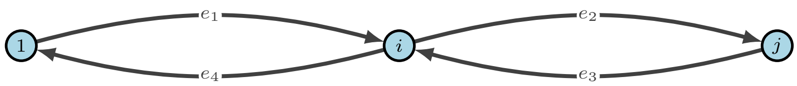

Figure 3: The walk corresponding to $\mathbb{E}[(E_m)_{1i}^2(E_m)_{ij}^2]$.

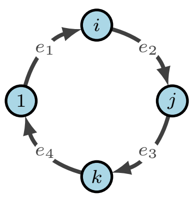

Figure 4: The walk corresponding to $\mathbb{E}[(E_m)_{1i}(E_m)_{ij}(E_m)_{jk}(E_m)_{k1}]$.

Using the fact that off-diagonal entries of E_m are identically distributed, uniform bound on $\mathbb{E}[(\frac{\langle R_1,R_2\rangle}{\sqrt{n}})^4]$, one can see that $\lim_{n\to\infty}\sum_{i\neq 1}\mathbb{E}[(E_m)_{1i}^4]=0$. Using a similar argument as above it can be seen that

$$\lim_{n\to\infty}\sum_{i,j\neq 1}\mathbb{E}\big[(E_m)_{1i}^2(E_m)_{1j}^2\big]=\lim_{n\to\infty}\sum_{i,j\neq 1}\mathbb{E}\big[(E_m)_{1i}^2(E_m)_{ij}^2\big]=\mathbb{E}\big[\big(|Z_1|^\alpha-\ell_\alpha\big)^2\big(|Z_2|^\alpha-\ell_\alpha\big)^2\big], \tag{3.5}$$

where Z_1,Z_2 are i.i.d. standard Gaussians. If we prove that

$$\lim_{n\to\infty}\sum_{i,j,k\neq 1}\mathbb{E}\big[(E_m)_{1i}(E_m)_{ij}(E_m)_{jk}(E_m)_{k1}\big]=0, \tag{3.6}$$

then using (3.5), (3.6), we would have proved that fourth moments of $\bar{\mu}_{E_m}$ are uniformly bounded and we would be done with the proof of Lemma 3.1. Note that

$$\lim_{n\to\infty}\sum_{i,j,k\neq 1}\mathbb{E}\big[(E_m)_{1i}(E_m)_{ij}(E_m)_{jk}(E_m)_{k1}\big]=$$
$$\lim_{n\to\infty}m\mathbb{E}\Bigg[\Bigg(\bigg|\frac{\langle R_1,R_2\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\Bigg(\bigg|\frac{\langle R_2,R_3\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\Bigg(\bigg|\frac{\langle R_3,R_4\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\Bigg(\bigg|\frac{\langle R_4,R_1\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\Bigg] \tag{3.7}$$

Let $\mathcal{F}_{1,3}$ denote the sigma algebra generated from the 1st row and 3rd row of $X_{m,n}$ and

$$Y_{1,3}:=\mathbb{E}\Bigg[\Bigg(\bigg|\frac{\langle R_1,R_2\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\Bigg(\bigg|\frac{\langle R_2,R_3\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\;\Bigg|\;\mathcal{F}_{1,3}\Bigg].$$

Note that using independence of 2nd row and 4th row of $X_{m,n}$, RHS of (3.7) can be written as, $\lim_{n\to\infty}m\mathbb{E}[Y_{1,3}^2]$.

Using Lemma 3.5 it follows that $\lim_{n\to\infty}m\mathbb{E}[Y_{1,3}^2]=0$ and hence the fourth moments of $\bar{\mu}_{E_m}$ are uniformly bounded.

This proves that the fourth moments are uniformly bounded. This completes the proof of Lemma 3.3. ☐

Lemma 3.5. $\mathbb{E}[(nY_{1,3})^k]$ *is uniformly bounded by* $M_k,\forall n,k\in\mathbb{N}$*, where* $M_k>0$ *are some constants dependent on* k*.*

Proof of Lemma 3.5.

$$Y_{1,3}=\mathbb{E}\Bigg[\Bigg(\bigg|\frac{\langle R_1,R_2\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\Bigg(\bigg|\frac{\langle R_2,R_3\rangle}{\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)\;\Bigg|\;\mathcal{F}_{1,3}\Bigg]=$$
$$\mathbb{E}\Bigg[\sigma_1^\alpha\Bigg(\Bigg(\bigg|\frac{\langle R_1,R_2\rangle}{\sigma_1\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)+\Bigg(\ell_\alpha-\frac{\ell_\alpha}{\sigma_1^\alpha}\Bigg)\Bigg)\sigma_3^\alpha\Bigg(\Bigg(\bigg|\frac{\langle R_2,R_3\rangle}{\sigma_3\sqrt{n}}\bigg|^\alpha-\ell_\alpha\Bigg)+\Bigg(\ell_\alpha-\frac{\ell_\alpha}{\sigma_3^\alpha}\Bigg)\Bigg)\;\Bigg|\;\mathcal{F}_{1,3}\Bigg]$$
$$=\sigma_1^\alpha\sigma_3^\alpha\mathbb{E}\big[\big(|Z_1|^\alpha-\ell_\alpha\big)\big(|Z_3|^\alpha-\ell_\alpha\big)\big]+\ell_\alpha^2\big(\sigma_1^\alpha-1\big)\big(\sigma_3^\alpha-1\big).$$

Here Z_1,Z_3 are standard normal random variables (after conditioning on $\mathcal{F}_{1,3}$) with correlation coefficient ρ_{13}. Note that almost surely $0<|\rho_{13}|<1$ and hence (Z_1,Z_3) have joint density.

Define a function of correlation coefficient as below,

$$I(\rho):=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\big(|x|^\alpha-\ell_\alpha\big)\big(|y|^\alpha-\ell_\alpha\big)\exp\Bigg(-\frac{1}{2(1-\rho^2)}\big(x^2+y^2-2xy\rho\big)\Bigg)dxdy.$$

Note that $I(0)=0,\ I(\rho)=I(-\rho)$ and $I(\rho)$ is a smooth function. Above given expansion of $Y_{1,3}$ can be written as

$$Y_{1,3}=\sigma_1^\alpha\sigma_3^\alpha\rho_{13}^2\frac{I(\rho_{13})}{\rho_{13}^2}+\ell_\alpha^2\big(\sigma_1^\alpha-1\big)\big(\sigma_3^\alpha-1\big).$$

We now show $I(\rho)/\rho^2$ is a bounded function. Fix $t > 0$. For $|\rho| > t$, note that $I(\rho)$ is Gaussian expectation and therefore $I(\rho)/\rho^2$ is bounded. We use L'Hospital's rule to get a bound on $\frac{I(\rho)}{\rho^2}$ when $|\rho| < t$. Using differentiation under integral sign, and using L'Hospital's rule twice, it can be seen that $I(\rho)/\rho^2$ is a bounded function. Hence we can write, $|Y| \le M\sigma_1^\alpha\sigma_3^\alpha|\rho_{13}^2| + m_\alpha^2|\sigma_1^\alpha - 1||\sigma_3^\alpha - 1|$. As $\forall\alpha < 2$,

$$|\sigma_1^\alpha - 1| \le \left|\frac{\langle R_1, R_1\rangle}{n} - 1\right| \le \frac{1}{\sqrt{n}}\left|\frac{\langle R_1, R_1\rangle - n}{\sqrt{n}}\right|. \tag{3.8}$$

As a result we can write,

$$|nY| \le M\sigma_1^\alpha\sigma_3^\alpha n\rho_{13}^2 + \left|\frac{\langle R_1, R_1\rangle - n}{\sqrt{n}}\right|\left|\frac{\langle R_3, R_3\rangle - n}{\sqrt{n}}\right|.$$

One can see that, the k-th moments of $\sigma_1^\alpha, \sigma_3^\alpha, n\rho_{13}^2, |\frac{\langle R_1,R_1\rangle - n}{\sqrt{n}}|$ are uniformly bounded by some constant, $\forall n \in \mathbb{N}$ and hence k-th moments of nY are also uniformly bounded. This completes the proof of Lemma 3.5. □

3.2 Proof of Theorem 1.1 for the range $\alpha > s$

In this subsection we consider the range $\alpha > s$. We prove Lemma 3.6 which immediately implies Theorem 1.1 for the range $\alpha > s$. For this we define the following matrices. For ease of notation, we write $B_{n,\alpha,s}$ as B_m. Define a diagonal matrix D_m such that $D_m(i,i) = B_m(i,i) - \frac{\ell_\alpha}{n^{\alpha/2}}$. Let $C_m := B_m - (\frac{\ell_\alpha}{n^{\alpha/2}})J_m - D_m$. Note that $C_m(i,j) = \frac{1}{n^{\alpha/2}}(|\frac{\langle R_i, R_j\rangle}{\sqrt{n}}|^\alpha - \ell_\alpha)$ and the diagonal entries of C_m are zero. We show that $\mathbb{E}[\mathrm{Tr}(C_m^{2k})] \to 0$. By Markov inequality and Weyl's inequality this implies Lemma 3.6.

Lemma 3.6. *Fix $s < \alpha$ and $0 < \varepsilon < 1/2$. Then $\mathbb{P}(\lambda_{\min}(B_{n,\alpha,s}) > \varepsilon) \to 1$, as $n \to \infty$.*

Proof of Lemma 3.6. We first show that $\mathbb{P}(\lambda_{\min}(C_m) \le -1 + 2\varepsilon) \to 0$ implies Lemma 3.6. Note that, using Lemma 2.1, we have

$$\mathbb{P}\left(\bigcup_{i=1}^m \big((D_m)_{ii} \le 1 - \varepsilon\big)\right) \le 2m\exp(-c_3 n), \tag{3.9}$$

for some constant $c_3 > 0$ depending on α. To get the matrix B_m, we add C_m with $D_m + (\ell_\alpha/n^{\alpha/2})J_m$. Using Weyl's inequality (Theorem 4.3.1 of [6]), we get

$$\mathbb{P}\big(\lambda_{\min}(B_m) - \lambda_{\min}(C_m) < 1 - \varepsilon\big) \le 2m\exp(-c_3 n). \tag{3.10}$$

The above inequality shows that the eigenvalues of B_m are at least $1 - \varepsilon$ more than that of C_m, with high probability. Hence $\mathbb{P}(\lambda_{\min}(C_m) \le -1 + 2\varepsilon) \to 0$ implies Lemma 3.6. This completes the proof if we prove $\mathbb{P}(\lambda_{\min}(C_m) \le -1 + 2\varepsilon) \to 0$. Choose k such that $\alpha > (\frac{k+1}{k})s$.

$$\mathbb{P}\big(\lambda_{\min}(C_m) \le -1 + 2\varepsilon\big) \le \mathbb{P}\Big(\big(\lambda_{\min}(C_m)\big)^{2k} \ge (1 - 2\varepsilon)^{2k}\Big) \le \frac{\mathbb{E}[\mathrm{Tr}(C_m^{2k})]}{(1 - 2\varepsilon)^{2k}}.$$

We prove that $\mathbb{E}[\mathrm{Tr}(C_m^{2k})] \to 0$, where $\alpha > (\frac{k+1}{k})s$. This completes the proof of the lemma.

Computation of $\mathbb{E}[\mathrm{Tr}(C_m^{2k})]$ Consider a closed walk of length $2k$ on complete graph K_m. Let $i_1 i_2 \ldots i_{2k-1} i_1$ be the closed walk. This corresponds to the term $\mathbb{E}[C_{i_1 i_2} \ldots C_{i_{2k-1} i_1}]$ in expansion of $\mathbb{E}[\mathrm{Tr}(C_m^{2k})]$. Thus terms in expansion of $\mathbb{E}[\mathrm{Tr}(C_m^{2k})]$ correspond to closed walks of length $2k$ (starting point can be any of the m vertices). As the diagonal entries are zero, the paths cannot have loops at any vertices. We first look at walks without "leaf vertices". By "leaf vertices" we mean the vertices, like "3" and "1", which are of degree 2 and have only one neighbour, as shown in Figure 5 (In the graph generated due to closed walk, such vertices are leafs).

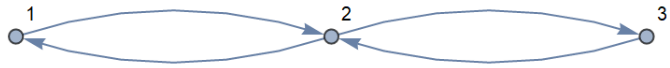

Figure 5: The vertices $1, 3$ are leaf vertices.

So we look at closed walks of length $2k$ without loops and leaf vertices. As the off-diagonal entries of C_m are of the order $1/n^{\alpha/2}$ and $\alpha > (\frac{k+1}{k})s$, the sums of expectations corresponding to paths visiting $k+l$ vertices with $l \leq 1$ (each vertex can be chosen in at most $\lfloor n^s \rfloor$ ways), goes to 0. So it is enough to look at paths visiting at least $k+2$ vertices.

Closed walks of length $2k$, visiting $k+l,\ l \geq 2$ vertices, must have at least $2l$ vertices of degree 2 (none of which are leaf vertices) as shown below. This is due to the fact that since it is a closed walk, degree of every vertex is even and sum of degrees of vertices must equal twice the total number of edges.

There would be $C_{i,j}C_{j,k}$ term when expanding $\mathrm{Tr}(C_m^{2k})$ as sum of product of entries of C_m. This factor shows up due to the vertex j having degree 2. We would like to condition on the rows i, k of $X_{m,n}$ and use Lemma 3.5. It could happen that more than 1, say t, degree-2 vertices come together in series as shown in Figure 6. In such a case we condition as shown below.

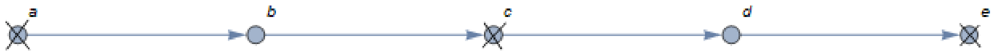

Figure 6: The rows corresponding to a, c, e are conditioned on.

Suppose there is a path traversing vertices a through e, as shown above, where degrees of both a, e are at least 4 and b, c, d are all degree-2 vertices. Here degrees are calculated in the graph generated by the closed walk of length $2k$. In such a case we will have the factor $C_{a,b}C_{b,c}C_{c,d}C_{d,e}$ in the expansion of $\mathrm{Tr}(C_m^{2k})$ corresponding to that path. In the expectation term corresponding to such a path, we condition on rows a, c, e and use independence to get 2 conditional expectations $Y_{a,c}, Y_{c,e}$ mentioned in Section 3.1. The "x" mark denotes the rows which we are going to condition on. If there are even number of degree-2 vertices coming together, we condition as in Figure 7.

Figure 7: The rows corresponding to a, c, d are conditioned on.

In the case shown above, vertices a, d have degree at least 4 and b, c are degree-2 vertices. We condition of rows a, c, d. All other rows corresponding to vertices with degrees greater than 2 will also be conditioned.

Now we look at $\mathbb{E}[\mathrm{Tr}(C_m^{2k})]$ and the walks of length $2k$, without loops and leaf vertices, visiting $k+l$ vertices. The $k+l$ vertices can be chosen in $\lfloor n^s \rfloor^{k+l}$ ways and taking the order of $C_{i,j}$ into account we can write,

$$\frac{\lfloor n^s \rfloor^{k+l}}{n^{k\alpha}} \mathbb{E}\left[\left(\left|\frac{\langle R_1, R_2\rangle}{\sqrt{n}}\right|^{\alpha} - \ell_\alpha\right)\dots\right]$$

corresponding to the walks we are interested in. Using Independence and conditioning on the rows corresponding to the vertices with degree at least 4 and those appropriate vertices when more than 1 degree-2 vertices come together, we get product of at least l number of conditional expectations like $Y_{i,j}$. Using Lemma 3.5, $n^l\mathbb{E}[(|\frac{\langle R_1,R_2\rangle}{\sqrt{n}}|^{\alpha} - \ell_\alpha)\dots]$ is uniformly bounded. As l was arbitrary and as $\alpha > (\frac{k+1}{k})s$, we can see that the expectation corresponding to the walks without loops and leaf vertices goes to 0 with n.

Now we look at paths without loops but have leaf vertices. If initially we had a closed walk of length $2g$ without leaf vertices and visited l different vertices. Note that each leaf vertex attached increases length of walk by 2 and number of vertices visited by 1. Adding t leaf vertices such that $g + t = k$ gives corresponding expectation terms like

$$\frac{\lfloor n^s \rfloor^{g+l+t}}{n^{(g+t)\alpha}} \mathbb{E}\left[\left(\left|\frac{\langle R_1, R_2\rangle}{\sqrt{n}}\right|^{\alpha} - \ell_\alpha\right)\dots\right]. \tag{3.11}$$

If such a leaf vertex or multiple leaf vertices can be attached to a vertex which is degree-2 originally, then we condition on the rows corresponding to all the leaf vertices and the vertices whose rows we were conditioning on originally, as shown in Figure 8.

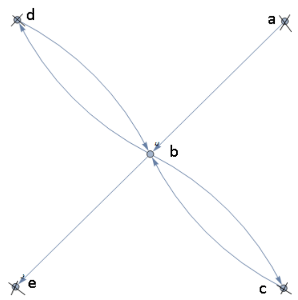

Figure 8: The rows corresponding to a, c, d, e are conditioned on.

The vertices d, c are leaf vertices attached to vertex b. Without the vertices d, c and edges between them and b, the vertex b would be of degree-2. After addition of vertices d, c and the edges, the conditioning will be done on rows corresponding to a, c, d, e. This is where Lemma 3.7 is used. Such conditioning gives conditional expectation factor like G in Lemma 3.7 for every vertex which get attached at least one leaf vertex to it.

If leaf vertices are attached to a vertex which is of degree 4 or more originally, then again we condition on rows corresponding to all leaf vertices along with the previous vertices we were conditioning on (Lemma 3.7 is not needed here). As G is of the order of $1/n$ and $\alpha > (\frac{k+1}{k})s$, limit of (3.11) is 0. This shows that $\mathbb{E}[\mathrm{Tr}(C_n^{2k})] \to 0$, as $n \to \infty$. Taking k arbitrarily large completes the proof of Lemma 3.6. □

Lemma 3.7. *Let $p \in \mathbb{N}_{\geq 3}$ and $\mathcal{F}_{2,3,\dots,p}$ denote the sigma algebra generated from the $2, 3, \dots, p$-th rows of $X_{m,n}$. Define*

$$G := n^{(2(p-3)+2)\alpha/2}\mathbb{E}\left[C_{12}C_{13}C_{14}^2C_{15}^2\dots C_{1p}^2|\mathcal{F}_{2,3,\dots,p}\right]$$

$\mathbb{E}[(nG)^k]$ is uniformly bounded by constant M_k for all $k \in \mathbb{N}$.

Proof. Let $W_{12} := (|\frac{\langle R_1,R_2\rangle}{\sigma_2\sqrt{n}}|^\alpha - \ell_\alpha) + \frac{\ell_\alpha}{\sigma_2^\alpha}(\sigma_2^\alpha - 1)$. Then

$$G = \sigma_2^\alpha\sigma_3^\alpha\sigma_4^{2\alpha}\sigma_5^{2\alpha}\dots\sigma_p^{2\alpha}\mathbb{E}\big[W_{12}W_{13}W_{14}^2W_{15}^2\dots W_{1p}^2|\mathcal{F}_{2,3,\dots,p}\big].$$

Due to (3.8), the term $(\sigma_2^\alpha - 1)$ is of the order of $1/\sqrt{n}$. All moments of σ_2^α are uniformly bounded. So for $\mathbb{E}[(nG)^k]$ to be uniformly bounded, it is enough to prove that k-th moments of

$$n\mathbb{E}\left[\left(\left|\frac{\langle R_1,R_2\rangle}{\sigma_2\sqrt{n}}\right|^\alpha - \ell_\alpha\right)\left(\left|\frac{\langle R_1,R_3\rangle}{\sigma_3\sqrt{n}}\right|^\alpha - \ell_\alpha\right)W_{14}^2\dots W_{1p}^2\;\middle|\;\mathcal{F}_{2,3,\dots,p}\right]$$

and

$$\sqrt{n}\mathbb{E}\left[\left(\left|\frac{\langle R_1,R_2\rangle}{\sigma_2\sqrt{n}}\right|^\alpha - \ell_\alpha\right)W_{14}^2\dots W_{1p}^2\;\middle|\;\mathcal{F}_{2,3,\dots,p}\right]$$

are uniformly bounded, $\forall n \in \mathbb{N}$. We will prove that k-th moment of first quantity is uniformly bounded. For the second quantity, similar argument works.

Note that conditional on $\mathcal{F}_{2,3,\dots,p}$, the conditional expectation G is a function of standard Gaussian random variables, say, $Z_2, Z_3, \dots, Z_p$, with the correlation matrix being $\tilde{\Sigma} = A_{p-1}A_{p-1}^T$, where A_{p-1} is $(p-1)\times n$ matrix with $A_{p-1}(i,j) = \frac{X_{m,n}(i+1,j)}{\sqrt{n}\sigma_{i+1}}$. It can be seen easily that almost surely rank$(A_{p-1}) = p-1$ and hence $\tilde{\Sigma}$ is invertible. For any symmetric invertible matrix Σ with 1s on diagonal, define

$$h(\Sigma) := \frac{1}{\sqrt{(2\pi)^{p-1}|\Sigma|}}\int\big(|x_1|^\alpha - \ell_\alpha\big)\big(|x_2|^\alpha - \ell_\alpha\big)\dots\big(|x_{p-1}|^\alpha - \ell_\alpha\big)^2\exp\left(\frac{-x^T\Sigma^{-1}x}{2}\right)dx_1\dots$$

Here h is a function of the entries above the diagonal of Σ. Using symmetry and independence $h(I_{p-1}) = 0$. Expanding $W_{12}W_{13}W_{14}^2W_{15}^2\dots W_{1p}^2$ and using the fact that $(\sigma_2^\alpha - 1)$ is of order $1/\sqrt{n}$, to prove that k-th moments of

$$n\mathbb{E}\left[\left(\left|\frac{\langle R_1,R_2\rangle}{\sigma_2\sqrt{n}}\right|^\alpha - \ell_\alpha\right)\left(\left|\frac{\langle R_1,R_3\rangle}{\sigma_3\sqrt{n}}\right|^\alpha - \ell_\alpha\right)W_{14}^2\dots W_{1p}^2\;\middle|\;\mathcal{F}_{2,3,\dots,p}\right]$$

are uniformly bounded, it is enough to prove that k-th moments of $nh(\tilde{\Sigma})$ are uniformly bounded.

It is easy to see that h is a differentiable function. We make use of the multi-variable mean value theorem $|f(y) - f(x)| \le |\nabla f(cx + (1-c)y)||y - x|$, for some $0 \le c \le 1$. Using the fact that $\sum_{i<j}\tilde{\Sigma}_{i,j}^2$ is of order of $1/n$, it is enough to show $h(\Sigma)/\sum_{i<j}\Sigma_{i,j}^2$ is bounded.

For Σ bounded away from the origin, using Gaussian integrals, it can be seen that $\frac{h(\Sigma)}{\sum_{i<j}\Sigma_{i,j}^2}$ is bounded. As $h(I_{p-1}) = 0$ at the origin, mean value theorem and basic computations gives boundedness of $h(\Sigma)/\sum_{i<j}\Sigma_{i,j}^2$ in a neighbourhood of the origin. This completes the proof of Lemma 3.7. □

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