Electron. Commun. Probab. **28** (2023), article no. 41, 1–9.
https://doi.org/10.1214/23-ECP545
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Moment characterization of the weak disorder phase for directed polymers in a class of unbounded environments

Ryoki Fukushima* Stefan Junk †‡

Abstract

For a directed polymer model in random environment, a characterization of the weak disorder phase in terms of the moment of the renormalized partition function has been proved in [S. Junk: Communications in Mathematical Physics 389, 1087–1097 (2022)]. We extend this characterization to a large class of unbounded environments which includes many commonly used distributions.

Keywords: directed polymers; random environment.
MSC2020 subject classifications: 60K37.
Submitted to ECP on March 9, 2023, final version accepted on August 27, 2023.
Supersedes arXiv:2303.01918.

1 Introduction

We consider a model of a directed polymer in random environment. Let $(X=(X_j)_{j\geq 0}, P^{\mathrm{SRW}})$ be the simple random walk on $\mathbb{Z}^d$ starting at the origin and $((\omega_{j,x})_{(j,x)\in\mathbb{N}\times\mathbb{Z}^d},\mathbb{P})$ be a sequence of independent and identically distributed random variables satisfying

$$e^{\lambda(\beta)} := \mathbb{E}\big[e^{\beta\omega_{0,0}}\big] < \infty \text{ for all } \beta \geq 0. \tag{1.1}$$

Then we define the law of the polymer of length n at inverse temperature $\beta\geq 0$ by

$$\mathrm{d}\mu^{\beta}_{\omega,n}(\mathrm{d}X) = \frac{1}{Z^{\beta}_n(\omega)}\exp\Big(\beta\sum_{j=1}^{n}\omega_{j,X_j}\Big)P^{\mathrm{SRW}}(\mathrm{d}X), \tag{1.2}$$

where $Z^{\beta}_n(\omega) = E^{\mathrm{SRW}}[\exp(\beta\sum_{j=1}^{n}\omega_{j,X_j})]$ is the normalizing constant, called the *partition function* of the model. Under this measure, the random walk is attracted by the sites where ω is positive, and repelled by the sites where it is negative. Thus we expect that the behavior of the polymer is strongly affected by the environment when β is large.

This intuition is made precise in [2, 3] under the assumption $\mathbb{E}\big[e^{\beta\omega_{0,0}}\big]<\infty$ for all $\beta\in\mathbb{R}$. In spatial dimension $d\geq 3$, there exists $\beta_{cr}\in(0,\infty)$ such that for $0<\beta<\beta_{cr}$,

$$e^{-n\lambda(\beta)}Z^{\beta}_n(\omega) \xrightarrow{n\to\infty} W^{\beta}_{\infty}(\omega) > 0, \quad \mathbb{P}\text{-a.s.}, \tag{1.3}$$

*Department of Mathematics, University of Tsukuba, Japan. E-mail: ryoki@math.tsukuba.ac.jp
†Advanced Institute for Materials Research, Tohoku University, Japan. E-mail: sjunk@tohoku.ac.jp
‡*Current address*: Department of Mathematics, Gakushuin University, Japan. E-mail: sjunk@math.gakushuin.ac.jp

whereas for $\beta > \beta_{cr}$,

$$e^{-n\lambda(\beta)} Z_n^\beta(\omega) \xrightarrow{n\to\infty} 0, \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

As one can readily verify that the annealed partition function satisfies $\mathbb{E}[Z_n^\beta] = e^{n\lambda(\beta)}$, the above shows that the quenched and annealed partition functions are comparable for $\beta < \beta_{cr}$ and contrary for $\beta > \beta_{cr}$. This indicates that the effect of disorder is weak in the former phase and strong in the latter phase with a drastic change in behavior across β_{cr}. We refer the interested reader to [2, 3].

The proof of the aforementioned results relies on the fact that $W_n^\beta(\omega) := e^{-n\lambda(\beta)} Z_n^\beta(\omega)$ is a non-negative martingale under $\mathbb{P}$ with the filtration $\mathcal{F}_n := \sigma(\omega_{j,x} \colon j \le n, x \in \mathbb{Z}^d)$, and one can further show that the phase (1.3) is characterized by the uniform integrability of $W_n^\beta(\omega)$. But in order to further analyze the weak disorder phase, it is desirable to have a stronger property for $(W_n^\beta(\omega))_{n\ge 0}$. The second author has recently proved in [6] that for $\beta < \beta_{cr}$, the martingale $(W_n^\beta(\omega))_{n\ge 0}$ is L^p-bounded for some $p > 1$, under the assumption that the random potential ω is bounded from above. The main result of this paper extends this characterization to a large class of unbounded environments.

2 Main result

We introduce the following condition for the environment ω.

Condition 1. For $\beta > 0$, there exist $A_1 = A_1(\beta) > 1$ and $c_1 = c_1(\beta) > 0$ such that, for all $A > A_1$,

$$\mathbb{E}\left[e^{2\beta\omega} \;\middle|\; \omega > A\right] \le c_1 e^{2\beta A}. \tag{2.1}$$

This condition strengthens the assumption (1.1) of finite exponential moments by requiring a control on the overshoot when ω is conditioned to be large. It does not seem to be very restrictive and holds for many commonly used distributions, although we stress that there are distributions that satisfy (1.1) but not Condition 1. We elaborate on these matters in Section 5.

The following is the main result of this paper.

Theorem 2.1. *Let β be such that $\mathbb{P}(W_\infty^\beta > 0) > 0$ and assume that ω satisfies (1.1) and Condition 1. Then there exists $p = p(\beta) > 1$ such that*

$$\sup_{n\in\mathbb{N}} \|W_n^\beta\|_p < \infty. \tag{2.2}$$

Moreover, the set of $p > 1$ such that (2.2) holds is open.

Remark 2.2. If $\lim_{n\to\infty} W_n^\beta = 0$, then $(W_n^\beta)_{n\in\mathbb{N}}$ is not uniformly integrable and hence (2.2) necessarily fails. Thus the weak disorder is characterized by the finiteness of a p-th moment.

Remark 2.3. In [6], it was further shown that if ω is bounded from below, then $\sup_n \mathbb{E}[W_n^{-\varepsilon}] < \infty$ for some $\varepsilon > 0$. The argument in this paper can easily be generalized to show that the same holds whenever ω satisfies the straightforward generalization of Condition 1 to the negative tail.

Remark 2.4. It is an interesting problem to describe the dependence of the optimal exponent $p^*(\beta) := \sup\{p \colon (W_n^\beta)_{n\in\mathbb{N}} \text{ is } L^p\text{-bounded}\}$ as a function of β. For bounded environments, it has been shown in [5] that $p^*(\beta) \ge 1 + 2/d$ whenever $W_\infty^\beta > 0$, so that $\beta \mapsto p^*(\beta)$ has a discontinuity at β_{cr}. It is natural to expect that the same holds in general.

3 Extension of Condition 1

As will be explained in detail below, the main step in proving Theorem 2.1 is to control the overshoot of W_τ at a stopping time τ, which takes the form

$$\frac{W_\tau^\beta}{W_{\tau-1}^\beta} = \sum_x \alpha_x e^{\beta\omega_{\tau,x}-\lambda(\beta)} \tag{3.1}$$

for a certain choice of probability weights $(\alpha_x)_{x\in\mathbb{Z}^d}$. The purpose of the current section is to translate Condition 1 on ω into a statement on such convex combinations.

First, we state a condition satisfied by $e^{\beta\omega-\lambda(\beta)}$ whenever ω satisfies Condition 1. In the following, the random variable Y plays the role of $e^{\beta\omega-\lambda(\beta)}$.

Condition 2. The random variable Y is non-negative with $\mathbb{E}[Y]=1$, $\mathbb{E}[Y^2]<\infty$ and there exist $A_2>1$ and $c_2>0$ such that, for all $p\in[1,2]$ and $A\geq A_2$,

$$\mathbb{E}\left[Y^p \mid Y>A\right] \leq c_2 A^p. \tag{3.2}$$

The next condition requires additionally that (3.2) extends to convex combinations.

Condition 3. The random variable Y is non-negative with $\mathbb{E}[Y]=1$, $\mathbb{E}[Y^2]<\infty$ and there exist $A_3>1$ and $c_3>0$ such that the following holds: If I is countable, $(Y_i)_{i\in I}$ are i.i.d. copies of Y and $(\alpha_i)_{i\in I}$ is a collection of non-negative numbers with $\sum_{i\in I}\alpha_i=1$, then for all $p\in[1,2]$ and $A\geq A_3$

$$\mathbb{E}\left[\Big(\textstyle\sum_{i\in I}\alpha_i Y_i\Big)^p \,\Big|\, \sum_{i\in I}\alpha_i Y_i > A\right] \leq c_3 A^p. \tag{3.3}$$

We now show that both conditions follow from Condition 1.

Lemma 3.1. *(i) If ω satisfies Condition 1, then $Y:=e^{\beta\omega-\lambda(\beta)}$ satisfies Condition 2.*

(ii) If a random variable Y satisfies Condition 2, then it also satisfies Condition 3.

Proof. The proof of **part (i)** is simple. For $A\geq A_2:=e^{\beta A_1(\beta)-\lambda(\beta)}$, we can use Condition 1 to get

$$\begin{aligned}\mathbb{E}[Y^2|Y>A] &= \mathbb{E}\Big[e^{2\beta\omega}\,\Big|\,\omega>\frac{1}{\beta}(\log A+\lambda(\beta))\Big]e^{-2\lambda(\beta)}\\ &\leq c_1(2\beta)e^{2\beta\frac{1}{\beta}(\log A+\lambda(\beta))}e^{-2\lambda(\beta)}\\ &=: c_2A^2.\end{aligned}$$

The extension to $p\in[1,2)$ follows from Jensen's inequality.

The proof of **part (ii)** is more involved. In the following, we use C for positive constants depending only on $\mathbb{E}[Y_i^2]$, A_2 and c_2, whose values may change from line to line. Let $A\geq A_3:=A_2$ and $N:=\sum_i \mathbb{1}_{\{\alpha_iY_i>A\}}$. We separately consider the case where all the summands are small ($N=0$) and the cases where the event $\sum_i\alpha_iY_i>A$ is realized due to a single large summand ($N\geq 1$). In the first case, we have

$$\mathbb{E}\Big[\Big(\textstyle\sum_i\alpha_iY_i\Big)^2\mathbb{1}_{\{N=0\}}\,\Big|\,\sum_i\alpha_iY_i>A\Big] \leq \mathbb{E}\Big[\Big(\textstyle\sum_i\alpha_iY_i\mathbb{1}_{\{\alpha_iY_i\leq A\}}\Big)^2\,\Big|\,\sum_i\alpha_iY_i>A\Big] \tag{3.4}$$

since $Y_i=Y_i\mathbb{1}_{\{\alpha_iY_i\leq A\}}$ for all i on $\{N=0\}$. Let $\tau:=\inf\{i\colon\sum_{j\leq i}\alpha_jY_j>A\}$ and observe that on $\{\sum_i\alpha_iY_i>A\}=\{\tau<\infty\}$,

$$\sum_{i\leq\tau}\alpha_iY_i\mathbb{1}_{\{\alpha_iY_i\leq A\}} \leq \sum_{i<\tau}\alpha_iY_i+\alpha_\tau Y_\tau\mathbb{1}_{\{\alpha_\tau Y_\tau\leq A\}} \leq 2A.$$

Note also that conditioned on $\tau = i$, the remaining variables $(Y_{j+i})_{j\geq 1}$ obey the unconditioned law $\mathbb{P}$. Therefore,

$$\begin{aligned}\mathbb{E}\left[\Big(\sum_i \alpha_i Y_i\Big)^2 \mathbb{1}_{\{N=0\}} \,\Big|\, \sum_i \alpha_i Y_i > A\right] &\leq \mathbb{E}\left[\Big(2A + \sum_{i>\tau} \alpha_i Y_i\Big)^2 \,\Big|\, \tau < \infty\right] \\ &\leq \mathbb{E}\left[\Big(2A + \sum_{i\in I} \alpha_i Y_i\Big)^2\right] \\ &\leq C(A^2+1),\end{aligned} \tag{3.5}$$

where in the last line, we have used $\sum_{i\in I} \alpha_i = 1$ and that Y_1 has a finite second moment. In the second case $N \geq 1$, we use $\{N \geq 1\} \subseteq \{\sum_i \alpha_i Y_i > A\}$ to obtain

$$\mathbb{E}\Big[\Big(\sum_i \alpha_i Y_i\Big)^2 \mathbb{1}_{\{N\geq 1\}} \,\Big|\, \sum_i \alpha_i Y_i > A\Big] \leq \mathbb{E}\Big[\Big(\sum_i \alpha_i Y_i\Big)^2 \,\Big|\, N \geq 1\Big]. \tag{3.6}$$

Let $Q_i := \{\alpha_i Y_i > A\}$ and $q_i := \mathbb{P}(Q_i \mid N \geq 1)$ and observe that

$$\begin{aligned}\alpha_i^2 \mathbb{E}[Y_i^2 \mid N \geq 1] &= \alpha_i^2 \mathbb{E}[Y_i^2 \mid Q_i] q_i + \alpha_i^2 \mathbb{E}[Y_i^2 \mid Q_i^c, \textstyle\bigcup_{j\neq i} Q_j] \mathbb{P}(Q_i^c \mid N \geq 1) \\ &= \alpha_i^2 \mathbb{E}[Y_i^2 \mid Q_i] q_i + \alpha_i^2 \mathbb{E}[Y_i^2 \mid Q_i^c](1-q_i) \\ &\leq c_2 A^2 q_i + \alpha_i^2 \mathbb{E}[Y_i^2].\end{aligned} \tag{3.7}$$

The first equality uses $\{Q_i^c, \bigcup_{j\neq i} Q_j\} = \{Q_i^c, N \geq 1\}$, and the second equality uses that Y_i and Q_i^c are independent of $\bigcup_{j\neq i} Q_j$. For the inequality, we have used Condition 2 for the first term (note that $A/\alpha_i \geq A_2$) and the negative correlation between Y_i^2 and $\mathbb{1}_{Q_i^c}$ for the second term.

Similarly, for $i \neq j$, let $q_{i,j} := \mathbb{P}(Q_i, Q_j \mid N \geq 1)$ and observe that

$$\begin{aligned}&\alpha_i \alpha_j \mathbb{E}[Y_i Y_j \mid N \geq 1] \\ &= \alpha_i \alpha_j \Big(q_{i,j} \mathbb{E}[Y_i Y_j \mid Q_i, Q_j] + \mathbb{P}(Q_i, Q_j^c \mid N \geq 1) \mathbb{E}[Y_i Y_j \mid Q_i, Q_j^c] \\ &\quad + \mathbb{P}(Q_i^c, Q_j \mid N \geq 1) \mathbb{E}[Y_i Y_j \mid Q_i^c, Q_j] + \mathbb{P}(Q_i^c, Q_j^c \mid N \geq 1) \mathbb{E}[Y_i Y_j \mid Q_i^c, Q_j^c]\Big) \\ &\leq C\big(q_{i,j} A^2 + \alpha_j q_i A + \alpha_i q_j A + \alpha_i \alpha_j\big).\end{aligned} \tag{3.8}$$

In the equality, we have used $\{Q_i^c, Q_j^c, \bigcup_{k\neq i,j} Q_k\} = \{Q_i^c, Q_j^c, N \geq 1\}$ and that Y_i, Y_j and Q_i^c, Q_j^c are independent of $\bigcup_{k\neq i,j} Q_k$ for the last term. In the inequality, we have first dropped events Q_i^c and Q_j^c from the probability factors and factorized the conditional expectations by using independence. Then we have used Condition 2 for terms of the form $\mathbb{E}[Y_k|Q_k]$ and $\mathbb{E}[Y_k|Q_k^c] \leq \mathbb{E}[Y_k] = 1$, which is again due to the negative correlations.

Now, to bound the right-hand side of (3.6), we are going to sum (3.7) over i and (3.8) over $i \neq j$. Note that $\sum_i q_i = \mathbb{E}[N \mid N \geq 1]$ and $\sum_{i\neq j} q_{i,j} \leq \mathbb{E}[N^2 \mid N \geq 1]$. We will prove the following bounds on these quantities in Lemma 3.2 below:

$$\mathbb{E}[N \mid N \geq 1] \leq 2 \text{ and } \mathbb{E}[N^2 \mid N \geq 1] \leq 5. \tag{3.9}$$

Summing (3.7) over i and (3.8) over $i \neq j$ and then using (3.9), we find that the left-hand side of (3.6) is bounded by $C(A^2+1)$. Combining this with (3.5) and recalling $A \geq 1$, we get

$$\mathbb{E}\Big[\Big(\sum_i \alpha_i Y_i\Big)^2 \,\Big|\, \sum_i \alpha_i Y_i > A\Big] \leq C(A^2+1) \leq 2CA^2.$$

Finally, the claim for $p \in [1,2)$ follows as before by applying Jensen's inequality to the above. □

Lemma 3.2. *In the above setup, it holds that* $\mathbb{E}[N \mid N \geq 1] \leq 2$ *and* $\mathbb{E}[N^2 \mid N \geq 1] \leq 5$.

Proof. Let $\sigma = \inf\{i\colon \alpha_i Y_i > A\}$ and write $N = \mathbb{1}_{\{\sigma<\infty\}} + \sum_{i>\sigma} \mathbb{1}_{\{\alpha_i Y_i > A\}}$. Conditioned on $\sigma = i$, the random variables $(Y_{i+j})_{j\geq 1}$ obey the unconditioned law $\mathbb{P}$. Therefore,

$$\mathbb{E}[N \mid N \geq 1] = \mathbb{E}[N \mid \sigma < \infty] \leq 1 + \mathbb{E}[N],$$
$$\mathbb{E}[N^2 \mid N \geq 1] = \mathbb{E}[N^2 \mid \sigma < \infty] \leq \mathbb{E}[(1+N)^2],$$

and hence it suffices to prove that $\mathbb{E}[N] \leq 1$ and $\mathbb{E}[N^2] \leq 2$. Both follow from the Markov inequality:

$$\begin{aligned}
\mathbb{E}[N] &= \sum_i \mathbb{P}(Y_i > A/\alpha_i) \leq \mathbb{E}[Y_1] \sum_i \frac{\alpha_i}{A} = \frac{\mathbb{E}[Y_1]}{A}, \\
\mathbb{E}[N^2] &= \sum_i \mathbb{P}(Y_i > A/\alpha_k) + \sum_{i\neq j} \mathbb{P}(Y_i > A/\alpha_i)\mathbb{P}(Y_j > A/\alpha_j) \\
&\leq \frac{\mathbb{E}[Y_1]}{A} + \frac{\mathbb{E}[Y_1]^2}{A^2}.
\end{aligned}$$

Recalling $\mathbb{E}[Y_1] = 1$ and $A \geq 1$, these imply the desired bounds. □

4 Proof of the main result

4.1 Outline of the argument from [6]

Before proving Theorem 2.1, we think it is helpful to recall the argument from [6, Theorem 1.1(ii)] and explain the role of the boundedness of the environment.

The first step is to prove that $\mathbb{E}[\sup_n W_n^\beta] < \infty$, which is done in [6, Theorem 1.1(i)] and does not require any assumptions on the environment. Note that for the exceedance time defined by

$$\tau(t) := \inf\left\{n \in \mathbb{N}\colon W_n^\beta > t\right\}, \tag{4.1}$$

we have $\mathbb{P}(\sup_n W_n^\beta > t) = \mathbb{P}(\tau(t) < \infty)$. Thus from the integrability of $\sup_n W_n^\beta$, we conclude that $t \mapsto \mathbb{P}(\tau(t) < \infty)$ is integrable, and in particular

$$\text{for every } \varepsilon > 0 \text{ there exists } t = t(\varepsilon) > 1 \text{ such that } \mathbb{P}(\tau(t) < \infty) \leq \frac{\varepsilon}{t}. \tag{4.2}$$

Next, we define the pinned version of W_n^β as follows:

$$W_{n,x}^\beta := E^{\mathrm{SRW}}\left[\exp\left(\sum_{t=1}^n (\beta\omega_{t,X_t} - \lambda(\beta))\right); X_n = x\right]. \tag{4.3}$$

Then, by using the Markov property for the simple random walk, we write on $\{\tau(t) \leq n\}$

$$\begin{aligned}
W_n^\beta &= \sum_{x\in\mathbb{Z}^d} W_{\tau(t),x}^\beta \left(W_{n-\tau(t)}^\beta \circ \theta_{\tau(t),x}\right) \\
&= W_{\tau(t)}^\beta \sum_{x\in\mathbb{Z}^d} \mu_{\omega,\tau(t)}^\beta (X_{\tau(t)} = x) \left(W_{n-\tau(t)}^\beta \circ \theta_{\tau(t),x}\right),
\end{aligned} \tag{4.4}$$

where $\theta_{k,x}$ stands for the time-space shift of the environment, defined by $(\theta_{k,x}\omega)(l,y) = \omega(k+l, x+y)$. By (4.4) and Jensen's inequality, we have, for any $k \leq n$,

$$\begin{aligned}
\mathbb{E}\Big[\big(W_n^\beta\big)^p \mathbb{1}_{\{\tau(t)=k\}}\Big] &\leq \mathbb{E}\left[\big(W_k^\beta\big)^p \mathbb{1}_{\tau(t)=k} \mathbb{E}\Big[\sum_x \mu_{\omega,k}^\beta (X_k = x) \big(W_{n-k}^\beta \circ \theta_{k,x}\big)^p \Big| \mathcal{F}_k\Big]\right] \\
&= \mathbb{E}\left[\big(W_k^\beta\big)^p \mathbb{1}_{\{\tau(t)=k\}}\right] \mathbb{E}\left[\big(W_{n-k}^\beta\big)^p\right].
\end{aligned} \tag{4.5}$$

The argument up to this point works for general environments. To continue the argument, we needed the following bound on the first factor, uniformly in $t > 1$ and $p \in [1, 2]$:

$$\mathbb{E}\left[\left(W_k^\beta\right)^p \mathbb{1}_{\{\tau(t)=k\}}\right] \le Ct^p \mathbb{P}(\tau(t) = k). \tag{4.6}$$

In [6], the assumption $\omega_{t,x} \le K$ was used to ensure that $(W_k^\beta)^p \le e^{2\beta K} t^p$ on $\{\tau(t) = k\}$.

Now, to conclude we sum (4.5) over $k \le n$ and apply (4.6) together with the fact that $k \mapsto \mathbb{E}[(W_k^\beta)^p]$ is increasing, which gives

$$\mathbb{E}[(W_n^\beta)^p] \le t^p + Ct^p \mathbb{P}(\tau(t) < \infty)\mathbb{E}[(W_n^\beta)^p].$$

Since t is arbitrary, we can choose the value $t(\frac{1}{4C})$ from (4.2) and then choose $p > 1$ small enough that $t^{p-1} \le 2$. The previous display becomes

$$\mathbb{E}[(W_n^\beta)^p] \le t^p + \frac{1}{2}\mathbb{E}[(W_n^\beta)^p].$$

Since n is arbitrary, we obtain $\sup_n \mathbb{E}[(W_n^\beta)^p] \le 2t^p$ by rearranging.

4.2 Proof of Theorem 2.1

We stress that the assumption of boundedness was only used in one place, namely, (4.6), and it is thus enough to replace this part of the argument. We will do so by using Lemma 3.1.

Proof. Let c_3 and A_3 be the constants obtained by applying Lemma 3.1(i)–(ii). We now bound the left-hand side in (4.6) by considering the cases $W_k^\beta \le A_3 t$ and $W_k^\beta > A_3 t$ separately. The first case is simple:

$$\mathbb{E}\left[\left(W_k^\beta\right)^p \mathbb{1}_{\{\tau(t)=k, W_k^\beta \le A_3 t\}}\right] \le (A_3 t)^p \mathbb{P}\big(\tau(t) = k, W_k^\beta \le A_3 t\big). \tag{4.7}$$

In the second case, we consider the conditional expectation given $\mathcal{F}_{k-1}$ to write

$$\begin{aligned} &\mathbb{E}\left[\left(W_k^\beta\right)^p \mathbb{1}_{\{\tau(t)=k, W_k^\beta > A_3 t\}}\right] \\ &= \mathbb{E}\left[\left(W_{k-1}^\beta\right)^p \mathbb{1}_{\{\tau(t)>k-1\}} \mathbb{E}\left[\left(W_k^\beta / W_{k-1}^\beta\right)^p \mathbb{1}_{\{W_k^\beta > A_3 t\}} \,\middle|\, \mathcal{F}_{k-1}\right]\right]. \end{aligned} \tag{4.8}$$

We further rewrite[1]

$$W_k^\beta / W_{k-1}^\beta = \textstyle\sum_x \alpha_x Y_x \text{ and } \left\{W_k^\beta > A_3 t\right\} = \left\{\sum_x \alpha_x Y_x > A\right\},$$

where $\alpha_x := \mu_{\omega,k-1}^\beta(X_k = x)$, $Y_x := e^{\beta\omega_{k,x} - \lambda(\beta)}$ and $A := A_3 t / W_{k-1}^\beta$. Then, noting that

- $(e^{\beta\omega_{k,x} - \lambda(\beta)})_{x \in \mathbb{Z}^d}$ is independent of $\mathcal{F}_{k-1}$,
- $\mu_{\omega,k-1}^\beta(X_k = x)$ is an $\mathcal{F}_{k-1}$-measurable probability measure on $\mathbb{Z}^d$ and
- $t / W_{k-1}^\beta \ge 1$ on $\{\tau(t) > k - 1\}$,

we can apply Lemma 3.1 under $\mathbb{P}(\cdot \mid \mathcal{F}_{k-1})$ to obtain

$$\mathbb{E}\left[\left(W_k^\beta / W_{k-1}^\beta\right)^p \mathbb{1}_{\{W_k^\beta > A_3 t\}} \,\middle|\, \mathcal{F}_{k-1}\right] \le c_3 \big(A_3 t / W_{k-1}^\beta\big)^p \mathbb{P}\left(W_k^\beta / W_{k-1}^\beta > A_3 t / W_{k-1}^\beta \,\middle|\, \mathcal{F}_{k-1}\right).$$

Substituting this into (4.8) yields

$$\mathbb{E}\left[\left(W_k^\beta\right)^p \mathbb{1}_{\{\tau(t)=k, W_k^\beta > A_3 t\}}\right] \le c_3 A_3^2 t^p \mathbb{P}(\tau(t) = k, W_k^\beta > A_3 t). \tag{4.9}$$

Combining this bound with (4.7), we obtain (4.6).

As explained above, the conclusion now follows from the same argument as in [6]. □

[1] In the following equation, we regard $\mu_{\omega,k-1}^\beta$ as a measure on the space of *infinite* path while the interaction with the environment is restricted to time interval $[0, k-1]$.

5 Discussion on Condition 1

In this section, we discuss Condition 1. First, although it looks natural, it does not hold in general. For example, if ω is supported on $\{k^2\}_{k\in\mathbb{N}}$, then regardless the concrete form of the distribution of ω, we have

$$\mathbb{E}\left[e^{2\beta\omega} \;\middle|\; \omega > k^2\right] = \mathbb{E}\left[e^{2\beta\omega} \;\middle|\; \omega \geq (k+1)^2\right] \geq e^{2\beta(k+1)^2}$$

and hence Condition 1 fails.

Next, we see that Condition 1 is valid under a one-sided tail regularity assumption, which holds under certain upper and lower bounds on the tail.

Proposition 5.1. *Let ω be a real-valued random variable.*

(i) Assume that there exist $K > 0$ and $M > 2\beta$ such that

$$\limsup_{x\to\infty} \sup_{y\geq K} \frac{\mathbb{P}(\omega > x+y)e^{My}}{\mathbb{P}(\omega > x)} < \infty. \tag{5.1}$$

Then Condition 1 holds.

(ii) Assume that there exist $c > 0$ and a convex function f satisfying $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$ such that, for x large enough,

$$c^{-1}e^{-f(x)} \leq \mathbb{P}(\omega > x) \leq ce^{-f(x)}. \tag{5.2}$$

Then Condition 1 holds for all values of β.

(iii) Assume that there exist $c > 0$ and an increasing function f satisfying $f(x+y) \geq f(x)f(y)$ such that, for x large enough,

$$c^{-1}e^{-cf(x)} \leq \mathbb{P}(\omega > x) \leq ce^{-f(x)/c}. \tag{5.3}$$

Then Condition 1 holds for all values of β.

This proposition covers many commonly used distributions.

- If ω has a logarithmically concave Lebesgue density, then $x \mapsto \mathbb{P}(\omega > x)$ is also logarithmically concave (see [7, Theorem 2]) and hence (5.2) holds with $f(x) := -\log \mathbb{P}(\omega > x)$. Note also that $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$ already follows from (1.1). This covers, for example, the Gaussian distribution or the Weibull distribution (with $\mathbb{P}(\omega > x) = ce^{-c'x^\alpha}$ for $\alpha > 1$).
- For the Poisson distribution, it is not hard to check (5.1) directly.
- The (negative) Gumbel distribution, with $\mathbb{P}(\omega > x) = \exp(-e^{(x-c)/c'})$, further satisfies (5.3). More generally, we can take $f(x) = e^{x^\alpha}$ with $\alpha \geq 1$ in (5.3).

Proof. **Part (i)**: We first observe

$$\begin{aligned}
\mathbb{E}[e^{2\beta\omega}\mathbb{1}_{\omega>A}] &= \int_0^\infty \mathbb{P}(e^{2\beta\omega}\mathbb{1}_{\omega>A} > t)\mathrm{d}t \\
&= \int_0^\infty \mathbb{P}\big(\omega > A \vee \log(t)/2\beta\big)\mathrm{d}t \\
&\leq e^{2\beta(A+K)}\mathbb{P}(\omega > A) + \int_{e^{2\beta(A+K)}}^\infty \mathbb{P}(\omega > \log(t)/2\beta)\mathrm{d}t.
\end{aligned} \tag{5.4}$$

To estimate the second term, note that by (5.1), there exist $A_1 > 1$ and $C > 0$ such that, for $y \geq K$, $A > A_1$ and $u > K + A$,

$$\mathbb{P}(\omega > u) \leq C\mathbb{P}(\omega > A)e^{-Mu+MA}.$$

Thus,

$$\begin{aligned}
&\int_{e^{2\beta(A+K)}}^{\infty} \mathbb{P}(\omega > \log(t)/2\beta)\mathrm{d}t \\
&\leq C\mathbb{P}(\omega > A)e^{MA} \int_{e^{2\beta(A+K)}}^{\infty} e^{-M\log(t)/2\beta}\mathrm{d}t \\
&= \frac{C}{M/2\beta - 1}\mathbb{P}(\omega > A)e^{MA}e^{2\beta(A+K)(1-M/2\beta)} \\
&\leq \frac{C}{M/2\beta - 1}\mathbb{P}(\omega > A)e^{2\beta(A+K)},
\end{aligned}$$

where we have used the assumption $M > 2\beta$ to ensure the convergence of the last integral. Together with (5.4), we see that Condition 1 holds with $c_1 := (1 + \frac{C}{M/2\beta-1})e^{2\beta K}$.

For **part (ii)**, it is now enough to verify (5.1). The convexity and the assumption on superlinear growth imply that there exists $x_0 > 0$ such that the right derivative $D_+f(x_0) \geq 3\beta$. Then for $x \geq x_0$ and $y > 0$, we have $f(x+y) - f(x) \geq 3\beta y$ and hence

$$\frac{\mathbb{P}(\omega > x+y)}{\mathbb{P}(\omega > x)} \leq c^2 e^{-(f(x+y)-f(x))} \leq c^2 e^{-3\beta y}.$$

This implies (5.1).

For **part (iii)**, note that by the super-additive theorem there exists $C > 0$ such that $f(x) \geq e^{Cx}$, hence for $y > 2\log(c)/C$ and x large enough,

$$\frac{\mathbb{P}(\omega > x+y)}{\mathbb{P}(\omega > x)} \leq c^2 \exp\Big(- f(x)\Big(\frac{f(x+y)}{cf(x)} - c\Big)\Big) \leq c^2 e^{-f(x)(f(y)/c-c)} \leq c^2 e^{-3\beta y}.$$

This again implies (5.1) and we are done. □

Remark 5.2. In Section 3, we rephrased Condition 1 in terms of the random variable $Y := e^{\beta\omega - \lambda(\beta)}$. Since some authors use this Y as the random potential in the directed polymer model (see, for example, [4, 8]), it might be of interest to rephrase also (5.1), which reads

$$\text{there exist } K > 1 \text{ and } M > 2 \text{ such that } \limsup_{y\to\infty} \sup_{\lambda \geq K} \lambda^M \frac{\mathbb{P}(Y > \lambda y)}{\mathbb{P}(Y > y)} < \infty. \tag{5.5}$$

This is a one-sided regular variation condition. It appears, for example, in [1, Theorem 2.0.1] and inspecting its proof, one can see that (5.5) follows from

$$\text{there exist } K > 1, M > 2 \text{ and } \rho < K^{-M} \text{ such that } \sup_{\lambda\in[K,K^2]} \limsup_{y\to\infty} \frac{\mathbb{P}(Y > \lambda y)}{\mathbb{P}(Y > y)} < \rho. \tag{5.6}$$

There are plenty of distributions that satisfy (5.5). For instance, if there exist $c, C > 0$ and $\gamma > 0$ such that

$$c\exp(-Cy^\gamma) \leq \mathbb{P}(Y > y) \leq C\exp(-cy^\gamma) \tag{5.7}$$

holds for all sufficiently large y, then

$$\frac{\mathbb{P}(Y > \lambda y)}{\mathbb{P}(Y > y)} \leq \frac{C}{c}\exp(-(c\lambda^\gamma - C)y^\gamma),$$

and (5.5) follows. A similar argument applies to the case where y^γ in (5.7) is replaced by $\exp(y^\gamma)$ ($\gamma > 0$) or $\exp(\log^\alpha y)$ ($\alpha > 1$).

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987. MR898871

[2] Francis Comets, Tokuzo Shiga, and Nobuo Yoshida, *Directed polymers in a random environment: path localization and strong disorder*, Bernoulli **9** (2003), no. 4, 705–723. MR1996276

[3] Francis Comets and Nobuo Yoshida, *Directed polymers in random environment are diffusive at weak disorder*, Ann. Probab. **34** (2006), no. 5, 1746–1770. MR2271480

[4] John Z. Imbrie and Thomas Spencer, *Diffusion of directed polymers in a random environment*, J. Statist. Phys. **52** (1988), no. 3–4, 609–626. MR968950

[5] Stefan Junk, *Fluctuations of partition functions of directed polymers in weak disorder beyond the L^2-phase*, *arXiv preprint* arXiv:2202.02907, 2022.

[6] Stefan Junk, *New Characterization of the Weak Disorder Phase of Directed Polymers in Bounded Random Environments*, Comm. Math. Phys. **389** (2022), no. 2, 1087–1097. MR4369727

[7] András Prékopa, *Logarithmic concave measures with application to stochastic programming*, Acta Scientiarum Mathematicarum **32** (1971), 301–316. MR0315079

[8] Timo Seppäläinen, *Scaling for a one-dimensional directed polymer with boundary conditions*, Ann. Probab. **40** (2012), no. 1, 19–73. MR2917766

Acknowledgments. The authors are grateful to the referees for their careful reading and constructive comments. This work was supported by KAKENHI 21K03286, 22H00099, 18H03672 and 23K12984.

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS[1])
- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4], ProjectEuclid[5]
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

[1]LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2]EJMS: Electronic Journal Management System: https://vtex.lt/services/ejms-peer-review/
[3]IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4]BS: Bernoulli Society http://www.bernoulli-society.org/
[5]Project Euclid: https://projecteuclid.org/
[6]IMS Open Access Fund: https://imstat.org/shop/donation/