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A subperiodic tree whose intermediate branching number is strictly less than the lower intermediate growth rate*

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Abstract

We construct an example of a subperiodic tree whose intermediate branching number is strictly less than the lower intermediate growth rate. This answers a question of Amir and Yang (2022) in the negative.

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1 Introduction and main result

There are several ways to measure the branching structure of an infinite locally finite tree. An important and successful one is the branching number introduced by Lyons [5]. For instance the branching number is the critical parameter for Bernoulli percolation and homesick random walk on trees. However the branching number is not so effective for trees with sub-exponential growth. Later Collevecchio, Kious and Sidoravicius [3] introduced a branching-ruin number which works well for trees with polynomial growth. Inspired by these previous work, recently Amir and Yang [1] introduced the intermediate branching number and showed that it is crucial for several probability models on trees with intermediate growth rate.

Our focus here is a special family of infinite locally finite trees—the subperiodic trees. For a subperiodic tree, the branching number actually equals the exponential growth rate—this result is due to Furstenberg [4]; see Theorem 3.8 in [6] for a proof. Amir and Yang [1] then asked whether the corresponding equality holds for the intermediate branching number and the lower intermediate growth rate on subperiodic trees. In the present note we construct an example of a subperiodic tree whose intermediate branching number is strictly less than its lower intermediate growth rate, answering their question in the negative.

1.1 Various branching numbers and growth rates of infinite trees

Suppose $T = (V, E)$ is an infinite locally finite tree with a distinguished vertex o, which will be called the **root** of T. We imagine the tree T as growing upward from the

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root o. For $x, y \in V$, we write $x \leq y$ if x is on the shortest path from o to y; and T^x for the **subtree** of T containing all the vertices y with $y \geq x$. For a vertex $x \in V$ we denote by $|x|$ the graph distance from o to x. For an edge $e \in E$, we write $e = (e^-, e^+)$ where $|e^+| = |e^-| + 1$ and define $|e| = |e^+|$. Write $T_n := \{e \in E\colon |e| = n\}$. Write $B(n) = \{x\colon x \in V, |x| \leq n\}$ for the ball of radius n centered at o.

A **cutset** π separating o and infinity is a set of edges such that every infinite path starting from o must include an edge in π. For instance T_n is a cutset separating o and infinity for every $n \geq 1$. We write $\Pi(T)$ for the collection of cutsets separating o and infinity. The **branching number** of T is defined as

$$\mathrm{br}(T) := \sup \left\{ \lambda > 0\colon \inf_{\pi \in \Pi(T)} \sum_{e \in \pi} \lambda^{-|e|} > 0 \right\}. \tag{1.1}$$

We recommend the readers Chap. 3 of [6] for backgrounds on branching numbers. The **lower exponential growth rate** of T is defined as

$$\underline{\mathrm{gr}}(T) := \liminf_{n \to \infty} |T_n|^{1/n}. \tag{1.2}$$

The **upper exponential growth rate** of T is defined as $\overline{\mathrm{gr}}(T) := \limsup_{n\to\infty} |T_n|^{1/n}$ similarly. Note that $\underline{\mathrm{gr}}(T)$ can be rewritten in a similar form as (1.1):

$$\underline{\mathrm{gr}}(T) = \sup \left\{ \lambda > 0\colon \liminf_{n \to \infty} \sum_{e \in T_n} \lambda^{-|e|} > 0 \right\}$$

and in particular

$$1 \leq \mathrm{br}(T) \leq \underline{\mathrm{gr}}(T).$$

The **branching-ruin number** introduced by Collevecchio, Kious and Sidoravicius [3] is defined as

$$\mathrm{brr}(T) := \sup \left\{ \lambda > 0\colon \inf_{\pi \in \Pi(T)} \sum_{e \in \pi} |e|^{-\lambda} > 0 \right\}, \tag{1.3}$$

where we use the convention of $\sup \emptyset = 0$. This branching-ruin number is a natural way to measure trees with polynomial growth rate and turned out be the critical parameter of some random processes [2] (in particular the once-reinforced random walk [3]). One can define corresponding **lower (upper) polynomial growth rates** by

$$\underline{\mathrm{grr}}(T) := \liminf_{n \to \infty} \frac{\log |T_n|}{\log n} \quad \text{and} \quad \overline{\mathrm{grr}}(T) := \limsup_{n \to \infty} \frac{\log |T_n|}{\log n}. \tag{1.4}$$

Note that

$$\underline{\mathrm{grr}}(T) = \sup \left\{ \lambda > 0\colon \liminf_{n \to \infty} \sum_{e \in T_n} |e|^{-\lambda} > 0 \right\}.$$

and in particular $\mathrm{brr}(T) \leq \underline{\mathrm{grr}}(T)$.

Recently Amir and Yang [1] introduced the **intermediate branching number**

$$\mathrm{Ibr}(T) := \sup \left\{ \lambda > 0\colon \inf_{\pi \in \Pi(T)} \sum_{e \in \pi} \exp\left(-|e|^{\lambda} \right) > 0 \right\} \tag{1.5}$$

and the **lower (upper) intermediate growth rates**

$$\underline{\mathrm{Igr}}(T) := \liminf_{n \to \infty} \frac{\log \log |T_n|}{\log n} \quad \text{and} \quad \overline{\mathrm{Igr}}(T) := \limsup_{n \to \infty} \frac{\log \log |T_n|}{\log n}. \tag{1.6}$$

Similarly,

$$\mathrm{lbr}(T) \le \underline{\mathrm{lgr}}(T) = \sup\left\{\lambda > 0\colon \liminf_{n\to\infty} \sum_{e\in T_n} \exp\left(-|e|^{\lambda}\right) > 0\right\}.$$

Amir and Yang [1] proved that the intermediate branching number is the critical parameter for certain random walk, percolation and firefighting problems on trees with intermediate growth, where a tree T was said to be of **intermediate (stretched exponential) growth** if $0 < \underline{\mathrm{lgr}}(T) \le \overline{\mathrm{lgr}}(T) < 1$.

We remark that these numbers $\mathrm{br}(T), \underline{\mathrm{gr}}(T), \overline{\mathrm{gr}}(T), \mathrm{brr}(T), \underline{\mathrm{grr}}(T), \overline{\mathrm{grr}}(T), \mathrm{lbr}(T), \underline{\mathrm{lgr}}(T)$ and $\overline{\mathrm{lgr}}(T)$ do not depend on the choice of the root of T.

1.2 Subperiodic trees

We first recall the definition of subperiodic trees from p 82 of [6]; see Example 3.6 and 3.7 there for some examples of subperiodic trees.

Definition 1.1. *Let $N \in \{0, 1, 2, 3, \ldots\}$. An infinite tree T is called N**-periodic** (resp., N-**subperiodic**) if $\forall\, x \in T$ there exists an adjacency-preserving bijection (resp. injection) $f : T^x \to T^{f(x)}$ with $|f(x)| \le N$. A tree is **periodic** (resp. **subperiodic**) if there is some N for which it is N-periodic (resp., N-subperiodic).*

As mentioned earlier $\mathrm{br}(T) = \underline{\mathrm{gr}}(T) = \overline{\mathrm{gr}}(T)$ for any subperiodic tree T ([6, Theorem 3.8]). Amir and Yang noticed that there exist subperiodic trees such that $\underline{\mathrm{lgr}}(T) < \overline{\mathrm{lgr}}(T)$ (see Sect. 4.1 of [1]) and asked[1] whether $\mathrm{lbr}(T) = \underline{\mathrm{lgr}}(T)$ for subperiodic trees with intermediate growth rate. Our main result gives a negative answer to their question.

Theorem 1.2. *There exists a subperiodic tree T with intermediate growth rate and*

$$\mathrm{lbr}(T) < \underline{\mathrm{lgr}}(T).$$

2 Proof of the main result

We will prove Theorem 1.2 via a concrete example (see Example 2.4).

2.1 Coding by trees

Our example will be a subtree of the 3**-ary tree** $\mathbb{T}_3$ and we view $\mathbb{T}_3$ as a **labelled** tree with the root labelled as $\emptyset$, the three children of the root labelled $0, 1, 2$ respectively from left to right, and so on. Write $\mathscr{D}(\mathbb{T}_3)$ for the set of infinite labelled subtrees of $\mathbb{T}_3$ which contain the root and have no leaf and write $\mathscr{R}(\mathbb{T}_3)$ for the set of labelled rays starting from the root. In particular $\mathscr{R}(\mathbb{T}_3) \subset \mathscr{D}(\mathbb{T}_3)$.

For each element $a = (a_1, a_2, a_3, \ldots) \in \{0, 1, 2\}^{\mathbb{N}}$, we associate it with a ray $\Phi(a) \in \mathscr{R}(\mathbb{T}_3)$ with the $(n+1)$-th vertex on the ray labelled as $a_1 a_2 \cdots a_n$. (The first vertex is just the root labelled as $\emptyset$.) Obviously Φ is a bijection between $\{0, 1, 2\}^{\mathbb{N}}$ and $\mathscr{R}(\mathbb{T}_3)$. We now extend Φ as a mapping from all nonempty subsets of $\{0, 1, 2\}^{\mathbb{N}}$ to $\mathscr{D}(\mathbb{T}_3)$.

Definition 2.1 (Coding by trees)**.** *For a nonempty subset E of $\{0, 1, 2\}^{\mathbb{N}}$, the tree $\Phi(E) \in \mathscr{D}(\mathbb{T}_3)$ is defined as the union of the rays (each ray is viewed as a labelled subtree of $\mathbb{T}_3$) $\Phi(E) = \bigcup_{x\in E} \Phi(x)$, where the union means the vertex set of $\Phi(E)$ is the union of the vertex set of $\Phi(x)$ and the same for the edge set.*

Remark 2.2. One can define a natural metric d on $\{0, 1, 2\}^{\mathbb{N}}$: the distance between two elements $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in \{0, 1, 2\}^{\mathbb{N}}$ is given by

$$d(x, y) := \frac{1}{e^k}, \quad \text{where } k = k(x, y) = \inf\{i\colon x_i \ne y_i\}.$$

[1] See (2.12) on page 4 of version 3 of the paper [1] on arXiv.

The coding by trees in Definition 2.1 comes from the canonical coding of closed subsets of the interval $[0,1]$ by trees; see Sect. 1.10 and 15.2 of [6] for background. We simply replace $[0,1]$ by the metric space $\left(\{0,1,2\}^{\mathbb{N}}, d\,\right)$ for convenience.

It is straightforward to verify that

- For each nonempty subset $E \subset \{0,1,2\}^{\mathbb{N}}$ and its closure $\overline{E}$ in the metric space $\left(\{0,1,2\}^{\mathbb{N}}, d\,\right)$, one has that $\Phi(E) = \Phi\big(\overline{E}\big)$.
- Moreover the map Φ is a bijection if its domain is restricted to the collection of all nonempty **closed** subsets of $\{0,1,2\}^{\mathbb{N}}$.

We also define the **shift map** $\mathcal{S} : \{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{N}}$ by

$$\mathcal{S}\big((a_1, a_2, a_3, \ldots)\big) = (a_2, a_3, a_4, \ldots).$$

The following observation is a rephrasing of Example 3.7 in [6] in the case $b = 3$ and it is crucial for our construction later.

Observation 2.3. If a nonempty closed subset $E \subset \{0,1,2\}^{\mathbb{N}}$ is invariant under the shift map in the sense that $\mathcal{S}(E) \subset E$, then the tree $\Phi(E)$ is 0-subperiodic.

2.2 The construction of our example

We first review the 1-3 tree $T_{1,3}$ [6, Example 1.2]: the root has two children; and $|T_n| = 2^n$; and for each $n \geq 1$, the left half vertices at distance n from the root will each have only 1 child, the right half will each have 3 children. We view $T_{1,3}$ as a labelled subtree of $\mathbb{T}_3$ according to the following **labeling rule**: the root is labelled as $\emptyset$ and if a vertex with label $a_1 a_2 \cdots a_n$ has k children, then its k children are labelled as $a_1 a_2 \cdots a_n 0, \ldots, a_1 a_2 \cdots a_n (k-1)$ respectively from left to right. See Fig. 1 for $T_{1,3}$ and its labeling.

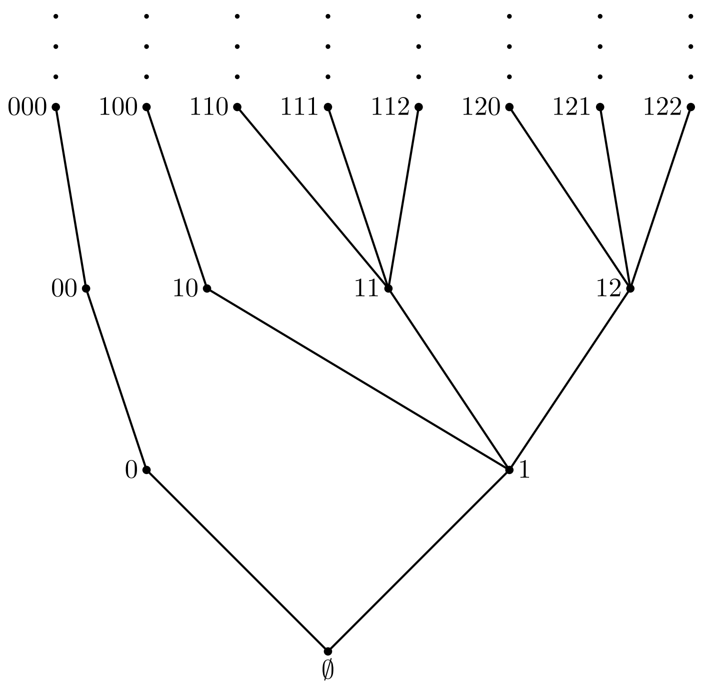

Figure 1: The 1-3 tree $T_{1,3}$ and its labeling.

Example 2.4. Let T_0 be the tree obtained by replacing each edge e of the 1-3 tree $T_{1,3}$ by a path of length $|e|$ and view it as a subtree of $\mathbb{T}_3$ labelled according to the labeling rule we used for $T_{1,3}$ (see Fig. 2). As already noted by Amir and Yang [1], the tree T_0 satisfies

$$\mathrm{Ibr}(T_0) = 0 \quad \text{and} \quad \underline{\mathrm{Igr}}(T_0) = \frac{1}{2}. \tag{2.1}$$

However T_0 is not subperiodic.

Let E_0 be the closed subset of $\{0,1,2\}^{\mathbb{N}}$ such that $\Phi(E) = T_0$. Define $E_j = \mathcal{S}(E_{j-1})$ for $j \geq 1$ and let $\widetilde{E} = \bigcup_{j=0}^{\infty} E_j$. Our example is just the tree $\widetilde{T} := \Phi\big(\widetilde{E}\big)$.

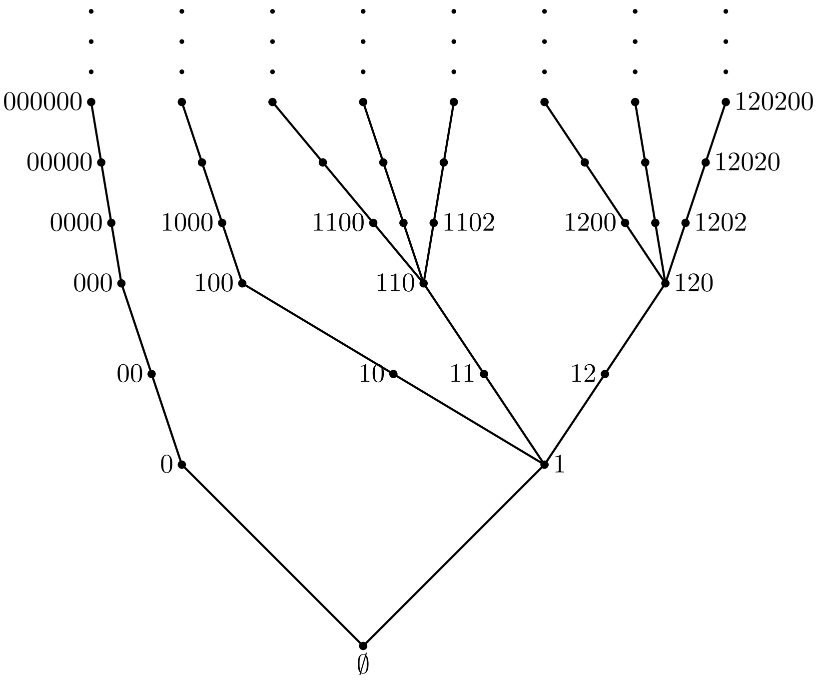

Figure 2: The tree T_0 and its labeling.

Recall that $\mathscr{D}(\mathbb{T}_3)$ denotes the set of infinite labelled subtrees of $\mathbb{T}_3$ which contain the root and have no leaf. For a vertex $v \in V(T_0)$ labelled as $a_1 a_2 \cdots a_n$, we will view the subtree T_0^v as a labelled subtree of $\mathbb{T}_3$ rooted at $\emptyset$, i.e., view it as the tree

$$\Phi\big(\mathcal{S}^{\circ n}\{x = (x_1, x_2, x_3, \ldots)\colon x \in E_0, x_i = a_i \text{ for } i = 1, \ldots, n\}\big) \in \mathscr{D}(\mathbb{T}_3).$$

Since $E_n = \bigcup_{v \in V(T_0), |v| = n} \mathcal{S}^{\circ n}\{x = (x_1, x_2, x_3, \ldots)\colon x \in E_0, v \text{ is labelled as } x_1 x_2 \cdots x_n\}$, we have the following equivalent description of $\widetilde{T}$:

Observation 2.5. As labelled subtrees of $\mathbb{T}_3$, the tree $\widetilde{T}$ is just the union of T_0^v over all $v \in V(T_0)$.

2.3 The intermediate branching number and the intermediate growth rate of our example

By construction the set $\widetilde{E}$ is invariant under the shift map. Thus by Observation 2.3 the tree $\widetilde{T} = \Phi\big(\widetilde{E}\big)$ is subperiodic. We will show that $0 = \mathrm{Ibr}\big(\widetilde{T}\big) < \underline{\mathrm{Igr}}\big(\widetilde{T}\big) = \frac{1}{2}$ which then proves Theorem 1.2.

Proposition 2.6. *For the tree $\widetilde{T} = \Phi\big(\widetilde{E}\big)$ constructed in Example 2.4, one has that*

$$\underline{\mathrm{Igr}}\big(\widetilde{T}\big) = \overline{\mathrm{Igr}}\big(\widetilde{T}\big) = \frac{1}{2} \quad \text{and} \quad \mathrm{Ibr}\big(\widetilde{T}\big) = 0.$$

Proof. First of all since $T_0 = \Phi(E_0)$ is a subtree of $\widetilde{T} = \Phi\big(\widetilde{E}\big)$, one has that

$$\overline{\mathrm{Igr}}\left(\widetilde{T}\right) \geq \underline{\mathrm{Igr}}\left(\widetilde{T}\right) \geq \underline{\mathrm{Igr}}(T_0) \overset{(2.1)}{=} \frac{1}{2}.$$

On the other hand, note that $\big|\widetilde{T}_n\big|$ equals the cardinality of the set $\{(x_1, \ldots, x_n)\colon x = (x_1, x_2, \ldots) \in \widetilde{E}\}$—the first n-bits of $\widetilde{E}$. Also observe that a ray γ in $T_{1,3}$ coding the sequence $(a_1, a_2, a_3, \ldots)$ becomes a ray γ' in T_0 coding the sequence

$$(a_1, a_2, 0, a_3, 0, 0, a_4, 0, 0, 0, a_5, 0, 0, 0, 0, a_6, 0, \ldots).$$

Hence by our construction an element $a \in \widetilde{E}$ always has the form

$$a = \big(\underbrace{0, \ldots, 0}_{m}, a_j, \underbrace{0, \ldots, 0}_{=j-1}, a_{j+1}, \underbrace{0, \ldots, 0}_{=j}, a_{j+2}, 0, \ldots\big), \tag{2.2}$$

where $(a_1, a_2, a_3, \ldots) \in \Phi^{-1}(T_{1,3})$ and $m \leq \max(j-2, 0)$. Note that there exists a constant $c > 0$ such that there are at most $c\sqrt{n} + 1$ nontrivial entries $a_j, a_{j+1}, \ldots, a_{j+c\sqrt{n}}$ in the first n-bits of a. If $j \geq n+1$, then there is at most one nonzero entry in the first n-bits and this would contribute at most $2n+1$ to the set $\{(x_1, \ldots, x_n)\colon x = (x_1, x_2, \ldots) \in \widetilde{E}\}$. If $j \leq n$, then there are at most $\max(n-2, 0) \leq n$ choices for m—the number of zeroes before a_j; once m and j are fixed, the positions of $a_j, a_{j+1}, \ldots, a_{j+c\sqrt{n}}$ are fixed and each element of $\{a_j, a_{j+1}, \ldots, a_{j+c\sqrt{n}}\}$ has at most 3 choices, hence this contributes at most $n^2 * 3^{c\sqrt{n}+1}$ to the set $\{(x_1, \ldots, x_n)\colon x = (x_1, x_2, \ldots) \in \widetilde{E}\}$. In sum we have $\big|\widetilde{T}_n\big| \leq 3^{C\sqrt{n}}$ for some constant $C > 0$. Therefore one has the other direction

$$\underline{\mathrm{Igr}}\left(\widetilde{T}\right) \leq \overline{\mathrm{Igr}}\left(\widetilde{T}\right) \leq \frac{1}{2}.$$

Next we proceed to show that $\mathrm{Ibr}\left(\widetilde{T}\right) = 0$. Fixing an arbitrary $\lambda > 0$, we will show that for any $\varepsilon > 0$ there exists a cutset π of $\widetilde{T}$ such that

$$\sum_{e \in \pi} \exp\big(-|e|^{\lambda}\big) \leq 2\varepsilon. \tag{2.3}$$

Since $\mathrm{Ibr}(T_0) \overset{(2.1)}{=} 0$, one has $\mathrm{Ibr}(T_0^v) = 0$ for any $v \in V(T_0)$. In particular one can choose cutsets π_v for T_0^v (viewed as a subtree of $\mathbb{T}_3$ rooted at $\emptyset$) such that

$$\sum_{v \in V(T_0)} \sum_{e \in \pi_v} \exp\big(-|e|^{\lambda}\big) \leq \varepsilon.$$

Since $\widetilde{T}$ is the union of T_0^v over $v \in V(T_0)$ (Observation 2.5), one might hope the set $\bigcup_{v \in V(T_0)} \pi_v$ is a cutset of $\widetilde{T}$. But it might not be the case since there might exist a ray γ in $\widetilde{T}$ such that its edges come from $T_0^{v_i}$ for infinitely many different v_i's and γ is not blocked from infinity by $\bigcup_{v \in V(T_0)} \pi_v$. To rescue this, we add some additional edges in the following way. Choose $N = N(\lambda, \varepsilon)$ large enough so that $9N\exp(-N^{\lambda}) \leq \varepsilon$. Let β be the collection of all edges in $\widetilde{T}_{N+1}$ with the form

$$(v, vj)\colon v = (v_1 v_2 \cdots v_N) \text{ with at most one nonzero entry and } j = 0, 1, 2.$$

In particular

$$\sum_{e \in \beta} \exp\big(-|e|^{\lambda}\big) \leq 9N\exp(-N^{\lambda}) \leq \varepsilon.$$

Now we set

$$\pi = \Big(\bigcup_{v \in V(T_0)} \pi_v\Big) \cup \beta.$$

and claim that π is a cutset of $\widetilde{T}$. In fact since $\widetilde{T}$ is just the union of T_0^v over all $v \in V(T_0)$, we can choose $M \geq 100N^2$ large enough so that all the edges e of $\widetilde{T}$ with $|e| \leq N$ appear in some T_0^v with $|v| \leq M$. Now if a ray γ of $\widetilde{T}$ does not use any edge outside $\bigcup_{v \in V(T_0), |v| \leq M} T_0^v$, then there must exist some $v \in V(T_0)$ with $|v| \leq M$ such that γ is just a ray in T_0^v. Hence in this case γ has a nonempty intersection with π_v. Otherwise γ must use some edge e' of $\widetilde{T}$ which is not in the union $\bigcup_{v \in V(T_0), |v| \leq M} T_0^v$. By our choice of M, one must have $|e'| > N$ and e' is coming from some T_0^v with $|v| > M \geq 100N^2$. For such a vertex v, in the first N levels of T_0^v there is at most one vertex with three children because of the long pieces of zeroes (see (2.2) and Fig. 2). Therefore the edge e' must be a descendant of some edge from the set β and so γ has a nonempty intersection with β. Hence π is a cutset of $\widetilde{T}$.

By our choice of π_v and β the cutset π satisfies (2.3). By (2.3) one obtains that $\mathrm{Ibr}\left(\widetilde{T}\right) \leq \lambda$. Since this is true for any $\lambda > 0$ one has that $\mathrm{Ibr}\left(\widetilde{T}\right) = 0$. □

3 Concluding remarks

In the construction of T_0 we replace an edge e by a path of length $f(|e|)$ where the function $f : \mathbb{N} \to \mathbb{N}$ is given by $f(x) = x$. If we use some other increasing functions, say $f(x) = \lceil x^s \rceil$ with $s \in (0, \infty)$, then we can obtain a family of subperiodic trees using the procedure in Example 2.4 so that for each $\alpha \in (0,1)$ there are some trees T in the family with the property that $0 = \mathrm{Ibr}(T) < \underline{\mathrm{Igr}}(T) = \alpha$.

We also note that there exist periodic trees T with polynomial growth that satisfy $\mathrm{brr}(T) < \underline{\mathrm{grr}}(T)$. For instance consider the following **lexicographically minimal spanning tree** of $\mathbb{Z}^2$ illustrated in Fig. 3; see Sect. 3.4 in [6] for definitions of Cayley graphs and their lexicographically minimal spanning trees. We don't know whether there exists a Cayley graph G of a finitely generated countable group with intermediate growth and a lexicographically minimal spanning tree T of G such that $\mathrm{Ibr}(T) < \underline{\mathrm{Igr}}(T)$.

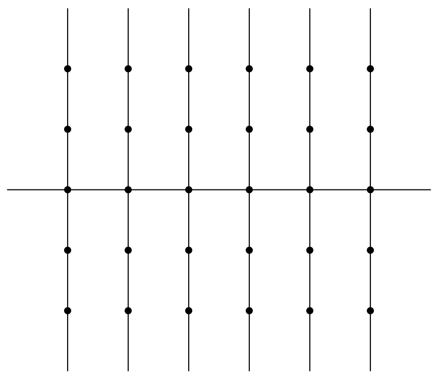

Figure 3: A lexicographically minimal spanning tree of $\mathbb{Z}^2$.

However there are no periodic trees with intermediate growth rate.

Proposition 3.1. *Suppose T is an infinite periodic tree. Then either $\mathrm{br}(T) > 1$ or there exists an integer $d \geq 1$ such that $\left|B(n)\right| = \Theta(n^d)$. Here $\left|B(n)\right| = \Theta(n^d)$ means that the ratio $\left|B(n)\right|/n^d$ is bounded away from zero and infinity.*

Proof. We give a sketch here and leave the details to interested readers.

First of all, the periodic tree T is the directed cover of some finite directed graph $G = (V, E)$ based at some vertex $x_0 \in V$; see p 82-83 in [6] for a proof of this fact.

Let $C_1, \ldots, C_m$ be the strongly connected components of G (if for a vertex v there is no directed path from v to itself, then we say v does not belong to any strongly connected component). If there exist some C_i and some $v \in V(C_i)$ such that v has at least two out-going edges in C_i, then it is easy to see $\mathrm{br}(T) > 1$. Otherwise, each C_i is either a single vertex with a self-loop, or it is a directed cycle. In this case one can prove $\left|B(n)\right| = \Theta(n^d)$ by induction on the size of $V(G)$ (Exercise 3.30 in [6] would be a good warm-up). We omit the details of the induction and just point out that in this case

$$d = \max\{C(\gamma)\colon \gamma \text{ is a self-avoiding directed path in } G \text{ starting from } x_0\},$$

where $C(\gamma)$ is the number of strongly connected components visited by γ. □

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